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Article

Statistical Power Law due to Reservoir Fluctuations and the Universal Thermostat Independence Principle

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Abstract: Certain fluctuations in particle number, n, at fixed total energy, E, lead exactly to a cut-power law distribution in the one-particle energy, ω, via the induced fluctuations in the phase-space volume ratio, $\Omega_n(E-\omega)/\Omega_n(E) = (1-\omega/E)^n$. The only parameters are $1/T = \langle\beta\rangle = \langle n\rangle/E$ and $q = 1 - 1/\langle n\rangle + \Delta n^2/\langle n\rangle^2$. For the binomial distribution of n one obtains $q = 1 - 1/k$, for the negative binomial $q = 1 + 1/(k+1)$. These results also represent an approximation for general particle number distributions in the reservoir up to second order in the canonical expansion $\omega \ll E$. For general systems the average phase-space volume ratio $\langle e^{S(E-\omega)}/e^{S(E)}\rangle$ to second order delivers $q = 1 - 1/C + \Delta\beta^2/\langle\beta\rangle^2$ with $\beta = S'(E)$ and $C = dE/dT$ heat capacity. However, $q \neq 1$ leads to non-additivity of the Boltzmann–Gibbs entropy, S. We demonstrate that a deformed entropy, $K(S)$, can be constructed and used for demanding additivity, *i.e.*, $q_K = 1$. This requirement leads to a second order differential equation for $K(S)$. Finally, the generalized q-entropy formula, $K(S) = \sum p_i K(-\ln p_i)$, contains the Tsallis, Rényi and Boltzmann–Gibbs–Shannon expressions as particular cases. For diverging variance, $\Delta\beta^2$ we obtain a novel entropy formula.

Keywords: generalized q-entropy; fluctuations; hadronization

1. Introduction

We have been studying generalizations of the Boltzmann–Gibbs–Shannon (BGS) entropy formula [1–6] since decades. Our studies included the investigation of the role of multiplicative noise [7,8], kinetic theory [9–11], non-extensive equilibration [12–14] and thermodynamical compatibility [15–18], also with respect to infinite repetitions of abstract composition rules [19]. Recently, in the quest for mechanisms explaining the occurence of a statistical power law distribution in canonical ensembles, we emphasized the role of finite reservoir effects in the mathematical derivation [20–23]. The majority attitude to nonextensive physics is in general to start with the presentation of a formula for the entropy and then deriving mathematical relations from it, in order to demonstrate that the traditional requirements, like concavity, unique equilibrium state or the Lagrange multiplier handling of secondary constraints, are fulfilled as well as in the original approach [24–28]. Comparisons to experimental data then usually supplement the results of such investigations [29–31].

Our present approach reveals a different path: We start with the traditional postulates and formulas, and then try to show why and how a "deformation" of the original classical BGS entropy formula becomes unavoidable. As a by-product of such a procedure we obtain the physical background interpretation for the parameters T and q, characterizing the ubiquitous cut power law probability distribution. In the limit $q = 1$ the BGS framework is reconstructed [17,28,32,33].

As we shall demonstrate below, the common physical cause of $q \neq 1$ is the finiteness of the physical environment, a finite heat bath [22,32,34–36]. Whether the finite size corrections may become negligible is a case by case problem, entangled with the physical properties of the system under study. Some systems, called "non-extensive", may behave as finite ones in this respect even in large volumes—since some effects behind $q \neq 1$ depend on ratios of large quantities [6]. To gain a feeling about the magnitude of such effects we remind that besides the Avogadro number $\mathcal{O}\left(10^{24}\right)$, considered in classical thermodynamics of atomic matter, complex networks, like, e.g., the human brain include about the square root of this number of elements $\mathcal{O}\left(10^{12}\right)$. The internet contains approximately 10^7 hubs and 10^{10} connections. On the other hand a relativistic heavy ion collision produces a fireball of several $\mathcal{O}\left(10^3\right)$ new hadrons (strongly interacting particles), while in a more elementary pp collision about $\mathcal{O}\left(10\right)$ particles are detected[37–39]. Since one expects that the relative (scaled) fluctuations grow with the decreasing number of participants, it is evident that the high energy physics experiments are able to reveal finite reservoir effects quantitatively [21,22,26,27,33,40–45].

In this paper we seek answer to the following two questions: (i) What is the physics behind $q \neq 1$ and (ii) what $K(S)$ deformation of the entropy S is necessary to achieve $q_K = 1$? We note that $q = 1$ signalizes an additive composition rule, so the second question is equivalent for seeking an additive (*"K-additive"*) description in case of non-negligible finite size corrections on the classical thermodynamics [16,18,22].

However, before going into the details, we would like to present our particular view on (non-)extensivity and (non-)additivity. The cut power-law formula as a form approaching the exponential in a given limit has already been established by Euler,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x. \tag{1}$$

Therefore, any time when seeking to establish a physical model of almost thermally behaving (complex) systems *finite size effects* already induce a deviation from the exponential law. It is known that the ideal gas, with a fixed finite number of particles and therefore with a fixed heat capacity, leads to a cut power-law distribution in the single particle energy in a microcanonical treatment [20]. Also, a nice example of superstatistics, given by Beck, refers to a distribution of the sum of the squares of random Gaussian variables (e.g., the classical kinetic energy sum for a finite number of particles) as being Euler Gamma distributed *for a finite number* of variables [17,46,47]. While the product formula of the cut power-law requires a *non-additivity* of its arguments,

$$x_{12} = x_1 + x_2 + \frac{1}{n} x_1 x_2 \quad \text{due to} \quad \left(1 + \frac{x_{12}}{n}\right)^n = \left(1 + \frac{x_1}{n}\right)^n \cdot \left(1 + \frac{x_2}{n}\right)^n, \tag{2}$$

it does not mean non-extensivity. The latter can be established only then if in the large system size limit, in this example in the $n \to \infty$ limit, the non-additivity of the arguments still holds. The distinction between non-additivity and non-extensivity is important, as it has been emphasized in [5,6]. In the physical models, we shall be discussing in this paper, we leave the question of this limit open and seek for general solutions and approximations valid both for finite non-additive and for infinite non-extensive systems. This classification scheme approach then should be specialized case by case, and the large system limits has to be considered in particular applications.

2. Finite Heat Bath and Fluctuation Effects

In this section we review the traditional, Boltzmann–Gibbs–Planck–Einstein thermodynamical approach to the thermodynamical statistical weight assuming a uniform phase-space distribution of microstates [48]. At the beginning we present a very simple model of particle production, where the total energy, E, is fixed (in experiments $\Delta E/E \lesssim 10^{-3}$), but the number of produced particles, n, fluctuates appreciably. Its distribution will be considered first in terms of the simplest possible assumptions about combining occupied and unoccupied phase-space cells in a *finite observed section of the available total phase-space*. Following this analysis more general n distributions and finally a general heat bath, described by its equation of state, $S(E)$, is considered. During this chain of models we seek answer for the question: *What is the physics behind the parameter q ?*

Our starting point is an ideal gas in a finite phase-space [20,30,36,48]. We describe the microcanonical statistical weight for having a one-particle energy, ω, out of total energy, E. In a one-dimensional relativistic jet it is distributed according to the ratio of corresponding phase-space volumes as

$$P_1(\omega) = \frac{\Omega_1(\omega)\,\Omega_n(E-\omega)}{\Omega_{n+1}(E)} = \rho(\omega) \cdot \frac{(E-\omega)^n}{E^n}. \tag{3}$$

Here $\Omega_{n+1}(E)$ is the total phase-space, while $\Omega_n(E-\omega)$ is the phase-space for the reservoir, missing one particle with energy ω. The number of particles, n, itself can have a distribution (based on the physical model of the reservoir and on the event by event detection of the spectra).

However, already by fixed n and E the statistical weight formula for such an ideal gas (the factor besides $\rho(\omega)$ in the above formula),

$$w_{E,n}(\omega) = \frac{\Omega_n(E-\omega)}{\Omega_{n+1}(E)} = \left(1 - \frac{\omega}{E}\right)^n, \tag{4}$$

represents a finite size (microcanonical) approach to the Euler exponential. Only in the large number, large energy limit, $n \to \infty$, and $E \to \infty$, while the ratio is kept constant according to an equipartition law, $E = nT$, does it approach the traditional canonical exponential

$$\lim_{n\to\infty} \lim_{E\to nT} w_{E,n}(\omega) = e^{-\omega/T}. \tag{5}$$

Non-extensive systems must undermine (or circumvent) this basic correspondence. Either the scaling in the equipartition law is different, or the cut power law is deformed, or the large n limit is effectively never achieved; e.g., because of some irregular fluctuation properties of the quantity n or E.

We consider now the fluctuation of n by fixed E in statistically ideal reservoirs. In the case of hadronization in high energy physics experiments n is distributed according to a negative binomial n-distribution, obtained from the following, statistically simplest argumentation. We distribute n particles among k cells: bosons in $\binom{n+k}{n}$ ways, fermions in $\binom{k}{n}$ ways. By observation we detect a subspace (n, k) out of a bigger (N, K) reservoir. The limit $K \to \infty$, $N \to \infty$ with a fixed average occupancy $f = N/K$, constitutes the traditional canonical limit. However, we keep here several finite size factors. We obtain

$$B_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{n+k}{n}\binom{N-n+K-k}{N-n}}{\binom{N+K+1}{N}} = \binom{n+k}{n} f^n \,(1+f)^{-n-k-1} \tag{6}$$

for bosons and

$$F_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{k}{n}\binom{K-k}{N-n}}{\binom{K}{N}} = \binom{k}{n} f^n \,(1-f)^{k-n} \tag{7}$$

for fermions.

Since most hadrons produced in high-energy experiments are pions, which are bosons, we consider first the bosonic reservoir described by $B_{n,k}(f)$. The average statistical weight factor, $w_E(\omega)$, with fixed E and the negative binomial distribution (NBD) of n becomes

$$w_E^{\rm NBD}(\omega) = \sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n B_{n,k}(f) = \left[(1+f) - f\left(1 - \frac{\omega}{E}\right)\right]^{-k-1} = \left(1 + f\frac{\omega}{E}\right)^{-k-1}. \tag{8}$$

Note that $\langle n \rangle = (k+1)f$ for NBD. Then with the notation $T = E/\langle n \rangle$ and $q - 1 = \frac{1}{k+1}$ we get

$$w_E^{\rm NBD}(\omega) = \left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}. \tag{9}$$

This is *exactly* a $q > 1$ Tsallis–Pareto distribution. The opposite correspondence, namely that an assumed Tsallis–Pareto distribution leads to an NBD multiplicity distribution, has been pointed out by Wilk and Wlodarczyk [27,28,49]. Experimental NBD distributions of total charged hadron multiplicites stemming from Au + Au collisons at $\sqrt{s}_{NN} = 62$ and 200 GeV can be inspected, e.g., in reference [37]. Characteristically $k \approx 10 - 20$, therefore $q \approx 1.05 - 1.10$, which roughly agrees with fits to p_T spectra in the same experiments [37–39].

For a fermionic reservoir n is distributed according to the Bernoulli distribution (BD). The average phase-space volume ratio becomes

$$w_E^{\rm BD}(\omega) = \sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n F_{n,k}(f) = \left[(1-f) + f\left(1 - \frac{\omega}{E}\right)\right]^k = \left(1 - f\frac{\omega}{E}\right)^k. \tag{10}$$

Note that $\langle n \rangle = kf$ for BD. Then with $T = E/\langle n \rangle$ and $q - 1 = -\frac{1}{k}$ we obtain *exactly* a $q < 1$ Tsallis–Pareto distribution,

$$w_E^{\rm BD}(\omega) = \left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}.$$

It is enlightening to consider the Boltzmann–Gibbs limit of the above. In case of low occupancy in the phase-space, $k \gg n$ and both the BD and NBD distributions approach a Poissonian:

$$\Pi_n = \frac{\langle n \rangle^n}{n!} \, \mathrm{e}^{-\langle n \rangle} \quad \text{with} \quad \langle n \rangle = k\,\frac{f}{1 \pm f} \quad \text{fixed.} \tag{11}$$

The resulting statistical factor is *exactly* the Boltzmann–Gibbs exponential with $T = E/\langle n \rangle$,

$$w_E^{\rm BG}(\omega) = \sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n \Pi_n(\langle n \rangle) = \mathrm{e}^{(1-\omega/E)\langle n \rangle} \, \mathrm{e}^{-\langle n \rangle} = \mathrm{e}^{-\langle n \rangle \omega/E} = \mathrm{e}^{-\omega/T}. \tag{12}$$

In all of the three above cases the parameter T is defined by the (one-dimensional, extreme relativistic) equipartition, and q is related to the scaled variance of the produced particle number:

$$T = \frac{E}{\langle n \rangle}, \quad \text{and} \quad q = 1 - \frac{1}{\langle n \rangle} + \frac{\Delta n^2}{\langle n \rangle^2}. \tag{13}$$

For general n-fluctuations, P_n, not necessarily BD or NBD or Poissonian ones, the above result also applies, albeit *only as an approximation*. In the above detailed philosophy of the microcanonical approaching the canonical for large systems, we expand our formulas for $\omega \ll E$. The Tsallis–Pareto distribution as an approximation reads as

$$\left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q\frac{\omega^2}{2T^2} - \ldots. \tag{14}$$

The ideal reservoir phase-space ratio up to second order in this limit results in

$$w_E(\omega) = \sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n P_n = 1 - \langle n \rangle \frac{\omega}{E} + \langle n(n-1) \rangle \frac{\omega^2}{2E^2} - \ldots. \tag{15}$$

Comparing the corresponding coefficients one concludes that Equation (13) as an approximation holds also for a general n-distribution, and the "non-extensivity parameter" q is related to the second order subleading term.

Finally, we deal with a general environment, given by its equation of state, $S(E)$. In the expansion for $E \gg \omega$ the phase-space volume ratio becomes

$$w_E(\omega) = \left\langle \frac{\Omega_n(E-\omega)}{\Omega_n(E)} \right\rangle = \left\langle e^{S(E-\omega)-S(E)} \right\rangle = \left\langle e^{-\omega S'(E)+\omega^2 S''(E)/2-\ldots} \right\rangle$$

$$= 1 - \omega \left\langle S'(E) \right\rangle + \frac{\omega^2}{2} \left\langle S'(E)^2 + S''(E) \right\rangle - \ldots . \tag{16}$$

Comparing it with the expansion of the Tsallis–Pareto distribution, Equation (14), one concludes

$$\frac{1}{T} = \langle \beta \rangle = \langle S'(E) \rangle , \qquad q = 1 - \frac{1}{C} + \frac{\Delta\beta^2}{\langle \beta \rangle^2}. \tag{17}$$

This is the final interpretation of the parameters T and q for a general reservoir in the framework of the Einstein postulate averaged over reservoir fluctuations. Later, in the next chapter, we shall aim to construct an additive deformed entropy, $K(S)$, which renders not only the form (15), but also the more general form (16) to be multiplicative. Note that due to $\langle S''(E) \rangle = -1/CT^2$, our result is expressed via the heat capacity of the reservoir, defined as $1/C = \mathrm{d}T/\mathrm{d}E$. In general we have opposite sign contributions from $\langle S'^2 \rangle - \langle S' \rangle^2$ and from $\langle S'' \rangle$. In the light of this result one realizes that

- $q > 1$ and $q < 1$ are both possible,
- for any relative variance $\Delta\beta / \langle \beta \rangle = 1/\sqrt{C}$ it is exactly $q = 1$,
- and for fixed $E \propto n/\beta$ we have $\Delta\beta / \langle \beta \rangle = \Delta n / \langle n \rangle$.

In this way the n-fluctuations represent a particular case of the more general reservoir fluctuations.

At the end of this section we sketch the relation of our approach to *superstatistics*[46,47,50–52]. In its original formulation superstatistics dealt with fluctuations of the Lagrange multiplier β. Demanding that we describe the same non-exponential statistics, only in two different ways, one arrives at the relation

$$\int e^{-\beta\omega} \gamma(\beta) \, \mathrm{d}\beta = \sum_n P_n(E) \left(1 - \frac{\omega}{E}\right)^n . \tag{18}$$

Note that $e^{-\beta\omega} = e^{\left(1-\frac{\omega}{E}\right)\beta E} e^{-\beta E}$. Using now the Taylor series of the first exponential one obtains

$$P_n(E) = \int \frac{(\beta E)^n}{n!} e^{-\beta E} \gamma(\beta) \, \mathrm{d}\beta \, . \tag{19}$$

The converting factor is a Poissonian with the parameter $\overline{n} = \beta E$. Inverting the above procedure one seeks for a superstatistics from the n-distribution. Applying the correspondence Equation (18) for $\omega = E$:

$$\int e^{-\beta E} \gamma(\beta) \, \mathrm{d}\beta = P_0(E). \tag{20}$$

Inverse Laplace transformation then, in principle, delivers the superstatistical factor

$$\gamma(\beta) = \mathcal{L}^{-1} \left[P_0(E) \right] . \tag{21}$$

Expanding for $E \gg \omega$, however, one gets $\langle \beta \rangle = \langle n \rangle / E$ and $\langle \beta^2 \rangle = \langle n(n-1) \rangle / E^2$, leading to

$$q = 1 + \frac{\Delta\beta^2}{\langle \beta \rangle^2} = 1 + \frac{\Delta n^2}{\langle n \rangle^2} - \frac{1}{\langle n \rangle}. \tag{22}$$

One immediately realizes that for some n-distributions, alike the BD, $\Delta\beta^2$ would have to be negative. It is impossible. This problem is also reflected in the fact that there is no guarantee that an inverse Laplace transformation results in an overall positive function. In this way the superstatistics due to n-fluctuations, $P_n(E)$, seems to be more general, than the approach with solely a β-distribution, $\gamma(\beta)$. In particular a superstatistical β-distribution cannot ever match a $q < 1$ result.

3. Deformation of the Entropy

Once we understood how and why finite reservoir effects lead to $q \neq 1$, and emerging from this to a non-exponential statistical weight, the need for mending this salient feature arises. Generalizing the Boltzmann–Gibbs exponential to another formula, containing finite reservoir corrections, also abandons the remarkable basic property of the exponential: the additivity of the arguments by the product. Since this property connected the dynamical independence (energy additivity) with the statistical independence (probability factorization or equivalently entropy additivity), its missing is a severe conundrum.

In this section we show that if the original logarithmic definition due to Boltzmann or equivalently its exponential inverse due to Einstein, postulating the phase-space volume to be proportional to the exponential of the entropy, fails to some degree, then one may search for another expression of the entropy, $K(S)$, in order to restore "K-additivity". We comprise our quest into the simple question: If S leads to $q \neq 1$, what $K(S)$ achieves $q_K = 1$?

3.1. The Additive Entropy $K(S)$

We call "deformed entropy" the quantity $K(S)$, being additive while S was non-additive. In the basic postulate we use $K(S)$ instead of S in the exponential in order to gain more flexibility for handling the subleading terms in the $E \gg \omega$ expansion discussed above and shown to interpret the parameter q. It has been already realized that in the non-extensive statistical physics the product formula inherent in the Einstein postulate can only be viewed as a deformed exponential of a q-summed expression of the entropy [53]. We have shown [16] that such deformed addition rules and deformed functions satisfy the zeroth law of thermodynamics only then if one uses a $K(S)$ function of the original entropy variable, add these, and considers the inverse K-function of the result. In this manner we consider

$$w_E^{\mathrm{K}}(\omega) = \left\langle e^{K(S(E-\omega))-K(S(E))} \right\rangle = 1 - \omega \left\langle \frac{\mathrm{d}}{\mathrm{d}E} K(S(E)) \right\rangle + \frac{\omega^2}{2} \left\langle \frac{\mathrm{d}^2}{\mathrm{d}E^2} K(S(E)) + \left(\frac{\mathrm{d}}{\mathrm{d}E} K(S(E)) \right)^2 \right\rangle + \ldots . \tag{23}$$

Note that $\frac{\mathrm{d}}{\mathrm{d}E}K(S(E)) = K'S'$ and $\frac{\mathrm{d}^2}{\mathrm{d}E^2}K(S(E)) = K''S'^2 + K'S''$. Now we compare this expression with the Tsallis–Pareto power-law. Using previous average notations and assuming that $K(S)$ is independent of the reservoir fluctuations (a certain *universality*) one obtains:

$$\frac{1}{T_K} = K'\frac{1}{T}, \qquad \frac{q_K}{T_K^2} = \left(K'' + K'^2\right)\frac{1}{T^2}\left(1 + \frac{\Delta\beta^2}{\langle\beta\rangle^2}\right) - K'\frac{1}{CT^2}. \tag{24}$$

By choosing a particular $K(S)$ *we shall manipulate* q_K. In order to simplify the differential equation posed on $K(S)$ by requiring a given value for q_K we introduce the notations $F = 1/K' = T_K/T$ and $\Delta\beta^2/\langle\beta\rangle^2 = \lambda/C$. Then the q_K parameter for the $K(S)$ entropy is expressed as

$$q_K = \left(1 + \frac{\lambda}{C}\right)(1 - F') - \frac{1}{C}F. \tag{25}$$

Re-arranged this represents a very simple differential equation with $q = 1 + (\lambda - 1)/C$:

$$(\lambda + C)F' + F = \lambda + C(1 - q_K) = 1 + C(q - q_K). \tag{26}$$

From this form it is easy to realize that two special choices are worth to be considered: $q_K = q$ and $q_K = 1$. Since we seek for entropy deformations with the property $K(0) = 0$ and $K'(0) = 1$, one fixes the condition $F(0) = 1$. In this case the only solution for $q_K = q$ is $F = 1, K(S) = S$. It is obvious that the other choice, $q_K = 1$, is the only purposeful deformation for reaching K-additivity [16,18]. Equation (26) becomes then easily solvable. We call this form of the $q_K = 1$ requirement the *"Additivity Restoration Condition"* (ARC):

$$(\lambda + C)\,F' + F = \lambda. \tag{27}$$

3.2. Classification by Fluctuation Models

$q_K = 1$ also means a re-exponentialization of the ω-expansion of the statistical weight based on the deformed entropy phase-space, $w_E^{\mathrm{K}}(\omega)$. In this way the effective equilibrium condition, the common temperature, least depends on the one-particle subsystem energy, ω. In earlier publications we called this the "Universal Thermostat Independence" (UTI) principle [22].

Now we explore the solutions of the ARC Equation (27) under different assumptions about the heat capacity and the reservoir fluctuations. In the simplest case we do not consider reservoir fluctuations at all, we put $\Delta\beta^2 = 0$ and therefore $\lambda = 0$. Applying our previous general result for this value we have to solve

$$F' + \frac{1}{C}F = 0. \tag{28}$$

Replacing back the definition $F = 1/K'$, one arrives at the original UTI equation [22]:

$$\frac{K''}{K'} = \frac{1}{C}. \tag{29}$$

For ideal gas $C = 1/(1-q)$ is constant, and the solution of Equation (29) with $K(0) = 0,\ K'(0) = 1$ delivers [20,32,35]

$$K(S) = C\left(e^{S/C} - 1\right). \tag{30}$$

From this result one arrives upon using $K(S) = \sum_i p_i K(-\ln p_i)$ at the statistical entropy formulas of Tsallis and Rényi: [1–4,6]

$$K(S) = \frac{1}{1-q}\sum_i \left(p_i^q - p_i\right), \qquad S = \frac{1}{1-q}\ln\sum_i p_i^q. \tag{31}$$

Next we obtain the deformed entropy formula with C and λ constant. Using Equation (27) one obtains the general differential equation

$$\lambda K'^2 - K' + C_\Delta K'' = 0 \tag{32}$$

with $C_\Delta = C + \lambda$. Its first integral,

$$K'(S) = \frac{1}{(1-\lambda)e^{-S/C_\Delta} + \lambda} \tag{33}$$

and second integral,

$$K(S) = \frac{C_\Delta}{\lambda}\ln\left(1 - \lambda + \lambda e^{S/C_\Delta}\right), \tag{34}$$

represent the optimal deformation of the entropy formula in this case. With the above result (34) the $K(S)$-additive composition rule, $K(S_{12}) = K(S_1) + K(S_2)$, is equivalent to

$$h(S_{12}) = h(S_1) + h(S_2) + \frac{\lambda}{C_\Delta}h(S_1)h(S_2) \tag{35}$$

with

$$h(S) = C_\Delta\left(e^{S/C_\Delta} - 1\right). \tag{36}$$

This is a combination of the ideal gas entropy-deformation, $h(S)$ and an original Tsallis composition law [6,54] with $q-1 = \lambda/C_\Delta$. Using the auxiliary function, $h_C(S) = C(e^{S/C}-1)$, we have $h_\infty(S) = S$ and the entropy deformation function can also be written as

$$K_\lambda(S) \;=\; h^{-1}_{C_\Delta/\lambda}\left(h_{C_\Delta}(S)\right). \tag{37}$$

- For $\lambda = 1$ it is obviously $K_1(S) = S$. This is the Gaussian fluctuation model, considered in several textbooks, and also believed to lead to the smallest physically possible variance due to a "thermodynamical uncertainty" principle [55–58]. Since $\beta = S'(E)$, the variances are related as $\Delta\beta = |S''(E)|\,\Delta E = \Delta E/CT^2$. Then from $\Delta\beta\cdot\Delta E \geq 1$ it follow $\Delta E \geq T\sqrt{C}$ and $\Delta\beta \geq 1/T\sqrt{C}$. A straightforward consequence of this is $\lambda/C = \Delta\beta^2/\left\langle\beta\right\rangle^2 \geq 1/C$ and therefore $\lambda \geq 1$. We note, that if this "uncertainty" principle were correct, then only $q > 1$ canonical distributions of ω would exist in Nature.
- For no fluctuations $\lambda = 0$ and we get $K_0(S) = h_C(S)$. We regain the Tsallis and Rényi formulas presented above in Equation (31).

- It is also very intriguing to inspect the following particular limit: $C \to \infty, \lambda \to \infty$ but $\lambda/C_\Delta \to \tilde{q}-1$ finite. In this non-extensive limit the fluctuations are much larger than the normal Gaussian ones, and we obtain a nontrivial entropy deformation:

$$K_{NE}(S) = h^{-1}_{1/(\tilde{q}-1)}\left(h_\infty(S)\right) = \frac{1}{\tilde{q}-1}\ln\left(1+(\tilde{q}-1)S\right). \tag{38}$$

  The K-additivity, $K(S_{12}) = K(S_1) + K(S_2)$, in this case leads to the non-additivity formula $S_{12} = S_1 + S_2 + (\tilde{q}-1)S_1S_2$, – investigated formerly in depth by Tsallis and Abe [6,54,59–64].

In the finite heat capacity, finite temperature variance case we arrive at a *Generalized Tsallis Formula* based on $K(S) = \sum_i p_i\, K(-\ln p_i)$:

$$K_\lambda(S) = \frac{C_\Delta}{\lambda}\sum_i p_i \,\ln\left(1-\lambda+\lambda p_i^{-1/C_\Delta}\right). \tag{39}$$

- For normal fluctuations $K_1(S) = -\sum_i p_i\, \ln p_i$ is exactly the Boltzmann entropy.
- Without fluctuations $K_0(S) = C\sum_i\left(p_i^{1-1/C} - p_i\right)$ is the Tsallis entropy with $q = 1-1/C$ and S is the corresponding Rényi entropy.
- Finally considering extreme large fluctuations and a finite heat capacity, $C(S)$ which however may be an arbitrary function of the total entropy, S, we obtain the non-extensive result Equation (38) with $\tilde{q} = 2$:

$$K_\infty(S) = \ln(1+S) = \sum_i p_i \ln(1-\ln p_i). \tag{40}$$

  The canonical p_i distribution maximizing this parameterless deformed entropy is also expressed in terms of Lambert-W function, it shows tails like the Gompertz distribution [65–67], known from extreme value statistics and nonequlibrium growth models for demography and tumors.

The extreme large fluctuation case, modeled by $\lambda = \infty$ and presented in Equation (40), leads to an allover concave entropy for the two state system: $K_\infty(S[p, 1-p])$ is shown on the Figure 1 by the full line, while the back–deformed "Rényi-type" entropy formula is plotted by the dashed line. These curves are close to each other for the very low (and very high) probability, and maximally differ at the equiprobability point $p = 0.5$. However, in the latter point $S[p, 1-p]$ has the same value as the traditional Boltzmann–Gibbs curve, for which S and $K(S)$ coincide due to $\lambda = 1$.

For investigating the concavity of the Equation (40) the second derivatives of the respective expressions, $K_\infty(S_\infty)$, S_∞ and the Boltzmann–Gibbs formula $K_1(S_1) = S_1$ for the two-state system have to be inspected. This quantity is plotted on Figure 2 demonstrating that its value is negative in the whole range.

Figure 1. The general entropy $K(S)$ (full line) and S (dashed line) are plotted for $\lambda = \infty$, meaning divergently large fluctuations with respect to the Gaussian model. For comparison the $\lambda = 1$ case, the traditional Boltzmann–Gibbs formula is indicated by the dotted line.

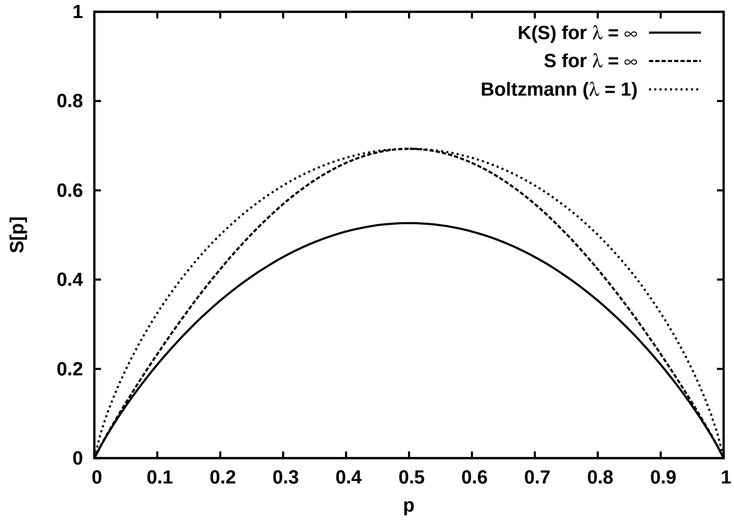
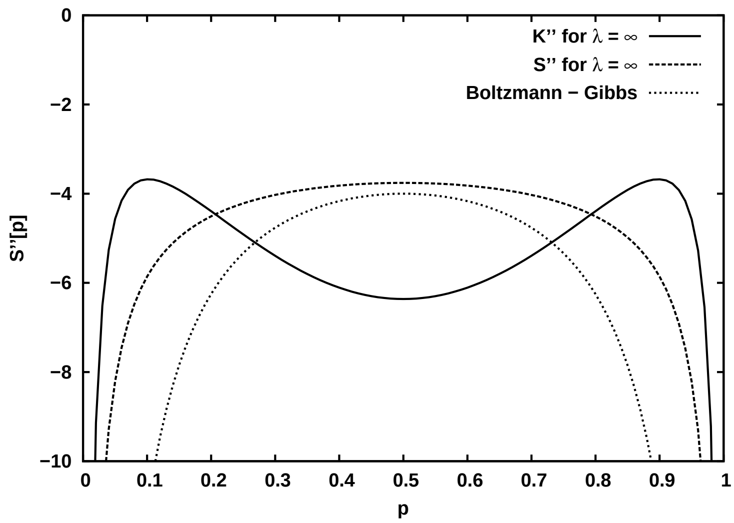


Figure 2. The second derivatives of the general entropy $K(S)$ (full line) and S (dashed line) are plotted for $\lambda = \infty$. For comparison the same derivative for the $\lambda = 1$ (Boltzmann) case is indicated by the dotted line.



Let us briefly discuss the canonical distribution following from and the concavity of the general entropy formula given in Equation (39). The deformed entropy is of a form $K(S) = \sum_i g(p_i)$, its properties are determined by those of the function $g(p)$. First we note that $g(0) = 0$ and $g(1) = 0$ rendering the totally ordered state, $\{p_i\} = \{1, 0, 0, \ldots\}$, to zero entropy. The canonical solution is obtained by its derivative:

$$g'(p) = -\frac{C_\Delta}{\lambda} \ln w - \frac{1}{\lambda} - w \left(1 - \frac{1}{\lambda}\right) = \alpha + \beta\omega \tag{41}$$

with

$$w = \frac{1}{1 - \lambda + \lambda p^{-1/C_\Delta}}. \tag{42}$$

The quantity w defined this way to simplify the algebraic manipulations is always between zero and one for probabilities with the same property. From the above canonical condition it follows

$$\ln w + \frac{\lambda - 1}{C_\Delta} w \;=\; -\frac{1}{C_\Delta}\left(1 + \lambda(\alpha + \beta\omega)\right). \tag{43}$$

As a check, for $\lambda = 1$ one obtains $w = p^{1/C_\Delta}$ and $\ln p = -(1 + \alpha + \beta\omega)$ as usual. In the non-extensive limit, $\lambda \gg C$, it reduces to

$$\ln w + w \;=\; -(\alpha + \beta\omega), \tag{44}$$

having a solution in terms of the Lambert-W function, $L(z)$:

$$w = L\left(\mathrm{e}^{-(\alpha+\beta\omega)}\right). \tag{45}$$

In the general case the solution of Equation (43) for w can also be expressed by the Lambert-W function. It is straightforward to derive, as follows: The Lambert-W function of a variable, say z, is defined by the $L(z)$ satisfying $L\,\mathrm{e}^L = z$. Replacing $z = \mathrm{e}^{-x}$ and taking the logarithm of both sides leads to

$$\ln L + L = -x. \tag{46}$$

Seeking for a solution of $\ln w + aw = -x$, one considers $L = aw$, and achieves the canonical solution of Equation (43) as

$$w = \frac{1}{a} L\left(a\,\mathrm{e}^{-x}\right) \tag{47}$$

with

$$a = \frac{\lambda - 1}{C_\Delta}, \quad \text{and} \quad x = \frac{1}{C_\Delta}\left(1 + \lambda(\alpha + \beta\omega)\right). \tag{48}$$

The corresponding probability is then reconstructed by inverting Equation (42) as

$$p = \left[1 + \frac{1}{\lambda}\left(\frac{1}{w} - 1\right)\right]^{-C_\Delta}. \tag{49}$$

The stability of the entropy formula can be tested by the sign of the second derivative $g''(p)$. Using the above notations

$$g''(p) = \frac{\mathrm{d}g'(p)}{\mathrm{d}w} \cdot \frac{\mathrm{d}w}{\mathrm{d}p} = -\left[\frac{1}{w} + \frac{\lambda - 1}{C_\Delta}\right] \frac{w^2}{p} p^{-1/C_\Delta}. \tag{50}$$

One has to investigate the sign of the expression between the square brackets only. For $\lambda > 1$ it is obviously positive (and hence the general entropy formula concave). For $\lambda < 1$ (but positive, since it scales the variance) the content of the square bracket is positive if $C_\Delta \geq (1 - \lambda)w$. But since $w \leq 1$ and $C_\Delta \geq C \geq 1$ for any reservoir containing at least one unit of heat capacity (at least one relativistic particle in at least one dimension or at least one massive particle in at least two dimensions) this requirement is fulfilled for all parameter values.

Before summarizing the main points let us make some remarks on the equipartition properties of our general ansatz. In the microcanonical maximal entropy state all states have equal probability, due to the normalization this value is the reciprocial of the number of all microstates: $p_i = 1/W$ for all $i = 1, \dots W$. Due to construction then we have

$$K\left(S_{\text{eqp}}\right) = W\,\frac{1}{W}\,K(\ln W) = K(\ln W), \tag{51}$$

from which $S_{\text{eqp}} = \ln W$ follows, whatever the $K(S)$ function was. In this way the Rényi-entropy type expression $S(p_i)$ at equiprobability is additive for multiplicative numbers of states; it is by construction extensive. The same cannot be told about $K(S)$, this deformed entropy may become non-extensive. On the other hand, assuming $K(S)$-additivity, as we did by prescribing the additivity restoration condition (ARC, Equation (27)), the product rule for the equipartioned probabilities is deformed—signalling statistical entanglement.

4. Conclusions and Outlook

In conclusion we have shown that in terms of traditional phase-space models the statistical cut power-law behavior can be interpreted as being primarily a particle number fluctuation effect during hadronization in high energy collisions. The $q > 1$ and $q < 1$ Tsallis–Pareto distributions are exact for NBD and BD distributions of the particle number, respectively, in a one-dimensional phase-space characteristic for high energy jets. The Boltzmann–Gibbs exponential weight factor is restored for the common limiting case of these distributions, for the Poissonian, leading to $q = 1$.

For general particle number distributions with fixed energy the Tsallis–Pareto cut power-law is only an approximation to subleading order in the expansion for large system energy, $E \gg \omega$. We have obtained and interpreted the parameters T and q by comparing coefficients of the respective expansions and concluded that $T = E/\langle n\rangle$ is an equipartition temperature, while $q = 1 + \Delta n^2/\langle n\rangle^2 - 1/\langle n\rangle$ reflects both the particle number variance and due to its expectation value the size of the reservoir. This formula also explains why both $q > 1$ and $q < 1$ cases can be observed in natural phenomena.

Further generalization towards the thermodynamical treatment considers the reservoir environment described by a simplified equation of state, $S(E)$. Repeating the above described approximations one concludes that $1/T = \langle\beta\rangle = \langle S'(E)\rangle$, *i.e.*, the parameter T also plays the role of a thermodynamical temperature. The parameter q is again related both to the size (total heat capacity, C) of the reservoir and to the variance of the fluctuating quantity $\beta = S'(E)$. The general formula follows the structure obtained in the high energy model, $q = 1 + \Delta\beta^2/\langle\beta\rangle^2 - 1/C$, with $1/C = \mathrm{d}T/\mathrm{d}E = -T^2\langle S''(E)\rangle$.

It is, however, known for long that the cut power-law does not follow the product rule, as the Boltzmann–Gibbs exponential does, for additive energy. The root of this behavior is the non-additivity of the Boltzmannian entropy, S, for finite and fluctuating reservoirs. $S(E_1 + E_2) \neq S(E_1) + S(E_2)$ for $q \neq 1$ is a weakness of the classical thermodynamics which has to be cured. Our approach here was to look for a function, $K(S)$, which restores additivity by leading to $q_K = 1$. This requirement for such a function concludes in the additivity restoring condition, ARC, in a differential equation satisfied by $K(S)$. Finally the usual canonical treatment must then be based on the additivity of $K(S)$, applied to an ensemble of configurations, which in turn provides the general formula $K(S) = \sum p_i K(-\ln p_i)$ (*cf.* Equation (40) and [20]).

The Boltzmann–Gibbs–Shannon formula is restored for $q = 1$ (when also $K(S) = S$ is the only solution), in particular for the traditional Gaussian approach to fluctuations when $\Delta\beta/\langle\beta\rangle = 1/\sqrt{C}$ is taken for granted. When the fluctuations are negligible, the Tsallis entropy formula arises for $K(S)$ and the corresponding Rényi formula for S with $q = 1 - 1/C$. In the extreme large variance limit a new, up to now not considered entropy – probability formula arises.

The canonical distribution belonging to this extreme case is given in terms of the Lambert-W function of single particle energy, defined as the $L(z)$ function, which satisfies $Le^L = z$. Recently, the Lambert-W function was met in relation to a study of scaling properties of a general two-parameter ansatz for non-additive entropy formulas by Hanel and Thurner [68] and in a non-extensive diffusion model by Andrade *et al*. [69]. Although these and ours are quite different contexts, a far analogy on the level of similar behavior at extreme low probabilities cannot be ab initio excluded.

As to the general scaling property, the $\lambda \gg C \to \infty$ case might be related to the $c = 1$, $d = 0$ point in Thurner's and Hanel's scheme [68]. However we have to emphasize that our present approach aimed and achieved the construction of a deformed entropy, which compensates finite reservoir size non-additivity as well as fluctuation effects even before taking the large system limit.

These initial results are encouraging for further pursuit of such a theoretical approach. The research of large systems, where $\lambda = C\Delta\beta^2/\langle\beta\rangle^2 \gg C \gg 1$ with a finite limit for λ/C, shall deal with genuine non-extensivity of the $K(S)$ entropy. The physical modelling of the reservoir environment, in particular with emphasis on the variable number of particles relevant for high energy physics, leads to more complex descriptions than presented here: a dependence like $C(S)$ and $\lambda(S)$ can be quite common. In such cases the ARC differential equation leads to further entropy formulas. Our approach provides a procedure to find the optimal entropy – probability relation from the viewpoint of the non-additive composition of two (or gradually more) subsystems. In addition, the superstatistics, originally conceptualized as a β-distribution behind non-Gibbsean factors in the statistics, may be extended to studies considering physical systems which cannot be described simply by an overall positive weight factor $\gamma(\beta)$ under an integral.

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Conflicts of Interest

The authors declare no conflict of interest.

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