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The Property of Chaotic Orbits with Lower Positions of Numerical Solutions in the Logistic Map

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Abstract: In this paper, we introduce an iterative method with lower positions of true numerical solutions located in the real orbit in order to investigate the property of the logistic map. The basic structure of the logistic map is presented, which consists of the root gene position, the common gene position and the individual gene position. The ergodicity and randomness of the logistic map are dependent on the individual gene position. We find that the lower positions of the true numerical solutions in the real orbits have the property of a half-life.

Keywords: logistic map; iterative method; numerical simulation; chaotic orbit; lower position; half-life

1. Introduction

Chaotic systems have been widely studied in physics, mathematics, biology, information technology, and so on. When the real chaos is realized in the discretized one with the finite computational precision, the continuous chaotic systems may collapse in digital computers, due to the sensitivity of chaotic systems to initial conditions and control parameters. If the precision of a numerical solution in iterating chaotic maps is greater than the specified precision of the computer, the value of the numerical solution will be truncated for the next iteration.

The truncation errors of arithmetic and round-off errors in numerical solutions are inevitable. Computation errors in digital computers are introduced into iterations of chaotic maps at every discrete step. The discrete orbits with the finite precision are different from the theoretical ones after several iterations [1,2]. Therefore, the computational precision has close links with the ergodicity and other dynamical properties of chaotic systems. More unseen phenomena hidden in the continuous and discrete chaos deserve further exploration. There are many research reports of theoretical and experimental analysis for the precision-dependent behavior of chaotic systems [3–7]. The theory and approach are used to study the problem, which include the unstable periodic orbit in the attractor, the shadowing lemma, the number theory, the statistical analysis of probability theory and the combination of these.

Unstable periodic orbits in the chaotic attractor play a key role in determining dynamics [8]. An important problem is to estimate the effect of the finite precision on chaotic orbits. Since discretized chaotic iterations with finite precision are constrained in the limited discrete space, every chaotic orbit will eventually become periodic [9]. Grebogi [10] studied the effect of the finite precision on the average period of chaotic maps.

The shadowing lemma is used to create a bridge between the true trajectory and the pseudo-trajectory. To explain in what sense the noisy trajectory reflects the true dynamics of the actual system, the procedure of containment and refinement was developed to show that some true trajectories remain close to the noisy one for long times [11].

The approach of number theory was used to reveal the precision-dependent behavior [12]. A rational number is converted to the binary equivalent. The authors made the point that increasing precision does not certainly lead to improving accuracy. Although the computation errors are deterministic, it is difficult to deal with them during iterations of chaotic maps.

Moreover, statistical experiments are used to explore this issue [13–15]. Li [13] proposed a series of dynamical indicators for the 1D piecewise linear chaotic map, which were used to investigate computable and measurable indicators of the dynamical properties of the digital chaotic system.

The Lyapunov exponent is the main indicator of chaos in dynamical systems. The combination of statistical analysis, the attractor size and Lyapunov exponent is used to study the relationship among computation errors, the attractor and the dynamical expansion. Shi [16] found that the relation on the round-off error α , the attractor size γ and the maximum of local dynamical expansion β in a driven or coupled logistic map system satisfies $\alpha\beta < \gamma$. Therefore, computation errors play important roles in iterations of chaotic maps. However, computation errors make the discretized chaotic orbits deviate from the continuous ones in a complex and intractable manner.

The numerical solutions with computation errors might interfere with the nature of nonlinear dynamics. The approaches mentioned above exploit approximations, which derive from the upper positions of numerical solutions. The lower positions of the numerical solutions in the orbit are completely ignored due to the floating-point arithmetic of the computer. However, some natural characteristics of the logistic map actually hide in lower positions of true numerical solutions in the real chaotic orbits. Liu [17] reported the property of the logistic map with scalable precision.

In this paper, we introduce the iterative method with lower positions of numerical solutions to explore the precision-dependent behavior of chaotic maps. As we will see later, the method can more accurately reflect the basic structure of true numerical solutions in a real chaotic orbit of the logistic map. In comparison with the traditional existing approaches to analyze chaotic systems, the basic structure of the logistic map is independent of the computational errors in computers. To the best of our knowledge, the basic structure of the logistic map is reported for the first time. It can really reflect the property of real chaotic orbits of the logistic map. It is able to help us to deeply understand the property of chaos.

The rest of this paper is organized as follows. Section 2 describes the property of real chaotic orbits of the logistic map. Section 3 introduces the iterative method, which gets lower positions of true numerical solutions in real chaotic orbits by using the finite computation precision. Section 4 presents the basic structure of the logistic map and the natural characteristic of lower positions of true numerical solutions in the real chaotic orbits. The last section concludes the paper.

2. Real Orbits of the Logistic Map

The logistic map is defined as:

$$x_{n+1} = a x_n (1 - x_n), n = 0, 1, 2, \dots \quad (1)$$

where x_n is a real number in the interval $(0, 1)$, n acts as the n -th iteration and x_0 is an initial value of x_n . The control parameter a is a real number in the interval $(3, 4)$.

Assume that a real number can be described as $\tau = r_0.r_1r_2 \dots r_i \dots r_n, r_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, *i.e.*, $\tau = \sum_{i=0}^n (r_i \times 10^{-i})$. When $r_n \neq 0$, n is the precision of τ , r_n is called the last position of τ . When $r_0 = 0$ or $r_0 = 3$, $\tau \in (0, 1)$ or $\tau \in (3, 4)$, respectively.

A true numerical solution (TNS) in a real chaotic orbit of the logistic map is computed in absolute accuracy. A real chaotic orbit of the logistic map consists of the true numerical solutions. We take an example to show a real chaotic orbit of the logistic map.

When $x_0 = 0.1$ and $a = 3.9$, the true numerical solutions in a real orbit are listed as follows.

$$x_1 = 0.351.$$

$$x_2 = 0.888416\underline{1}.$$

$$x_3 = 0.38661843971708\underline{1}.$$

$$x_4 = 0.924864024972462138613439673812\underline{1}.$$

$$x_5 = 0.27101318510837636620001408664012488249844635480944697847018500\underline{1}.$$

$$x_6 = 0.770503650562578242010028850180072467300258215809084121933740850721435498498331883296601240471717068336699090686871224455056996\underline{1}.$$

$$x_7 = 0.689628322626042399346162417789631574331673792279087727295587249959700088953695903028503904227316732329007279116817939583585410360955275942688611619532554186417$$

689465749874585911734826353859703733156498976808571405387443824495937854094564832442
251733880681.

We can observe that the last two positions of the true numerical solutions have regularity, which is as follows.

$$\begin{aligned}
 x_1 &= 0.351 \\
 x_2 &= 0. \dots \underline{61} \quad x_3 = 0. \dots 81 \quad x_4 = 0. \dots 21 \quad x_5 = 0. \dots 01 \\
 x_6 &= 0. \dots \underline{61} \quad x_7 = 0. \dots 81
 \end{aligned}$$

Note that the last two positions of the true numerical solutions become periodic from x_2 to x_6 . We can deduce that the last two positions of the true numerical solution x_9 in the real chaotic orbit are equal to 01.

The precision of the true numerical solution x_9 is equal to 1023. We provide the upper ten positions and the lower ten positions of the true numerical solution x_9 . The middle positions of the true numerical solution x_9 are omitted.

The true numerical solution x_9 is equal to 0. $\underbrace{5379486456}_{\text{Upper 10 positions}} \dots \underbrace{4865207401}_{\text{Lower 10 positions}} .$

The real chaotic orbits of the logistic map are not periodic. The lower positions of true numerical solutions in the real orbits have regularity. An interesting question is: what is the characteristic of the lower positions of the true numerical solutions in real chaotic orbits?

The precision of the true numerical solutions of x_7 , x_{10} and x_{20} is equal to 255, 2 047 and 2 097 151, respectively. The precision difference is close to 1000 times after 10 iterations.

When the values of the last position of x_0 and a belong in the integer set $Z^d = \{1, 2, 3, 4, 6, 7, 8, 9\}$, the relationship between the iteration and the precision satisfies:

$$L_{x_{n+1}} = 2 \times L_{x_n} + L_a \tag{2}$$

where L_{x_n} and L_a are the precision of x_n and a , respectively. Equation (2) implies that the greater the iteration is, the larger the precision of the true numerical solutions in the real orbit will be.

Computing systems of computers perform the floating-point computation. A common standardized format for the floating-point computation is called the IEEE 754 format. It comes in three forms, all of which are very similar in procedure: single precision (32-bit), double precision (64-bit) and extended precision (80-bit).

With the double precision of the fraction significantly appearing in the memory format, the total precision is approximately 15 decimal digits. The principle of the floating-point arithmetic in computers is that the upper digit positions are stored in the memory.

On the one hand, when the precision of a true numerical solution exceeds the maximum precision with double precision in computers, the lower positions of the true numerical solution will be truncated. The lower positions of the numerical solution could be difficult to obtain by using the arithmetic operation of floating-point computation. Therefore, the floating-point arithmetic in computers cannot get the lower positions of true numerical solutions in the real orbits of the logistic map. This is the reason why the property of the lower positions of the numerical solutions in the orbit is completely ignored.

On the other hand, according to Equation (2), the precision of true numerical solutions in the real orbit could tremendously vary after several iterations. The finite computational precision has an influence on

the dynamics of chaotic systems. The computational errors will gradually accumulate, since the precision in computers is limited. Furthermore, the true numerical solutions in a real chaotic orbit of the logistic map are obtained when the computing procedure does not include any computational error.

The problem is to obtain the lower positions of true numerical solutions in the real orbits by using the finite computational precision in computers.

3. Iterative Method with Lower Positions

We use an iterative method with lower positions of numerical solutions in order to overcome the limit of precision. Every step of the computing procedure does not include the accumulated round-off error. In other words, the logistic map is computing in absolute precision without computational errors.

Assume that we need to get the lower p_n positions starting from the last position of true numerical solutions in the real orbits, where p_n is a positive integer. In order to guarantee the absolute precision, the logistic map can be divided into two parts for computation. The iterative method with lower positions is described as following:

Step 1: computing $\mu = x_n(1 - x_n)$ in absolute accuracy;

Step 2: computing $\xi = a \mu$ in absolute accuracy;

Step 3: if the total number of positions of ξ is less than or equal to p_n , all positions of the numerical solution will be returned for the next iteration; otherwise, the lower p_n positions starting from the last position are returned.

We take an example to explain the iterative method with lower positions of numerical solutions. Assume $x_0 = 0.1$ and $a = 3.9$; we need to get the lower 12 positions of true numerical solutions in the real orbit, *i.e.*, $p_n = 12$.

The computing procedure is shown as follows.

$\hat{x}_1 = ax_0(1 - x_0) = 3.9 \times 0.1 \times (1 - 0.1) = 0.351$. The total number of positions of \hat{x}_1 is less than p_n . All positions will be returned for the next iteration.

Thus, $x_1^l = 0.351$.

$\hat{x}_2 = ax_1^l(1 - x_1^l) = 3.9 \times 0.351 \times (1 - 0.351) = 0.8884161$. \hat{x}_2 has the same situation as \hat{x}_1 .

Thus, $x_2^l = 0.8884161$.

$\hat{x}_3 = ax_2^l(1 - x_2^l) = 3.9 \times 0.8884161 \times (1 - 0.8884161) = 0.386618439717081$. The total number of positions of \hat{x}_3 is greater than p_n . The lower twelve positions of \hat{x}_3 are returned.

Thus, $x_3^l = 0.618439717081$.

$\hat{x}_4 = ax_3^l(1 - x_3^l) = 3.9 \times 0.618439717081 \times (1 - 0.618439717081) = 0.9202909303293134396738121$. \hat{x}_4 has the same situation as \hat{x}_3 .

Thus, $x_4^l = 0.134396738121$.

The true numerical solution x_4 in the real orbit is equal to 0.9248640249724621386 134396738121.

The iterative method can get the lower p_n positions of the corresponding true numerical solution of x_4 .

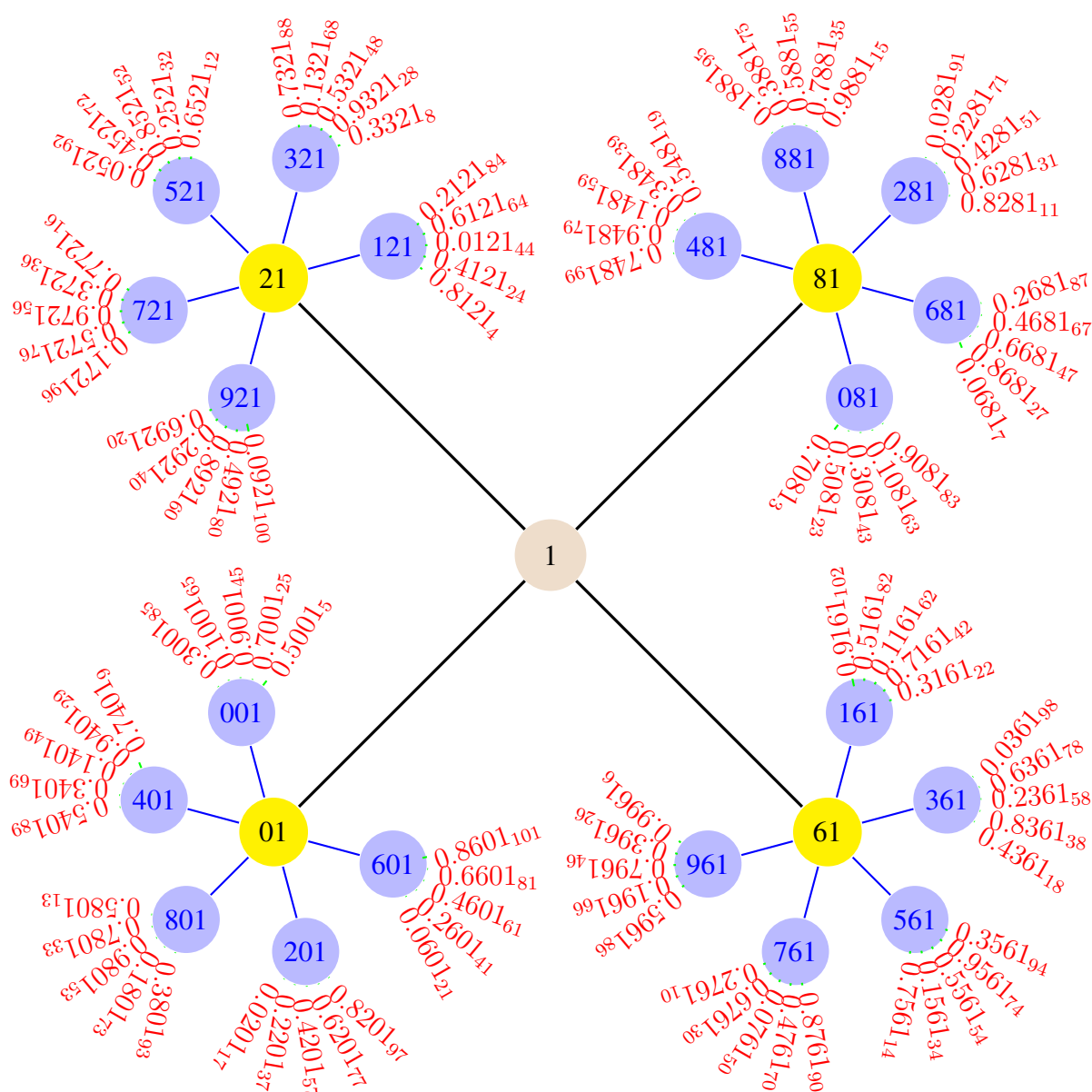
$\hat{x}_5 = ax_4^l(1 - x_4^l) = 3.9 \times 0.134396738121 \times (1 - 0.134396738121) = 0.4537035941233978470185001$.

Lower 12 positions

We increase the value of p_n in order to further explore the property of lower positions. Assume that $x_0 = 0.1$, $a = 3.9$ and $p_n = 3$; the lower p_n positions of true numerical solutions are listed as follows.

				$x_1 = 0.351$
$x_2 = 0.161$	$x_3 = 0.081$	$x_4 = 0.121$	$x_5 = 0.001$	
$x_6 = 0.961$	$x_7 = 0.681$	$x_8 = 0.321$	$x_9 = 0.401$	
$x_{10} = 0.761$	$x_{11} = 0.281$	$x_{12} = 0.521$	$x_{13} = 0.801$	
$x_{14} = 0.561$	$x_{15} = 0.881$	$x_{16} = 0.721$	$x_{17} = 0.201$	
$x_{18} = 0.361$	$x_{19} = 0.481$	$x_{20} = 0.921$	$x_{21} = 0.601$	
$x_{22} = 0.161$	$x_{23} = 0.081$	$x_{24} = 0.121$	$x_{25} = 0.001$	

Figure 1. The structure of the lower four positions for $x_0 = 0.1$ and $a = 3.9$ is plotted. The four digit positions represent the lower four positions of TNSs in the real orbit. The subscript $i(3 \leq i \leq 102)$ stands for the i -th iteration.



The length of the period is equal to 20 for $x_0 = 0.1$, $a = 3.9$ and $p_n = 3$. Figure 1 shows the structure of the lower four positions of TNSs for $x_0 = 0.1$ and $a = 3.9$.

The position located in the last positions of TNSs is called the root gene position, when the value of the root gene positions in the cycle part is invariable. For example, when $x_0 = 0.1$, $a = 3.9$ and $p_n = 4$, the root gene position is the last position of TNSs. In Figure 1, the root gene position is located in the center. Its value is equal to one.

When the period of the lower positions is equal to four, the positions are called the common gene position (CGP). For example, in Figure 1, the last two positions are the common gene positions, *i.e.*, “21”, “81”, “01” and “61”. Others are called the individual gene position (IGP).

In Figure 1, every individual gene position varies in the set $Z^{odd} = \{1, 3, 5, 7, 9\}$ and the set $Z^{even} = \{0, 2, 4, 6, 8\}$. For example, the individual gene position with the common gene positions “81” is “081”, “681”, “281”, “881” and “481”, which belong in the set Z^{even} . The individual gene position with the common gene positions “21” is “121”, “321”, “521”, “721” and “921”, which belong in the set Z^{odd} .

Note that every iteration with the individual gene position moves from one common gene position to another. For example, for $x_0 = 0.1$, $a = 3.9$ and $p_n = 3$, $x_{17} = 0.201$ with the individual gene position “2” and the common gene positions “01” moves to $x_{18} = 0.361$ with the common gene positions “61”. In Figure 1, $x_9 = 0.7401$ with the individual gene positions “74” and the common gene positions “01” moves to $x_{10} = 0.2761$ with the common gene positions “61”.

We take another example to investigate the property of lower positions. Assume that $x_0 = 0.3$ and $a = 3.4$; the true numerical solutions are listed as follows.

- $x_1 = 0.714.$
- $x_2 = 0.6942936.$
- $x_3 = 0.721649989796736.$
- $x_4 = 0.682 \dots 5727374336.$
- $x_5 = 0.736 \dots 1171009536.$
- $x_6 = 0.660 \dots 4300199936.$
- $x_7 = 0.762 \dots 4470260736.$

We provide the upper three positions and the lower ten positions of the true numerical solutions from x_4 to x_7 . Some middle positions are omitted, because the situation has no effect on observing lower positions of the true numerical solutions. As seen above, the lower three positions of TNSs are listed as follows.

$$\begin{array}{rcccc}
 & & & & x_1 = 0.714 \\
 & & & & x_2 = 0.936 \quad x_3 = 0.736 \quad x_4 = 0.336 \quad x_5 = 0.536 \\
 & & & & x_6 = 0.936 \quad x_7 = 0.736 \quad \dots
 \end{array}$$

Note that the regularity occurs in the lower three positions. The length of period is equal to four. For $x_0 = 0.3$ and $a = 3.4$, the root gene positions are the last two positions, *i.e.*, “36”. The common gene positions are “936”, “736”, “336” and “536”.

We increase the value of lower p_n positions. Assume that $x_0 = 0.3$, $a = 3.4$ and $p_n = 4$. The lower four positions of TNSs are listed as follows.

$$\begin{array}{cccc}
 & & x_1 = 0.714 & x_2 = 0.2936 \\
 x_3 = 0.6736 & x_4 = 0.4336 & x_5 = 0.9536 & x_6 = 0.9936 \\
 x_7 = 0.0736 & x_8 = 0.2336 & x_9 = 0.5536 & x_{10} = 0.1936 \\
 x_{11} = 0.4736 & x_{12} = 0.0336 & x_{13} = 0.1536 & x_{14} = 0.3936 \\
 x_{15} = 0.8736 & x_{16} = 0.8336 & x_{17} = 0.7536 & x_{18} = 0.5936 \\
 x_{19} = 0.2736 & x_{20} = 0.6336 & x_{21} = 0.3536 & x_{22} = 0.7936 \\
 x_{23} = 0.6736 & x_{24} = 0.4336 & x_{25} = 0.9536 & x_{26} = 0.9936
 \end{array}$$

As seen above, the individual gene position with the common gene positions “736” is “6736”, “0736”, “4736”, “8736” and “2736”, which belong in the set Z^{even} . The individual gene position with the common gene positions “936” is “1936”, “3936”, “5936”, “7936” and “9936”, which belong in the set Z^{odd} .

Table 1. The length of period in the iterative method with lower positions for the initial conditions of $x_0 = 0.1$ and $a = 3.9$.

p_n	Length of period	$c_{p_n=\eta+1}/c_{p_n=\eta}$
1	1	
2	4	$c_{p_n=2}/c_{p_n=1} = 4$
3	20	$c_{p_n=3}/c_{p_n=2} = 5$
4	100	$c_{p_n=4}/c_{p_n=3} = 5$
5	500	$c_{p_n=5}/c_{p_n=4} = 5$
6	2 500	$c_{p_n=6}/c_{p_n=5} = 5$
7	12 500	$c_{p_n=7}/c_{p_n=6} = 5$
8	62 500	$c_{p_n=8}/c_{p_n=7} = 5$
9	312 500	$c_{p_n=9}/c_{p_n=8} = 5$
10	1 562 500	$c_{p_n=10}/c_{p_n=9} = 5$
11	7 812 500	$c_{p_n=11}/c_{p_n=10} = 5$
12	39 062 500	$c_{p_n=12}/c_{p_n=11} = 5$
13	195 312 500	$c_{p_n=13}/c_{p_n=12} = 5$
14	976 562 500	$c_{p_n=14}/c_{p_n=13} = 5$
15	4 882 812 500	$c_{p_n=15}/c_{p_n=14} = 5$

Assume that $c_{p_n=\eta}$ is the length of the period with the lower p_n positions, where p_n is equal to η . From Table 1, we can see that when the length of the period is greater than four and the individual gene position increases by one digit position, the new length of period will be five-times as long as the original length of period.

In order to further verify the regularity, we take values of the control parameter of $a \in \{3.1, 3.2, 3.3, 3.4, 3.6, 3.7, 3.8, 3.9\}$, and initial values of $x_0 \in \{0.1, 0.2, 0.3, 0.4, 0.6, 0.7, 0.8, 0.9\}$.

When a is specified, the value of x_1 is the same for the initial values of $x_0 = 0.1$ and $x_0 = 0.9$, $x_0 = 0.2$ and $x_0 = 0.8$, $x_0 = 0.3$ and $x_0 = 0.7$, $x_0 = 0.4$ and $x_0 = 0.6$, respectively. Therefore, we give, for example, $(x_0 = 0.1, a)$ and omit $(x_0 = 0.9, a)$.

Table 2. The length of the period in the iterative method with lower positions for the precision 1 of the control parameter and initial value x_0 . c^1 , c^2 and c^3 represent the length of period for initial conditions in Group 1, Group 2 and Group 3, respectively.

p_n	c^1	c^2	c^3
1	1	1	1
2	1	1	4
3	4	1	20
4	20	4	100
5	100	20	500
6	500	100	2 500
7	2 500	500	12 500
8	12 500	2 500	62 500
9	62 500	12 500	312 500
10	312 500	62 500	1 562 500
11	1 562 500	312 500	7 812 500
12	7 812 500	1 562 500	39 062 500
13	39 062 500	7 812 500	195 312 500
14	195 312 500	39 062 500	976 562 500
15	976 562 500	195 312 500	4 882 812 500

The length of the period is summarized in Table 2. Experimental results by the initial conditions are divided into three groups in the form of (x_0, a) :

Group 1 = $\{(0.1, 3.1), (0.4, 3.2), (0.1, 3.3), (0.2, 3.3), (0.3, 3.4), (0.3, 3.8)\}$;

Group 2 = $\{(0.2, 3.1), (0.3, 3.7)\}$;

Group 3 = $\{(0.1, 3.2), (0.1, 3.4), (0.1, 3.6), (0.1, 3.7), (0.1, 3.8), (0.2, 3.2), (0.2, 3.4), (0.2, 3.6), (0.2, 3.7), (0.2, 3.8), (0.2, 3.9), (0.3, 3.1), (0.3, 3.2), (0.3, 3.3), (0.3, 3.6), (0.3, 3.9), (0.4, 3.1), (0.4, 3.3), (0.4, 3.4), (0.4, 3.6), (0.4, 3.7), (0.4, 3.8), (0.4, 3.9)\}$.

For $x_0 = 0.2$ and $a = 3.1$ in Group 2, the lower four positions of TNSs are listed as follows.

$$\begin{aligned}
 x_1 &= 0.496 \\
 x_2 &= 0.9504 \quad x_3 = 0.3504 \quad x_4 = 0.1504 \quad x_5 = 0.7504 \\
 x_6 &= 0.9504 \quad x_7 = 0.3504 \quad \dots
 \end{aligned}$$

For $x_0 = 0.2$, $a = 3.1$ and $p_n = 4$, the root gene positions are the last three positions, *i.e.*, “504”. When the length of the period is equal to four, it is called the critical point of the period.

From Table 2, the situation is similar in that the new length of the period is five-times as long as the original length of the period when the individual gene position increases by one digit position. This

implies that the length of the period will enlarge for the same initial conditions and control parameter when the individual gene positions increase.

When lower p_n positions are specified, the less the number of the root gene position is, the larger the length of period will be. For example, assume that the lower positions are set to four. For $x_0 = 0.2$, $a = 3.1$ and $p_n = 4$ in Group 2, the length of the period is equal to four. The root gene position occupies three digital positions, *i.e.*, “504”. For $x_0 = 0.3$, $a = 3.4$ and $p_n = 4$ in Group 1, the length of the period is equal to 20. The root gene position occupies two digital positions, *i.e.*, “36”. For $x_0 = 0.1$, $a = 3.9$ and $p_n = 4$, the length of the period is equal to 100. The root gene position occupies a digital position, *i.e.*, “1”.

Table 3. The individual gene positions with the same common gene positions for $x_0 = 0.2$, $a = 3.9$ and $p_n = 4$. CGP, common gene position.

CGP =“36”	CGP =“56”	CGP =“96”	CGP =“76”
$x_2 = 0.0336$	$x_3 = 0.7056$	$x_4 = 0.1696$	$x_5 = 0.9776$
$x_6 = 0.3136$	$x_7 = 0.4656$	$x_8 = 0.4896$	$x_9 = 0.8176$
$x_{10} = 0.7936$	$x_{11} = 0.6256$	$x_{12} = 0.6096$	$x_{13} = 0.2576$
$x_{14} = 0.4736$	$x_{15} = 0.1856$	$x_{16} = 0.5296$	$x_{17} = 0.2976$
$x_{18} = 0.3536$	$x_{19} = 0.1456$	$x_{20} = 0.2496$	$x_{21} = 0.9376$
$x_{22} = 0.4336$	$x_{23} = 0.5056$	$x_{24} = 0.7696$	$x_{25} = 0.1776$
$x_{26} = 0.7136$	$x_{27} = 0.2656$	$x_{28} = 0.0896$	$x_{29} = 0.0176$
$x_{30} = 0.1936$	$x_{31} = 0.4256$	$x_{32} = 0.2096$	$x_{33} = 0.4576$
$x_{34} = 0.8736$	$x_{35} = 0.9856$	$x_{36} = 0.1296$	$x_{37} = 0.4976$
$x_{38} = 0.7536$	$x_{39} = 0.9456$	$x_{40} = 0.8496$	$x_{41} = 0.1376$
$x_{42} = 0.8336$	$x_{43} = 0.3056$	$x_{44} = 0.3696$	$x_{45} = 0.3776$
$x_{46} = 0.1136$	$x_{47} = 0.0656$	$x_{48} = 0.6896$	$x_{49} = 0.2176$
$x_{50} = 0.5936$	$x_{51} = 0.2256$	$x_{52} = 0.8096$	$x_{53} = 0.6576$
$x_{54} = 0.2736$	$x_{55} = 0.7856$	$x_{56} = 0.7296$	$x_{57} = 0.6976$
$x_{58} = 0.1536$	$x_{59} = 0.7456$	$x_{60} = 0.4496$	$x_{61} = 0.3376$
$x_{62} = 0.2336$	$x_{63} = 0.1056$	$x_{64} = 0.9696$	$x_{65} = 0.5776$
$x_{66} = 0.5136$	$x_{67} = 0.8656$	$x_{68} = 0.2896$	$x_{69} = 0.4176$
$x_{70} = 0.9936$	$x_{71} = 0.0256$	$x_{72} = 0.4096$	$x_{73} = 0.8576$
$x_{74} = 0.6736$	$x_{75} = 0.5856$	$x_{76} = 0.3296$	$x_{77} = 0.8976$
$x_{78} = 0.5536$	$x_{79} = 0.5456$	$x_{80} = 0.0496$	$x_{81} = 0.5376$
$x_{82} = 0.6336$	$x_{83} = 0.9056$	$x_{84} = 0.5696$	$x_{85} = 0.7776$
$x_{86} = 0.9136$	$x_{87} = 0.6656$	$x_{88} = 0.8896$	$x_{89} = 0.6176$
$x_{90} = 0.3936$	$x_{91} = 0.8256$	$x_{92} = 0.0096$	$x_{93} = 0.0576$
$x_{94} = 0.0736$	$x_{95} = 0.3856$	$x_{96} = 0.9296$	$x_{97} = 0.0976$
$x_{98} = 0.9536$	$x_{99} = 0.3456$	$x_{100} = 0.6496$	$x_{101} = 0.7376$
$x_{102} = 0.0336$	$x_{103} = 0.7056$	$x_{104} = 0.1696$	$x_{105} = 0.9776$

For $a = 3.9$, $x_0 = 0.2$ and $p_n = 4$, the lower four positions of the true numerical solutions in the real orbit become the period starting from x_2 to x_{102} . The length of period is equal to 100. The individual gene positions with the common gene positions are shown in Table 3.

The numerical solution represents $0.r_1r_2r_3r_4$ for $a = 3.9$, $x_0 = 0.2$ and $p_n = 4$. The root gene position is the digit position r_4 , which is equal to six. The combined digit positions of r_3 and r_4 are the common gene positions, which are equal to “36”, “56”, “96” and “76”. The digit positions r_1 and r_2 stand for the first position and the second position of the individual gene positions.

Table 4. Stochastic individual gene positions of the true numerical solutions in the real orbit of the logistic map.

$r_1 \backslash r_2$	0	1	2	3	4	5	6	7	8	9
0	x_{92}	x_{29}	x_{71}	x_2	x_{80}	x_{93}	x_{47}	x_{94}	x_{28}	x_{97}
1	x_{63}	x_{46}	x_{36}	x_{41}	x_{19}	x_{58}	x_4	x_{25}	x_{15}	x_{30}
2	x_{32}	x_{49}	x_{51}	x_{62}	x_{20}	x_{13}	x_{27}	x_{54}	x_{68}	x_{17}
3	x_{43}	x_6	x_{76}	x_{61}	x_{99}	x_{18}	x_{44}	x_{45}	x_{95}	x_{90}
4	x_{72}	x_{69}	x_{31}	x_{22}	x_{60}	x_{33}	x_7	x_{14}	x_8	x_{37}
5	x_{23}	x_{66}	x_{16}	x_{81}	x_{79}	x_{78}	x_{84}	x_{65}	x_{75}	x_{50}
6	x_{12}	x_{89}	x_{11}	x_{82}	x_{100}	x_{53}	x_{87}	x_{74}	x_{48}	x_{57}
7	x_3	x_{26}	x_{56}	x_{101}	x_{59}	x_{38}	x_{24}	x_{85}	x_{55}	x_{10}
8	x_{52}	x_9	x_{91}	x_{42}	x_{40}	x_{73}	x_{67}	x_{34}	x_{88}	x_{77}
9	x_{83}	x_{86}	x_{96}	x_{21}	x_{39}	x_{98}	x_{64}	x_5	x_{35}	x_{70}

Table 5. The property of the half-life of the individual gene positions. c_{max} represents the maximum length of period. c_{IGP} represents the length of the period with the same common gene position. IGP, individual gene position.

Number of IGP	c_{max}	c_{IGP}	c_{IGP}/c_{max}
1	10^1	5	50.0%
2	10^2	25	25.0%
3	10^3	125	12.5%
4	10^4	625	6.25%
5	10^5	3 125	3.125%
6	10^6	15 625	1.5625%
7	10^7	78 125	0.78125%
8	10^8	390 625	0.390625%
...
n	10^n	5^n	$1/2^n$

The randomness of IGPs of the true numerical solutions in the real orbit is shown in Table 4. The permutation and combination values of r_1 and r_2 range from 00 to 99. It is obvious that the combined

individual gene positions of r_1 and r_2 are different from each other and impinge on all points in the given space. In fact, the property of the logistic map derives from the individual gene positions. In other words, the individual gene positions exhibit the ergodicity and randomness of the logistic map.

Table 5 shows the property of the half-life of the logistic map. For example, the first column of Table 4 shows the length of the period with the common gene position “36” equal to 25. The same situation happens in the second, third and fourth columns of Table 4. From Table 5, we can see that when the individual gene positions increase, the length of the period with the same common gene positions has the property of the half-life. The reason is that each digital position of the individual gene positions varies in the sets Z^{odd} and Z^{even} .

The property of the half-life is the nature of the logistic map, which is independent of computational errors in computers. Therefore, the numerical solutions with computational errors could interfere with the nature of nonlinear dynamics.

5. Discussion and Conclusions

In summary, we introduce the iterative method with lower positions of numerical solutions in order to explore the property of the real orbits in the logistic map. The iterative method can observe the property of the lower positions of true numerical solutions in the real chaotic orbit of the logistic map.

Every real chaotic orbit includes the same root gene position, which is constant and invariable in iterations of the logistic map. The root gene position represents the basic characteristic of a real chaotic orbit. It seems that the gene of a parent is passed on to their offspring. In the evolution, they could possess the same characteristic.

When the length of the period of lower p_n positions is larger than the critical point of the period, the individual gene position will appear. The ergodicity and randomness of the logistic map depend on the individual gene positions. The basic structure of the logistic map is presented, which reflects the nature of the logistic map.

The interesting finding is that the lower positions of the true numerical solutions in the real orbits of the logistic map exhibit the property of the half-life. The property of the half-life is independent of computation errors. In other words, it is the natural characteristic of the logistic map.

These observations can help us to deeply understand chaos. The greatest mystery in this topic still remains open. It may be worthwhile to reveal whether other chaotic systems have this property.

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Author Contributions

Jiahui Liu designed the iterative method with lower positions of true numerical solutions and discovered the basic structure of the logistic map. He finished the program for iterating the logistic map based on the iterative method with lower positions in the C programming language. Hongli Zhang finished the data analysis of the length of the period and gave three groups according to the initial conditions. Dahua Song proposed the definition of the root gene position, the common gene position and individual gene positions. She summarized the property of the logistic map in genetics. Besides, she finished the data analysis for the property of the half-life of the individual gene positions. All authors have read and approved the final paper.

Conflicts of Interest

The authors declare no conflict of interest.

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