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Article

Complex Modified Hybrid Projective Synchronization of Different Dimensional Fractional-Order Complex Chaos and Real Hyper-Chaos

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Abstract: This paper introduces a type of modified hybrid projective synchronization with complex transformation matrix (CMHPS) for different dimensional fractional-order complex chaos and fractional-order real hyper-chaos. The transformation matrix in this type of chaotic synchronization is a non-square matrix, and its elements are complex numbers. Based on the stability theory of fractional-order systems, by employing the feedback control technique, necessary and sufficient criteria on CMHPS are derived. Furthermore, CMHPS between fractional-order real hyper-chaotic Rössler system and other two different dimensional fractional-order complex Lorenz-like chaotic systems is provided as two examples to discuss reduced order and increased order synchronization, respectively.

Keywords: complex modified hybrid projective synchronization; chaos with complex variable; different dimension; fractional-order

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1. Introduction

With the development of interdisciplinary applications, it was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives, for instance, viscoelastic systems [1], dielectric polarization [2], quantitative finance [3], quantum evolution of complex systems [4], and so forth. As is well known, different chaotic characteristics, e.g., the largest Lyapunov exponent, Kolmogorov entropy and correlation dimension, can represent different nonlinear features. In particular, Kolmogorov entropy [5,6] evaluates the chaotic degree of a system, or the average velocity at which new information is generated by the system, or equivalently, which current information about the system is lost. If a dynamic system exhibits a constant positive value of Kolmogorov entropy, it denotes that the system has chaotic characteristics. Chaos has been a focus of intensive discussion in numerous fields during the last four decades. Meanwhile, it has been proven that some fractional-order differential systems can behave chaotically. For example, Hartley *et al.* [7] discussed chaos in a fractional-order Chua system; Li and Chen [8] studied chaos and hyperchaos in fractional-order Rössler systems; and Daftardar-Gejji and Bhalekar [9] investigated chaos in a fractional-order Liu system.

In recent years, synchronization of fractional-order chaotic systems has attracted great attention. For example, Srivastava *et al.* [10] studied anti-synchronization between identical and non-identical fractional-order chaotic systems using the active control method; Zhao and Wang [11] discussed global outer synchronization between two fractional-order complex networks coupled in a drive-response configuration; and Sun *et al.* [12] investigated compound synchronization for four chaotic systems of integer order and fractional order. Projective synchronization (PS) has been especially extensively studied, because it can be used to obtain faster communication with its proportional feature, and the unpredictability of the scaling factor can additionally enhance the security of communication. In [13], Wu and Lu presented a generalized projective synchronization (GPS) method for fractional-order Chen hyper-chaotic systems, which associates with the projective synchronization and the generalized one, where the drive and response systems could be synchronized up to scaling factors δ_i. Liu *et al.* [14] introduced modified generalized projective synchronization (MGPS) of fractional-order chaotic systems with different structures, where the drive and response systems could be asymptotically synchronized up to a desired non-diagonal transformation matrix.

In applied sciences and engineering, there are a lot of problems involving complex variables, which are described by these complex systems, for example, when amplitudes of electromagnetic fields and atomic polarization are involved. In 2013, Luo and Wang introduced a fractional-order Lorenz system [15] and Chen system [16] in complex space and investigated their application to digital secure communication, where the complex variables increase the content of transmitting information signals and enhance their security further.

However, all of the scaling factors in the above synchronization are real numbers. In fact, for complex dynamical systems, the scaling factors can be complex [17,18], and the drive and response systems may evolve in different directions with a constant intersection angle; for example, $\zeta = \rho e^{j\gamma}\eta$, where $\rho e^{j\gamma} = \rho(\cos\gamma + j\sin\gamma)$, ζ and η denote the complex state variables of drive and response systems, respectively, $\rho > 0$ denotes the zoom rate and $\gamma \in [0, 2\pi)$ denotes the rotate angle. In addition, different dimensional drive and response systems could be synchronized in practical applications [19]. Modified hybrid projective synchronization with complex state transformation matrix $\Theta = \Theta^r + j\Theta^i$ (CMHPS) considers both different dimensions and the complex scaling factors. By means of complex state transformation, every state variable in a response system will be involved in multiple state variables of the drive system, which will increase the complexity of the synchronization and further increase the diversity and the security of communications [20]. Therefore, it is interesting and significant to study CMHPS of different dimensional fractional-order complex chaos and real hyper-chaos. Up till now, to the best of my knowledge, all of the works involved in complex scaling factors focus on the integer-order complex chaotic systems, and there is almost no paper about this type of CMHPS for fractional-order chaotic systems.

Inspired by the above discussion, in this paper, CMHPS is addressed between different dimensional fractional-order complex chaotic systems and real hyper-chaotic systems based on the stability theory of fractional-order systems. In addition, as a generalization of synchronization, depending on the form of complex transformation matrix, CMHPS will contain MGPS of fractional-order chaotic systems with a real constant scaling matrix and MHPS of fractional-order chaotic systems with a real transformation matrix, extending previous works.

The remainder of this paper is organized as follows. In Section 2, a brief review of the fractional derivative and the stability theory of a fractional-order system is given. General methods of CMHPS for different dimensional fractional-order complex and real hyper-chaotic (chaotic) systems are presented in Section 3, Section 4 and Section 5, respectively. Two numerical examples are presented in Section 6. Finally, some conclusions are given in Section 7.

Notations: $\mathbb{R}^n$ and $\mathbb{C}^n$ stand for n dimensional real and complex vector space, respectively. If z is a complex vector (or complex number), then $z = z^r + jz^i$, $j = \sqrt{-1}$ is the imaginary unit, superscripts r and i stand for the real and imaginary parts of z, z^{T} and $\bar{z}$ are the transpose and the complex conjugate of z, respectively. $\|z\|$ implies the two-norm of z, defined by $\|z\| = \sqrt{z^T \bar{z}}$.

Assume $\alpha > 0$, then $\lceil \alpha \rceil$ is just the value α rounded up to the nearest integer, J^α denotes the Riemann–Liouville-type fractional integral of order α, D^α denotes the Riemann–Liouville-type fractional derivative of order α, D_*^α denotes the Caputo-type fractional derivative of order α, $\Gamma(\cdot)$ and denotes the gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, x > 0$.

2. Preliminaries

2.1. The Definition of Fractional Derivative

There are many definitions of fractional derivative [21]. The definition of the Riemann-Liouville derivative is given as:

$$D^\alpha f(t) = \frac{d^m}{dt^m} J^{m-\alpha} f(t), \tag{1}$$

where $\alpha > 0$, $m := \lceil \alpha \rceil$, J^β is the β-order Riemann-Liouville integral operator as described by:

$$J^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau, \tag{2}$$

where $0 < \beta \leq 1$.

The Caputo fractional derivative is defined as:

$$D_*^\alpha f(t) = \frac{d^m}{dt^m} J^{m-\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{(q-m+1)}} d\tau, & m-1 < q < m, \\ \frac{d^m}{dt^m} f(t), & q = m, \end{cases} \tag{3}$$

where $\alpha > 0$, $m := \lceil \alpha \rceil$.

Here, the Caputo version is chosen, and an improved predictor-corrector algorithm, *i.e.*, the Adams-Bashforth-Moulton predictor-correctors scheme, is adopted for fractional differential equations, where the numerical approximation is a time-domain approach that is more accurate and the computational cost is greatly reduced [22].

2.2. The Stability of Fractional-Order Systems

For a given fractional-order linear time-invariant system:

$$D_*^\alpha x = Ax \tag{4}$$

with $x(0) = x_0$, where $0 < \alpha < 1$ and $x \in R^n$, A is a constant matrix.

Now, the stability of fractional-order systems [23] is recalled briefly.

Lemma 1. *System (4) is:*

(i) Asymptotically stable if and only if:

$$|\arg(\lambda_\ell(A))| > \frac{\alpha\pi}{2}, \quad (\ell = 1, 2, \cdots, n), \tag{5}$$

where $arg(\lambda_\ell(A))$ denotes the argument of the eigenvalue λ_ℓ of A. In this case, each component of the states decays toward zero like $t^{-\alpha}$.

(ii) Stable if and only if:

$$|\arg(\lambda_\ell(A))| \geq \frac{\alpha\pi}{2}, \quad (\ell = 1, 2, \cdots, n), \tag{6}$$

and those critical eigenvalues λ_i that satisfy $|\arg(\lambda_\ell(A))| = \alpha\pi/2 \quad (\ell = 1, 2, \cdots, n)$, have geometric multiplicity one.

Fractional-order differential equations are at least as stable as their integer-order counterpart, because systems with memory are typically more stable than those without memory [22].

3. CMHPS Scheme of a Different Dimensional Fractional-Order Real Hyper-Chaotic (Chaotic) Drive System and Complex Chaotic Response System

3.1. Mathematical Model and Problem Descriptions

First, a class of n-dimensional fractional-order real hyper-chaotic (chaotic) drive systems is considered as:

$$D_*^\alpha y = Cy + h(y), \tag{7}$$

and the m-dimensional fractional-order complex chaotic response system with the controller is written as:

$$D_*^\alpha z = D_*^\alpha z^r + jD_*^\alpha z^i = Pz + \Phi(z) + v, \tag{8}$$

where $y = (y_1, y_2, \cdots, y_n)^{\mathrm{T}} \in \mathbb{R}^n$ is a real state vector, $z = z^r + jz^i \in \mathbb{C}^m$ is a complex state vector, $C \in \mathbb{R}^{n\times n}$ and $P \in \mathbb{R}^{m\times m}$ are the coefficient matrices of y and z, while $h = (h_1, h_2, \cdots, h_n)^{\mathrm{T}}$ and

$\Phi = (\phi_1, \phi_2, \cdots, \phi_m)^{\mathrm{T}}$ are the nonlinear parts, respectively, and $v = (v_1, v_2, \cdots, v_m)^{\mathrm{T}}$ is the controller to be designed.

Next, the definition of CMHPS with a complex transformation matrix is introduced between fractional-order systems (8) and (7) based on that of the integer-order counterpart [18,20].

Definition 1. *For the fractional-order complex chaotic drive system (7) and response system (8), it is said to be CMHPS with* $\Theta = \Theta^r + j\Theta^i$ *between* $z(t)$ *and* $y(t)$*, if there exists a complex controller* $v = v^r + jv^i \in \mathbb{C}^m$*, such that:*

$$\lim_{t\to+\infty} \|z(t) - \Theta y(t)\| = 0, \tag{9}$$

i.e.,

$$\lim_{t\to+\infty} \|z^r(t) - \Theta^r y(t)\| = 0,$$

and:

$$\lim_{t\to+\infty} \|z^i(t) - \Theta^i y(t)\| = 0,$$

while the matrix $\Theta \in \mathbb{C}^{m\times n}$ *is defined as a complex transformation matrix of the fractional-order real hyper-chaotic (chaotic) drive system (7).*

If CMHPS error of systems (8) and (7) are defined as:

$$\delta(t) = \delta^r(t) + j\delta^i(t) = z(t) - \Theta y(t), \tag{10}$$

then:

$$\begin{cases} \delta^r(t) = z^r(t) - \Theta^r y(t), \\ \delta^i(t) = z^i(t) - \Theta^i y(t), \end{cases} \tag{11}$$

The objective of this section is to design a controller v to ensure that synchronization error tends to zero asymptotically, *i.e.*, $\lim_{t\to+\infty} \|\delta^r(t)\| = 0$, and $\lim_{t\to+\infty} \|\delta^i(t)\| = 0$.

Remark 1. *Lots of classical fractional-order real hyper-chaotic (chaotic) systems can be formed as system (7), such as the fractional-order real Chua system [7], the fractional-order real hyper-chaotic Rössler system [8], the fractional-order real Liu system [9] and other fractional-order real Lorenz-like systems [10]. Lots of classical fractional-order complex chaotic systems can be formed as system (8), such as fractional-order complex Lorenz system [15] and fractional-order complex Chen system [16].*

Remark 2. *Several types of synchronization are special cases of CMHPS, such as complex generalized projective synchronization (CGPS), complex projective synchronization (CPS), modified hybrid projective synchronization (MHPS), modified generalized projective synchronization (MGPS), generalized projective synchronization (GPS), projective synchronization (PS), complete synchronization (CS), anti-synchronization (AS); see Table 1.*

Table 1. Types of synchronization.

Settings the Matrix Θ	Synchronization Type
$\Theta=\Theta^r+j\Theta^i\in\mathbb{C}^{m\times n},m\neq n$	CMHPS
$\Theta=diag\{\theta_1,\theta_2,\cdots,\theta_n\}\in\mathbb{C}^{n\times n},m=n$	CGPS
$\Theta=diag\{\theta,\theta,\cdots,\theta\}\in\mathbb{C}^{n\times n},m=n$	CPS
$\Theta\in\mathbb{R}^{m\times n},m\neq n$	MHPS
$\Theta\in\mathbb{R}^{m\times n},m=n$	MGPS
$\Theta=diag\{\theta_1,\theta_2,\cdots,\theta_n\}\in\mathbb{R}^{n\times n},m=n$	GPS
$\Theta=diag\{\theta,\theta,\cdots,\theta\}\in\mathbb{R}^{n\times n},m=n$	PS
$\Theta=diag\{1,1,\cdots,1\},m=n$	CS
$\Theta=diag\{-1,-1,\cdots,-1\},m=n$	AS

Therefore, the CMHPS contains most existing works and extends previous works [10,13–16].

3.2. General Method of CMHPS

Theorem 1. *Given complex transformation matrix* $\Theta = \Theta^r + j\Theta^i$ *and initial conditions* $y(0), z(0)$*, if the complex controller is designed as:*

$$\begin{aligned} v &= v^r + jv^i \\ &= -\Phi(z) + \Theta(Cy + h(y)) - P\Theta y - K\delta \\ &= (-\Phi^r(z) + \Theta^r(Cy + h(y)) - P\Theta^r y - K\delta^r) + j(-\Phi^i(z) + \Theta^i(Cy + h(y)) - P\Theta^i y - K\delta^i), \end{aligned} \tag{12}$$

then CMHPS between the different dimensional fractional-order complex chaotic response system (8) and fractional-order real hyper-chaotic (chaotic) drive system (7) is achieved with desired complex transformation matrix Θ *asymptotically if and only if all of the eigenvalues of* $P - K$ *satisfy* $|\arg(\lambda_\ell(P - K))| > \frac{\alpha\pi}{2}$, $(\ell = 1, 2, \cdots, n)$*, where* $K \in \mathbb{R}^{m\times m}$ *is the control gain matrix.*

Proof. Equation (10) can be written as:

$$\begin{aligned} \delta(t) &= \delta^r(t) + j\delta^i(t) \\ &= (z^r(t) - \Theta^r y(t)) + j(z^i(t) - \Theta^i y(t)). \end{aligned} \tag{13}$$

Substituting Equation (7) and Equation (8) into Equation (13), one can get the derivative of the error system:

$$\begin{aligned} D_*^\alpha\delta(t) &= D_*^\alpha\delta^r(t) + jD_*^\alpha\delta^i(t) \\ &= (D_*^\alpha z^r(t) - \Theta^r D_*^\alpha y(t)) + j(D_*^\alpha z^i(t) - \Theta^i D_*^\alpha y(t)) \\ &= (Pz^r + \Phi^r(z) - \Theta^r(Cy + h(y)) + v^r) + j(Pz^i + \Phi^i(z) - \Theta^i(Cy + h(y)) + v^i). \end{aligned} \tag{14}$$

Insertion of Equation (12) into Equation (14) and separation of the real and imaginary parts give:

$$\begin{cases} D_*^\alpha\delta^r(t) = (P - K)\delta^r(t), \\ D_*^\alpha\delta^i(t) = (P - K)\delta^i(t), \end{cases} \tag{15}$$

Due to Lemma 1, the error system (15) is asymptotically stable if and only if all of the eigenvalues of $P - K$ satisfy $|\arg(\lambda_\ell(P - K))| > \frac{\alpha\pi}{2}(\ell = 1, 2, \cdots, n)$, where $K \in \mathbb{R}^{n\times n}$ is the control gain matrix.

That is, $\lim_{t\to+\infty}\|\delta^r(t)\| = 0$, and $\lim_{t\to+\infty}\|\delta^i(t)\| = 0$. Therefore, $\lim_{t\to+\infty}\|\delta(t)\| = 0$; CMHPS between the systems (8) and (7) is realized. This completes the proof. □

4. CMHPS Scheme of Different Dimensional Fractional-Order Complex Chaotic Drive Systems and Real Hyper-Chaotic (Chaotic) Response Systems

4.1. Mathematical Model and Problem Descriptions

Now, an n-dimensional fractional-order complex chaotic drive system is considered as:

$$D_*^\alpha w = D_*^\alpha w^r + jD_*^\alpha w^i = Qw + \Psi(w), \tag{16}$$

and an m-dimensional fractional-order real hyper-chaotic (chaotic) response system is written as:

$$D_*^\alpha x = Bx + p(x) + v, \tag{17}$$

where $w = w^r + jw^i \in \mathbb{C}^n$ is the complex state vector, $x = (x_1, x_2, \cdots, x_m)^{\mathrm{T}} \in \mathbb{R}^m$ is the real state vector, $Q \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ are the coefficient matrices of w and x, while $\Psi = (\psi_1, \psi_2, \cdots, \psi_n)^{\mathrm{T}}$ and $p = (p_1, p_2, \cdots, p_m)^{\mathrm{T}}$ are the nonlinear parts and $v = (v_1, v_2, \cdots, v_m)^{\mathrm{T}}$ is the controller to be designed, respectively.

If the error of CMHPS with a complex transformation matrix $\Theta = \Theta^r + j\Theta^i$ between systems (17) and (16) is defined as:

$$\delta(t) = x(t) - \Theta^r w^r(t) + \Theta^i w^i(t), \tag{18}$$

then the objective of this section is to design a controller v to ensure that synchronization error tends to zero asymptotically, *i.e.*, $\lim_{t\to+\infty}\|\delta(t)\| = 0$.

4.2. General Method of CMHPS

Theorem 2. *Given complex transformation matrix $\Theta = \Theta^r + j\Theta^i$ and initial conditions $w(0), x(0)$, if the designed controller is real as:*

$$v = \Theta^r\Psi^r(w) - \Theta^i\Psi^i(w) + (\Theta^r Q - B\Theta^r)w^r + (B\Theta^i - \Theta^i Q)w^i - p(x) - K\delta, \tag{19}$$

then CMHPS between the different dimensional fractional-order real hyper-chaotic (chaotic) response system (17) and complex chaotic drive system (16) is achieved with desired complex transformation matrix Θ asymptotically if and only if all of the eigenvalues of $B - K$ satisfy $|\arg(\lambda_\ell(B-K))| > \frac{\alpha\pi}{2}, (\ell = 1, 2, \cdots, n)$, where $K \in \mathbb{R}^{m\times m}$ is the control gain matrix.

Proof. Substituting Equation (16) and Equation (17) into Equation (18), one can get the derivative of the error system:

$$\begin{aligned} D_*^\alpha\delta &= D_*^\alpha x - \Theta^r D_*^\alpha w^r + \Theta^i D_*^\alpha w^i \\ &= Bx + p(x) + v - \Theta^r(Qw^r + \Psi^r(w)) + \Theta^i(Qw^i + \Psi^i(w)) \\ &= Be + B(\Theta^r w^r - \Theta^i w^i) + p(x) + v - \Theta^r(Qw^r + \Psi^r(w)) + \Theta^i(Qw^i + \Psi^i(w)). \end{aligned} \tag{20}$$

Insertion of Equation (19) into Equation (20) gives:

$$D_*^\alpha \delta(t) = (B - K)\delta(t). \tag{21}$$

Due to Lemma 1, the error system (21) is asymptotically stable if and only if all of the eigenvalues of $B - K$ satisfy $|\arg(\lambda_\ell(B-K))| > \frac{\alpha\pi}{2}(\ell = 1, 2, \cdots, n)$, where $K \in \mathbb{R}^{n\times n}$ is the control gain matrix. That is, $\lim_{t\to+\infty} \|\delta(t)\| = 0$; CMHPS between the fractional-order real hyper-chaotic (chaotic) response system (17) and fractional-order complex chaotic drive system (16) is realized. This completes the proof. □

5. CMHPS Scheme of Different Dimensional Fractional-Order Complex Chaotic Systems

Now, the case of CMHPS with a complex transformation matrix is considered between different dimensional fractional-order complex chaotic drive system (16) and response system (8).

If the error of CMHPS with complex transformation matrix $\Theta = \Theta^r + j\Theta^i$ is defined as:

$$\delta(t) = \delta^r(t) + j\delta^i(t) = z(t) - \Theta w(t), \tag{22}$$

then:

$$\begin{cases} \delta^r(t) = z^r(t) - \Theta^r w^r(t) + \Theta^i w^i(t), \\ \delta^i(t) = z^i(t) - \Theta^r w^i(t) - \Theta^i w^r(t), \end{cases} \tag{23}$$

The objective of this section is to design a controller v to ensure that synchronization error tends to zero asymptotically, *i.e.*, $\lim_{t\to+\infty} \|\delta^r(t)\| = 0$, and $\lim_{t\to+\infty} \|\delta^i(t)\| = 0$.

Based on Lemma 1, the following results can be obtained.

Theorem 3. *Given complex transformation matrix* $\Theta = \Theta^r + j\Theta^i$ *and initial conditions* $w(0), z(0)$*, if the complex controller is designed as:*

$$\begin{aligned} v &= v^r + jv^i \\ &= \Theta\Psi(w) + (\Theta Q - P\Theta)w - \Phi(z) - Ke \\ &= (\Theta^r\Psi^r(w) - \Theta^i\Psi^i(w) + (\Theta^r Q - P\Theta^r)w^r + (P\Theta^i - \Theta^i Q)w^i - \Phi^r(z) - Ke^r) \\ &\quad + j(\Theta^r\Psi^i(w) + \Theta^i\Psi^r(w) + \Theta^i Q - P\Theta^i)w^r + (\Theta^r Q - P\Theta^r)w^i - \Phi^i(z) - Ke^i), \end{aligned} \tag{24}$$

then CMHPS between the different dimensional fractional-order complex chaotic response system (8) and drive system (16) is achieved with desired complex transformation matrix Θ *asymptotically if and only if all of the eigenvalues of* $P - K$ *satisfy* $|\arg(\lambda_\ell(P-K))| > \frac{\alpha\pi}{2}$, $(\ell = 1, 2, \cdots, n)$*, where* $K \in \mathbb{R}^{m\times m}$ *is the control gain matrix.*

Proof. This is similar to the proof in Theorem 1 and, thus, is omitted. □

6. Numerical Examples

Now, two examples are worked out to illustrate the theoretical results in this paper.

6.1. Reduced Order CMHPS

In order to illustrate reduced order CMHPS, it is assumed that a four-dimensional fractional-order real hyper-chaotic Rössler system [8] drives a three-dimensional fractional-order complex Chen system [16]. Therefore, the drive system is given in the form as:

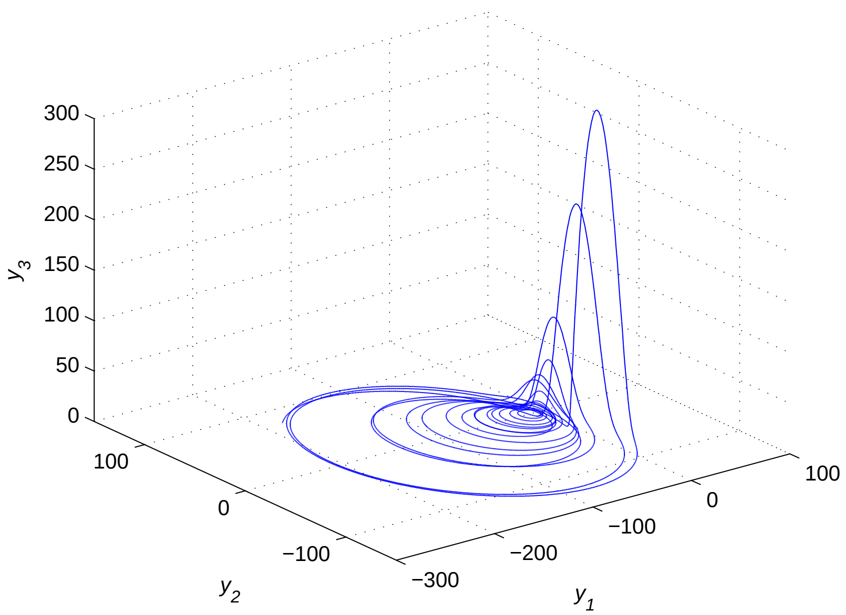
$$\begin{cases} D_*^\alpha y_1 = -(y_2 + y_3), \\ D_*^\alpha y_2 = y_1 + c_1 y_2 + y_4, \\ D_*^\alpha y_3 = c_2 + y_1 y_3, \\ D_*^\alpha y_4 = -c_3 y_3 + c_4 y_4, \end{cases} \tag{25}$$

where:

$$C = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & c_1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c_3 & c_4 \end{pmatrix}, h(y) = \begin{pmatrix} 0 \\ 0 \\ c_2 + y_1 y_3 \\ 0 \end{pmatrix},$$

and $y = (y_1, y_2, y_3, y_4)^T \in \mathbb{R}^4$ is real state vector. The system (25) is hyper-chaotic when $c_1 = 0.32$, $c_2 = 3$, $c_3 = 0.5$, $c_4 = 0.05$, $\alpha = 0.95$ in Figure 1; see [8] for more details.

Figure 1. The hyper-chaotic attractor of the fractional-order real Rössler system (25) for $c_1 = 0.32, c_2 = 3, c_3 = 0.5, c_4 = 0.05, \alpha = 0.95$.



The response system with the controller is written in the form as:

$$\begin{cases} D_*^\alpha z_1 = p_1(z_2 - z_1) + v_1, \\ D_*^\alpha z_2 = (p_2 - p_1) z_1 + p_2 z_2 - z_1 z_3 + v_2, \\ D_*^\alpha z_3 = -p_3 z_3 + (1/2)(\bar{z}_1 z_2 + z_1 \bar{z}_2) + v_3, \end{cases} \tag{26}$$

where:

$$P = \begin{pmatrix} -p_1 & p_1 & 0 \\ p_2 - p_1 & p_2 & 0 \\ 0 & 0 & -p_3 \end{pmatrix}, \Phi(z) = \begin{pmatrix} 0 \\ -z_1 z_3 \\ (1/2)(\bar{z}_1 z_2 + z_1 \bar{z}_2) \end{pmatrix},$$

and $z_1 = z_1^r + jz_1^i, z_2 = z_2^r + jz_2^i$ are complex state variables and z_3 is a real state variable. The system (26) is chaotic when $p_1 = 35, p_2 = 28, p_3 = 3, \alpha = 0.95$ and in the absence of the controller $v = v^r + jv^i$ in Figure 2; see [16] for more details.

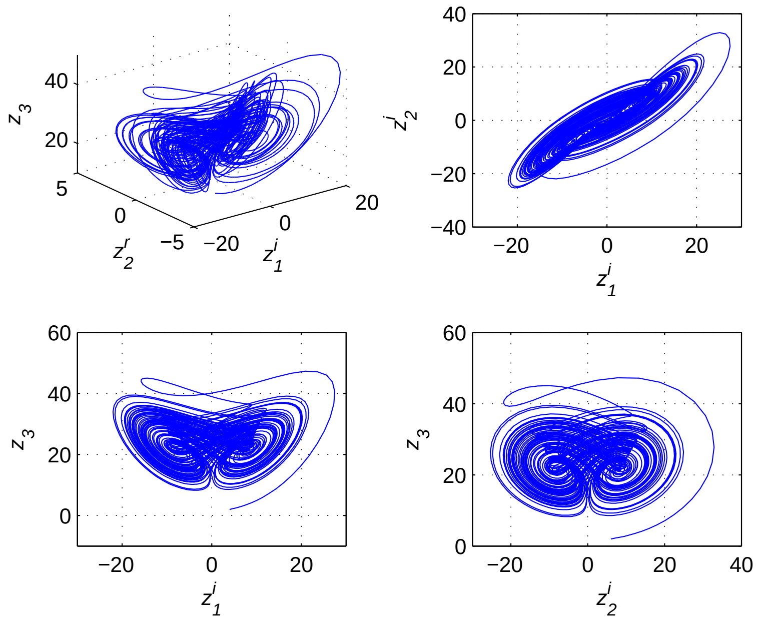
The complex transformation matrix can be taken as:

$$\Theta = \begin{pmatrix} 1-j & 0 & 0 & 0 \\ 0 & -1+2j & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \tag{27}$$

and the error system $\delta(t) = z(t) - \Theta y(t)$ is obtained as:

$$\begin{cases} \delta_1 = z_1 - (1-j)y_1 = (z_1^r - y_1) + j(z_1^i + y_1), \\ \delta_2 = z_2 - (-1+2j)y_2 = (z_2^r + y_2) + j(z_2^i - 2y_2), \\ \delta_3 = z_3 - y_3 + y_4. \end{cases}$$

Figure 2. Chaotic attractor projections of fractional-order complex Chen system (26) for $p_1 = 35, p_2 = 28, p_3 = 3, \alpha = 0.95$.



The control gain matrix is chosen as:

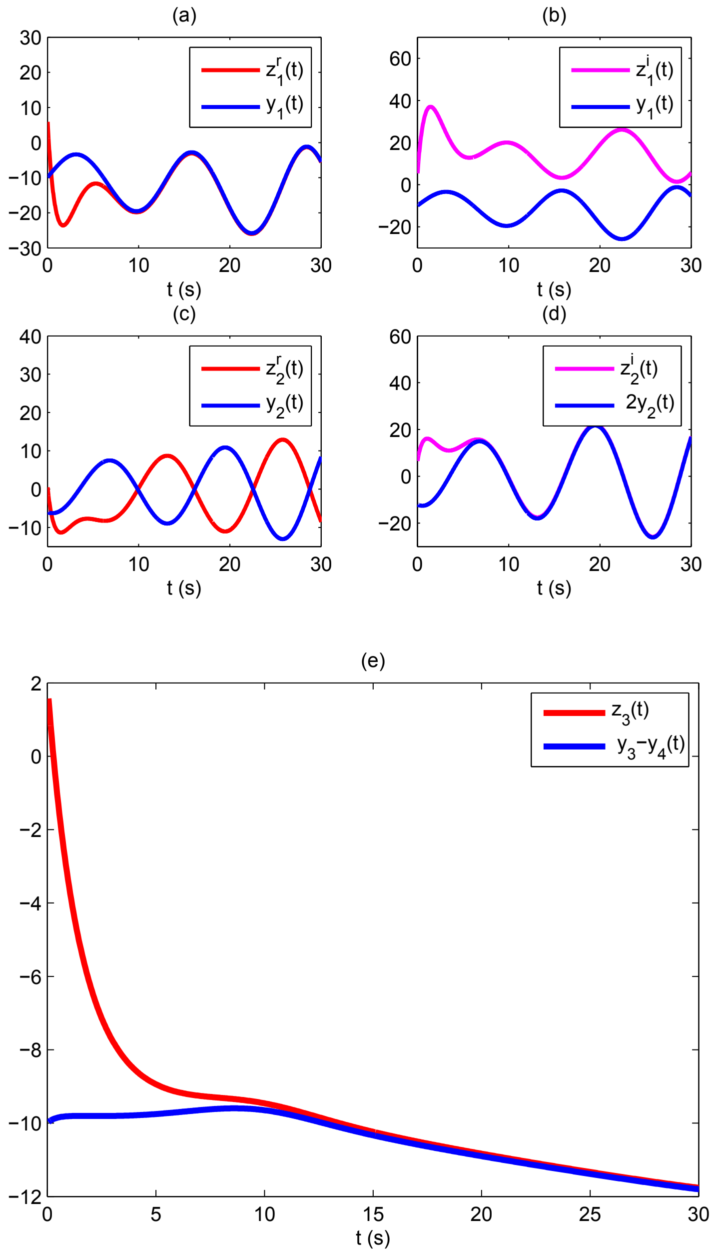
$$K = \begin{pmatrix} -31 & 30 & 0 \\ -5 & 26 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the complex controller is designed according to (12) in Theorem 1 as follows,

$$\begin{aligned} v &= v^r + jv^i \\ &= \begin{pmatrix} p_1y_1 + (p_1-1)y_2 - y_3 + 31\delta_1^r - 30\delta_2^r \\ (p_1 - p_2 - 1)y_1 + (p_2 - c_1)y_2 - y_4 + z_1^r z_3 + 5\delta_1^r - 26\delta_2^r \\ (p_3 + c_3)y_3 - (p_3 + c_4)y_4 + y_1y_3 + c_2 - z_1^r z_2^r - z_1^i z_2^i \end{pmatrix} \end{aligned}$$

$$+ \quad j \begin{pmatrix} -p_1 y_1 + (1-2p_1)y_2 + y_3 + 31\delta_1^i - 30\delta_2^i \\ (2-p_1+p_2)y_1 + 2(c_1-p_2)y_2 + 2y_4 + z_1^i z_3 + 5\delta_1^i - 26\delta_2^i \\ 0 \end{pmatrix} \tag{28}$$

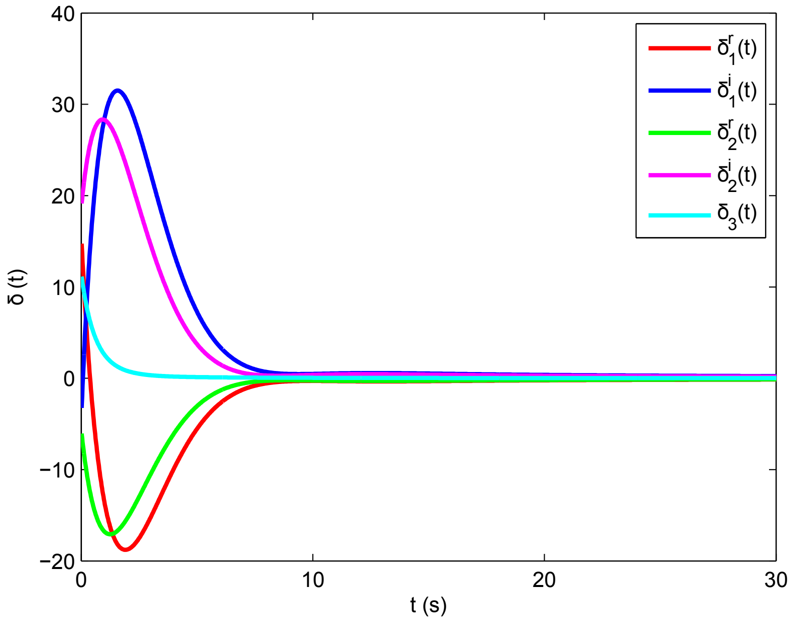
Figure 3. Reduced order synchronization-CMHPS between four-dimensional fractional-order real hyper-chaotic Rössler drive system (25) and three-dimensional fractional-order complex chaotic Chen response system (26) with the controller (28). (**a**) z_1^r synchronizes y_1; (**b**) z_1^i anti-synchronizes y_1; (**c**) z_2^r anti-synchronizes y_2; (**d**) z_1^i synchronizes $2y_2$; (**e**) z_3 synchronizes $y_3 - y_4$.



The parameters of drive system (25) and response system (26) are chosen as $\alpha = 0.95$, $c_1 = 0.32$, $c_2 = 3, c_3 = 0.5, c_4 = 0.05$ and $p_1 = 35, p_2 = 28, p_3 = 3$, respectively. The initial values are randomly chosen as $y_0 = (-10, -6, 0, 10)^T$ and $z_0 = z_0^r + jz_0^i = (7+4j, 1+6j, 2)^T$, respectively. Therefore, all of the eigenvalues of $P - K$ are $\lambda_1 = -1-j, \lambda_2 = -1+j, \lambda_3 = -3$, which satisfies

$|\arg(\lambda_\ell(P-K))| > \frac{\alpha\pi}{2}$, $(\ell = 1, 2, 3)$. The simulation results are demonstrated in Figure 3, where the blue line presents the states of drive system (25) and the red (pink) line presents the real (imaginary) parts of the states in the response system (26). The errors of CMHPS converge asymptotically to zero as in Figure 4. Hence, CMHPS has been achieved between fractional-order real hyperchaotic Rössler drive system (25) and fractional-order complex chaotic Chen response system (26).

Figure 4. The CMHPS error dynamic of fractional-order real hyper-chaotic Rössler drive system (25) and fractional-order complex chaotic Chen response system (26) with the controller (28).



6.2. Increased Order CMHPS

In order to illustrate increased order CMHPS, it is assumed that a three-dimensional fractional-order complex Lorenz system [15] drives a four-dimensional fractional-order real hyper-chaotic Rössler system (25). Therefore, the drive system is given in the form as:

$$\begin{cases} D_*^\alpha w_1 = q_1(w_2 - w_1), \\ D_*^\alpha w_2 = q_2 w_1 - w_2 - w_1 w_3, \\ D_*^\alpha w_3 = -q_3 w_3 + (1/2)(\bar{w}_1 w_2 + w_1 \bar{w}_2), \end{cases} \tag{29}$$

where:

$$Q = \begin{pmatrix} -q_1 & q_1 & 0 \\ q_2 & -1 & 0 \\ 0 & 0 & -q_3 \end{pmatrix}, \Psi(w) = \begin{pmatrix} 0 \\ -w_1 w_3 \\ \frac{1}{2}(w_1 \bar{w}_2 + \bar{w}_1 w_2) \end{pmatrix},$$

and $w_1 = w_1^r + jw_1^i, w_2 = w_2^r + jw_2^i$ are complex state variables and w_3 is real state variable. The system (29) is chaotic when $q_1 = 10, q_2 = 180, q_3 = \frac{8}{3}, \alpha = 0.95$ in Figure 5; see [15] for more details.

The response system with the controller is written in the form as:

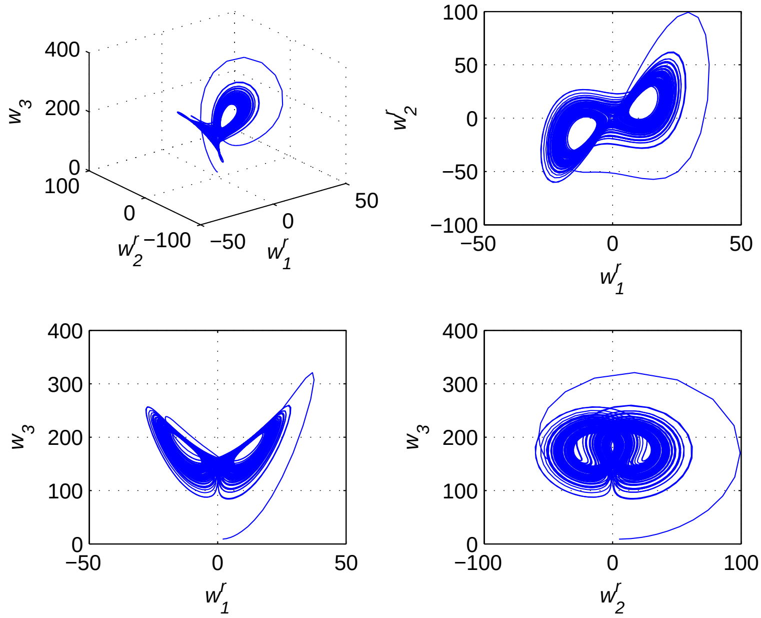
$$\begin{cases} D_*^\alpha x_1 = -(x_2 + x_3) + v_1, \\ D_*^\alpha x_2 = x_1 + b_1 x_2 + x_4 + v_2, \\ D_*^\alpha x_3 = b_2 + x_1 x_3 + v_3, \\ D_*^\alpha x_4 = -b_3 x_3 + b_4 x_4 + v_4, \end{cases} \tag{30}$$

where:

$$B = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & b_1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -b_3 & b_4 \end{pmatrix}, p(x) = \begin{pmatrix} 0 \\ 0 \\ b_2 + x_1 x_3 \\ 0 \end{pmatrix},$$

$x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ is real state vector and $v = (v_1, v_2, v_3, v_4)^T$ is the controller to be designed.

Figure 5. The chaotic attractor projections of fractional-order complex Lorenz system (29) for $q_1 = 10, q_2 = 180, q_3 = \frac{8}{3}, \alpha = 0.95$.



The complex transformation matrix can be taken as:

$$\Theta = \begin{pmatrix} -j & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \tag{31}$$

and the error system $\delta(t) = x(t) - \Theta^r w^r(t) + \Theta^i w^i(t)$ is obtained as:

$$\begin{cases} \delta_1 = x_1 - w_1^i, \\ \delta_2 = x_2 + w_2^i, \\ \delta_3 = x_3 - 2w_3, \\ \delta_4 = x_4 + w_3. \end{cases}$$

The control gain matrix is chosen as:

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.32 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2.05 \end{pmatrix},$$

and the real controller is designed according to (19) in Theorem 2 as follows,

$$\begin{cases} v_1 = -q_1 w_1^i + (q_1 - 1) w_2^i + 2w_3 - \delta_1. \\ v_2 = -(1 + q_2) w_1^i + (1 + b_1) w_2^i + w_1^i w_3 + w_3 - 1.32\delta_2. \\ v_3 = 2(w_1^r w_2^r + w_1^i w_2^i) - 2q_3 w_3 - b_2 - x_1 x_3 - 3\delta_3. \\ v_4 = -(w_1^r w_2^r + w_1^i w_2^i) + (2b_3 + b_4 + q_3) w_3 - 2.05\delta_4. \end{cases} \tag{32}$$

The parameters of drive system (29) and response system (30) are chosen as $\alpha = 0.95, q_1 = 10, q_2 = 180, q_3 = \frac{8}{3}$ and $b_1 = 0.32, b_2 = 3, b_3 = 0.5, b_4 = 0.05$, respectively. The initial values are randomly selected as $w_0 = w_0^r + jw_0^i = (2 + 3j, 5 + 6j, 9)$ and $x_0 = (-10, -6, 0, 10)^T$, respectively. Therefore, all of the eigenvalues of $B - K$ are $\lambda_1 = -1 + j, \lambda_2 = -1 - j, \lambda_3 = -2, \lambda_4 = -3$, which satisfies $|\arg(\lambda_\ell(B - K))| > \frac{\alpha\pi}{2}$, $(\ell = 1, 2, 3, 4)$. The simulation results are demonstrated in Figure 6, where the blue line presents the states of response system (30) and the red (pink) line presents the real (imaginary) parts of the states in the drive system (29). The errors of CMHPS converge asymptotically to zero as in Figure 7. Hence, CMHPS has been achieved between three-dimensional fractional-order complex chaotic Lorenz drive system (29) and four-dimensional fractional-order real hyper-chaotic Rössler response system (30).

Figure 6. Increased order synchronization-CMHPS between three-dimensional fractional-order complex chaotic Lorenz drive system (29) and four-dimensional fractional-order real hyper-chaotic Rössler response system (30) with the controller (32). (**a**) x_1 synchronizes w_1^i; (**b**) x_2 anti-synchronizes w_2^i; (**c**) x_3 synchronizes $2w_3$; (**d**) x_4 anti-synchronizes w_3.

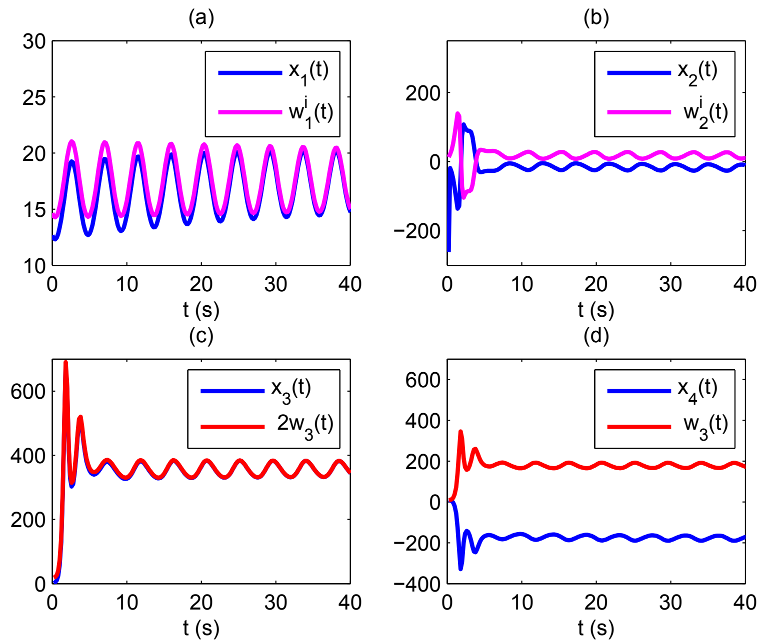
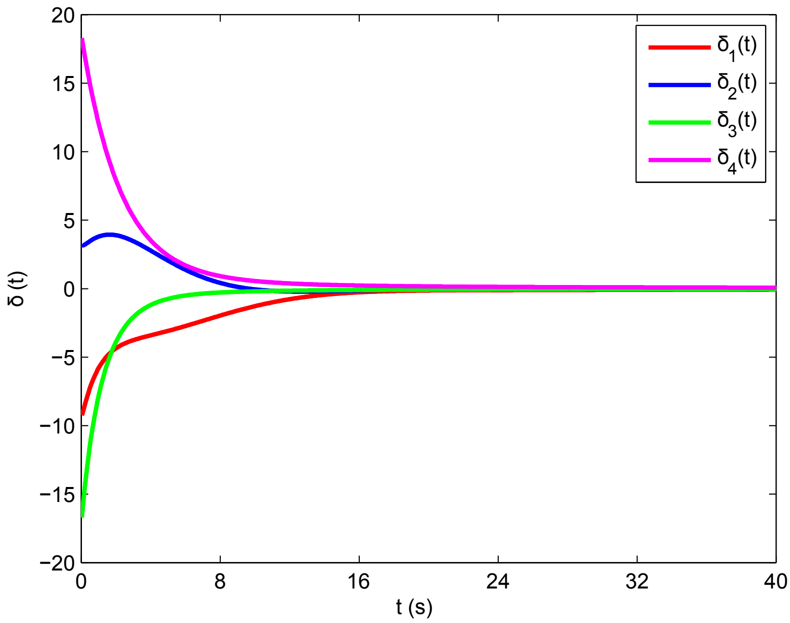


Figure 7. The CMHPS error dynamic of fractional-order complex chaotic Lorenz drive system (29) and fractional-order real hyper-chaotic Rössler response system (30) with the controller (32).



7. Conclusions

In this paper, CMHPS is introduced for different dimensional fractional-order complex chaos and fractional-order real hyper-chaos. The fractional-order real (complex) response system becomes a complex projection of different dimensional fractional-order complex (real) drive systems by the complex transformation matrix.

A general scheme of CMHPS is addressed based on the stability theory of fractional-order systems and the feedback control technique. It is worth noting that the Lyapunov function is not required to be calculated in this scheme; it is really simple and feasible in practical applications.

Moreover, CMHPS between a four-dimensional fractional-order real hyper-chaotic Rössler drive system and a three-dimensional fractional-order complex chaotic Chen response system is implemented as an example to discuss reduced order synchronization, and CMHPS between a three-dimensional fractional-order complex chaotic Lorenz drive system and a four-dimensional fractional-order real hyper-chaotic Rössler response system is implemented as an example to discuss increased order synchronization, as well. The proposed scheme clearly exhibits its simplicity, effectiveness and feasibility during applications and implementation. These theoretical and numerical results bridge the gap between fractional-order real hyper-chaos and fractional-order complex chaos with different dimensions.

Finally, it is also believed that the proposed scheme has applications in different fields of engineering, such as secure communication, encryption and control process, since the fractional-order system possesses memory.

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Conflicts of Interest

The author declares no conflict of interest.

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