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Article

On the Microscopic Perspective of Black Branes Thermodynamic Geometry

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Abstract: We study thermodynamic state-space geometry of the black holes in string theory and M-theory. For a large number of microstates, we analyze the intrinsic state-space geometry for (i) extremal and non-extremal black branes in string theory, (ii) multi-centered black brane configurations, (iv) small black holes with fractional branes, and (v) fuzzy rings in the setup of Mathur's fuzzballs and subensemble theory. We extend our analysis for the black brane foams and bubbling black brane solutions in M-theory. We discuss the nature of state-space correlations of various black brane configurations, and show that the notion of state-space manifolds describes the associated coarse-grained interactions of the corresponding microscopic CFT data.

Keywords: black hole physics; thermodynamic state-space geometry; higher-dimensional black objects; string theory; M-theory

Classification: **PACS** 04.70.-s Physics of black holes; 04.70.Bw Classical black holes; 04.70.Dy Quantum aspects of black holes, evaporation, thermodynamics; 04.50.Gh Higher-dimensional black holes, black strings, and related objects.

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1. Introduction

String theory has made significant progress towards the understanding of the microstates for the extremal and near extremal black branes. In particular, we can count the microstates for certain black brane configuration that carries the same charges and energy as the black brane with an entropy S_{micro} that equals the Bekenstein-Hawking entropy, $S_{BH} = \frac{A}{4G}$ of the corresponding black brane [1,2]. The concept of AdS/CFT correspondence [3–5] suggests that S_{micro} counts the states of the branes in dual conformal field theory description, in correspondence to the gravitational description. In fact, the conventional understanding of the entropy is based on coarse graining of large number of microstates. Thus, such an ensemble of equilibrium microstates turns out to be a crucial ingredient in the realization of thermodynamic configuration.

The thermodynamics of black holes, black rings, black strings and black brane solutions in general plays the central role in understanding certain properties of string theory and M-theory [1,6,7,47–50]. In particular, the AdS/CFT correspondence involves equality of partition functions of a string theory (or M-theory) on a space with asymptotics of form $AdS_n \times K$. For instance, the compact space K can be thought of as the Calabi-Yau manifold, and the holographic dual conformal field theory to be defined on the conformal boundary of AdS_n [51–56]. The present paper concentrates on the macroscopic-microscopic aspects of a class of consistent black brane solutions and discusses the underlying state-space geometry arising from the entropy obtained as certain bound states in the theory of either D-branes or M-branes. Our study is therefore well suited for the original application of AdS/CFT correspondence and phase-transitions, if any. As explained in the next section, the state-space manifold of an underlying equilibrium black brane configuration is characterized by a set of invariant electric/ magnetic charges and possible angular momenta.

Thermodynamic state-space geometry is one of the main tools in studying statistical structures of black holes or black brane solutions, see for instance [57,58,62–64,66–72]. In particular, the thermodynamic geometry analyzes the equilibrium microstates of such black branes. In fact, an idea of interactions or phase transitions present in associated state-space configuration may be further analyzed from the perspective of macroscopic-microscopic duality. In this article, we shall consider various specific cases of extremal and non-extremal black holes, D-brane system like $D_6D_4D_2D_0$ black holes, fuzzy black rings and bubbling black brane configurations for three charged foamed black branes. We find that such thermodynamic systems have a regular state-space geometry which turns out to be non-degenerate, and corresponds in general to an interacting statistical systems, whenever there exists a non-zero state-space scalar curvature. Recently, we have explored the complete set of non-trivial relative state-space correlation functions for string theory and M-theory black brane configurations [60,61]. Similar considerations are valid over the black holes in general relativity [67–70], attractor black holes [6–9,13,16] and Legendre transformed finite parameter chemical configurations [57,58], quantum field theory and QCD backgrounds [59].

The state-space geometry may further be shown to exhibit certain associated microscopic properties and, in turn, one may analyze the geometric nature of the correlation functions and correlation volume of the boundary dual conformal field theory. This indicates an accordance with the AdS/CFT correspondence. This is because the components of the state-space metric tensor that are related to the set of two point correlation functions of the boundary conformal field theory. This follows from the fact that the thermodynamic metric tensor deals with Gaussian fluctuations in the state-space of the brane configuration. We may thus explain how the two point correlation functions of certain microstates characterizing a black brane behave on the state-space. In particular, it is interesting to investigate whether they are regular or singular functions in the state-space. In the case of singular state-space scalar curvature, we have analyzed the nature of the singularities present in the underlying state-space manifold. For example, we have shown that there may exists certain critical points or critical (hyper)-surfaces on which the state-space scalar curvature diverges.

In this article, we shall explicitly analyze the state-space configurations for, (i) the two and three charge extremal black holes, (ii) the four and six charge non-extremal black branes arising from the string theory solutions one by one. We further extend our analysis for the D_6-D_4-D_2-D_0 multi-centered black branes, fractional small black branes and two charge rotating fuzzy rings in the setup of Mathur's fuzzballs. The state-space pair correlations and nature of stabilities will be investigated for three charged bubbling black brane foams, and thereby the M-theory solutions will be brought into the considerations. Interestingly, all these black brane systems are characterized by the charges (electric and magnetic), masses, and angular momenta. Thus, the state-space configurations are described in terms of the correlations in the parameters.

Before we move on with the analysis, we shall briefly describe the physical aspects of the parameters forming the state-space geometry. It is known that the charged extremal black holes in $D = 4, \mathcal{N} = 2$ supergravity may be characterized by certain electric and magnetic charges q_J and p^I arising from usual flux integrals of the field strength tensors and their Poincaré duals. On the other hand, the scalar fields arising from the compactification of either string theory or M-theory serve as the moduli which in fact parameterize the compact internal manifold. The extremal charged black hole solutions are BPS solitons, which interpolate between asymptotic infinity and the near horizon geometry. The spherical symmetry in turn determines this interpolation to be a radial evolution of the scalar moduli which encodes the consequent changes in the underlying internal compact manifold. Moreover, one has flat Minkowskian manifold at the asymptotic infinity and thus the asymptotic ADM mass is described by the associated complex central extension Z_∞ of the $\mathcal{N} = 2$ supersymmetry algebra. For given scalar moduli going to certain arbitrary values, it tuns out [6–9] that the ADM mass, $M(p, q, \phi^a) = |Z_\infty|$.

In such cases, the near horizon geometry of an extremal black hole turns out to be an $AdS_2 \times S^2$ manifold which describes the concerned Bertotti-Robinson vacuum. In D=4, the area of the horizon A and hence the macroscopic entropy [6] is given as $S_{macro} = \pi |Z_{fix}|^2$. However, the radial variation of the moduli is described by a damped geodesic equation which flows to an attractive fixed point at the horizon. Thus, they may solely be determined by the charges carried by the extremal black hole. Such attractors may further be studied from the perspective of criticality of black hole effective potential in $\mathcal{N} = 2, D = 4$ supergravities coupled to n_V abelian vector multiplets for an asymptotically flat extremal black hole background, which in turn is described by $(2n_V + 2)$-dyonic charges and the n_V-complex scalar fields parameterizing the n_V-dimensional special Kähler manifold, see [10–15] and references therein for further details.

In the study of attractor horizon geometries of $D = 4$ dyonic extremal black holes in certain supergravity theories, it turns out that such a hole have a non-vanishing Bekenstein-Hawking entropy, see for example [16,17]. In these articles, the authors have outlined such analysis from 1/2-BPS and non-BPS (non-supersymmetric) attractors with non-vanishing central charge in the context of the $\mathcal{N}=2, D=4$ ungauged supergravity coupled to the n_V-number of abelian vector multiplets. Further, they have demonstrated a complete classification of the orbits in the symplectic representation of the classical U-duality group, as well as that of the moduli spaces associated with the non-BPS attractors in the context of symmetric special Kähler geometries. Furthermore, the case of non-extremal black branes may be considered similarly by adding the corresponding antibranes to the extremal black brane configurations. Thus, the computation of associated brane entropy may either be performed in the microscopic framework or by macroscopic considerations. References [2,18] show a match between the counting entropy S_{micro} and attractor horizon entropy S_{macro} for a given set of brane charges and some total mass added to the extremal configuration.

On other hand, the role of multi-centered black holes in the context of D-brane systems has recently been discussed in [19,20]. In these developments the authors have considered a special class of multi-centered black brane molecular configurations for which there is single centered black hole solution whose degeneracy is predicted by an exact formula to be non-zero, and thus showed how such states of D-branes may be represented as 2-centered configurations. Here, we shall focus our attention on a special class of black brane configurations, which may be characterized by the bound states of D_6, D_4, D_2 and D_0-branes. However, it turns out that the concerned entropies for each of these two configurations characterized by the charge vector considered in [19,20] are of different orders, and hence their role in producing the correct contribution(s) to the degeneracy formulas. This in turn implies that the nature of the correlations present in the state-space is not manifest.

Thus, we intend to analyze an associated intrinsic Riemannian geometric model, whose metric in the entropy representation may be defined as the negative Hessian matrix of the entropy with respect to the extensive D-brane charges. In turn, one can calculate the state-space correlation functions and state-space correlation volume directly from the concerned entropy, with such two centered black brane configurations, by using standard intrinsic geometric techniques. Thus, one may compare it with that of the standard leading order single centered black brane configuration. We find here that the associated geometric results are in perfect agreement with the fact that the multi-centered configurations are more weakly interacting than the most entropic minimal energy single centered supergravity configurations.

Next, we extend our study of the state-space geometry of a given D-brane system into the perspective of the fractionation of branes which in turn deals with the counting of certain chiral primaries. For simplicity, we shall consider the example of two charge extremal small black holes in type IIA string theory compactified on $T^2 \times \mathcal{M}$, where $\mathcal{M}$ can be either K_3 or T^4, see for examples [21–24]. The microscopic understanding of entropy of the black holes in string theory has been estimated in recent times at least for the case of two charge extremal black holes [25–27]. Thus, there has been remarkable progress in the understanding of black holes having zero horizon area in supergravity approximation. However, they have non-zero statistical entropy from independent string state counting arguments.

Such black hole space-time geometries are expected to develop non-zero horizon sizes after including certain higher derivative corrections and in turn these black holes are referred as small black holes. The statistical entropy for a supersymmetric two charge black hole coming from counting the bound state degeneracy has been checked to agree with that calculated by using Wald's macroscopic entropy [28–30] in higher-derivative supergravities [24]. The most studied interesting extremal black hole solutions are the small black holes, which are the ones made from the long heterotic string wrapping on a circle carrying winding charge w and momentum charge n. For simplicity, we consider two charge small black holes in the duality frame which is described by D_0D_4-brane configurations in the type-IIA string theory compactified on $K_3 \times T^2$.

The microscopic understanding of this system is governed by superconformal quantum mechanics describing $D0$-branes in the $AdS_2 \times S^2 \times CY_3$ attractor geometry of a black hole in type IIA with $D4$-branes on the CY_3, see for details [31]. This quantum mechanics has a class of chiral primaries which may be identified as the microstates of the black hole. It was shown in [31] that the microscopic entropy for a small black hole with vanishing triple intersection number could be reproduced from the leading order asymptotic degeneracy of the associated chiral primaries. Further from the microscopic perspective as shown in [25], the degeneracy in counting the microstates arises from the combinatorics of the total N units of D_0-brane charge that splits into certain k-small clusters having n_i units of the D_0-brane charge such that on each cluster the sum $\sum_{i=1}^{k} n_i = N$ remains true. This in turn corresponds to the wrapped D_2-branes residing on either of the $24p$ bosonic chiral primary states. It is thus instructive to ask: what are the possible interactions among the brane microstates in a given k-cluster which are arising from the degeneracy of the two charged extremal D_0D_4 small black holes? In particular, we shall investigate such interactions from the perspective of the brane fractionation in any finite number of arbitrary clusters and thus shed light on the geometric perspective of the counting chiral primaries associated with the two charge extremal small black holes.

In order to understand how to count the microstates of a brane, Mathur [32] proposed an idea of the brane fractionation which led to the fuzzball structure of black brane interior. He has further shown [32] that the different vibration states of the string make different fuzzballs which do not have a singularity or a horizon. Notice that Kostas Skenderis and Marika Taylor recently wrote a review article for Physics Reports, summarizing the status of the fuzzball proposal [33] and in a series of earlier papers completed the set of 2 charge fuzzball solutions ([34] and [35]: Lunin and Mathur only had a subset of 2 charge solutions) and used AdS/CFT technology to propose a precise dictionary between geometries and black hole microstates [35–37]. However, if we draw a bounding sphere corresponding to the size of a typical fuzzball then the area A of the sphere satisfies $\frac{A}{G} \sim \sqrt{n_1 n_p} \sim S_{micro}$ [38]. In this consideration, the information contained inside a black hole is distributed throughout the interior of horizon sized fuzzballs whose vicinity is not empty of information. However, there exists radiation from an excited fuzzball leaving the surface of such quantum fuzzball. Thus, the fuzzballs carry information of the states of the black hole. In fact, in any string theory construction, the bound states with certain momentum cause the string to spread out over a horizon sized transverse region which in turn generates the horizon sized fuzzball having sizable quantum fluctuations.

Yet another important view point follows from the subensemble theory. One may carry out state-space investigations for the rotating D1-D5 system by considering a family of subensembles of states for which each boundary area of the fuzzball satisfies a Bekenstein type relation with an enclosed entropy [39]. To explore further the relation between state-space interaction in a subensemble with an enclosed entropy, we may restrict our states to a given subensemble. The simplest case of this interaction may be described as follows. Let us now consider a string with given momentum charge.Then the corresponding string states can be either in high wave numbers with small transverse amplitude, or in low wave numbers with large transverse amplitude. Thus the entropy reaches an optimal value for some wave number and amplitude, and such a typical state has infinite throat horizon, which corresponds to the classical picture of the black hole configuration. However, if we restrict ourselves to the subset of states which have certain wave number higher than the optimal value and an amplitude of transverse vibration smaller than the amplitude of the generic states. Then the microscopic entropy of the states restricted to this smaller amplitude is smaller than the standard microscopic entropy by a factor M, *i.e.*, the number of the subensembles of the considered ensemble.

In this case, the string in the space-time description lives inside a smaller ball, and thus the space-time geometry before capping off has a narrower and deeper throat. It has been shown that the boundary of the fuzzball region in leading order computations also decreases exactly by the same factor M, which further remains the same for the case of with or without rotation. These subensembles with smaller entropies have their fuzzball boundaries at deeper points in the throat with smaller transverse areas. Thus, we find weaker state-space interactions for two charge extremal systems with an angular momentum whose boundary is given by certain fuzzy regions, with the associated area satisfying $\frac{A}{G} \sim S_{micro}$ [39]. It is just for simplicity that we consider the state-space of two charged rotating system. But there are several evidences for three charge as well as four charge systems that there exists cap structures analogous to that of the two charge systems. Thus, one can easily describe the state-space manifolds associated with such black branes in any subensemble under consideration.

As a final example, one of the bubbling solutions whose state-space we offer to analyze is the case of $D = 11$ black brane solutions with given three charges having required amount of supersymmetry [44]. In certain cases [40–43], it is known further that such solutions describe three charged BPS-black holes. As the next step towards this objective, it is instructive to explore the state-space geometry for black brane foam configurations. The example that we shall consider is the case of the M-theory compactified on T^6. This configuration possesses far separated unit Gibbons-Hawking charges with vanishing total Gibbons-Hawking charge. It is important to investigate whether the state-space metric tensor is non-degenerate, for such configurations. Thus, we wish to establish the behavior of associated scalar curvature of the state-space manifold for the above mentioned, well separated Gibbons-Hawking charges with vanishing total Gibbons-Hawking charge. This construction would clearly elucidate the thermodynamical issues of a black foam and further provides a geometrical realization of the equilibrium thermodynamical structures of the microstates of the foam configurations. We demonstrate that the state-space geometry of the black brane foam, arising from the associated leading order topological entropy of the foam is non-degenerate and everywhere regular for all well-defined three parameter topological black brane foams. Our results are thus in agreement with the fact that the black brane foams are stable thermodynamic objects. Furthermore, it may be argued that the nature of state-space of the given black brane foams remains the same under subleading corrections to the considered foam configurations.

The rest of the article has been organized into several sections. The first section offers the motivations to study the state-space geometry obtained from the Gaussian fluctuations of the entropy for given consistent brane configurations in the string theory and M-theory. In particular, we have motivated interesting implications of the state-space configurations. In section 2, we have introduced very briefly what is the black brane thermodynamic geometry, based on the consideration of large number of equilibrium microstates. In section 3, we analyze the state-space of extremal and non-extremal black holes in string theory. In section 4, we focus our attention on the state-space geometry of multi-centered black brane configurations. Explicitly, we discuss here the state-space geometry defined as an intrinsic Riemannian geometry obtained from the entropy of $D_6D_4D_2D_0$ black brane configuration. In section 5, we investigate the implications arising from the fractionation of D-branes on the state-space geometry of the two charged small black holes. In section 6, we explore the meaning of state-space geometry from the view-points of Mathur's fuzzballs and subensemble theory. Here, we have demonstrated it explicitly for the example of fuzzy rings. In section 7, we have extended our study of the state-space geometry to the M-theory bubbling black brane solutions. We have explained in this case that the state-space geometry obtained from the entropy of a black brane foam is well-defined and pertains to an interacting statistical system, for a range of brane charges. Finally, section 8 contains certain concluding issues and discussions for the state-space geometry of various black branes thus considered and their implications to the boundary CFT data, and in particular Mathur's fuzzball proposal of brane microstates.

2. Thermodynamic Geometry

In this section, we provide a brief review of the n-dimensional state-space manifold M_n which describes the so called thermodynamic Ruppeiner geometry [62–64,66,70,71]. We begin by considering an intrinsic Riemannian geometric model whose covariant metric tensor is defined as a negative Hessian matrix of the entropy of the black brane configuration. In particular, the components of the state-space metric tensor are defined to be

$$g_{ij} := -\frac{\partial^2 S(\vec{x})}{\partial x^j \partial x^i} \tag{1}$$

where the state-space vector $\vec{x} = (p^i, q_i, J_i) \in M_n$. The state-space geometry thus defined introduces the fact that the thermodynamic interactions can be considered as the function of charges, angular momenta, and the mass of the given black brane configuration. Thus, the microscopic nature of the intrinsic state-space geometry may be considered by an ensemble of equilibrium microstates characterizing underlying statistical configurations. Explicitly, we notice for the simplest two dimensional state-space geometry that the components of the Ruppeiner metric tensor are given by

$$g_{qq} = -\frac{\partial^2 S}{\partial q^2}, \quad g_{qp} = -\frac{\partial^2 S}{\partial q \partial p}, \quad g_{pp} = -\frac{\partial^2 S}{\partial p^2} \tag{2}$$

Under this consideration, the components of the state-space metric tensor are related to respective statistical pair correlation functions which may be defined in terms of the parameters describing the microscopic conformal field theory on the boundary of the dual space-time solution. This is because the underlying state-space metric tensor comprising Gaussian fluctuations in the entropy defines an intrinsic geometric configuration. In fact, it is immediate to perceive, for the chosen black brane solution, that the local stability of the fluctuating statistical configuration requires that both principle components of the state-space metric tensor, $\{g_{x_a x_a} \mid x_a = (q,p)\}$, signifying possible heat capacities of the system, should remain positive definite

$$g_{x_i x_i} \ > \ 0, \ i = \ q, p \tag{3}$$

The existence of positive definite volume form on the concerned state-space manifold (M_2, g) imposes condition on the Gaussian fluctuations over chosen equilibrium statistical configurations. In order to have stability on (M_2, g), one must have positive determinant of the state-space metric tensor. In fact, it is easy to express in this case that the determinant of the metric tensor is

$$\|g\| \;=\; S_{qq}S_{pp} - S_{qp}^2 \tag{4}$$

Consequently, it may be noticed, for the two charge configurations, that the associated geometric quantities corresponding to a chosen state-space elucidates typical features of the Gaussian fluctuations about an ensemble of equilibrium D-brane (or M-brane) microstates. Now, we can easily calculate the Christoffel connection Γ_{ijk}, Riemann curvature tensor R_{ijkl}, Ricci tensor R_{ij} and Ricci scalar R for the above thermodynamic state-space geometry. Furthermore, we may thus procure that it is easy to determine a global invariant on two dimensional state-space manifold. In this case, it is nevertheless important to notice that one may actually explain the expected nature of underlying microscopic configurations simply by computing the state-space scalar curvature. In fact, the intrinsic scalar curvature is an interesting invariant which accompanies information of the correlation volume of underlying statistical systems. Explicitly, we may see for all (M_2, g) that the scalar curvature R is given by

$$R = \frac{1}{2}\Big(\frac{S_{pp}S_{qqq}S_{qpp} + S_{qp}S_{qqp}S_{qpp} + S_{qq}S_{qqp}S_{ppp} - S_{qp}S_{qqq}S_{ppp} - S_{qq}S_{qpp}^2 - S_{pp}S_{qqp}^2}{(S_{qq}S_{pp} - S_{qp}^2)^2}\Big) \tag{5}$$

Essentially, it turns out that the zero scalar curvature indicates certain bits of information on the event horizon fluctuating independently of each other, while the divergent scalar curvature signals some phase transitions indicating highly correlated pixels of the information. In this concern, Ruppeiner has interpreted the assumption "that all the statistical degrees of freedom of a black hole live on the black hole event horizon" so that the scalar curvature signifies the average number of correlated Planck areas on the event horizon of the black brane [62–64,66]. Moreover, Bekenstein has long before introduced an elegant picture for the quantization of the area of the event horizon being defined as an integral multiple of Planck areas [73]. The present analysis may thus be anticipated by saying that it involves the fundamental picture of well-celebrated Mathur's fuzzballs [101] or similar framework of the microscopic state counting examinations. Further, the relation between thermodynamic state-space scalar curvature and the Ruppeiner curvature tensor is given (see [71] for details) by

$$R(q,p) = \frac{2}{\|g\|}R_{qpqp} \tag{6}$$

Such a relation is quite usual for any two dimensional intrinsic Riemannian manifold (M_2, g). The state-space geometry thus characterizes the global nature of an ensemble of equilibrium microstates of the black hole statistical configurations. Our geometric perspective in effect discloses an appropriate set of grounds and geometric statements that the state-space scalar curvature of interest has not only exiguous microscopic knowledge of the black hole configurations, but also it has an intriguing intrinsic geometric structures which can be explicated in terms of the parameters of the chosen black brane space-time configuration. Incrementally, one may expose that the configurations under present analysis are effectively attractive or repulsive, and weakly interacting in general; while they are stable, only if at least one of the parameter remains fixed. Specifically, finding statistical mechanical models with like behavior might yield further insight into the microscopic properties of black branes in string theory. Thus, the hope is that our state-space analysis would provide a set of conclusive physical interpretation encoded in the state-space quantities, scalar curvature and related intrinsic geometric invariants for the black brane solutions. The state-space formulation thus tacitly involves a statistical basis in terms of the chosen ensemble or subensemble of microstates of the considered black brane configuration in the thermodynamic limit.

Here it is worth to mention that the thermodynamic state-space scalar curvature captures the nature of the global correlation volume present in the underlying statistical system. This strongly suggests that in the context of the state-space manifold of a black brane arising from either a consistent string theory or M-theory solution as a closed systems, the non-zero scalar curvature might provide useful information regarding the nature of the interactions present between the microstates of the considered black brane solution. With this general geometric introduction to the thermodynamic of a general brane solution, let us now proceed to systematically analyze the state-space geometric structures of such branes characterized by a finite number of electric-magnetic charges with or without angular momenta as bound states of a large number of D-branes (or M-branes).

Thus, after gaining an appropriate understanding of the exact spectrum of a class of black branes in string theory and M-theory, we shall describe in the next section that there exists an asymptotic expansion of the degeneracy formula for large charges which not only reproduces the entropy of the corresponding black branes to the leading order, but also to the first few subleading orders as an expansion in inverse power of the charges [18]. Given this understanding of the microscopic spectra and macroscopic entropy of a black brane, a natural question to ask would be to understand the origin of the state-space interactions from the degeneracy formula and its correspondence to the black brane side. In this paper, we would indeed show from the perspective of an intrinsic Riemannian geometry that there exists a clear mechanism on the black brane side that describes the associated notion of state-space interactions among the microstates of the black brane solution or vice-versa. The coming sections deal with the various implications arising from the state-space geometry of extremal and non-extremal black branes in string theory, multi-centered black brane configurations, fractionation of electric branes, Mathur's subensemble theory and bubbling black brane solutions in M-theory.

3. Black Holes in String Theory

In this section, we shall study state-space geometry arising from the entropy of black branes as bound states of large number D-branes having a fixed number of electric-magnetic charges, angular momentum and may be mass in the case of non-extremal black brane solutions. It is well known that the extremal black branes do not Hawking radiate, since the Hawking temperature is proportional to the difference of the inner and outer horizon radius of the brane configuration. In the sequel, we analyze the case of R^2-corrected non-supersymmetric extremal small black holes in four space-time dimensions. It has been shown that the state-space configuration for such a simple black hole solutions turns out to be non-degenerate and corresponds to an everywhere non-interacting statistical system.

3.1. Extremal Black Holes

The simplest example of state-space geometry of a stringy black hole that may be studied is as follows. Consider at first to understand the state-space geometry arising from an extremal black hole whose microstates are the winding and the momentum modes of a string carrying n_1 number of winding and n_p number of momentum, then the large charge microscopic entropy turns out to be [74–77]

$$S_{micro} = 2\sqrt{2n_1 n_p} \tag{7}$$

Macroscopically, this two charged black hole entropy may be computed by considering D_4 and D_0 branes, with certain compactification to obtain a $M_{3,1}$ space-time. Of course in string theory there are higher order corrections like R^2 or R^4-corrections for instance to the standard Einstein action, and thus these corrections make a non-zero horizon area, as it get stretched by such higher derivative corrections. These computations of the macroscopic entropy are usually easy to solve with a spherically symmetric Ansatz for the non-compact directions [78]. The microscopic entropy may be counted by the consideration of weakly interacting D-brane ensemble [79], and one finds for n_4 number of D_4 branes and n_0 number of D_0 branes that

$$S_{micro} = 2\pi\sqrt{n_0 n_4} = S_{macro} \tag{8}$$

Now the state-space geometry associated with the entropy of D_0, D_4 system may easily be seen to be ill-defined. The argument simply follows from the determinant of the Hessian matrix of the entropy with respect to all the thermodynamic extensive variables which in this case are just the D_0, D_4 brane numbers or brane charges. For a given configuration entropy $S_0 := 2\pi c$, we see for brane numbers, *viz.*, n_0, n_4 which form coordinates on the state-space manifold that the constant entropy curve can be depicted as the rectangular hyperbola

$$n_0 n_4 = c^2 \tag{9}$$

However, it is possible to incorporate further higher derivative corrections to have a well defined Ruppeiner like geometry where the determinant of the Hessian matrix of the entropy may become negative definite in certain domain of the brane charges. The argument follows from the fact that the determinant of the state-space metric tensor depends on chosen attractor values of the scalar fields which arise from string compactification.

One of the important correction arrives due to the higher derivative quartic terms in the Riemann tensor which may be encoded in a scalar field Υ. In this case, one finds for the case of non-supersymmetric two charge small black holes [80,81] that the macroscopic entropy of R^2-corrected space-time solution to be

$$S(Q,P) = 2\pi PQ(1 + \frac{40C}{P^2}) \tag{10}$$

In order to take a closer look of the state-space geometry of the equilibrium microstates of the two charged small black holes, we shall now consider the implications of this R^2-corrected entropy expression. The Ruppeiner metric on the state-space may be easily read off from the negative Hessian matrix of such a corrected entropy of the small black holes to be

$$g_{QQ} = 0, \;\; g_{QP} = 2\pi(-1 + \frac{40C}{P^2}), \;\; g_{PP} = -\frac{160\pi CQ}{P^3} \tag{11}$$

We notice that the concerned magnetic heat capacity is positive quantity, if $C < 0$; On the other hand, the electric heat capacity vanishes identically for all electric and magnetic charges. Here, we see due to vanishing of g_{QQ} that there is only one independent non-zero relative state-space correlation function. A simple observation finds that the state-space pair correlation functions scale as

$$\frac{g_{QP}}{g_{PP}} = \frac{P}{80CQ}(40C - P^2) \tag{12}$$

It is worth to mention that the electric magnetic state-space pair correlation function and the determinant of the metric tensor vanish identically for the magnetic charge $P = \sqrt{40C}$. At this limiting value of the magnetic charge, we find that the magnetic state-space pair correlation function, g_{PP} takes the value of $\frac{2\pi Q}{\sqrt{10C}}$. We observe here that the inclusion of higher derivative R^2-corrections make the determinant of the metric tensor to be non-zero, as one can simply see that it is just given by

$$\|g\| = -4\pi^2(-1 + \frac{40C}{P^2})^2 \tag{13}$$

Along with the nature of principle components of state-space metric tensor, we notice again that the associated determinant of the metric tensor does not have a positive definite volume form. Thus, we notice for all physically allowed values of invariant electric-magnetic charges that the R^2-corrected two charge small black holes do not accomplish locally stable statistical system.

Moreover, the state-space configuration of non-supersymmetric two charge small black holes with quartic corrections corresponds to a non-interacting statistical system. In particular, we find in this case that the state-space scalar curvature is zero. Thus, it is certain that the thermodynamic stability criteria indicate the absence of phase transitions, and the underlying state-space configuration is everywhere non-interacting. Here, we see for given electric-magnetic charges that the constant entropy curve takes the form of the hyperbola

$$\frac{\frac{1}{PQ}}{\frac{2\pi}{k}} - \frac{\frac{1}{P^2}}{40C} = 1 \tag{14}$$

with coordinates $\frac{1}{\sqrt{PQ}}$ and $\frac{1}{P}$, where the real constant k can be is determined for some given entropy $S_0(Q, P)$. This curve may further be expressed in another form describing a value of the electric charge

$$Q = \frac{k}{2\pi}\frac{P}{(P^2 + 40C)} \tag{15}$$

To have a test of more complicated black hole solutions, we may add n_5 number of D_5 branes, and then leading order black hole entropy [1] obtained from the Einstein action takes the form of

$$S_{micro} = 2\pi\sqrt{n_1 n_5 n_p} = S_{macro} \tag{16}$$

In this case, one can easily calculate the associated Ruppeiner metric and thus finds again that the determinant of the metric tensor turns out to be a non-zero quantity. A similar fact follows as well, namely that the associated scalar curvature is everywhere regular. Both quantities scale as the inverse of the square root of the leading order $D_1 D_5$ branes entropy with n_p number of Kaluza-Klein momentum. Here the constant entropy curve in the state-space is some higher dimensional hyperbola

$$n_1 n_5 n_p = c^2 \tag{17}$$

with $c := S_0^2/4\pi^2$. Moreover, similar results hold for the four charge tree level extremal black holes whose state-space geometry has the same form of metric tensor and scalar curvature as that of the three charged extremal black hole configurations. See [82–84] for interesting details of the thermodynamic geometry applied to a large class of the other black brane systems arising from certain compactifications of string theory and M-theory.

3.2. Non-Extremal Black Holes

Next we shall consider the state-space geometry arising from the entropy of a non-extremal black hole. Such a solution can be simply realized by adding the corresponding antibranes to the extremal black brane solutions. The simplest example of such a system is a string having large amount of winding and D_5 brane charges n_1, n_5 with some extra energy, which in the microscopic description creates an equal amount of momenta running in the opposite direction of the S^1. In this case, the entropy has been calculated from both the macroscopic and microscopic perspective [2] which matches for the given total mass and charges. In particular, we have

$$S_{micro} = 2\pi\sqrt{n_1 n_5}(\sqrt{n_p} + \sqrt{\overline{n_p}}) = S_{macro} \tag{18}$$

As invoked earlier, we notice in this case also that the state-apace geometry describing the nature of equilibrium brane microstates may be constructed out of the brane charges and KK momenta of the non-extremal D_1-D_5 black holes. As proclaimed in the previous subsections, the covariant metric tensor may immediately be computed from the negative Hessian matrix of the concerned entropy resulting from the underlying horizon area of black hole configuration. Thus, the asymptotic invariant charges and momenta: $\{n_1, n_5, n_p, \overline{n_p}\}$ form the coordinate charts for the state-space manifold of our interest. The components of the state-space Ruppeiner metric may now easily be computed from the corrected entropy expression to be

$$\begin{aligned} g_{n_1 n_1} &= \frac{\pi}{2}\sqrt{\frac{n_5}{n_1^3}}(\sqrt{n_p} + \sqrt{\overline{n_p}}), \quad g_{n_1 n_5} = -\frac{\pi}{2\sqrt{n_1 n_5}}(\sqrt{n_p} + \sqrt{\overline{n_p}}) \\ g_{n_1 n_p} &= -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1 n_p}}, \quad g_{n_1 \overline{n_p}} = -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1 \overline{n_p}}} \\ g_{n_5 n_5} &= \frac{\pi}{2}\sqrt{\frac{n_1}{n_5^3}}(\sqrt{n_p} + \sqrt{\overline{n_p}}), \quad g_{n_5 n_p} = -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5 n_p}} \\ g_{n_5 \overline{n_p}} &= -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5 \overline{n_p}}}, \quad g_{n_p n_p} = \frac{\pi}{2}\sqrt{\frac{n_1 n_5}{n_p^3}} \\ g_{n_p \overline{n_p}} &= 0, \quad g_{\overline{n_p}\overline{n_p}} = \frac{\pi}{2}\sqrt{\frac{n_1 n_5}{\overline{n_p}^3}} \end{aligned} \tag{19}$$

We may thus describe the typical intrinsic geometric features of the non-extremal black holes with respect to the parameters of the concerned black hole. In this case, we thus see from the negative Hessian matrix of the entropy that the principle components of the state-space metric tensor $\{g_{n_i n_i} |\ i = 1, 5, n_p, \overline{n_p}\}$ essentially signify a set of definite heat capacities (or related compressibilities) whose positivity demonstrates that the four charge D_1-D_5-P black holes form an underlying locally stable equilibrium statistical configuration.

From the view-points of D_1-D_5 black hole solutions, it turns out that the local stability over an entire equilibrium statistical configuration may clearly be determined just by computing the determinant of the state-space metric tensor. As in the previous examples, it is easy to observe that the state-space metric tensor is a non-degenerate and everywhere regular function of the brane charges and underlying KK momenta. As a result, we observe that the determinant of the metric tensor is

$$\|g\| = -\frac{\pi^4}{4(n_p\overline{n_p})^{3/2}}(\sqrt{n_p}+\sqrt{\overline{n_p}})^2 \tag{20}$$

The determinant is non-zero for any set of given non-zero brane-antibrane charges. As before we obtain a non-degenerate state-space geometry of the equilibrium microstates of the four charged non-extremal D_1D_5P black hole system. Herewith, we may exhibit that the nature of the statistical interactions and the other global properties of the D_1-D_5 non-extremal configurations are indeed not difficult to analyze. Furthermore, we may work in the large charge limit in which the asymptotic expansion of the entropy of non-extremal D_1-D_5 system is valid.

In this concern, one may compute certain global invariants of the state-space manifold (M_4, g) which in the present case can easily be determined in terms of the parameters of underlying brane configurations. The scalar curvature of the underlying state-space may easily be computed using the Ruppeiner metric technology defined as the negative Hessian matrix of the entropy corrected by the non-extremal contributions. In particular, we find that the state-space curvature scalar is given by

$$R(n_1,n_5,n_p,\overline{n_p}) = \frac{9}{4\pi\sqrt{n_1n_5}}\left(\frac{n_p^{5/2}+10n_p^{3/2}\overline{n_p}+5n_p^{1/2}\overline{n_p}^2+5n_p^2\overline{n_p}^{1/2}+10n_p\overline{n_p}^{3/2}+\overline{n_p}^{5/2}}{(\sqrt{n_p}+\sqrt{\overline{n_p}})^6}\right) \tag{21}$$

The curvature scalar is thus non-zero, except for the set of roots of a two variable polynomial for the two momenta $(n_p, \overline{n_p})$ running in opposite directions of the S^1. Such a polynomial of charges and anticharges determining global state-space correlations may simply be defined as the numerator of the state-space curvature scalar, that is just the function

$$f(n_p,\overline{n_p}) := n_p^{5/2}+10n_p^{3/2}\overline{n_p}+5n_p^{1/2}\overline{n_p}^2+5n_p^2\overline{n_p}^{1/2}+10n_p\overline{n_p}^{3/2}+\overline{n_p}^{5/2} \tag{22}$$

Consequently, the underlying state-space manifold is everywhere regular and corresponds to a thermodynamic system whose statistical basis is interacting except at the roots of the $f(n_p,\overline{n_p})$. Thus, there exists a hypersurface in the state-space of the non-extremal D_1D_5P black hole system on which the latter becomes a non-interacting statistical system. Here, we can see that the constant entropy curve is a certain non-standard curve, and it is just given by

$$\frac{c^2}{n_1n_5} = (\sqrt{n_p}+\sqrt{\overline{n_p}})^2 \tag{23}$$

As stated in the case of two charge D_0-D_4 extremal black holes and D_1-D_5-P extremal black holes, we herewith see as well, for the non-extremal D_1-D_5-P-$\overline{P}$ black holes in string theory, that the constant c takes the same value of $c := S_0^2/4\pi^2$. Moreover, we see that the scalar curvature thus determined never vanishes for physically allowed values of the invariant charges and any KK momenta, except for $f(n_p,\overline{n_p}) = 0$. Thus, the non-extremal D_1-D_5 black holes have everywhere regular state-space manifold unless either one of the charges approaches infinity. Thus, for any non-zero value of the charges and KK momenta the D_1-D_5 black holes always corresponds to a weakly interacting statistical configuration. Furthermore, we see for a given state-space scalar curvature k that the constant state-space curvature curves take the form

$$f(n_p, \overline{n_p}) = k\sqrt{n_1 n_5}(\sqrt{n_p} + \sqrt{\overline{n_p}})^6 \tag{24}$$

Similarly, the present consideration educes from an ensemble of equilibrium microstates continues that the non-extremal D_1-D_5 black holes possess a non-singular intrinsic state-space manifold. In other words, the regularity of the scalar curvature invariant exposes that the intrinsic Riemannian geometry constructed out of the parameters of non-extremal D_1-D_5 configurations remains regular as an extremum of the entropy. Furthermore, the absence of divergences in the state-space scalar curvature implies that the underlying statistical systems of the four charge black holes live in unique thermodynamic domain under the Gaussian fluctuations. Thus, there are no vacuum phase transitions in the counterpoised microscopic configurations.

In order to take account of higher dimensional state-space configurations, we shall now consider six parameter non-extremal black hole solutions in string theory and thereby focus our attention to analyze state-space pair correlation functions and global stability analysis of concerned general D_1-D_5 black brane solutions in detail. One can extrapolate the above entropy expression to a non-large charge domain, where we are no longer close to a supersymmetric state. The leading order entropy [85] which includes all such special extremal and near-extremal cases may be written as a function of charges $\{n_i\}$-anticharges $\{m_i\}$ to be

$$S(n_1, m_1, n_2, m_2, n_3, m_3) := 2\pi(\sqrt{n_1} + \sqrt{m_1})(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3}) \tag{25}$$

It is again not difficult to explore the state-space geometry of the equilibrium microstates of the 6 charge anticharge non-extremal black hole in $D = 4$ arising from the entropy expression arising from just the Hilbert-Einstein action. As stated earlier, the Ruppeiner metric on the state-space is given by the negative Hessian matrix of the non-extremal entropy with respect to the extensive variables. These variables in this case are in turn the conserved charges-anticharges carried by the non-extremal black hole. Explicitly, we find that the components of covariant metric tensor under such non-large charge domains are easily obtained to be

$$\begin{aligned}
g_{n_1 n_1} &= \frac{\pi}{2n_1^{3/2}}(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3}), \;\; g_{n_1 m_1} = 0 \\
g_{n_1 n_2} &= -\frac{\pi}{2\sqrt{n_1 n_2}}(\sqrt{n_3} + \sqrt{m_3}), \;\; g_{n_1 m_2} = -\frac{\pi}{2\sqrt{n_1 m_2}}(\sqrt{n_3} + \sqrt{m_3}) \\
g_{n_1 n_3} &= -\frac{\pi}{2\sqrt{n_1 n_3}}(\sqrt{n_2} + \sqrt{m_2}), \;\; g_{n_1 m_3} = -\frac{\pi}{2\sqrt{n_1 m_3}}(\sqrt{n_2} + \sqrt{m_2}) \\
g_{m_1 m_1} &= \frac{\pi}{2m_1^{3/2}}(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3}), \;\; g_{m_1 n_2} = -\frac{\pi}{2\sqrt{m_1 n_2}}(\sqrt{n_3} + \sqrt{m_3}) \\
g_{m_1 m_2} &= -\frac{\pi}{2\sqrt{m_1 m_2}}(\sqrt{n_3} + \sqrt{m_3}), \;\; g_{m_1 n_3} = -\frac{\pi}{2\sqrt{m_1 n_3}}(\sqrt{n_2} + \sqrt{m_2}) \\
g_{m_1 m_3} &= -\frac{\pi}{2\sqrt{m_1 m_3}}(\sqrt{n_2} + \sqrt{m_2}), \;\; g_{n_2 n_2} = \frac{\pi}{2n_2^{3/2}}(\sqrt{n_1} + \sqrt{m_1})(\sqrt{n_3} + \sqrt{m_3}) \\
g_{n_2 m_2} &= 0, \;\; g_{n_2 n_3} = -\frac{\pi}{2\sqrt{n_2 n_3}}(\sqrt{n_1} + \sqrt{m_1})
\end{aligned}$$

$$
\begin{aligned}
g_{n_2m_3} &= -\frac{\pi}{2\sqrt{n_2m_3}}(\sqrt{n_1}+\sqrt{m_1}),\ \ g_{m_2m_2}=\frac{\pi}{2m_2^{3/2}}(\sqrt{n_1}+\sqrt{m_1})(\sqrt{n_3}+\sqrt{m_3}) \\
g_{m_2n_3} &= -\frac{\pi}{2\sqrt{m_2n_3}}(\sqrt{n_1}+\sqrt{m_1}),\ \ g_{m_2m_3}=-\frac{\pi}{2\sqrt{m_2m_3}}(\sqrt{n_1}+\sqrt{m_1}) \\
g_{n_3n_3} &= \frac{\pi}{2n_3^{3/2}}(\sqrt{n_1}+\sqrt{m_1})(\sqrt{n_2}+\sqrt{m_2}),\ \ g_{n_3m_3}=0 \\
g_{m_3m_3} &= \frac{\pi}{2m_3^{3/2}}(\sqrt{n_1}+\sqrt{m_1})(\sqrt{n_2}+\sqrt{m_2}) \qquad (26)
\end{aligned}
$$

Physically, we observe that the principle components of the state-space metric tensor $\{g_{n_in_i}, g_{m_im_i} \mid i = 1,2,3\}$ signify a set of heat capacities (or the associated compressibilities) whose positivity exhibits that the underlying black hole system is in a locally equilibrium statistical configuration of branes and antibranes. Incidentally, we see for given charges $\{n_i, m_i\}$ that the intrinsic state-space pair correlations turn out to be in precise accordance with the underlying macroscopic attractor configurations being disclosed in the special leading order limit of the non-extremal solutions. Moreover, it is not difficult to encourage the stability of the full state-space configuration of such non-extremal D_1-D_5 black branes. In fact, a straightforward computation yields that the determinant of the metric tensor is

$$
\begin{aligned}
\|g\| &= -\frac{\pi^6}{16}(n_1m_1n_2m_2n_3m_3)^{-3/2}(\sqrt{n_2}+\sqrt{m_2})^2(\sqrt{n_3}+\sqrt{m_3})^3(\sqrt{n_1}+\sqrt{m_1})^3 \\
&\quad (n_2\sqrt{m_1n_3}+n_2\sqrt{m_1m_3}+2\sqrt{n_2m_1m_2n_3}+2\sqrt{n_2m_1m_2m_3}+m_2\sqrt{m_1n_3} \\
&\quad +m_2\sqrt{m_1m_3}+n_2\sqrt{n_1n_3}+n_2\sqrt{n_1m_3}+2\sqrt{n_1n_2m_2n_3}+2\sqrt{n_1n_2m_2m_3} \\
&\quad +m_2\sqrt{n_1n_3}+m_2\sqrt{n_1m_3}) \qquad (27)
\end{aligned}
$$

The fact that the determinant of the metric tensor does not take a positive definite value shows that there is no positive definite volume form on the concerned state-space manifold (M_4, g) of the black holes. In fact, it is worth to note that the responsible equilibrium entropy tends to its maximum value on the entire state-space manifold. It may thus be envisaged in this description that these black holes do not correspond to an intrinsically stable statistical configuration, and thus it is very probable that the underlying ensemble of microstates may smoothly move into more stable brane configurations. This is however intelligible from the fact the concerned metric tensor defines as almost everywhere non-degenerate, well-defined, positive definite state-space manifold, which may solely be parameterized in terms of the brane and antibrane charges, *viz.*, $\{n_i, m_i \mid i = 1,2,3\}$. We further see for non-zero brane-antibrane charges that the determinant remains non-zero except for the set of brane-antibrane charges defined by

$$
\begin{aligned}
B &:= \{(n_1,n_2,n_3,m_1,m_2,m_3) \mid n_2\sqrt{m_1n_3}+n_2\sqrt{m_1m_3}+2\sqrt{n_2m_1m_2n_3}+2\sqrt{n_2m_1m_2m_3}+ \\
&\quad m_2\sqrt{m_1n_3}+m_2\sqrt{m_1m_3}+n_2\sqrt{n_1n_3}+n_2\sqrt{n_1m_3}+2\sqrt{n_1n_2m_2n_3}+2\sqrt{n_1n_2m_2m_3} \\
&\quad +m_2\sqrt{n_1n_3}+m_2\sqrt{n_1m_3}=0\} \qquad (28)
\end{aligned}
$$

and thus this state-space geometry is well-defined only on an intrinsic Riemannian manifold $N := M_6 \setminus B$. Furthermore, we may trace a certain behavior of the components of the covariant Riemann tensors and note for example that the component $R_{n_1n_2m_3m_4}$ diverges at the roots of the some polynomials of two variables. In particular, these polynomials as functions of the brane charges and anticharges are

$$\begin{aligned} f_1(n_2, m_2) &= n_2^4 m_2^3 + 2(n_2 m_2)^{7/2} + n_2^3 m_2^4 \\ f_2(n_3, m_3) &= m_3^{9/2} n_3^4 + n_3^4 m_3^{9/2} \end{aligned} \tag{29}$$

However, we observe that some components of the covariant Riemann tensors may diverge differently. For example the component R_{n_3,m_3,n_3,m_3} with equal number of brane and antibrane components diverges at a root of the single higher degree polynomial

$$\begin{aligned} f(n_1, m_1, n_2, m_2, n_3, m_3) \;:=\; & n_2^4 m_2^3 n_3^{9/2} m_3^4 + n_2^4 m_2^3 n_3^4 m_3^{9/2} + 2 n_2^{7/2} m_2^{7/2} n_3^{9/2} m_3^4 + \\ & 2 n_2^{7/2} m_2^{7/2} n_3^4 m_3^{9/2} + n_2^3 m_2^4 n_3^{9/2} m_3^4 + n_2^3 m_2^4 n_3^4 m_3^{9/2} \end{aligned} \tag{30}$$

Furthermore, we see that the components of the covariant Riemann tensors may become zero for certain values of charges-anticharges. A systematic calculation further shows that the Ricci scalar curvature diverges at the set of the roots of the determinant of the metric tensor B, and becomes null on some single higher degree polynomial. Exactly at these points in the state-space of the underlying extremal or near-extremal, the general black hole system becomes a non-interacting statistical system. The scalar curvature of this state-space manifold may now be computed using the intrinsic Riemannian metric based on the Hessian matrix of the entropy corrected by non-extremal contributions. The exact expression for the scalar curvature is somewhat involved but it is seen that the nature of the state-space manifold has a similar behavior to the earlier non-extremal case of the charge-anticharge KK-system of the D_1-D_5 branes.

There exists an akin single higher degree polynomial equation on which we find that the Ricci scalar curvature becomes null. Exactly at these points, the underlying state-space configuration of the (extremal or near-extremal or general) non-large charge black hole system corresponds to some non-interacting statistical system, where the state-space manifold (M_6, g) is curvature free. It also follows from detailed computation that the general expression for the Ricci scalar is quite involved, and even for the equal brane charges $n_1 := n$; $n_2 := n$; $n_3 := n$; and equal antibrane charges $m_1 := m$; $m_2 := m$; $m_3 := m$ the result does not sufficiently simplify. Nevertheless, we find for the identical large values of brane and antibrane charges $n := k$ and $m := k$ that there exists an attractive state-space basis for which the expression of the corresponding curvature scalar reduces to a particular small negative value of

$$R(k) = -\frac{15}{16}\frac{1}{\pi k^{3/2}} \tag{31}$$

Thus, there are no critical phenomena in the state-space manifold spanned by brane and antibrane charges, except for the roots of the determinant vanishing condition defined by the Equation (28). Herewith, we find that the regular state-space scalar curvature seems to be comprehensively universal for the given number of parameters of the non-large charge non-extremal D_1-D_5 configurations. In fact, the concerned perception turns out to be related with the typical form of the state-space geometry arising from negative Hessian matrix of the duality invariant expression of the interested black brane entropy. In this case, we may nevertheless easily observe for given entropy S_0 that the constant entropy curve is again some non-standard curve

$$(\sqrt{n_1} + \sqrt{m_1})(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3}) = c \tag{32}$$

where c is a real constant taking precise value of $S_0/2\pi$. The present analysis in fact indicates that the entropy of the black brane solution defines a non-degenerate embedding in the viewpoints of intrinsic state-space geometry. Thus being encouraged from the above computations, we may further assert that our state-space geometry determines an intricate set of physical properties of the statistical pair correlation functions and correlation volume which reveal possible nature of associated parameters prescribing an ensemble of microstates of the dual CFT living on the boundary of the chosen black brane solution. Furthermore, our expectation is that we can consider such an analysis for a class of general higher dimensional black brane configurations with multiple parameters, where the state-space geometric propositions of having ordinary computations might not be very feasible. However, one may possibly exhibit certain geometric acquisitions with an appropriate comprehension of the required parameters defining the state-space coordinate for the chosen configurations.

Finally, we would thus like to see the nature of the non-extremal black hole state-space, *i.e.*, what happens to it with the addition of KK-monopoles as a non-trivially fibered circle. In the past, several authors have calculated the entropy for extremal, near-extremal and general holes which follows the same pattern as before. For example, the case of four charge solutions as considered in [86,87] has the following entropy

$$S(n_1, m_1, n_2, m_2, n_3, m_3, n_4, m_4) := 2\pi(\sqrt{n_1} + \sqrt{m_1})(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3})(\sqrt{n_4} + \sqrt{m_4}) \quad (33)$$

This leading order entropy includes all the special extremal and near-extremal cases and has been written as a function of charges $\{n_i\}$-anticharges $\{m_i\}$. In this case too, one obtains the same pattern of the underlying state-space geometry and constant entropy curve as that of the three charge non-extremal black holes. However, the exact expression for the geometric invariants such as the determinant of the state-space metric tensor or that of the scalar curvature are of course much more involved. The conclusion to be drawn however remains the same and as usual the underlying state-space geometry remains well-defined as certain intrinsic Riemannian manifold $N := M_8 \setminus \tilde{B}$, where $\tilde{B}$ is the set of roots of the determinant of the metric tensor. The underlying state-space manifold involves the four charges and four anticharges of the non-extremal black holes with the inclusion of KK-monopole contributions.

4. Multi-centered Black Branes: $D_6D_4D_2D_0$ system

In this section as a first exercise of the state-space manifold containing both single centered black branes and multi-centered black branes, we study the state-space geometry defined in terms of the extensive four charges of $D_6D_4D_2D_0$ black brane configuration. Here, we shall explicitly present the analysis of the state-space geometry arising from the entropy of stationary single-centered as well as multi-centered black hole molecule configurations. Such multi-centered black hole configurations may be examined by the so called pin-sized D-brane systems [19,20] and thus we can realize the underlying state-space geometry arising from the counting entropy of the number of microstates of a zoo of entropically dominant multi-centered black hole configurations along with usual single centered black holes.

Although it is well known that the most entropic minimal energy supergravity solution should be a single centered black hole [6]. Thus we are interested in analyzing the state-space geometry arising from the leading D-brane entropy with the leading single centered black brane configurations. However, here our main focus is to study the state-space of most general supersymmetric solutions which are visualized as stationary, multi-centered, molecular black hole bound states determined by certain electric-magnetic charge centers $\Gamma_i := (p_i^\Lambda, q_{\Lambda,i})$ with respect to a certain gauge group with an index Λ and total charge $\Gamma = \sum_i \Gamma_i$, see [88,89] for detailed constructions of such brane configurations. The main difference between the single centered black hole configurations and multi-centered black hole configurations is that the first case of supersymmetric solutions is completely determined by the topological data of the string compactification, while the second case is typically given by genuine bound states where one cannot move the centers away from each other without supplying some energy to the system, which is the same as for some constrained supergravity solutions [20,90]. For example in the case of the two centered black hole solution, one has an equilibrium separation condition. Moreover, the existence of multi-centered black hole bound states depends on the choice of vacuum. In the weak string coupling limit, the zoo of multi-centered black hole configurations collapses to a single D-brane.

As the authors of [19,20] have shown, in the suitable moduli regimes, the multi-centered entropy dominates the single centered entropy in the uniform large charge limit. Thus, we investigate the state-space geometric implications for these well studied examples of $D_6D_4D_2D_0$ systems. Let us consider a charge Γ obtained by wrapping D_4, D_2 and D_0 branes around various cycles of a compact space X which are scaled up as $\Gamma \to \Lambda\Gamma$. Then there exists a two centered brane solution with horizon entropy scaling as Λ^3, while that of the single centered entropy scales as Λ^2. More properly, let us consider type IIA string theory compactified on the product of three two-tori $X := T_1^2 \times T_2^2 \times T_3^2$, see [19,20] for the detailed consideration of multi-centered space-time geometries. Then the entropy for a given charge vector Γ corresponding to $p0$ D_6 branes on X, p D branes on $(T_1^2 \times T_2^2) + (T_2^2 \times T_3^2) + (T_3^2 \times T_1^2)$, q D_2 branes on $(T_1^2 + T_2^2 + T_3^2)$ and $q0$ D_0 branes is given by

$$S(\Gamma) := \pi\sqrt{-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2} \tag{34}$$

The metric tensor of the state-space geometry in the entropy representation may be obtained as before from the negative Hessian matrix of the entropy with respect to all extensive thermodynamic variables, which in this case are just the D_6, D_4, D_2 and D_0-brane charges. Explicitly, the components of the metric tensor are given as

$$\begin{aligned}
g_{p0p0} &= -4\pi\frac{-3p^2q^2q0^2 + 3pq^4q0 - q^6 + p^3q0^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{p0p} &= 6\pi\frac{-p^3q0^2q + 2p^2q0q^3 + p^2q0^3p0 - pq^5 - 2pq^2p0q0^2 + p0q^4q0}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{p0q} &= -12\pi\frac{2p^3q^2q0 + p^2qq0^2p0 - 2pq^3q0p0 - q^4p^2 + q^5p0 - p^4q0^2}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{p0q0} &= -\pi\frac{-6p^4qq0 + 3p^2q^2q0p0 - 9pqq0^2p0^2 + 5q^3p^3 - 6q^4p0p + 6q^3p0^2q0 + 6p0q0^2p^3 + p0^3q0^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{pp} &= -12\pi\frac{p^4q0^2 - p^3q^2q0 - 3p^2qq0^2p0 + 4pq^3q0p0 - p0^2q0^2q^2 + p0^2q0^3p - q^5p0}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{pq} &= 3\pi\frac{2p^4qq0 - 2p0q0^2p^3 + 3p^2q^2q0p0 - 3q^3p^3 + 2q^4p0p - pqq0^2p0^2 - 2q^3p0^2q0 + (p0q0)^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{pq0} &= -12\pi\frac{p^5q0 - 2p^3q0p0q - p^4q^2 + 2p^2q^3p0 + pq^2p0^2q0 - p0^2q^4}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}
\end{aligned}$$

$$
\begin{aligned}
g_{qq} &= -12\pi\frac{4p^3q0p0q - p^2q^3p0 - p^2q0^2p0^2 - 3pq^2p0^2q0 + p0^2q^4 - p^5q0 + p0^3qq0^2}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{qq0} &= 6\pi\frac{-p^5q + 2p^3q^2p0 - 2p^2qp0^2q0 + p0p^4q0 - p0^2q^3p + p0^3q^2q0}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{q0q0} &= -4\pi\frac{-p^6 + 3p^4p0q - 3p0^2p^2q^2 + p0^3q^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}
\end{aligned} \tag{35}
$$

¿From the above expressions of intrinsic metric tensor, we may easily visualize that the principle components of state-space metric tensor $\{g_{ii}(\Gamma) \mid \forall i \in \{p0, p, q, q0\}\}$ signify a set of definite heat capacities (or the related compressibilities). Essentially, the positivity of $g_{ii}(\Gamma)$ thus apprises for a domain of brane charges that the D_6-D_4-D_2-D_0 black brane solution comply an underlying locally equilibrium statistical configuration. This is indeed admissible because the brane configuration divulges physically stable system for definite set of brane charges associated with the single and double centered black holes.

Furthermore, we find that it is easy to inquire about the complete local stability of the full phase-space configuration. Such a notion may in effect be acclaimed by computing the determinant of the state-space metric tensor. Nevertheless, it is not difficult to enumerate a compact formula for the determinant of the metric tensor. For the different possible values of brane charges, *viz.*, $\{p0, p, q, q0\}$, it may apparently be discovered from the intrinsic geometric analysis of the D_6-D_4-D_2-D_0 black holes. In fact, we observe that the determinant of the state-space metric tensor is

$$\|g\| = 9\pi^4 \tag{36}$$

which is a positive definite constant over the entire state-space configuration. The determinant of the metric tensor thus calculated is non-zero for any set of given non-zero brane charges, and thus provides a non-degenerate state-space geometry for this configuration. One may thus illustrate the order of statistical correlations between the equilibrium microstates of the multicentered charged black hole system in string theory. Furthermore, the positivity of the determinant of state-space metric tensor indicates that the underlying systems globally endure a stable statistical basis. This condition in effect connotes that there exists a positive definite volume form on (M_4, g), and thus one may conclude that the D_6-D_4-D_2-D_0 system finds definite stable brane configurations.

In order to conclusively analyze the nature of interaction and the other concerned properties of the statistical configurations, one needs to determine certain global invariants on the parametric state-space manifold (M_4, g). One may thus ascertain that such a simple invariant is the state-space scalar curvature, which may now be computed in a straightforward fashion by applying our previously advertised intrinsic geometric technology. Over a range of brane charges, we may easily see that the non-degenerate metric tensor defines a well-defined state-space configuration which is parameterized in terms of the D-brane charges $\{\Gamma := (p0, p, q, q0)\}$. The scalar curvature corresponding to the state-space geometry of the equilibrium D-brane microstates is now easily be determined to be

$$R(\Gamma) = \frac{8}{3\pi}(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{-1/2} \tag{37}$$

We observe that the scalar curvature is a non-zero, positive and everywhere regular function of Γ. This fact seems to be universal and is related to the typical form for the Ruppeiner geometry as the negative Hessian matrix of the black brane entropy. The standard interpretation of scalar curvature of the state-space geometry is to describe the interactions of the underlying statistical system, which thus is non-zero for D_6-D_4-D_2-D_0 black holes. Note that the absence of divergences in the scalar curvature indicates, this system to be thermodynamically stable everywhere. Thus, there are no phase transitions or such critical phenomena in the underlying state-space manifold of this D-brane system. Furthermore, we may easily see that the constant entropy curve and constant scalar curvature curve are defined as

$$4p^3q0 - 3p^2q^2 - 6p0pqq0 + 4p0q^3 + (p0q0)^2 = c \tag{38}$$

In effect, the Equation (38) determines the polynomial curve for an intrinsic geometric view-point of the state-space basis for the D_6-D_4-D_2-D_0 multi-center black brane configuration in string theory. Both the above conditions for some given constant charge Γ_0 easily find that the respective constant $c := (c_S, c_R)$ are given by

$$\begin{aligned} c_S &:= -(\frac{S(\Gamma_0)}{\pi})^2 \; for\ constant\ entropy \\ c_R &:= -(\frac{8}{3\pi R(\Gamma_0)})^2 \; for\ constant\ scalar\ curvature \end{aligned} \tag{39}$$

In the sequel, we shall now analyze the associated state-space geometric curvature scalar for the cases of single and double centered black hole configurations. Let us take the same cases of charge centers as that of [19,20]. Let us consider a total charge $\Gamma := (p0, p, q, q0) = \Lambda(0, 6, 0, -12)$ in a background in which the area of each T^2 is v. Then, it is known that for all v there exists a single centered black hole solution. In particular, we see that $S(\Gamma = \Lambda(0, 6, 0, -12)) = \pi\sqrt{10368}\Lambda^2$. It is easy to find that the scalar curvature remains non zero, positive and takes the value

$$R(\Gamma = \Lambda(0, 6, 0, -12)) = \frac{\sqrt{2}}{54\pi\Lambda^2} \tag{40}$$

Furthermore, as indicated by Denef and Moore for $v > \sqrt{12}\Lambda$ there exists a two centered bound state with charge centers $\Gamma_1 = (1, 3\Lambda, 6\Lambda^2, -6\Lambda)$ and $\Gamma_2 = (-1, 3\Lambda, -6\Lambda^2, -6\Lambda)$. It is apparent that the two entropies match for the Γ_1, Γ_2 with $S(\Gamma_1) = S(\Gamma_2) = \pi\sqrt{108\Lambda^6 - 36\Lambda^2} \sim \Lambda^3$ for large Λ. We see further that the respective state-space scalar curvatures remains non-zero, positive quantities and take the same values for the both charge centers. and in particular, we see that the scalar curvatures are given by

$$R(\Gamma_1) = R(\Gamma_2) = \frac{4}{9\pi}\frac{1}{\sqrt{3\Lambda^6 - \Lambda^2}} \sim \frac{1}{\Lambda^3} \tag{41}$$

¿From this information one may predict that the correlation volume shall be identical for these two charge centers of a double centered black brane configurations.

It is instructive to note for large Λ that the state-space curvature scalar for the case of a single centered black brane configurations goes as $R(\Lambda) \sim \Lambda^{-2}$, whereas for the two centered black brane configurations it behaves as $R(\Lambda) \sim \Lambda^{-3}$. Therefore, we can analyze the nature of interactions present in the underlying state-space corresponding to both the single and doubled centered black hole configurations. In order to do so, let us consider large charge limit with $\Lambda \to \infty$ for some fixed v. Then, we see that the state-space scalar curvature of a double centered black hole goes to zero, faster than that of the single centered black hole configuration. It may further be indicated that the two point correlations become very small in the large charge limit with fixed moduli at infinity, and thus it should indicate a weakly interacting statistical system. This is indeed the correct picture because the calculation of the entropy of $D_6D_4D_2D_0$ black hole is based on the consideration of weakly bound states of these D-branes over different cycles of the compactifying space X.

5. Fractionation of Branes: Small Black Holes

In this section, we analyze the state-space geometry of a given D-brane system from the perspective of the fractionation of branes and counting the associated chiral primaries. For simplicity, let us consider the example of two charge extremal small black holes in type IIA string theory compactified on $T^2 \times \mathcal{M}$, where $\mathcal{M}$ can be either K_3 or T^4, [21–24]. This four dimensional black hole solution is made up out of some D_0 branes and D_4 branes wrapping over $\mathcal{M}$. It is further well-known that this system has the near horizon geometry of $AdS_2 \times S^2$, see for details [91–94].

Next, in order to make contact of our state-space geometry with the microscopic perspective, let us consider the chiral primaries of $SU(1,1 \mid 2)_Z$ and the associated supersymmetric ground states of $\mathcal{N} = 4$ supersymmetric quantum mechanics [31]. In this consideration, we may see easily that there are $24p$ bosonic chiral primaries with total D_0 brane charge N in the background with fixed magnetic D_4 charge p. The counting degeneracy may arise from the combinatorics of total N number of the D_0 brane charge splitting into k-small clusters with n_i number of D_0 branes such that the sum $\sum_{i=1}^{k} n_i = N$ on each cluster corresponds to the wrapped D_2 branes residing on any of the $24p$ bosonic chiral primary states. Here, the counting is done with the degeneracy d_N of the states having level N in a $(1+1)$ CFT with $24p$ bosons, and thus one arrives at the celebrated microscopic entropy

$$S = \ln d_N \simeq 4\pi\sqrt{Np} = 4\pi\sqrt{\sum_{i=1}^{k} n_i p} \tag{42}$$

One can see [78,95–98] for further details. In the following, we shall explain the state-space geometry for a few electric clusters, and in this concern, we shall then explain the state-space geometry for higher specific values of the number of the clusters in which a total N number of fluctuating D_0 brane charge splits. First of all, we see for $k = 2$ that the entropy takes the form

$$S(n_1, n_2, p) = 4\pi\sqrt{p(n_1 + n_2)} \tag{43}$$

We thus find that there are two sets of charges which form coordinate charts on the state-space configuration of underlying small black holes. We shall take the first set of state-space variables to be the fractionated D_0 brane numbers $\{n_1, n_2\}$ which are simply proportional to the available fraction of electric charges present in respective clusters, while the second state-space variable is the number p representing the D_4 brane magnetic charge. In this case, the components of the state-space metric associated with small black holes can easily be computed from the negative Hessian matrix of the fractional brane entropy as

$$\begin{aligned} g_{pp} &= \frac{\pi}{p}\sqrt{\frac{n_1 + n_2}{p}} \\ g_{pn_i} &= -\frac{\pi}{\sqrt{p(n_1 + n_2)}} \end{aligned}$$

$$g_{n_i n_j} = \frac{\pi}{(n_1+n_2)}\sqrt{\frac{p}{n_1+n_2}} \tag{44}$$

For given $\{n_1, n_2\}$ and p, we observe that the required set of positivity conditions on the state-space configuration is satisfied. Hitherto, we see apparently that the principle components of state-space pair correlations remain positive definite quantities for all admissible values of fractionated electric charges and underlying magnetic charge of the two clustered black brane configuration. The principle components of state-space metric tensor $\{g_{n_i n_i}, g_{pp} | \ i = 1, 2\}$ signify a set of positive definite heat capacities (or the related compressibilities) of the two cluster configurations. In fact, the positivity constraint apprises that the D_0-D_4 black branes comply an underlying locally equilibrium statistical configuration. Furthermore, it is worth to mention that the non diagonal component, *viz.*, $g_{n_i n_j}$ also takes a positive value on the state-space manifold (M_3, g) of the two cluster D_0-D_4 black brane configurations.

Moreover, it is not difficult to investigate the global stability on the full state-space configuration, and in fact it can be easily carried out by computing the determinant of the state-space metric tensor of the two cluster D_0-D_4 black hole configurations. Here, we may nevertheless check for the determinant of the metric tensor and, indispensably, our intrinsic geometric analysis shows that the expression for the determinant of the metric tensor at the leading order entropy analysis does not find an intrinsic state-space value for any desired brane clusters and magnetic charge. In particular, we may notice that the determinant of the metric tensor is identically zero for such a microscopic entropy calculation. Such an analysis in turn holds in the limit of large charges on the branes where one finds as asymptotic expansion for the entropy. We may further see for some given fractionation entropy S_0 that the constant entropy curve is given by

$$p(n_1+n_2) = c \tag{45}$$

where the real constant c takes the value of $c := S_0^2/4\pi^2$. Such a curve in fact explicates that the entropy of the black branes defines an embedding in the view-points of the intrinsic state-space geometry. In the subsequent analysis, we shall restrict the number of electric clusters in which the total N number of D_0 brane charge splits into specific three partitions. Now, we would like to investigate the state-space geometry for three electric clusters. In this case, we observe that the entropy of the D_0-D_4 black holes takes the form

$$S(n_1, n_2, n_3, p) = 4\pi\sqrt{p(n_1+n_2+n_3)} \tag{46}$$

Thus, the intrinsic Riemannian geometry as the equilibrium state-space configuration may immediately be introduced as before from the negative Hessian matrix of the entropy of three electric charges and one magnetic charge extremal small black holes with D_0 brane fractionations. Once again, one may see that the components of the associated Ruppeiner metric are

$$\begin{aligned} g_{pp} &= \frac{\pi}{p}\sqrt{\frac{n_1+n_2+n_3}{p}} \\ g_{pn_i} &= -\frac{\pi}{\sqrt{p(n_1+n_2+n_3)}} \\ g_{n_i n_j} &= \frac{\pi}{(n_1+n_2+n_3)}\sqrt{\frac{p}{n_1+n_2+n_3}} \end{aligned} \tag{47}$$

We thus observe that the statistical pair correlations just accounted may in turn be simply ascertained by microscopic descriptions which are being expressed in terms of large integers (or associated brane charges) of the D_0-D_4 small black brane solutions connoting an ensemble of microstates. Furthermore, it is evident for the small black brane configurations that the principle components of the statistical pair correlations are positive definite for all allowed values of the concerned parameters. Our analysis further complies that the positivity of pure state-space correlations involving either electric or magnetic charges obliges that the associated dual conformal field theory living on the boundary should possess a non-vanishing duality frame of the electric-magnetic charge defining an associated degeneracy of large number of conformal field theory microstates. It is worth to mention for given $\{n_1, n_2, n_3\}$ and p that the inter cluster state-space electric correlation functions are again non-trivial in nature, and they all in turn take definite positive values.

In addition, it is likewise evident that the local stability of full D_0-D_4 black brane configurations can be determined by computing the determinant of the concerned state-space metric tensor. We thus again see that in this case too the determinant of the metric tensor remains zero. It is in effect not difficult to see that the determinant of the metric tensor vanishes identically for all finite values of the number of branes carried by the small black ring configurations at this order and thus the corresponding state-space geometry turns out to be an ill-defined degenerate intrinsic Riemannian manifold. For given constant entropy S_0, we see from the view-points of intrinsic state-space geometry that the constant entropy curve defining an embedding function is given by

$$p(n_1 + n_2 + n_3) = c \tag{48}$$

where the real constant c, as we have demonstrated in the previous cases, takes the same value of $S_0^2/16\pi^2$. Finally, we shall explore the state-space geometry for the most general brane fractionation of finite clusters of D_0-D_4 black brane configurations, and thereby present our analysis from the view-points of associated microscopic entropy obtained for arbitrary k-clusters. In order to so, let us now consider the most general case of fractionated small black hole microscopic entropy

$$S(n_1, n_2, \ldots, n_k, p) = 4\pi \sqrt{\sum_{i=1}^{k} n_i p} \tag{49}$$

The Ruppeiner geometry describing the local state-space pair correlations between the equilibrium microstates of multi-clustered charged extremal D_0-D_4 black holes resulting from the degeneracy of the microstates are thence obtained as earlier from the Hessian matrix of the fractionated entropy, with respect to the parameters of the black brane configuration. In particular, the nature of state-space pair correlations can be easily computed for given N D_0 electric branes having respective charges $\{n_1, n_2, \ldots n_k\}$ and the magnetic D_4 brane with magnetic charge p. Herewith, one may again find that the components of associated state-space metric tensor with respect to the electric charges $\{n_1, n_2, \ldots, n_k\}$ and the magnetic charge p are

$$\begin{aligned} g_{pp} &= \frac{\pi}{p}\sqrt{\frac{\sum_{i=1}^{k} n_i}{p}} \\ g_{pn_i} &= -\frac{\pi}{\sqrt{p(\sum_{i=1}^{k} n_i)}} \end{aligned}$$

$$g_{n_i n_j} \;=\; \frac{\pi}{(\sum_{i=1}^{k} n_i)}\sqrt{\frac{p}{\sum_{i=1}^{k} n_i}},\ \forall\, i,j = 1,2,\ldots,k \tag{50}$$

It is again not difficult to see that the determinant of the metric tensor remains zero for any finite number of clusters corresponding to wrapped D_2 branes residing on any of the $24p$ bosonic chiral primary states. Hence the state-space geometry of the equilibrium microstates based on the large charge small black hole entropy obtained from the fractionation of D_0 branes is degenerate for any finite number of clusters and thus the corresponding state-space geometry turns out to be ill-defined. Our conclusion that the state-space geometry of N fractional branes with electric charges $\{n_i\}_{i=1}^{k}$ corresponding to k clusters in the background of D_4 branes with magnetic charge p is everywhere ill-defined, is in very well accordance with the principle of mathematical induction. The determinant of the metric of the state-space geometry is trivially zero for $i = 1$. We have shown that it remains zero for $i = 2$ and $i = 3$. Let the result holds for $i = j$, then we may easily check that it holds for $i = j + 1$, and thus the result holds for $i = k, \forall k \in Z$. Further, it is easy to see that the curve

$$(n_1 + n_2 + \ldots + n_k)p = c \tag{51}$$

describes the constant entropy curve on the state-space of k-clustered small black holes. It is worth to mention that the real constant c takes the same value of $c := S_0^2/4\pi^2$, and thus we notice against an arbitrary electric brane fractionation that the constant entropy curve remains independent of the number of clusters for the D_0-D_4 small black holes. This is because the microscopic calculation of the entropy for n_0 number of D_0-branes and n_4 number of D_4-branes deals with counting of the weakly interacting D-brane ensemble. Furthermore, it seems in accordance with the fact that there are certain problems with the application of the OSV formula to original non-fractionated two charged small black holes [99,100]. We may thus anticipate that their brane fractionations need some further investigations in order to have a well-defined state-space configuration.

Such a consideration may be envisaged to be achieved via certain definite higher derivative α'-contributions [99] in to the entropy. It may be expected that there would probably exist a general α'-corrected entropy formula which would offer an intrinsic state-space account for its microscopic CFT models. The equilibrium statistical configurations may thence be geometrically exposed in the limit of Gaussian approximations. At this point, it is however worth to mention that the interesting state-space inter-relations and their moduli space details are beyond the scope of the present set-up and thus are matter of a future exploration.

6. Mathur's Fuzzball Proposal and Subensemble Theory: Fuzzy Rings

Mathur's fuzzball conjecture says that the throat geometry of an extremal black hole ends in a finite quantum fuzz, which is in contrast to the classical picture that the throat ends with an infinite horizon length [39]. Furthermore, the author suggests an intriguing way how one may bypass the information paradox. For simplicity, let us consider the case of two charge extremal black hole, which is microscopically characterized by certain string winding and momentum states. Then the microscopic entropy comes from the number of different possible ways in which the string carries the momentum as the traveling waves. This is due to the fact that the associated fundamental string in the heterotic picture does not carry any longitudinal waves and thus all vibrations must be transverse. Thus, the string carrying a momentum charge departs from its central location $r = 0$ and in turn gets spread over a certain transverse region which makes a fuzzball.

Moreover, it thus suggests that there exists a relation between the capped throat structure of microstates geometries and that of the extremal black hole geometries having zero entropy in the leading order gravity computations. Note however that the actual microstates of the system in fuzzball conjecture do not have horizon, but the boundary of the underlying region where the microstates start differing from each other satisfies a Bekenstein type relation with the entropy enclosed inside this boundary whose state-space geometry we are going to investigate in this article. Further we know that a black hole in general have a large entropy which is a measure of the associated phase space volume and thus the size of a typical fuzzball cannot be too small, otherwise the phase space of the fuzzball cannot account for the large entropy of the black hole.

6.1. The Fuzzball Proposal

It is the associated state-space geometry of the entropy of this rotating two charge system that we would like to construct, whose dimension is equal to the actual number of parameters characterizing the brane microstates. We shall study this state-space of a black ring whose CFT deals with n_1 number of D_1 branes having charge Q and n_5 number of D_5 branes having charge P. In the case of two charge extremal holes carrying an angular momentum J which are quite natural in string theory, one finds the stretched horizon entropy to be $S(n_1, n_5, J) = c\sqrt{n_1 n_5 - J}$. In other words, as $Q \sim n_1$ and $P \sim n_5$. See [39,102,103] for the detail that for some definite value of c we may write the ring entropy in terms of the charges Q and P to be

$$S(Q, P, J) = C\sqrt{QP - J} \tag{52}$$

The state-space geometry constructed out of the equilibrium state of this rotating two charged extremal black ring resulting from the entropy may now be computed as earlier from the negative Hessian matrix of the entropy with respect to the charges and angular momentum. Note that the understanding of the state-apace geometry based on the stretched horizon requires the classical time scale limit of fuzzball. This is because the size of the fuzzball is made by the generic states such that its surface area in leading order satisfies a Bekenstein type relation with the entropy of the fuzzball whose boundary surface becomes like a horizon only over classical time scales. We see that the components of the metric tensor are explicitly given as

$$\begin{aligned}
g_{PP} &= \frac{1}{4}CQ^2(PQ - J)^{-3/2}, \;\; g_{PQ} = -\frac{1}{4}C(PQ - 2J)(PQ - J)^{-3/2} \\
g_{PJ} &= -\frac{1}{4}CQ(PQ - J)^{-3/2}, \;\; g_{QQ} = \frac{1}{4}CP^2(PQ - J)^{-3/2} \\
g_{QJ} &= -\frac{1}{4}CP(PQ - J)^{-3/2}, \;\; g_{JJ} = \frac{1}{4}C(PQ - J)^{-3/2}
\end{aligned} \tag{53}$$

We thus see that there exists an intriguing intrinsic geometric enumeration which describes the possible nature of statistical pair correlations in the fuzzball configurations. Our framework affirms that the concerned state-space pair fluctuations determined in terms of the brane charges (or brane numbers) of the D_1-D_5 black holes demonstrate the expected behavior of the underlying heat capacities, *i.e.*, that all of them are positive valued functions of the parameter of Mathur's fuzzy ring. Hence, we may apparently demonstrate that the principle components of state-space pair correlation functions remain positive definite quantities for all admissible values of underlying electric-magnetic charges and angular momentum of the fuzzy black ring configurations.

In the view-points of the simplest D_1-D_5 fuzzy solutions, it turns out that the local stability on the entire equilibrium statistical configurations may now be determined just by computing the determinant of the underlying state-space metric tensor. Radically, we may easily determine the associated determinant of the metric tensor with Gaussian statistical fluctuations. In order to have stability of the full ring configuration, one should thereby demand that the determinant must be a positive definite function of the ring parameters. As explicated in the previous examples, it is easy to observe, in this case as well, that the determinant of the Ruppeiner state-space metric tensor is

$$\|g\| = -\frac{1}{16}C^3(PQ - J)^{-5/2} \tag{54}$$

We thus see for non-zero charges that the determinant of the metric does not vanish and is everywhere a regular function of P, Q, J, and thus defines a non-degenerate intrinsic state-space geometry. Herewith, we observe that the determinant of the metric tensor does not take a well-defined positive definite form, and thus there is no positive definite globally well-defined volume form on the state-space manifold (M_3, g) for the concerned systems. Moreover, the negative state-space determinant $g(P, Q, J)$ thus constructed indicates that the D_1-D_5-J fuzzy rings may decay into certain degenerate vacuum CFT state configurations procuring the same corresponding microscopic entropy. In turn, we may thus deduce that the D_1-D_5-J black rings when they are considered as the bound state of D_1-D_5-brane microstates do not correspond to an intrinsically stable statistical configuration. It is worth to mention that the conclusions thus drawn remain independent of the microscopic descriptions.

In order to examine important global properties in the fuzzy black rings state-space configurations, one is required to further determine the associated invariants of the underlying state-space geometry. For the present configurations, the simplest invariant turns out to be the state-space scalar curvature, which may as well be easily computed by using the intrinsic geometric technology defined earlier, as the negative Hessian matrix of the ring entropy captured by the Mathur's fuzzball contributions. The state-space scalar curvature in the large charge limit in which the asymptotic expansion of the rotating two charge ring entropy is valid, may easily be computed to be

$$R(P, Q, J) = -\frac{5}{2C}(PQ - J)^{-1/2} \tag{55}$$

Notice that the scalar curvature $R(P, Q, J)$ is a regular function of the (P, Q, J), and thus this system corresponds to an everywhere interacting statistical system. Here, it is worth to mention that the scalar curvature is a regular function of electric-magnetic charges and angular momentum. In the large charge limit, in which the asymptotic expansion of the concerned BPS-entropy is valid, the scalar curvature inclines to a vanishingly small value. The negative sign of the state-space scalar curvature discloses that the underlying ring configuration is effectively an attractive system. Thus, the two charge black rings turn out to be less stable than an arbitrary positively curved state-space configuration. Furthermore, it is remarkable that the constant entropy curve in the state-space for any non-zero rotation takes the form

$$PQ - J = k \tag{56}$$

which is just a hyperbolic paraboloid on which the state-space geometry turns out to be a non-degenerate, interacting statistical system. Furthermore, it is easy to notice in the present case that the constant $k := (k_S, k_R)$ may respectively be defined as

$$\begin{aligned} k_S &:= \frac{S_0^2}{C^2} \, for \; constant \; entropy \\ k_R &:= \frac{25}{4C^2 R_0^2} \, for \; constant \; scalar \; curvature \end{aligned} \tag{57}$$

Note however that the vanishing angular momentum limit $J \to 0$ makes the state-space geometry to be ill-defined, which in turn is the same case as that of small black holes. Consequently, it may further be noticed with a different value of k describing the two charge rotating fuzzy rings that the same form of constant entropy curve holds as that of the constant state-space scalar curvature curve. Our analysis thus elucidates the typical feature of possible Gaussian fluctuations about an equilibrium configuration of the D-brane microstates. In the sequel, we shall discuss some definite relations between our state-space geometry and Mathur's fuzzball proposal of constructing brane microstates that define a subensemble of the rotating black holes and black rings.

In this section, we shall next study certain aspects of the state-space geometry for the case of two charge extremal black holes with an angular momentum J from the perspective of Mathur's fuzzball proposal and subensemble theory [101]. In this picture, one can construct the classical space-time geometry with a definite horizon when many quanta of the underlying D_1D_5P CFT lie in the same mode. But a generic state will not have all its quanta placed in a few modes, so the throat of the black hole space-time ends in a very quantum fuzzball in general [39,102,103]. It is however interesting to note that the actual microstates of the system in the fuzzball picture do not have horizon, but it is rather the area of the boundary of the region, where microstates start differing from each other that satisfies a Bekenstein type relation with the entropy inside the boundary. Moreover, different microstates according to string theory are 'cap off' before reaching the end of the infinite throat and thus give rise to different near horizon space-time geometries. In particular, the throat behaves as the inverse of the average radius of fuzzballs. Thus the Bekenstein entropy may be obtained from the area of the above stretched horizon with a statistical interpretation as a coarse graining entropy.

6.2. Subensemble Theory

As mentioned in connection with the Mathur's fuzzball proposal, the microstates of an extremal hole cannot have singularity. Here, we find in the state-space picture that even an intrinsic Riemannian manifold of the equilibrium microstates remains non-singular. In other words, the regularity of the state-space geometric curvature invariant indicates the fact that the intrinsic Riemannian geometry constructed out of the microstates of a rotating two charge ring as the maxima of the entropy remains also regular. The absence of divergences in the scalar curvature consequently implies that the underlying state-space of a rotating two charge ring is thermodynamically stable and thus there are no phase transitions. Moreover, the non-zero $R(P, Q, J)$ indicates that the state-space of the extremal D_1D_5J system corresponds to an underlying interacting statistical system.

We now indicate the nature of our state-space geometry from the viewpoints of the Mathur's subensemble theory. In this perspective, let us consider the conserved quantities of the ring, which are here the charges P, Q and the angular momentum J, and consider a subset of the states that defines the subensemble. Let there be M number of such subensembles in which the entropy of the ring is $\tilde{S}(n_1', n_5', J)$, for any given ensemble in which the total entropy of the ring is $S(n_1, n_5, J)$. Then, the entropy in each subensemble [39,102,103] is given by

$$\tilde{S}(n_1', n_5', J) = \frac{1}{M} S(n_1, n_5, J) \tag{58}$$

Therefore, under the considerations of the subensemble theory, we can define as before the state-space geometry to be the negative Hessian matrix of the ring entropy in any given subensemble, with respect to the conserved charges or number of branes with a rotation. Thus, as a result, we see that the state-space scalar curvature in each subset gets reduced by some large factor, which in turn is precisely the number of subensembles. In particular, as $M \to \infty$, we see that $R \to 0$. Therefore each subensemble with some given number of microstates corresponds to a non-interacting statistical system in the infinite subensemble limit. Of course, this conclusion remains true only in the large charge and large angular momentum limit, in which the computation of the ring entropy is valid. It has been shown [104] that the corrections like space-time higher derivative terms do not spoil the leading order entropy and thus the associated state-space geometry.

In the case of the subensemble of a two charge non-rotating extremal hole, we can easily find that the underlying state-space geometry remains ill-defined, as the norm of the Hessian matrix of the entropy with respect to the charges is zero. Thus the state-space geometry in each subensemble of D_1D_5 system without angular momentum is ill-defined at the leading order entropy. But it is known that the higher derivative corrections make the small black hole state-space geometry well-defined, as there is a non-zero subleading entropy arising from higher derivative contributions. What happens really is that actually in the limiting picture of angular momentum, these two rotating and non-rotating state-space geometries will marry each other in a given ergo-branch. In conclusion, the state-space geometry of a rotating two charged extremal ring in a given ensemble is curved and corresponds to an interacting statistical system, whereas it ceases to be non-interacting in each subensemble with a given number of brane microstates. However, this fact looses its meaning for the case of a non-rotating two charge extremal holes and the underlying state-space geometry becomes ill-defined in either an ensemble or in a subensemble of any given ensemble.

Physically, when many quanta of underlying D_1D_5P-CFT lying in the same mode are considered together, they define the classical black hole space-time geometry having definite horizon structure. In general, all the quanta may not be placed in a few modes for a generic state, so the throat of black hole space-time ends in a very quantum fuzzball. Note however that the actual microstates of the fuzzball system do not have horizon, but it is the boundary region of microstates, where they start differing from each other. Thus, the entropy enclosed inside the area of bounding surface satisfies a Bekenstein-Hawking type of entropy, with an interpretation of the coarse graining statistical entropy.

In the present consideration, this leads to the fact that there exists a non-singular intrinsic Riemannian manifold over an ensemble of equilibrium microstates of the solutions. This is in perfect accordance with the idea of stretched horizon, which arises from the fact that the different microstates are caped-off differently before they reach the end of the respective infinite throat of fuzzballs. Thus, they are allowed to have different near horizon space-time geometries.

7. Bubbling Black Brane Solutions: Black Brane Foams

In this section, we analyze the state-space geometry of equilibrium microstates of the foamed black branes having three charges which appear naturally as supergravity bubbling solutions in M-theory [44,49,105,106]. It is worth recalling the case of T^6 compactification where the N^{th} Gibbons-Hawking charge, far separated from the blob, is described by the associated 2-cycles Δ_{jN}, which must have large flux compared to the fluxes on each $\Delta_{ij}, \forall i,j < N$. In such cases, it has been shown that the flux parameters $\{k_i^I\}$ may further be expressed in terms of the charges over the blob, whose quantized nature arises from the gauge symmetry given by the shift: $k^I \to k^I + c^I V$, with k^I half-integer. We note that this shift symmetry may thus solely be determined by a function V which in fact defines the nature of the Gibbons-Hawking base metric. In the limit of large N, the leading order charges are given by

$$Q_1 \simeq 4N^2 k_0^2 k_0^3, \;\; Q_2 \simeq 4N^2 k_0^1 k_0^3, \;\; Q_3 \simeq 4N^2 k_0^1 k_0^2, \;\; J_R = 16N^3 k_0^1 k_0^2 k_0^3 \tag{59}$$

In the case of maximally symmetric BMPV black hole [44], it turns out that the angular momentum takes the following form: $J_R = 4Q_1Q_2Q_3$. Such a relation does not depend on the number of Gibbons-Hawking base points considered in the theory. Furthermore, it is also independent of the charges distributed on Gibbons-Hawking base points. Reference [44] shows the foaming of black ring solutions with the integer black ring dipole charge(s).

Thus, there exists a non-trivial entropy coming from the combinatorics of possibilities in the choices of $\{k_i^I | \, k_i^I > 0\}$, subject to a constraint that the summations over k_i^I and over both the worldvolume and supergravity (as well as the CFT) descriptions yield the same set of relations between the parameters of brane configurations. Actually, it is the combinatorics of laying out such quantized fluxes on the topologically non-trivial cycles Δ_{ij}, which give rise to the topological entropy of the black brane. Such an example appears in the large charge limit with the following angular momenta

$$J_1^2 = J_2^2 = Q_1Q_2Q_3 \tag{60}$$

This describes the associated leading order properties of bubbled supertubes [45]. One of the natural question which we would like to ask is: how many bubbled black brane configurations exist for some given set of brane charges? The other associated question that we may ask further is: what is the nature of the underlying state-space geometry? In particular, do certain instabilities in a bubbled black brane configuration exist? Notice that the partitioning of k_i^1, k_i^2 and k_i^3 are not independent, and thus the bubble configurations collapse, if any of the flux parameter is taken to be zero. To do so, we need to count the number of all possible ways of having non-zero partitions of the fluxes $\{k_i^I\}$ over N bubbles. Finally, the topological entropy of the black foam is obtained by summing up over the number of Gibbons-Hawking charges N. However, such effects are immaterial for leading order considerations. This is due to the fact that these effects contribute only to the subleading corrections to the black foam entropy [46].

In the rest of the section, we deal with the state-space geometry of large N black brane foam solutions. We analyze the stability conditions and global state-space properties for the fluxes distributed in (i) one direction and (ii) three directions. The leading entropy configuration with the single flux direction corresponding to the single Gibbons-Hawking (GH) center is illustrated in the first subsection. In the next subsection, we extend the state-space picture to the general case, when all possible sets of flux parameters fluctuate.

7.1. A Toy Model: Single GH-center

As the first toy model example of the state-space geometry arising from the entropy of black brane foam, we consider M-theory compactified on T^6, in the large N limit, then one has a set of flux parameters which may be written in terms of the brane charges. Thus, there is an associated entropy which is independent of the number and flux charges in the Gibbons-Hawking base points. The origin of this entropy lies on the possible number of choices of the positive quantized fluxes on topologically non-trivial cycles. These cycles satisfy a set of constraints, *viz.*, that the supergravity and worldvolume descriptions have the same relations between the brane parameters. In particular, this determines the entropy of the foam. Specifically, when one considers the flux parameters $\{k_i^1, k_i^2, k_i^3\}$ to be positive half integers [44], the leading order topological entropy coming from the contributions of partitioning $\{k_i^1\}$ can be solely expressed in terms of charges as

$$S(Q_1, Q_2, Q_3) := \frac{\pi}{3}\sqrt{6}(\frac{Q_2 Q_3}{Q_1})^{1/4} \tag{61}$$

As proclaimed earlier, we notice that the state-space geometry describing the nature of equilibrium brane microstates can be constructed out of the three charges of bubbled black brane foam. The covariant metric tensor can immediately be described from the negative Hessian matrix of the foam entropy, resulting from the underlying statistical configuration. Thus, the brane charges, *viz.*, $\{Q_1, Q_2, Q_3\}$ form the coordinate charts for the state-space manifold of interest. In effect, we may compute typical features of the intrinsic state-space geometry in terms of the brane parameters describing an ensemble of microstates of the single GH center bubbled black brane foam solution. Explicitly, we see that the components of the metric tensor are

$$\begin{aligned}
g_{Q_1Q_1} &= -\frac{5\pi\sqrt{6}}{48Q_1^2}(\frac{Q_2Q_3}{Q_1})^{1/4}, \quad g_{Q_1Q_2} = \frac{\pi\sqrt{6}Q_3}{48Q_1^2}(\frac{Q_1}{Q_2Q_3})^{3/4} \\
g_{Q_1Q_3} &= \frac{\pi\sqrt{6}Q_2}{48Q_1^2}(\frac{Q_1}{Q_2Q_3})^{3/4}, \quad g_{Q_2Q_2} = \frac{\pi\sqrt{6}}{16}(\frac{Q_3}{Q_1})^2(\frac{Q_1}{Q_2Q_3})^{7/4} \\
g_{Q_2Q_3} &= -\frac{\pi\sqrt{6}}{48Q_1}(\frac{Q_1}{Q_2Q_3})^{3/4}, \quad g_{Q_3Q_3} = \frac{\pi\sqrt{6}}{16}(\frac{Q_2}{Q_1})^2(\frac{Q_1}{Q_2Q_3})^{7/4}
\end{aligned} \tag{62}$$

We may thus appreciate for all $i, j, k \in \{1, 2, 3\}$ describing the single GH center bubbling brane configuration that the state-space geometry materializing from the leading order Bekenstein-Hawking entropy of the toroidally compactified M-theory configuration admits remarkably simple expressions in terms of physical charges. As enumerated earlier, we may nevertheless stress out for all non-zero values of the brane charges Q_1, Q_2, Q_3 that all the principle components of concerned state-space metric tensor do not satisfy positivity heat capacity conditions. It is instructive to note that the brane-brane state-space pair correlation, $g_{Q_1Q_1} < 0$, is asymmetric in contrast to other diagonal pair correlations. In fact, the negativity may be understood by arguing that an increment of the Q_1 brane charge reduces the entropy and thus corresponds to unstable local interactions, in contrast with the brane-brane state-space interactions involving either the Q_2 and Q_3 charges.

In order to investigate global stability of full state-space configuration of the foam, we now compute the determinant of the state-space metric tensor, and observe its positivity and functional dependence on the brane charges. As invoked earlier, it may easily be observed that the determinant of the intrinsic state-space metric tensor is a well behaved function of brane charges. In this case, we find that the determinant of the metric tensor is

$$\|g\| = -\frac{\pi^3\sqrt{6}}{384Q_1^4}(\frac{Q_1}{Q_2Q_3})^{-5/4} \tag{63}$$

As the determinant of the basic metric tensor is a small non-zero quantity, in the large charge limit. Our analysis further discovers that there exists a non-degenerate thermodynamic configuration with the $\{k_i^1\}$ contributions. However, it is worth to note that the determinant of the metric tensor does not take a positive definite form, which in turn shows that there is no positive definite volume form on the concerned state-space manifold (M_3, g) of the single GH-center M-theory foams. Now, one can obtain the associated Christoffel symbols Γ_{ijk}, covariant Riemann tensors R_{ijkl}, Ricci tensors R_{ij} in the usual ways. In fact, the components of the covariant state-space Ricci tensor are given by

$$\begin{aligned} R_{11} &= \frac{1}{24Q_1^2}, \;\; R_{1i} = -\frac{1}{48Q_1Q_2}, \\ R_{ii} &= 0, i = 2, 3; \;\; R_{23} = \frac{1}{48Q_2Q_3} \end{aligned} \tag{64}$$

In order to elucidate the universal nature of statistical interactions and the state-space properties concerning M-theory black brane foams, one needs to determine definite global geometric invariant quantities on the state-space manifold (M_3, g). Here, we notice that such an indicated simplest invariant is obtained just by computing the state-space scalar curvature. Nevertheless, we find that the scalar curvature may, in a straightforward fashion, be achieved by applying our formerly explained tools of intrinsic geometry. Finally, we see that the state-space Ricci scalar scales as the inverse of the entropy. It reduces to the following expression

$$R = -\frac{1}{2\pi\sqrt{6}}(\frac{Q_1}{Q_2Q_3})^{1/4} \tag{65}$$

We see that the Ricci scalar is a non-zero and an everywhere regular function of charges $\{Q_1, Q_2, Q_3\}$ and thus the statistical system of underlying unidirectional three charge black brane foams is well-defined and interacting for a range of $\{Q_1, Q_2, Q_3\}$. It is noteworthy to mention that the state-space scalar curvature becomes small in the limit of large brane charges $Q_i \to \infty$, and thus the bubbled configuration only with $\{k_i^1\}$ contributions approaches a weakly interacting statistical system. Furthermore, we see that the constant entropy curve defining the state-space corresponding to non-zero flux parameter in k-direction of the configuration is

$$Q_iQ_j = cQ_k \tag{66}$$

where c is a real constant. Notice further that the same curve describes the constant scalar curvature curve in the state-space, with a different real constant. In fact, a simple inspection finds the concerned state-space constant curves for given entropy S_0 and state-space scalar curvature R_0. As a matter of fact, we observe for the choice of $(i, i, k) = (Q_2, Q_3, Q_1)$ that the constants $c_i := \{c_S, c_R\}$ are defined by

$$c_S = \frac{9S_0^4}{4\pi^4}, \quad c_R = \frac{1}{576\pi^4 R_0^4} \tag{67}$$

It is worth to mention, from the general expression of the determinant of the metric tensor and, in addition, from the state-space scalar curvature signifying the global correlation volume of the underlying statistical system, that the single GH center bubbled systems are unstable and find an attractive statistical nature for given non zero foam entropy solution. Interestingly, we come up with the fact that the state-space scalar curvature, signifying global correlation length of an underlying statistical system, confirms that there is no divergence for all admissible values of the brane charges. Furthermore, it is evident that the state-space scalar curvature varies as an inverse function of the entropy of the chosen single center GH center bubbled configurations.

7.2. Black Brane Foams

In order to describe the state-space geometric implications of the generic foam solutions, we focus on the most exhaustively studied three charged black brane configurations, which naturally emerge from the equilibrium microstates of various bubbling supergravity solutions [44,49,105,106]. As we have encountered the state-space geometry of the single GH center bubbled black branes in the previous subsection, in this subsection, we shall demonstrate the state-space fluctuations for unrestricted 3-charge bubbled black brane foam configurations having positive half integer flux parameters [44]. Interestingly, we appreciate for all $i, j, k \in \{1, 2, 3\}$ describing the three GH center bubbling brane configuration that the state-space geometry materializing from the leading order Bekenstein-Hawking entropy of the toroidally compactified M-theory configuration admits remarkably simple expressions in terms of the physical charges.

The detailed construction of the M-theory configurations nevertheless confirms that the set of CFT microstates which are dual to the bubbled space-time geometries has interesting intrinsic state-space geometric features. In fact, it may be envisaged further that the nature of the equilibrium statistical configuration may be highlighted from the outset of quadratic fluctuations. In order to so, we may now consider all factors coming from the possible partitioning of the flux parameters $\{k_i^1, k_i^2, k_i^3\}$. Thence, one finds that the leading order topological entropy of the three charged black foam can be characterized by three charges Q_1, Q_2 and Q_3. A detailed analysis [44] shows that the concerned Bekenstein-Hawking entropy of general black brane foams may be expressed as

$$S(Q_1, Q_2, Q_3) := \frac{2\pi}{\sqrt{6}}\{(\frac{Q_2 Q_3}{Q_1})^{1/4} + (\frac{Q_1 Q_2}{Q_3})^{1/4} + (\frac{Q_1 Q_3}{Q_2})^{1/4}\} \tag{68}$$

As stated in the previous sections, it turns out that the state-space metric tensor thus characterizing the generic three charge foam configurations may again be given by a negative Hessian matrix of the foam entropy with respect to extensive brane charges. A straightforward computation yields the components of the state-space metric tensor of the foam microstates, which in this case are characterized by the three conserved charges carried by the black brane foam. In particular they are given by

$$\begin{aligned}
g_{Q_1Q_1} &= -\pi\{\frac{5\sqrt{6}}{48Q_1^2}(\frac{Q_2Q_3}{Q_1})^{1/4} - \frac{\sqrt{6}}{16Q_1}(\frac{Q_2}{Q_3Q_1})^{1/4} - \frac{\sqrt{6}}{16Q_1}(\frac{Q_3}{Q_2Q_1})^{1/4}\} \\
g_{Q_1Q_2} &= -\pi\{-\frac{\sqrt{6}Q_3}{48Q_1^2}(\frac{Q_1}{Q_2Q_3})^{3/4} + \frac{\sqrt{6}}{48Q_3}(\frac{Q_3}{Q_1Q_2})^{3/4} - \frac{\sqrt{6}Q_3}{48Q_2^2}(\frac{Q_2}{Q_1Q_3})^{3/4}\} \\
g_{Q_1Q_3} &= -\pi\{-\frac{\sqrt{6}Q_2}{48Q_1^2}(\frac{Q_1}{Q_2Q_3})^{3/4} - \frac{\sqrt{6}Q_2}{48Q_3^2}(\frac{Q_3}{Q_1Q_2})^{3/4} + \frac{\sqrt{6}}{48Q_2}(\frac{Q_2}{Q_1Q_3})^{3/4}\} \\
g_{Q_2Q_2} &= -\pi\{\frac{5\sqrt{6}}{48Q_2^2}(\frac{Q_1Q_3}{Q_2})^{1/4} - \frac{\sqrt{6}}{16Q_2}(\frac{Q_3}{Q_1Q_2})^{1/4} - \frac{\sqrt{6}}{16Q_2}(\frac{Q_1}{Q_3Q_2})^{1/4}\} \\
g_{Q_2Q_3} &= -\pi\{\frac{\sqrt{6}}{48Q_1}(\frac{Q_1}{Q_2Q_3})^{3/4} - \frac{\sqrt{6}Q_1}{48Q_3^2}(\frac{Q_3}{Q_1Q_2})^{3/4} - \frac{\sqrt{6}Q_1}{48Q_2^2}(\frac{Q_2}{Q_1Q_3})^{3/4}\} \\
g_{Q_3Q_3} &= -\pi\{\frac{5\sqrt{6}}{48Q_3^2}(\frac{Q_1Q_2}{Q_3})^{1/4} - \frac{\sqrt{6}}{16Q_3}(\frac{Q_2}{Q_1Q_3})^{1/4} - \frac{\sqrt{6}}{16Q_3}(\frac{Q_1}{Q_2Q_3})^{1/4}\}
\end{aligned} \tag{69}$$

It follows from the above expressions that the statistical pair correlations thus described may in turn be accounted for a simple geometric description expressed in terms of the brane charges, connoting an ensemble of fluxes for the general three centered bubbling black brane configurations. Furthermore, we may easily observe that the principle components of the underlying state-space configuration are positive definite, for a range of allowed values of the bubbling parameters of the multi GH center M-theory bubbling solutions. We thus notice that the corresponding microscopic nature of the state-space manifold may easily be read off from the position of the GH-points and the flux parameters associated with an irregular shape of the brane blob, under the scaling of true microstates of the black brane foams.

In this framework, it seems interesting to investigate the full bubbled configuration space of the M-theory black brane solutions. In particular, it is evident that the principle components of state-space metric tensor $\{g_{Q_iQ_i} \mid i = 1, 2, 3\}$ signify a set of definite heat capacities (or the related compressibilities), whose positivity holds for a range of involved charges. In this case, one may thus apprise that the bubbled black holes comply an underlying locally equilibrium statistical configurations. It is further intriguing to note that the positivity of $g_{Q_iQ_i}$ holds even if some of the brane charges of the associated brane charges become zero. This is clearly perceptible because of the fact that there exists a set of brane charges (Q_1, Q_2, Q_3) such that foams with general partitioning of the flux parameters remain physical and lead to locally stable statistical configurations.

Moreover, it is not difficult to investigate the global stability on the full state-space configuration, which may in fact be easily carried out by computing the determinant of the state-space metric tensor. Requiring positivity of the volume form on the state-space manifold of the foam solutions imposes a condition of stability on the Gaussian statistical fluctuations, *i.e.*, the state-space metric tensor of underlying foam configurations must be positive definite. In effect, we may specifically observe that the determinant of the metric tensor now takes the simple form

$$\|g\| = -\frac{\pi^3\sqrt{6}}{384}(Q_1Q_2Q_3)^{-13/4} f_1(Q_1, Q_2, Q_3) \tag{70}$$

where the function f_1 is defined by

$$f_1(Q_1, Q_2, Q_3) \;\; := \;\; -Q_1^{3/2}Q_2Q_3^2 - Q_1Q_2^{3/2}Q_3^2 + 3Q_1^{3/2}Q_2^{3/2}Q_3^{3/2} - Q_1^{3/2}Q_2^2Q_3$$

$$-Q_1Q_2^2Q_3^{3/2} - Q_1^2Q_2^{3/2}Q_3 + Q_1^2Q_2^{1/2}Q_3^2 + Q_1^2Q_2^2Q_3^{1/2}$$
$$-Q_1^2Q_2Q_3^{3/2} + Q_1^{1/2}Q_2^2Q_3^2 \quad (71)$$

For a domain of brane charges, we thus observe a non-degenerate metric for the state-space geometry at extremality, as an intrinsic Riemannian manifold. Moreover, one finds that the determinant of the metric tensor remains positive for all (Q_1, Q_2, Q_3), if the function $f_1(Q_1, Q_2, Q_3)$ takes a definite negative value. In this case, we thus find that the determinant of the intrinsic state-space metric tensor is a well behaved function of brane charges, and remains regular in general, unless one of the brane charges takes an infinite value. Note further that the $\Gamma_{ijk}, R_{ijkl}, R_{ij}$ are not difficult to obtain. Furthermore, the scalar curvature corresponding to this state-space geometry elucidates the typical feature of Gaussian fluctuations about an equilibrium brane microstates of the desired bubbling foam configurations.

As a standard interpretation, the state-space scalar curvature describes the nature of underlying statistical interactions of possible microscopic brane configurations, which in particular turn out to be non-zero and well defined for a large band of the parameters of the chosen foam solution. Furthermore, we may easily procure that the scalar curvature determines a global invariant on the three dimensional state-space manifold. In this case, it thus follows that one may explicate the average nature of underlying microscopic black brane configurations. Such an interesting invariant, which accompanies the information of the global correlation volume of underlying statistical systems, turns out to be the intrinsic state-space scalar curvature. In this geometric framework, the state-space curvature scalar may thence be easily read-off to be

$$R = -\frac{\sqrt{6}}{12\pi}(Q_1Q_2Q_3)^{7/4} f_2(Q_1, Q_2, Q_3) f_1(Q_1, Q_2, Q_3)^{-3} \quad (72)$$

where f_2 is defined by

$$\begin{aligned} f_2(Q_1, Q_2, Q_3) \ := \ & -10Q_1^4Q_2^2Q_3^2 - 10Q_1^2Q_2^2Q_3^4 - 10Q_1^2Q_2^4Q_3^2 + Q_1^4Q_2^4 + Q_1^4Q_3^4 + Q_2^4Q_3^4 \\ & +4Q_1^4Q_2^{5/2}Q_3^{3/2} - 27Q_1^3Q_2^2Q_3^3 + 4Q_1^{5/2}Q_2^{5/2}Q_3^4 + 4Q_1^{3/2}Q_2^{5/2}Q_3^4 \\ & +4Q_1^{5/2}Q_2^4Q_3^{3/2} - 4Q_1^4Q_2^{7/2}Q_3^{1/2} + 4Q_1^3Q_2^4Q_3 - 4Q_1^{7/2}Q_2^4Q_3^{1/2} \\ & -27Q_1^3Q_2^3Q_3^2 - 4Q_1^{1/2}Q_2^{7/2}Q_3^4 - 4Q_1^2Q_2^{7/2}Q_3^{5/2} + 6Q_1^3Q_2^{3/2}Q_3^{7/2} \\ & -4Q_1^{7/2}Q_2^2Q_3^{5/2} - 4Q_1^{7/2}Q_2^{5/2}Q_3^2 + 22Q_1^{5/2}Q_2^{5/2}Q_3^3 - 4Q_1^{5/2}Q_2^{7/2}Q_3^2 \\ & -4Q_1^{5/2}Q_2^2Q_3^{7/2} + 2Q_1Q_2^{7/2}Q_3^{7/2} + 2Q_1^{7/2}Q_2Q_3^{7/2} + 2Q_1^{7/2}Q_2^{7/2}Q_3 \\ & +22Q_1^{5/2}Q_2^3Q_3^{5/2} + 6Q_1^{7/2}Q_2^{3/2}Q_3^3 + 4Q_1^4Q_2Q_3^3 - 4Q_1^{7/2}Q_2^{1/2}Q_3^4 \\ & +4Q_1^3Q_2Q_3^4 + 4Q_1Q_2^4Q_3^3 + 4Q_1^{3/2}Q_2^4Q_3^{5/2} + 4Q_1^4Q_2^{3/2}Q_3^{5/2} \\ & +4Q_1Q_2^3Q_3^4 + 4Q_1^4Q_2^3Q_3 - 4Q_1^{1/2}Q_2^4Q_3^{7/2} - 27Q_1^2Q_2^3Q_3^3 \\ & +6Q_1^{7/2}Q_2^3Q_3^{3/2} + 6Q_1^3Q_2^{7/2}Q_3^{3/2} + 22Q_1^3Q_2^{5/2}Q_3^{5/2} - 4Q_1^2Q_2^{5/2}Q_3^{7/2} \\ & +6Q_1^{3/2}Q_2^3Q_3^{7/2} + 6Q_1^{3/2}Q_2^{7/2}Q_3^3 - 4Q_1^4Q_2^{1/2}Q_3^{7/2} \end{aligned} \quad (73)$$

The curvature scalar is thus finite and everywhere regular, for all well-defined foam charges $\{Q_1, Q_2, Q_3\}$, except for the case $f_1(Q_1, Q_2, Q_3) = 0$. It is however interesting to note that, in the large charge limit where the entropy computation is valid, and whenever $f_2(Q_1, Q_2, Q_3) \neq 0$, we see that the underlying scalar curvature is a non vanishing function of the brane charges, and thus indicating an underlying interacting statistical system. Furthermore, we observe that there are no phase transition(s) or critical lines or such commensurable phenomena in the underlying state-space manifold of three charge black brane foam solutions, except for the vanishing determinant configuration. We see that the underlying state-space geometry remains well-defined, only as an intrinsic Riemannian manifold, $M := M_3 \setminus B$, where B is the set of charges defined by

$$B := \{(Q_1, Q_2, Q_3) | f_1(Q_1, Q_2, Q_3) = 0\} \tag{74}$$

Furthermore, the entire state-space configuration of three GH center bubbled black brane foam solutions remains positive definite for a set of admissible values of the brane charges Q_i. In turn, we observe that the underlying state-space geometry of three charge foam configurations in M-theory are in good compliance with supergravity, indicating that they are BPS objects. Generically, it turns out that they correspond to a non-degenerate fluctuating statistical basis as an intrinsic Riemannian manifold $N := M_3 \setminus B$. We may note further that the components of covariant Riemann tensors become zero for definite values of charges on branes and antibranes. In addition, the Ricci scalar curvature diverges at the same set of points on the state-space manifold (M_3, g), as that of the roots of determinant of the metric tensor, which are defined by the set B. In this case, we observe for some entropy S_0 that the constant entropy curve is

$$(\frac{Q_2 Q_3}{Q_1})^{1/4} + (\frac{Q_1 Q_2}{Q_3})^{1/4} + (\frac{Q_1 Q_3}{Q_2})^{1/4} = c \tag{75}$$

where the real constant c takes the value $c := \sqrt{6} S_0 / 2\pi$. We easily see that this case is not the same as that of the previous subsection, but rather here the curve of constant curvature scalar is given by $f_1(Q_1, Q_2, Q_3)^3 = K(Q_1 Q_2 Q_3)^{7/4} f_2(Q_1, Q_2, Q_3)$. Ultimately, we find with non-zero three flux parameters that this foam system corresponds to a non-interacting statistical system on a hypersurface defined by the equation $f_2(Q_1, Q_2, Q_3) = 0$. Note however that the entropy considered here is the one arising from the usual two derivative terms in the low energy effective supergravity action, which is consistent with the Bekenstein-Hawking area law. Certainly, the expression of the foam entropy will be modified by contributions from higher derivative terms in the effective action or corrections arising from the topological Chern numbers. Reference [107] describes the case of associated microscopic considerations.

Consequently, such an analysis would modify the equilibrium microstates, and thus the corresponding state-space geometry, including the curvature scalar. In particular, the origin of the state-space geometry of the foam is rather independent of the number of charges and Gibbons-Hawking base points. Indeed, it solely relies on the number of possible choices of positive quantized fluxes on each topologically non-trivial cycles, defining associated brane charges. Such an analysis may also be accomplished in terms of extensive flux parameters considered to be either positive half integers or positive integers.

In this section, we thus observe the state-space geometric distinction between the family of smooth large charged BPS space-time geometries corresponding respectively to (i) unidirectional one family of fluxes and (ii) generic foam solutions in M-theory with three families of Gibbons-Hawking fluxes. In the case of a large number of two cycles, we find that a class of black brane foams, having correct charges and angular momenta, define a non-degenerate curved intrinsic Riemannian geometry. Thus, it is possible to realize state-space correlations among the equilibrium microstates of a three charged maximally spinning BPS black hole in five dimensions. This is achieved by computing the state-space metric tensor and the associated scalar curvature. It is well known that the foam solution [108] can be dualized to the frame of D_1D_5P charges, which is asymptotic to $AdS_3 \times S^3 \times T^4$. Thus, we may find a clue of the two point state-space correlation functions of dual D_1D_5P-CFT states. Furthermore, this indicates the typical nature of the correlations among CFT microstates of the black brane. These are dual to the thermodynamic state-space correlation functions, arising from Gaussian statistical fluctuations of foam entropy. The associated state-space geometric notions are also interesting in a different regime of the parameters of the bubbling solution, *i.e.*, the non-large charge limit. This task is however left for a future publication.

8. Discussion and Conclusions

The present paper has explored the state-space geometry of various interesting string theory, extremal and non-extremal black brane solutions, with or without higher derivative α'-corrections, multi-centered black brane configurations, fuzzy rings and bubbling black brane foam solutions in M-theory. We find in all such cases that the underlying state-space geometry is well-defined and is an regular intrinsic Riemannian manifold over a range of brane charges. The fact that the various state-space geometries turned out to be everywhere regular is in an agreement with the fact that the BPS black branes are stable objects under higher derivative corrections, brane fractionation, brane fractionation and Mathur's subensemble theory, or M-theory bubbling black brane configurations. We have analyzed the nature of the state-space geometry arising from the entropy of the (extremal) black brane solutions. These configurations are considered as the bound states of a large number of microstates, having finitely many electric-magnetic charges, angular momenta. This, in the case of non-extremal black brane configurations, requires an addition of the mass or the charges on respective antibranes.

The higher order stringy corrections modify the Einstein-Hilbert action and thus with such corrections, one finds that the horizon area is being stretched which leads to a non-zero entropy. Without the higher order stringy corrections, viz., space-time R^2 and R^4-corrections, we have shown that the state-space geometry associated with this system is ill-defined, which may easily be seen from the determinant of the negative Hessian matrix of the entropy, with respect to the charges carried by the branes. Furthermore, this result holds for any two parameter constant entropy curve, whose state-space coordinates lies on the rectangular hyperbola. However, upon inclusion of higher derivative R^2 corrections, we find, for non-supersymmetric electrically and magnetically charged small black holes, that this state-space geometry becomes a non-degenerate intrinsic Riemannian geometry. A closure look to such a corrected state-space geometry of equilibrium microstates of the two charged small black holes allows us to reaches the conclusion that the state-space of non-supersymmetric two charge small black holes with quartic corrections is a non-interacting statistical system. Thus, this it indicates that there are no thermodynamic instabilities and phase transitions anywhere in this state-space, whose constant entropy curve takes the form of a hyperbola. Moreover, similar results hold for the three and four charged tree level extremal black holes, whose state-space geometry takes the same form of the associated metric tensors, and in turn, yields everywhere regular scalar curvatures, corresponding to the charged extremal black hole configurations. In these cases, it turns out that the determinant of the state-space metric tensors and the scalar curvatures both scale as the inverse of the square root of the leading order entropies. The associated constant entropy curves take the form of some higher dimensional hyperbolas.

The thermodynamic geometry for non-extremal black holes turns out to be quite interesting from the view-point of the underlying state-space interactions. We anticipate that the interactions among bound states of the brane-antibrane pairs are non-trivial. This is due to the fact that a non-extremal black hole configuration can be obtained by adding the corresponding antibranes to the extremal black brane configuration. In particular, the microscopic description of a simplest non-extremal D_1, D_5 system requires a string with some extra energy having large amount of winding and D_5 brane charges, which creates equal amount of momenta running in the opposite direction of the S^1. In this case, the state-space metric tensor implies a non-zero determinant of the metric tensor, for a set of given non-zero brane-antibrane charges.

In particular, we find that the state-space geometry of the four charged non-extremal D_1D_5P black hole system is non-degenerate. The curvature scalar with such non-extremal contributions turns out to be non-zero, and thus interacting, except for the set of the roots of a two variable polynomial $f(n_p, \overline{n_p})$, as a function of the two momenta running in the opposite directions of the S^1. In this case, the constant entropy curve is no more a standard hyperbola as that of the extremal black branes. One can further extrapolate the above leading order entropy to a non-large charge domain, where the microstates are no longer close to the supersymmetric states. Thus, one deals in one stroke with all special extremal and near-extremal configurations. For example, it is not difficult to explore the state-space geometry arising from the leading order entropy expression of the equilibrium microstates of the six charge anticharge non-extremal black holes in $D = 4$. We see that the state-space geometry, defined as the negative Hessian matrix of the entropy with respect to the conserved charges-anticharges carried by the black hole, is non-degenerate for all non-zero brane-antibrane charges, except for a set B of brane-antibrane charges defined by the roots of the determinant of the state-space metric tensor. The state-space geometry thus remains well-defined as an intrinsic Riemannian manifold $N := M_6 \setminus B$. In this case, the scalar curvature has further been computed by using the intrinsic Riemannian metric of non-extremal corrected contributions. In turn, we have shown that the state-space geometry has a similar nature to that of the non-extremal charge-anticharge Kaluza-Klein (KK) observed system.

It is worth to note that some components of the covariant Riemann tensors may diverge at the roots of some polynomials of lower degrees, as a function of brane and antibrane charges, while those components of the covariant Riemann tensors, which come with an equal brane-antibrane composition, diverge differently as the roots of a single higher degree polynomial. We have also observed that certain components of the covariant Riemann tensors may become zero, for certain values of the charges-anticharges. Furthermore, the Ricci scalar curvature diverges on the set B of the roots of the determinant of the metric tensor, as well as it becomes null on some single higher degree polynomial. Exactly at these points in the state-space of the underlying extremal or near-extremal, or those of a general black hole, the system corresponds to a non-interacting statistical basis. The state-space nature of the non-extremal black hole with an addition of non-trivially fibered KK-monopoles defines the same pattern, and it can as before be determined as the function of charges-anticharges. In this case too, we see that there exists the same pattern of state-space constant curvature and constant entropy curves retain the expressions similar to that of the three charge non-extremal black holes.

The state-space geometry containing four charged single centered black brane configurations and that of multi-centered black brane molecule configurations may be examined under the consideration of pin-sized D-brane systems. In particular, it turns out that the state-space geometry arising from the counting entropy of the microstates of the entropically dominant multi-centered $D_6D_4D_2D_0$ black holes is a weakly interacting statistical system, in contrast to the case of single centered black holes. Note that this observation is consistent with fact that the existence of multi-centered black hole bound states depends on the choice of vacuum, and such multi-centered black hole configurations, in the weak string coupling limit, collapse to a single D-brane configuration. However, we see that in suitable parameter regimes, the uniform large charged multi-centered configurations dominate the single centered configurations. We have shown that the state-space geometric implication for the case of $D_6D_4D_2D_0$ system with a charge vector Γ obtained by wrapping D_4, D_2 and D_0 branes around various cycles of a three two-tori $T_1^2 \times T_2^2 \times T_3^2$, on which the type IIA string theory is being compactified, indicates that there exist two regions in the state-space, which respectively deal with single centered and multicentered solutions for a scaling $\Gamma \to \Lambda\Gamma$. In particular, the interactions present in the state-space for the case of two centered solutions scale as Λ^{-3}, while they scale as Λ^{-2} for single centered D-brane configurations.

It turns out that the state-space geometry obtained in the entropy representation has a positive definite constant determinant. Over the range of parameters, this state-space defines a well-defined non-degenerate intrinsic Riemannian geometry, which is parameterized by the D-brane charges. We have observed that the corresponding scalar curvature of this state-space manifold is a non-zero, positive and everywhere regular function of the D-brane charges. Moreover, this fact seems to be universal, which is related to the typical form for the state-space geometry arising from the negative Hessian matrix of the black brane entropy. As with the standard interpretation of the state-spaces geometry, the non-vanishing of the scalar curvature describes that the underlying statistical system is in general interacting for $D_6D_4D_2D_0$ configurations. Note that the absence of divergences in the scalar curvature indicates that this system is thermodynamically everywhere stable. Thus, there are no phase transitions or other such critical phenomena in the state-space manifold of D-brane systems.

It is worth to mention that in the case of a single centered black hole configuration and that of the double centered black hole configurations, which are respectively described by a total charge center $\Gamma := \Lambda(0, 6, 0, -12)$ and the two charge centers $\Gamma_1 = (1, 3\Lambda, 6\Lambda^2, -6\Lambda)$ and $\Gamma_2 = (-1, 3\Lambda, -6\Lambda^2, -6\Lambda)$, we find that the state-space scalar curvatures remain non-zero, positive quantities and, in particular, take the same numerical values for the case of the two charge centers Γ_1 and Γ_2. Therefore, we predict from this information that the correlation volumes remain identical in the two sub-configurations of a double centered black brane configuration. It is thus instructive to note that the nature of interactions present in the underlying state-space manifold corresponding to the single and doubled centered configurations behaves differently. In particular, we see that the state-space scalar curvature of the double centered configurations goes to zero faster than that of the single centered configurations, in the limit of $\Lambda \to \infty$. It may further be indicated that the two point correlations become very small, in the uniform large charge limit with fixed moduli at the infinity. Thus, both of them should indicate a weakly interacting statistical system. This is indeed the correct picture, as the calculation of the entropy is based on the consideration that such a brane configuration is a bound state of the weakly interacting D-branes.

On other hand, in order to make contact between our state-space geometry and the brane fractionation, we have considered the chiral primaries of $SU(1,1 \mid 2)_Z$ in a supersymmetric ground state of $\mathcal{N} = 4$ supersymmetric quantum mechanics of the D-branes, associated with the two charge extremal small black holes having $AdS_2 \times S^2$ near horizon geometry, which are obtained in the compactification of type IIA string theory, either on $K_3 \times T^2$ or on T^6. The state-space geometry thus computed from the fractional brane entropy, for an arbitrary finite number of the clusters in which the total N unit of the D_0-brane charge splits, turns out to be a degenerate intrinsic Riemannian manifold. Further, it is easy to see that this ill-defined state-space geometry of N fractional branes with $\{n_i\}_{i=1}^k$ electric charges corresponds to k-clusters in the background of D_4-branes having p unit of magnetic charges. The constant entropy curve for an arbitrary k-clustered small black hole configuration follows simply from the brane number conservation constraint.

In order to acquire further support with the recent microscopic studies of black hole physics, we have investigated that the state-space geometry of two charge extremal black holes with an angular momentum renders to be very illuminating from the perspective of the fuzzballs and subensemble theory. In particular, we have shown that the state-space constructed out of equilibrium microstates of an extremal black ring, whose CFT deals with a n_1 number of D_1 brane having charge Q and a n_5 number of D_5 brane having charge P carrying angular momentum J, is a non-degenerate, everywhere regular intrinsic Riemannian manifold. This is due to fact that the determinant of the metric tensor is non-zero and the state-space scalar curvature, in the large charge limit in which the asymptotic expansion of the entropy of the rotating two charge ring is valid, turns out to be a non-zero, regular function of (P, Q, J), which is in perfect accordance with the Mathur's fuzzball proposal that the microstates of an extremal hole cannot have singularity. The absence of divergences in the scalar curvature consequently implies that the underlying state-space of a rotating two charge ring is thermodynamically stable, and thus there are no phase transitions. Consequently, this system corresponds to an non-degenerate interacting statistical system, whose constant entropy curve in the state-space, for any non-zero rotation, takes the form of a hyperbolic paraboloid, on which the state-space geometry is defined. It is worth to mention that the underlying state-space geometry turns out to be ill-defined in the vanishing angular momentum limit, which in turn is the same case as that of small black holes.

We have further indicated the nature of state-space geometry of such a ring from the perspective of Mathur's subensemble theory. In particular, we considered the conserved charges and angular momentum of the ring and focused our attention on a subset of the microstates that define the subensembles, such that the total ring entropy as an additive thermodynamic quantity, remains conserved. Under this consideration, we find, as before, that the state-space geometry defined in any given subensemble, as the negative Hessian matrix of the ring entropy, in the large charge and large angular momentum limit with respect to the conserved charges or the number of branes, is a non-degenerate and everywhere regular, curved, intrinsic Riemannian manifold. Moreover, we see that the state-space scalar curvature in each subset gets precisely reduced by the number of subensembles. In particular, each subensemble with a large number of given microstates corresponds to a non-interacting statistical system in the infinite subensemble limit. From the perspective of the fuzzball geometries, it is worth to mention that higher derivative space-time corrections do not spoil our state-space geometric conclusions, as the associated extremal ring entropy remains intact under such corrections, see for details [104].

In the case of vanishing angular momentum, we have demonstrated that the state-space geometry, under any subensemble of a two charge non-rotating extremal holes, is ill-defined. This may easily be read off from the norm of the state-space tensor with respect to the brane charges. Thus, the state-space geometry in each subensemble of D_1D_5-CFT without angular momentum is ill-defined at the leading order entropy. However, it is known that the subsequent higher derivative contributions are nontrivial. The non-zero subleading small black hole entropy corresponds to a non-degenerate state-space geometry. An interesting aspect of our intrinsic geometric study is that the two rotating and non-rotating state-space geometries marry each other in the given ergo-branch. This happens in the limiting picture of vanishing angular momentum, holding either over an ensemble, or over a subensemble of the given ensemble.

As the final exercise, we have analyzed the state-space geometry of a three charged foamed bubbling black brane supergravity configurations, from the perspective of the large N limit of M-theory compactified on T^6. We have shown that the state-space geometry arising from the negative Hessian matrix of the foam entropy associated with the single center Gibbons-Hawking base points, with respect to the extensive brane charges, is a non-degenerate, everywhere regular, curved intrinsic Riemannian manifold. Over a rage of brane charges, the underlying statistical system of such a three charge unidirectional black foam is everywhere well-defined, and it is an interacting statistical system for the extensive brane charges. Up to the scaling, we may see further that both the constant entropy curve and the constant state-space curvature one are defined by the constraint $Q_iQ_j = cQ_k$, where the non-zero flux lies in the k-direction of the state-space of the three charged black foam.

In the most general consideration, when all flux factors coming from various partitionings of the flux parameters contribute to the leading order topological entropy of three charged black brane foam, we have shown that the state-space geometry characterized by the Hessian of the entropy, as a function of the conserved charges of the black foam, pertains to be a non-degenerate intrinsic Riemannian manifold, except at the zeros of the determinant of the metric tensor. In this case, it turns out further that the state-space scalar curvature is an everywhere regular and finite function of the foam charges, and thus it indicates an underlying interacting statistical system, except for the zeros of the curvature scalar. Furthermore, it is worth to mention that the underlying state-space geometry remains well-defined as an intrinsic Riemannian manifold $M := M_3 \setminus B$, where the set of charges B is defined as the set of the roots of the determinant of the state-space metric tensor. It is easy to see that this is not the same case as that of the a three charge unidirectional black brane foam, but rather the constant scalar curvature curve turns out to be a non-trivial constraint on the state-space manifold. In contrast, the state-space of the three charge unidirectional black foam is always an interacting statistical system.

Note that the state-space geometry of the black brane foam thus considered is the one arising from the entropy, obtained from the leading order contributions of a low energy effective supergravity action. However, certainly the entropy expression will be modified by the contributions coming from higher derivative corrections arising from topological Chern numbers, which would consequently modify the equilibrium state-space geometry of the black foam. Indeed, the state-space geometries explored, for a large family of smooth three charged BPS space-time geometries, indicate that the nature of the correlation volume and that of the two point correlation functions among the microstates of a three charged maximally spinning BPS black hole in five dimensions, as well as that of a large class of black brane foams having the correct charges and angular momenta, may easily be analyzed in the case of a large number of two cycles. It is further instructive to investigate the detailed nature of such correlations, *i.e*, whether the state-space perspective arising from the topological entropy of the bubbling configurations remains the same for the corresponding D_1D_5P dual CFT configurations, if one goes beyond the Gaussian statistical fluctuations.

The construction of any state-space manifold, whose constant entropy curve is a k-rectangular hyperbola in a given S, T-duality basis, is ill-defined. In turn, this is the case for the state-space of fractionated extremal D_0-D_4 holes with k-clusters of the electric D_0-branes. However, the underlying state-space becomes non-degenerate by adding one or more coordinates to the constant entropy curve in the state-space. The simplest example of such instances comes with the rotation contribution to the entropy. As a matter of fact, we have analyzed the fuzzy black ring configuration in detail. Here, the constant entropy curve is just the hyperbolic paraboloid which offers the nature of the state-space geometry. Therefore, further understanding of these matters is required, which will ultimately depend on how well the entropy of a given brane configuration can be understood. From the perspective of Weinhold geometry, these effects are directly related to the moduli space geometry and thus one may anticipate them from possible different viewpoints, to understand the microscopic origin of the thermodynamic geometry of extremal and non-extremal black branes. Such a notion may be expected to arise from the consideration of a large number of coincident D-branes or M-branes.

Finally, the study of covariant state-space geometric issues sheds light on the nature of two point state-space correlation functions and the correlation volume of the underlying boundary conformal field theory, *i.e.*, whether one would have a stable or unstable system from the viewpoints of a CFT. Therefore, it seems possible to understand the microscopic origin of thermodynamic singularities of non-extremal black branes, from the associated CFT data consisting of brane-antibrane pairs. In short, our geometric notions are important to understand the nature of statistical interactions present in an ensemble of weakly interacting dual CFT microstates. The concerned macroscopic notions of extremal and non-extremal black holes and general black branes arise from some compactifications of string theory and M-theory.

In certain cases of the chosen state-space geometries, it is observed that the determinant of the state-space metric tensor is negative, whereas for complex systems we find it to be positive definite. In particular, the foamed three charge black branes and fuzzy black rings have a negative determinant whereas, the latter is everywhere positive for the two centered $D_6D_4D_2D_0$ black holes. In the case of specific D-brane solutions, it seems thus that there are an attractor tree and topological flow tree data, but this is not the case with charged black brane foams or fuzzy rings. Thus, their entropy needs some further corrections, which may possibly be the case for example of the non-perturbative instanton corrections. In the considered state-space examples, it has been further shown that there are no singularity in the state-space geometry of a class of black brane systems. The state-space study thus supports the general conjecture that the interior of black branes is non-trivial, and there are no singularities on the state-space, except those which above from the roots of the determinant of the state-space metric tensor. This supports the Mathur's fuzzball proposal. It renders that generic state-space geometries of string theory and M-theory brane solutions are non-degenerate, smooth intrinsic Riemannian manifolds, for an appropriate entropy defined as a function of the charges of the black brane configuration. Moreover, this article shows that the state-space manifolds of the underlying black brane solutions in string theory and M-theory indicate an everywhere regular near equilibrium statistical configuration.

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References

1. Strominger, A.; Vafa, C. Microscopic origin of the Bekenstein-Hawking entropy. *Phys. Lett. B* **1996**, *379*, 99-104.
2. Callan, C. G.; Maldacena, J. M. D-brane approach to black hole quantum mechanics. *Nucl. Phys. B* **1996**, *472*, 591-610.
3. Maldacena, J. M. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.* **1998**, *2*, 231-252.
4. Bellucci, S.; Ivanov, E.; Krivonos, S. AdS/CFT equivalence transformation. *Phys. Rev. D* **2002**, *66*, 086001.
5. Bellucci, S.; Galajinsky, A.; Ivanov, E.; Krivonos, S. AdS_2/CFT_1, canonical transformations and superconformal mechanics. *Phys. Lett. B* **2003**, *555* , 99-106.
6. Ferrara, S.; Kallosh, R.; Strominger, A. N=2 extremal black holes. *Phys. Rev. D* **1995**, *52*, R5412-R5416.
7. Strominger, A. Macroscopic entropy of $N = 2$ extremal black holes. *Phys. Lett. B* **1996**, *383*, 39-43.
8. Ferrara, S.; Kallosh, R. N=2 extremal black holes. *Phys. Rev. D* **1996**, *54*, 1514-1524.
9. Ferrara, S.; Gibbons, G. W.; Kallosh, R. N=2 extremal black holes. *Nucl. Phys. B* **1997**, *500*, 75-93.
10. Bellucci, S.; Ferrara, S.; Marrani, A. *Supersymmetric Mechanics, Vol. 2 The Attractor Mechanism and Space Time Singularities*; Springer: Berlin/Heidelberg, Germany, 2006, pp. 1-225.
11. Bellucci, S.; Ferrara, S.; Marrani, A. Extremal black hole and flux vacua attractors. *Lect. Notes Phys.* **2008**, *755*, 1-77.
12. Bellucci, S.; Ferrara, S.; Gunaydin, M.; Marrani, A. SAM lectures on extremal black holes in d=4 extended supergravity. *arXiv* **2009**, 0905.3739.
13. Bellucci, S.; Ferrara, S.; Marrani, A. Attractor horizon geometries of extremal black holes. In Proceedings of the XVII SIGRAV Conference, Turin, Italy, 4-7 September 2006.
14. Bellucci, S.; Ferrara, S.; Marrani, A.; Shcherbakov, A. Quantum lift of non-BPS flat directions. *Phys. Lett. B* **2009**, *672*, 77-81.

15. Bellucci, S.; Ferrara, S.; Shcherbakov, A.; Yeranyan, A. Black hole entropy, flat directions and higher derivatives. *arXiv* **2009**, 0906.4910.
16. Bellucci, S.; Ferrara, S.; Marrani, A. Attractors in black. *Fortsch. Phys.* **2008**, *56*, 761-785.
17. Bellucci, S.; Shcherbakov, A.; Yeranyan, A. Black hole entropy, flat directions and higher derivatives. *Lect. Notes Phys.* **2008**, *755*, 115-191.
18. Cardoso, G.L.; de Wit, B.; Kappeli, J.; Mohaupt, T. Asymptotic degeneracy of dyonic N = 4 string states and black hole entropy. *JHEP* **2004**, *2004JHEP12*, JHEP122004075.
19. Denef, F.; Moore, G.W. How many black holes fit on the head of a pin? *Gen. Rel. Grav.* **2007**, *39* 1539-1544.
20. Denef, F., Moore, G.W.; Split states, entropy enigmas, holes and halos. *arXiv* **2007**, 0702146v2.
21. Andrianopoli, L.; D'Auria, R.; Ferrara, S. Flat symplectic bundles of N-extended supergravities, central charges and black-hole entropy. *arXiv* **1997**, 9707203v1.
22. Dabholkar, A.; Denef, F.; Moore, G.W.; Pioline, B. Precision counting of small black holes. *JHEP* **2005**, *2005JHEP10*, JHEP102005096.
23. Dabholkar, A.; Denef, F.; Moore, G.W.; Pioline, B. Exact and asymptotic degeneracies of small black holes. *JHEP* **2005**, *2005JHEP10*, JHEP102005021.
24. Sen, A. Stretching the horizon of a higher dimensional small black hole. *JHEP* **2005**, *2005JHEP10*, JHEP102005073.
25. Dabholkar, A. Exact counting of black hole microstates. *Phys. Rev. Lett.* **2005**, *94*, 241301.
26. Sen, A. How does a fundamental string stretch its horizon? *JHEP* **2005**, *2005JHEP10*, JHEP102005059.
27. Sen, A. Black holes and the spectrum of half-BPS states in N = 4 supersymmetric string theory. *Adv. Theor. Math. Phys.* **2005**, *9*, 527-558.
28. Wald, R.M. Black hole entropy in the Noether charge. *Phys. Rev. D* **1993**, *48*, R3427-R3431.
29. Jacobson, T.; Kang, G.; Myers, R.C. Black hole entropy in higher curvature gravity. *arXiv* **1995**, 95020091.
30. Jacobson, T.; Myers, R.C. Black hole entropy and higher curvature interactions. *Phys. Rev. Lett.* **1993**, *70*, 3684-3687.
31. Gaiotto, D.; Strominger, A.; Yin, X. Superconformal black hole quantum mechanics. *JHEP* **2005**, *2005JHEP10*, JHEP102005017.
32. Lunin, O.; Mathur, S.D. AdS/CFT duality and the black hole information paradox. *Nucl. Phys. B* **2002**, *623*, 342-394.
33. Skenderis, K.; Taylor, M. The fuzzball proposal for black holes. *Phys. Rept.* **2008**, *467*, 117-171.
34. Taylor, M. General 2 charge geometries. *JHEP* **2006**, *2006JHEP03*, JHEP032006009.
35. Kanitscheider, I.; Skenderis, K.; Taylor, M. Fuzzballs with internal excitations. *JHEP* **2007**, *2007JHEP06*, JHEP062007056.
36. Skenderis, K.; Taylor, M. Fuzzball solutions for black holes and D1-braneCD5-brane microstates. *Phys. Rev. Lett.* **2007**), *98*, 071601.
37. Kanitscheider, I.; Skenderis, K.; Taylor, M. Holographic anatomy of fuzzballs. *JHEP* **2007**, *2007JHEP04*, JHEP042007023.

38. Lunin, O.; Mathur, S.D. Statistical interpretation of Bekenstein entropy for systems with a stretched horizon. *Phys. Rev. Lett.* **2002**, *88*, 211303.
39. Mathur, S.D. Black hole size and phase space volumes. *arXiv* **2007**, 0706.3884v1.
40. Gauntlett, J.P.; Gutowski, J.B.; Hull, C.M.; Pakis, S.; Reall, H.S. All supersymmetric solutions of minimal supergravity in five dimensions. *Class. Quant. Grav.* **2003**, *20*, 4587-4634.
41. Gutowski, J.B.; Reall, H.S. General supersymmetric AdS5 black holes. *JHEP* **2004**, *2004JHEP04*, JHEP042004048.
42. Bena, I.; Warner, N.P. One ring to rule them all and in the darkness bind them? *Adv. Theor. Math. Phys. 2005*, *9*, 667-701.
43. Gauntlett, J.P.; Gutowski, J.B. General Concentric Black Rings. *Phys. Rev. D* **2005**, *71*, 045002.
44. Bena, I.; Wang, C.W.; Warner, N.P. Foaming three-charge black holes. *Phys. Rev. D* **2007**, *75*, 124026.
45. Bena, I.; Warner, N.P. Bubbling Supertubes and Foaming Black Holes. *Phys. Rev. D* **2006**, *74*, 066001.
46. Bena, I.; Wang, C.W.; Warner, N.P. Mergers and typical black hole microstates. *JHEP* **2006**, *2006JHEP11*, JHEP112004042.
47. Maldacena, J.M. *PhD Thesis: Black Holes in String Theory*; Princeton University: Princeton, NJ, USA, 1996, pp. 1-80 .
48. Maldacena, J.M.; Strominger, A.; Witten, E. Black hole entropy in M-theory. *JHEP* **1997**, *1997JHEP12*, JHEP121997002.
49. Elvang, H.; Emparan, R.; Mateos, D.; Reall, H.S. Supersymmetric black rings and three-charge supertubes. *Phys. Rev. D* **2005**, *71* 024033.
50. Emparan, R.; Reall, H.S. A rotating black ring in five dimensions. *Phys. Rev. Lett.* **2002**, *88*, 101101.
51. Aharony, O.; Gubser, S.S.; Maldacena, J.M.; Ooguri, H.; Oz, Y. Large N field theories, string theory and gravity. *Phys. Rept.* **2000**, *323*, 183-386.
52. Moore, G. Introduction to Modular Functions and Their Application to 2D CFT. In Proceedings of Spring School on Superstring Theory and Related Topics (ICTP), Trieste, Italy, 2008.
53. Gaberdiel, M.R.; Gukov, S.; Keller, C.A.; Moore, G.W.; Ooguri, H. Extremal N=(2,2) 2D conformal field theories and constraints of modularity. *arXiv*, **2008**, 0805.4216v1.
54. Sen, A. Entropy function and AdS(2)/CFT(1) correspondence. *JHEP* **2008**, *2008JHEP11*, JHEP112008075.
55. Gupta, K.R.; Sen, A. Ads(3)/CFT(2) to Ads(2)/CFT(1). *JHEP* **2009**, *2009JHEP04*, JHEP042009034.
56. Gaiotto, D.; Yin, X. Genus two partition functions of extremal conformal field theories. *JHEP* **2007**, *2007JHEP08*, JHEP082007029.
57. Weinhold, F. Metric geometry of equilibrium thermodynamics. *J. Chem. Phys.* **1975**, *63*, 2479-2484.
58. Weinhold, F. Metric geometry of equilibrium thermodynamics. II, Scaling, homogeneity, and generalized GibbsDuhem relations. *ibid J. Chem. Phys* **1975**, *63*, 2484-2488.

59. Bellucci, S.; Chandra, V.; Tiwari, B.N. On the thermodynamic geometry of hot QCD. *arXiv* **2008**, 0812.3792v1.
60. Bellucci, S.; Tiwari, B.N. State-space Correlations and Stabilities. *arXiv* **2009**, 0910.5309v1.
61. Bellucci, S.; Tiwari, B.N. An exact fluctuating 1/2-BPS configuration. *JHEP* **2010**, *2010JHEP05*, JHEP052010023.
62. Ruppeiner, G. Riemannian geometry in thermodynamic fluctuation theory. *Rev. Mod. Phys.* **1995**, *67*, 605-659.
63. Ruppeiner, G. Thermodynamics: A Riemannian geometric model. *Phys. Rev. A* **1979**, *20*,1608-1613.
64. Ruppeiner, G. Thermodynamic critical fluctuation theory? *Phys. Rev. Lett.* **1983**, *50*, 287-290.
65. Ruppeiner, G. New thermodynamic fluctuation theory using path integrals. *Phys. Rev. A* **1983**, *27*, 1116-1133.
66. Ruppeiner, G. Thermodynamic curvature of the multicomponent ideal gas. *Phys. Rev. A* **1990**, *41*, 2200-2202.
67. Shen, J.; Cai, R.G.; Wang, B.; Su, R.K. Thermodynamic geometry and critical behavior of black holes. *Int. J. Mod. Phys. A* **2007**, *22*, 11-27
68. Aman, J.E.; Bengtsson, I.; dokrajt, N. Flat information geometries in black hole thermodynamics. *Gen. Rel. Grav.* **2006**, *38*, 1305-1315.
69. Aman, J.E.; Bengtsson, I.; dokrajt, N. Geometry of black hole thermodynamics. *Gen. Rel. Grav.* **2003**, *35*, 1733-1743.
70. Aman, J.E.; Pidokrajt, N. Geometry of higher-dimensional black hole thermodynamics. *Phys. Rev. D* **2006**, *73*, 024017.
71. Sarkar, T.; Sengupta, G.; Tiwari, B.N. On the thermodynamic geometry of BTZ black holes. *JHEP* **2006**, *2006JHEP11*, JHEP112006015.
72. Balasubramanian, V.; De Boer, J.; Jejjala, V.; Simon, J. The Library of babel: On the origin of gravitational thermodynamics. *JHEP* **2005**, *2005JHEP12*, JHEP122005006.
73. Bekenstein, D. Information in the holographic universe. *Sci. Am.* **2003**, *289*, 58-65.
74. Susskind, L. Some speculations about black hole entropy in string theory. *arXiv* **1993**, 9309145v2.
75. Russo, J.G.; Susskind, L. Asymptotic level density in heterotic string theory and rotating black holes. *Nucl. Phys. B* **1995**, *437*, 611-626.
76. Sen, A. Black hole solutions in heterotic string theory on a torus. *Nucl. Phys. B* **1995**, *440*, 421-440.
77. Sen, A. Extremal black holes and elementary string states. *Mod. Phys. Lett. A* **1995**, *10*, 2081-2094.
78. Dabholkar, A. Exact counting of black hole microstates. *Phys. Rev. Lett.* **2005**, *94*, 241301.
79. Vafa, C. Instantons on D-branes. *Nucl. Phys. B* **1996**, *463*, 435-442.
80. de Wit, B.; Saueressig, F. Off-shell N=2 tensor supermultiplets. *JHEP* **2006**, *2006JHEP09*, JHEP092006062.
81. Cardoso, G.L.; de Wit, B.; Mahapatra, S. Black hole entropy functions and attractor equations. *JHEP* **2007**, *2007JHEP03*, JHEP032007085.
82. Tiwari, B.N. Sur les corrections de la géométrie thermodynamique des trous noirs. *arXiv* **2008**, 0801.4087v1 [hep-th].
83. Tiwari, B.N. On generalized uncertainty principle. *arXiv* **2008**, 0801.3402v1.

84. Sarkar, T.; Sengupta, G.; Tiwari, B.N. Thermodynamic geometry and extremal black holes in string theory. *JHEP* **2008**, 2008JHEP10, JHEP102008076.
85. Horowitz, G.; Maldacena, J.M.; Strominger, A. Nonextremal black hole microstates and U-duality. *Phys. Lett. B*, **1996**, *383*, 151-159.
86. Horowitz, G.T.; Lowe, D.A.; Maldacena, J.M. Statistical entropy of nonextremal four-dimensional black holes and U-duality. *Phys. Rev. Lett.* **1996**, *77*, 430-433.
87. Johnson, C.V.; Khuri, R.R.; Myers, R.C. Entropy of 4D extremal black holes. *Phys. Lett. B*, **1996**, *378*, 78-86.
88. Denef, F. Supergravity flows and D-brane stability. *JHEP* **2000**, *2000JHEP08*, JHEP082000050.
89. Cardoso, G.L.; de Wit, B.; Käppeli, J.; Mohaupt, T. Stationary BPS solutions in N=2 supergravity with R^2-Interactions. *JHEP* **2000**, *2000JHEP12*, JHEP122000019.
90. Bates, B.; Denef, F. Exact solutions for supersymmetric stationary black hole composites. *RUNHETC* **2003**, *10*, 1-13.
91. Horowitz, G.T.; Hubeny, V. E. Note on small black holes in $AdS_p \times S^q$. *JHEP* **2000**, *2000JHEP06*, JHEP062000031.
92. Zhou, J.G. Super 0-brane and GS superstring actions on $AdS_2 \times S^2$. *Nucl. Phys. B* **1999**, *559*, 92-102.
93. Mohaupt, T. Black hole entropy, special geometry and strings. *Fortsch. Phys.* **2001**, *49*, 3-161.
94. Mohaupt, T. Black Holes in Supergravity and String Theory, *Class. Quant. Grav.* **2000**, *17*, 3429-3482.
95. Dabholkar, A.; Harvey, J. A. Nonrenormalization of the superstring tension, *Phys. Rev. Lett.* **1989**, *63*, 478-481.
96. Dabholkar, A.; Gauntlett, J.P.; Harvey, J.A.; Waldram, D. Strings as solitons and black holes as strings. *Nucl. Phys. B* **1996**, *474*, 85-121.
97. Dabholkar, A.; Iizuka, N.; Iqubal, A.; Shigemori, M. Precision microstate counting of small black rings. *Phys. Rev. Lett.* **2006**, *96*, 071601.
98. Dabholkar, A.; Kallosh, R.; Maloney, A. Stringy cloak for a classical singularity. *JHEP* **2004**, *2004JHEP12*, JHEP122004059.
99. Cardosow, G.L. Black Holes, Entropy and String Theory. In Proceedings of RTN Meeting, Corfu, Greece, September 2005.
100. Ooguri, H.; Strominger, A.; Vafa, C. Black hole attractors and the topological string. *Phys. Rev. D* **2004**, *70*, 106007.
101. Mathur, S.D. The fuzzball proposal for black holes: an elementary review. *Fortsch. Phys.* **2005**, *53*, 793-827.
102. Lunin, O.; Mathur, S.D. AdS/CFT duality and the black hole information paradox. *Nucl. Phys. B* **2002**, *623*, 342-394.
103. Lunin, O.; Mathur, S.D. Statistical interpretation of Bekenstein entropy for systems with a stretched horizon. *Phys. Rev. Lett.* **2002**, *88*, 211303.
104. Giusto, S.; Mathur, S.D. Fuzzball geometries and higher derivative corrections for extremal holes. *Nucl. Phys. B* **2006**, *738*, 48-75.

105. Lin, H.; Lunin, O.; Maldacena, J. Bubbling AdS space and 1/2 BPS geometries. *JHEP* **2004**, *2004JHEP10*, JHEP102004025.
106. Bena, I.; Warner, N.P. Bubbling supertubes and foaming black holes. *Phys. Rev. D* **2006**, *74*, 066001.
107. Bena, I.; Kraus, P. R^2-Corrections to black ring entropy. *arXiv* **2005**, 0506015v2.
108. Bena, I.; Kraus, P. Microscopic Description of Black Rings in AdS/CFT. *JHEP* **2004**, *2004JHEP12*, JHEP122004070.