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Article

Robust H_∞ Finite-Time Control for Discrete Markovian Jump Systems with Disturbances of Probabilistic Distributions

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Abstract: This paper is concerned with the robust H_∞ finite-time control for discrete delayed nonlinear systems with Markovian jumps and external disturbances. It is usually assumed that the disturbance affects the system states and outputs with the same influence degree of 100%, which is not evident enough to reflect the situation where the disturbance affects these two parts by different influence degrees. To tackle this problem, a probabilistic distribution denoted by binomial sequences is introduced to describe the external disturbance. Throughout the paper, the definitions of the finite-time boundedness (FTB) and the H_∞ FTB are firstly given respectively. To extend the results further, a model which combines a linear dynamic system and a static nonlinear operator is referred to describe the system under discussion. Then by virtue of state feedback control method, some new sufficient criteria are derived which guarantee the FTB and H_∞ FTB performances for the considered system. Finally, an example is provided to demonstrate the effectiveness of the developed control laws.

Keywords: discrete nonlinear systems; finite-time control; H_∞ control; Markovian jumps; time delays; disturbances of probabilistic distributions

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1. Introduction

In the past few years, finite-time control issues have become a hot topic due to their wide applications in practical engineering, such as switched systems [1–3], Markovian jump systems [4,5], singular systems [6], *etc*. For real industrial systems, it is usually required that the values of the system states should not exceed some given level in a certain time interval, avoiding the saturations of the sensors or damages to the equipments caused by the excitation of nonlinear dynamics [7]. On the other hand, it is always necessary to observe some transient properties of the industrial systems such as overshoot, settlement time, *etc*. [8].

As for the finite-time control, there are mainly two kinds of issues, including finite-time boundedness (FTB) and finite-time stability (FTS). Given constraints on the initial values and the energy of disturbance inputs, a system is called FTB if its states remain under a given value over a fixed time interval [9]. While FTS [10] can be viewed as a special case of FTB where no disturbances are considered. In particular, recently the issue concerning input to output finite-time stability (IO-FTS) has also been investigated based on Differential/Difference LMIs [11], where necessary and sufficient conditions are achieved. Though IO-FTS is a more general type, we think the research on FTB is the basis of IO-FTS study and can be applied (if achieved) to it by introducing the IO-FTS concept. Hence this paper deals with the FTB as a primary issue.

Note that both of FTB and FTS deal with the short-time performances over a finite time interval, which is the key difference from the Lyapunov stability where the asymptotical behaviors over a infinite time interval are investigated. Though the asymptotical performances are usually sufficient for the industrial operations, the aforementioned situations make it necessary to pay close attention to FTB and FTS problems.

At present, topics about hybrid and stochastic systems also raise interest; for instance, lead-following consensus of multi-agent systems [12], synchronization of complex networks based on entropy measures [13,14]. As a typical type of hybrid and stochastic system, Markovian jump systems are largely employed to describe the practical industrial processes with random mode changes due to such as failures of the components, abrupt environment changes, variations of the operation point, *etc*. [15–17]. The mode change is ruled by a Markov chain subject to certain mode-to-mode transition probabilities. In addition, recently many interesting results concerning the finite-time control of Markovian jump systems have been achieved. For instance, in [18], FTB is investigated for a class of singular time-delayed neural networks with Markovian jumps. Authors of [19] make a research on the H_∞ finite-time control for Markovian jump systems. Further, subject to average dwell time and partially known transition probabilities, H_∞ filtering is studied to obtain the FTB performance for Markovian jump systems in [20].

Furthermore, due to finite speeds of the information transmission, time-delays always exist which are the resource of poor control performances and even system instabilities. Therefore it is more reasonable to add the delay terms into the controlled systems. In addition, disturbances in the environment also have a bad effect on the control performance. To minimize or reduce the effect of the external disturbance on the controlled system, it is popular to introduce the H_∞ control concept [21]. However, to the best of our knowledge, in most of the literatures it is usually assumed that the disturbance is incorporated into both the system states and the outputs with the same influence degree of 100%, which actually is not accurate enough to reflect the real system due to the case where the disturbance affects the system states and outputs with different percentages of the total degrees, respectively. In particular, as is shown in Figure 1, the disturbance enters both the system states and system outputs. Here two cases are considered. In Case I, the disturbance affects the system states and system outputs by 100%. While in Case II, the disturbance enters the system states and system outputs by $100q\%$ and $100(1-q)\%$, respectively, where q is called the distribution ratio ($0 \leq q \leq 1$), and $100q\%$ and$100(1-q)\%$ refer to the disturbance influence degree. In this paper, this work will be carried out as one of the few attempts.

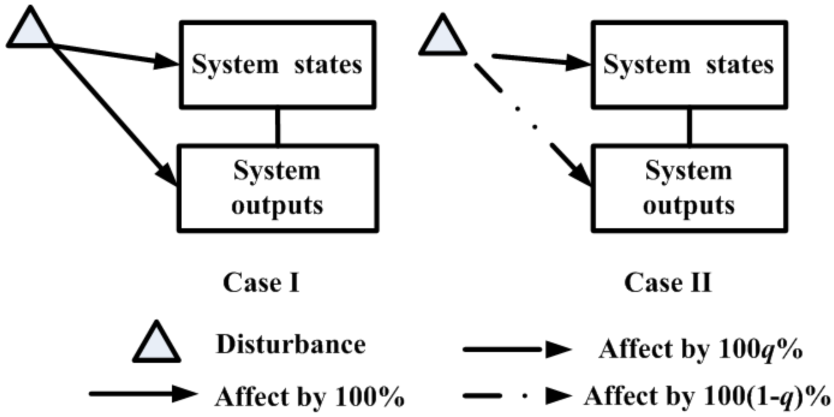

Figure 1. Concept of disturbance distribution.

Motivated by aforementioned facts, H_∞ *finite-time control for discrete delayed nonlinear systems with Markovian jumps and disturbances of probabilistic distributions* is addressed in this paper. Firstly, referring to the model in [22] which is the interconnection of a linear dynamic system and a static nonlinear operator, a new set of equations are established to describe the discrete-time delayed nonlinear system with Markovian jumps. By introducing the Bernoulli distribution and Binomial distribution sequences, the disturbance distributed into the system states and outputs by different influence degrees is incorporated into the model system. Then by employing the Lyapunov functions and state feedback control method, some new criteria are derived such that the robust H_∞ finite-time control performances are achieved for all possible Markovian jumps and disturbances of probabilistic distributions. Finally an example is provided to validate the developed control laws.

The contributions of this paper mainly lie in three aspects: (i) establish a more general model which helps extend the results into more nonlinear systems; (ii) attenuate the effect of the disturbance on the output with a prescribed level; (iii) model the disturbance influence degree between the system states and the outputs such that the effect of disturbances subject to certain probabilistic distribution on the control performance is investigated.

The rest of this paper is organized as follows. Problem formulations and preliminaries are given in Section 2. Section 3 presents the main results of the robust H_∞ finite-time controller design. In Section 4, an illustrative example is demonstrated to verify the effectiveness of the proposed control approaches. Finally, some conclusions are drawn in Section 5.

Notations: The superscript "T" stands for matrix transposition. $l_2[0,\infty)$ is the space of square integrable vector functions over $[0, \infty)$. $\Re^n$ denotes n dimensional Euclidean space, and $\Re^{n\times m}$ is the set of all $n\times m$ real matrices. I denotes identity matrix of appropriate orders. $*$ denotes the symmetric parts.

$diag\{\ldots\}$ stands for a block-diagonal matrix. $\|x\|$ denotes the Euclid norm of vector x. The notation $X > Y$, where X and Y are matrices of the same dimensions, means that the matrix $X - Y$ is positive definite. $Pr\{\cdot\}$ denotes the occurrence probability of event "$\cdot$". $Pr\{A|B\}$ represents the occurrence probability of event A on condition B. $E\{\cdot\}$ stands for the mathematical expectation of event "$\cdot$". If $X \in \Re^p$ and $Y \in \Re^q$, $C(X;Y)$ denotes the space of all continuous functions mapping $\Re^p \to \Re^q$. N_0 represents the set of nonnegative integers.

2. Problem Formulations and Preliminaries

Based on the model [22], we establish the following new sets of equations with Markovian jumps:

$$\begin{cases} x(k+1) = A(r_k)x(k) + A_d(r_k)x(k-\tau) + B_p(r_k)\phi(\xi(k)) + B_u(r_k)u(k) + B_w(r_k)w(k) \\ \xi(k) = C_q(r_k)x(k) + C_{qd}(r_k)x(k-\tau) + D_p(r_k)\phi(\xi(k)) + D_u(r_k)u(k) + D_w(r_k)w(k) \end{cases} \quad (1)$$

with the initial condition function $x(k) = \rho(k)\ \forall k \in [-\tau, 0]$, where $x(k) \in \Re^n$ is the system state, $u(k) \in \Re^m$ is the control input, $w(k) \in \Re^s$ is the external disturbance which belongs to $l_2[0,\infty)$. $\xi \in \Re^L$ is the input of the nonlinear function ϕ, $\phi \in C(\Re^L;\Re^L)$ is the nonlinear function satisfying $\phi(0) = 0$, $L \in N_0$ is the number of nonlinear functions. $A(r_k) \in \Re^{n\times n}$, $A_d(r_k) \in \Re^{n\times n}$, $B_p(r_k) \in \Re^{n\times L}$, $B_u(r_k) \in \Re^{n\times m}$, $B_w(r_k) \in \Re^{n\times s}$, $C_q(r_k) \in \Re^{L\times n}$, $C_{qd}(r_k) \in \Re^{L\times n}$, $D_p(r_k) \in \Re^{L\times L}$, $D_u(r_k) \in \Re^{L\times m}$, and $D_w(r_k) \in \Re^{L\times s}$ are mode-dependent matrices where $r(k)$ denotes the discrete-time Markov chain taking values from a finite set $V = \{1, 2, \cdots, s\}$ with the mode-to-mode transition probabilities as follows:

$$Pr\{r_{k+1} = j | r_k = i\} = \mu_{ij} \quad (2)$$

where $0 \le \mu_{ij} \le 1$, $\sum_{j=1}^{s} \mu_{ij} = 1\ \forall i \in V$.

For a better representation, here we denote $Q(r_k)$ as $Q_i\ \forall r_k = i$, $i \in V$, *i.e.*, $A(r_k)$ is denoted by A_i, $A_d(r_k)$ by A_{di}, and so on.

Here we adopt the following state feedback controller:

$$u(k) = K_i x(k) \quad (3)$$

where $K_i \in \Re^{m\times n}$, $i \in V$.

Remark 1. *We choose linear state feedback control here because it is a quite classical and effective method to stabilize the system. If nonlinear feedback is applied, better control performances may be achieved although, the implementation may become a little more complex or difficult than that of linear state feedback. Furthermore, once the state feedback is successfully applied to the desired issue, based on which Luenburger-like state estimator which is of nonlinear type can be constructed to investigate the current issue further. To this regard, we made the choice of linear state feedback for its important role in further study.*

Substituting (3) in (1), we obtain:

$$\begin{cases} x(k+1) = \bar{A}_i x(k) + A_{di}x(k-\tau) + B_{pi}\phi(\xi(k)) + B_{wi}w(k) \\ \xi(k) = \bar{C}_{qi}x(k) + C_{qdi}x(k-\tau) + D_{pi}\phi(\xi(k)) + D_{wi}w(k) \end{cases} \quad (4)$$

where $\bar{A}_i = A_i + B_{ui}K_i$, $\bar{C}_{qi} = C_{qi} + D_{ui}K_i$.

Definition 1. *(FTB): Given $0 \le c_1 \le \beta$, $c_2 \ge 0$, $R_i > 0$, $N \in N_0$ and the time delay τ, if*

$$\begin{cases} E\{x^T(k)R_i x(k)\} \le c_1^2 \\ (k \in [-\tau,\ 0]) \\ E\{\sum_{k=0}^{N} w^T(k)w(k)\} \le c_2^2 \end{cases} \Rightarrow E\{x^T(k)R_i x(k)\} \le \beta^2 \ \ \forall k = 1, \cdots, N. \tag{5}$$

system (4) is said to be FTB with respect to $(c_1, c_2, \beta, R_i, N)$.

Here we consider the following output:

$$z(k) = C_{zi}x(k) + D_{zwi}w(k) \tag{6}$$

where $C_{zi} \in \Re^{l\times n}$ and $D_{zwi} \in \Re^{l\times s}$ are both mode-dependent matrices.

Definition 2. *(H_∞ FTB): With the FTB control performance defined in* **Definition 1** *achieved, if the following index holds under zero initial conditions:*

$$J = E\left\{ \sum_{k=0}^{N} \left(z^T(k)z(k) - \gamma^2 w^T(k)w(k) \right) \right\} < 0 \tag{7}$$

system (4) is said to be H_∞ FTB for any nonzero $w(k)$, where $\gamma > 0$ is called the disturbance attenuation rate.

Assumption 1. [22]*: We assume the nonlinear functions in (1) are monotonically non-decreasing and globally Lipschitz,* i.e.*, the following relation holds:*

$$0 \le \frac{\phi_l(\varepsilon_1) - \phi_l(\varepsilon_2)}{\varepsilon_1 - \varepsilon_2} \le h_l \tag{8}$$

where $\forall \varepsilon_1, \varepsilon_2 \in \Re, \varepsilon_1 \ne \varepsilon_2, l = 1, \ldots, L, h_l > 0$.

And in our work, the disturbance process is described by a Bernoulli distribution $\alpha_0(k)$. According to (4) and (6), the disturbance enters the system states and the outputs respectively. Here we consider two main cases as follows, which is also shown in Figure 1:

Case I:

No distribution of the disturbance occurs between two parts (system states and outputs), in another word, the influence degree of the disturbance for two parts are the same by 100%.

Case II:

The disturbance affects two parts by $100q\%$ and $100(1-q)\%$ of the total influence degrees, respectively, where q $(0 \le q \le 1)$ is defined as the distribution ratio.

$$\alpha_0(k) = \begin{cases} 1, & Case\ I \\ 0, & Case\ II \end{cases}$$

where $Pr\{\alpha_0(k) = 1\} = E\{\alpha_0(k)\} = b$.

Consider the Binomial distribution sequences as follows:

$$\alpha_1(k) = v_1$$

we define $q = \frac{v_1}{v}, v_1 = 1, 2, \cdots, v$, where v denotes the total trial numbers, b_1 denotes $E\{q\}$. Hence $E\{\alpha_1(k)\} = vb_1$.

Remark 2. *The Markovian process denoted by the Markov chain r_k is independent of the Bernoulli distribution $\alpha_0(k)$.*

Remark 3. *For the system under discussion, there are two kinds of noises (disturbances),* i.e., *the process noise which enters the system states and the measurement noise which enters the system outputs. Usually, these two kinds of noises are taken as mutually uncorrelated white noises when dealing with control problems such as state estimation. However, in practical engineering, colored noises may occur which makes it difficult for the controlled system to guarantee this assumption, especially for a discrete time system sampled from a continuous time system where the process noise is correlated to the measurement noise* [23]. *Motivated by this kind of application or the like, in this paper we propose the idea of distributed disturbance subject to certain probabilistic distribution shown in Figure 1 to make an alternative research.*

Taking the probabilistic distributed disturbance into account, we obtain the following augmented system:

$$\begin{cases} x(k+1) = \bar{A}_i x(k) + A_{di}x(k-\tau) + B_{pi}\phi(\xi(k)) + \left(\alpha_0(k) + (1-\alpha_0(k))\dfrac{\alpha_1(k)}{v}\right) B_{wi}w(k) \\ \xi(k) = \bar{C}_{qi}x(k) + C_{qdi}x(k-\tau) + D_{pi}\phi(\xi(k)) + \left(\alpha_0(k) + (1-\alpha_0(k))\dfrac{\alpha_1(k)}{v}\right) D_{wi}w(k) \\ z(k) = C_{zi}x(k) + \left(\alpha_0(k) + (1-\alpha_0(k))(1-\dfrac{\alpha_1(k)}{v})\right) D_{zwi}w(k) \end{cases} \tag{9}$$

3. Main Results

In this section, the conditions for the FTB performance and the H_∞ FTB performance are derived in the first two theorems, respectively. To obtain the desired controller, Theorem 3 and Corollary 1 are displayed for time-delayed and non-delayed systems respectively.

Theorem 1. *Given $0 \le c_1 \le \beta$, $c_2 \ge 0$, $R_i > 0$, and $N \in N_0$, system (9) is said to be FTB with respect to $(c_1, c_2, \beta, R_i, N)$ provided there exist $\sigma_1^{-1} > 0$, $\sigma_2 > 0$, $\sigma_3^{-1} > 0$, $\alpha \ge 1$, symmetric positive definite matrices P_i, Γ, Q_i, a set of diagonal positive definite matrices Λ_i, and matrices K_i such that the following LMIs hold:*

$$G_i = \begin{bmatrix} G_{i11} & 0 & G_{i13} & G_{i14} & G_{i15} \\ * & G_{i22} & 0 & 0 & 0 \\ * & * & G_{i33} & G_{i34} & G_{i35} \\ * & * & * & G_{i44} & G_{i45} \\ * & * & * & * & G_{i55} \end{bmatrix} < 0 \tag{10}$$

$$R_i < P_i < \sigma_1^{-1} R_i \tag{11}$$

$$0 < Q_i < \sigma_2 I \tag{12}$$

$$0 < \Gamma < \sigma_3^{-1} R_i \tag{13}$$

$$d\sigma_2 c_2^2 + \tau\sigma_3^{-1} c_1^2 - \beta^2\alpha^{-N} + c_1^2\sigma_1^{-1} < 0 \tag{14}$$

where

$$
\begin{aligned}
&G_{i11} = \sum_{j=1}^{s} \mu_{ij}\bar{A}_i^T P_j \bar{A}_i - \alpha P_i + \Gamma,\ G_{i13} = \sum_{j=1}^{s} \mu_{ij}\bar{A}_i^T P_j A_{di}\\
&G_{i14} = \sum_{j=1}^{s} \mu_{ij}\bar{A}_i^T P_j B_{pi} + \bar{C}_{qi}^T H \Lambda_i,\ G_{i15} = n_1 \sum_{j=1}^{s} \mu_{ij}\bar{A}_i^T P_j B_{wi}\\
&G_{i22} = diag\{\underbrace{(1-\alpha)\Gamma, \cdots, (1-\alpha)\Gamma}_{\tau-1}\},\ G_{i33} = \sum_{j=1}^{s} \mu_{ij} A_{di}^T P_j A_{di} - \alpha\Gamma\\
&G_{i34} = \sum_{j=1}^{s} \mu_{ij} A_{di}^T P_j B_{pi} + C_{qdi}^T H \Lambda_i,\ G_{i35} = n_1 \sum_{j=1}^{s} \mu_{ij} A_{di}^T P_j B_{wi}\\
&G_{i44} = -2\Lambda_i + \Lambda_i H D_{pi} + D_{pi}^T H \Lambda_i + \sum_{j=1}^{s} \mu_{ij} B_{pi}^T P_j B_{pi},\ G_{i45} = n_1 \sum_{j=1}^{s} \mu_{ij} B_{pi}^T P_j B_{wi} + n_1 H \Lambda_i D_{wi}\\
&G_{i55} = (n_1^2 + \bar{n}_1) \sum_{j=1}^{s} \mu_{ij} B_{wi}^T P_j B_{wi} - d Q_i,\ n_1 = b + b_1(1-b)\\
&\bar{n}_1 = (1-b_1)^2 m + (\frac{1-b}{v})^2 m_1 + \frac{m m_1}{v^2},\ i = 1, 2, \cdots, s\\
&m = E\left\{(\alpha(k) - b)^2\right\} = b(1-b),\ m_1 = E\left\{(\alpha_1(k) - v b_1)^2\right\} = v b_1 (1-b_1)
\end{aligned}
$$

Proof. Take the following Lyapunov functional:

$$
V(k) = x^T(k) P_i x(k) + \sum_{\theta=-\tau}^{-1} x^T(k+\theta) \Gamma x(k+\theta) \tag{15}
$$

where $P_i > 0$, $\Gamma > 0$.

Remark 4. *Recently, the Differential/Difference LMIs (D/DLMIs) are applied to FTS of linear systems and deterministic hybrid systems [24,25], where necessary and sufficient conditions are derived. However, in this paper, due to the introduction of Markov jumps, P_i is a mode-dependent matrix which swifts between different values with the time instant going on. Since the number of modes is finite, P_i takes values from a finite set. If Difference Lyapunov functional is adopted, P will be derived by recursive algorithm rather than chosen from a finite set, which makes it hard to introduce the Markov jumps. Therefore in our opinions DLMIs cannot be applied to our current work directly. However, future work will concentrate on FTB issue of the nonlinear system without Markov jumps based on Difference Lyapunov functional such that less conservative criteria can be derived and output feedback control is also accessible.*

Construct the following function:

$$
\tilde{V}(k) = V(k+1) - \alpha V(k) - d w^T(k) Q_i w(k) \tag{16}
$$

where $Q_i > 0$, $\alpha \geq 1$, and $d > 0$.

Define $E\{V(k+1)\} = E\{V(k+1, r_{k+1} = j)|r_k = i\}$, then

$$
\begin{aligned}
&E\{\tilde{V}(k)\} \\
&= E\{V(k+1)\} - \alpha E\{V(k)\} - dE\{w^T(k)Q_i w(k)\} \\
&= \sum_{j=1}^{s} x^T(k+1)\mu_{ij}P_j x(k+1) + \sum_{\theta=-\tau}^{-1} x^T(k+1+\theta)\Gamma x(k+1+\theta) \\
&\quad - \alpha x^T(k)P_i x(k) - \sum_{\theta=-\tau}^{-1} x^T(k+\theta)\alpha\Gamma x(k+\theta) - dw^T(k)Q_i w(k) \\
&= \sum_{j=1}^{s}\left[\bar{A}_i x(k) + A_{di}x(k-\tau) + B_{pi}\phi(\xi(k)) + \left(\alpha_0(k) + (1-\alpha_0(k))\frac{\alpha_1(k)}{v}\right)B_{wi}w(k)\right]^T \mu_{ij}P_j \\
&\quad \times \left[\bar{A}_i x(k) + A_{di}x(k-\tau) + B_{pi}\phi(\xi(k)) + \left(\alpha_0(k) + (1-\alpha_0(k))\frac{\alpha_1(k)}{v}\right)B_{wi}w(k)\right] \\
&\quad - x^T(k)\alpha P_i x(k) - dw^T(k)Q_i w(k) + x^T(k)\Gamma x(k) - x^T(k-\tau)\alpha\Gamma x(k-\tau) \\
&\quad + \sum_{\theta=-1}^{-\tau+1} x^T(k+\theta)(1-\alpha)\Gamma x(k+\theta)
\end{aligned}
\tag{17}
$$

Since

$$
\begin{aligned}
&\alpha_0(k) + (1-\alpha_0(k))\frac{\alpha_1(k)}{v} \\
&= \alpha_0(k) - b + b + [1-(\alpha_0(k)-b)]\frac{\alpha_1(k)}{v} - b\frac{\alpha_1(k) - vb_1 + vb_1}{v} \\
&= (\alpha_0(k)-b) + b + [1-(\alpha_0(k)-b)]\frac{\alpha_1(k) - vb_1 + vb_1}{v} - \frac{b}{v}(\alpha_1(k) - vb_1) - bb_1 \\
&= (1-b_1)(\alpha_0(k)-b) + \frac{1-b}{v}(\alpha_1(k)-vb_1) - \frac{(\alpha_0(k)-b)(\alpha_1(k)-vb_1)}{v} + b + b_1(1-b)
\end{aligned}
\tag{18}
$$

using the following facts:

$$
\begin{cases}
E\{\alpha_0(k)\} = b,\ E\{(\alpha_0 - b)\} = 0,\ m = E\{(\alpha_0(k)-b)^2\} = b(1-b) \\
E\{\alpha_1(k)\} = b_1,\ E\{(\alpha_1(k) - vb_1)\} = 0,\ m_1 = E\{(\alpha_1(k)-vb_1)^2\} = vb_1(1-b_1) \\
n_1 = b + b_1(1-b),\ \bar{n}_1 = (1-b_1)^2 m + (\frac{1-b}{v})^2 m_1 + \frac{mm_1}{v^2}
\end{cases}
\tag{19}
$$

we have

$$
\begin{aligned}
&E\{\tilde{V}(k)\} \\
&= \sum_{j=1}^{s}[\bar{A}_i x(k) + A_{di}x(k-\tau) + B_{pi}\phi(\xi(k)) + n_1 B_{wi}w(k)]^T \mu_{ij}P_j[\bar{A}_i x(k) + A_{di}x(k-\tau) \\
&\quad + B_{pi}\phi(\xi(k)) + n_1 B_{wi}w(k)] + \sum_{j=1}^{s} w^T(k)B_{wi}^T[(1-\frac{b_1}{v})^2 m + (\frac{1-b}{v})^2 m_1 + \frac{mm_1}{v^2}]\mu_{ij} \\
&\quad \times P_j B_{wi}w(k) - x^T(k)\alpha P_i x(k) - dw^T(k)Q_i w(k) + x^T(k)\Gamma x(k) - x^T(k-\tau)\alpha\Gamma x(k-\tau) \\
&\quad + \sum_{\theta=-1}^{-\tau+1} x^T(k+\theta)(1-\alpha)\Gamma x(k+\theta) \\
&= x^T(k)(\sum_{j=1}^{s}\mu_{ij}\bar{A}_i^T P_j \bar{A}_i - \alpha P_i + \Gamma)x(k) + x^T(k-\tau)(\sum_{j=1}^{s} A_{di}^T P_j A_{di} - \alpha\Gamma)x(k-\tau) \\
&\quad + \phi^T(\xi(k))\sum_{j=1}^{s}\mu_{ij}B_{pi}^T P_j B_{pi}\phi(\xi(k)) + w^T(k)\left((\bar{n}_1 + n_1^2)\sum_{j=1}^{s}\mu_{ij}B_{wi}^T P_j B_{wi} - dQ_i\right)w(k) \\
&\quad + 2x^T(k)\sum_{j=1}^{s}\mu_{ij}\bar{A}_i^T P_j A_{di}x(k-\tau) + 2x^T(k)\sum_{j=1}^{s}\mu_{ij}\bar{A}_i^T P_j B_{pi}\phi(\xi(k)) + 2n_1 x^T(k) \\
&\quad \times \sum_{j=1}^{s}\mu_{ij}\bar{A}_i^T P_j B_{wi}w(k) + 2x^T(k-\tau)\sum_{j=1}^{s}\mu_{ij}A_{di}^T P_j B_{pi}\phi(\xi(k)) + 2n_1 x^T(k-\tau)\sum_{j=1}^{s}\mu_{ij}A_{di}^T \\
&\quad \times P_j B_{wi}w(k) + 2n_1\phi^T(\xi(k))\sum_{j=1}^{s}\mu_{ij}B_{pi}^T P_j B_{wi}w(k) + \sum_{\theta=-1}^{-\tau+1} x^T(k+\theta)(1-\alpha)\Gamma x(k+\theta)
\end{aligned} \tag{20}
$$

According to **Assumption 1**, the inequality (8) can be rewritten as follows:

$$\phi_l^2(\xi_l(k)) - h_l\phi_l(\xi_l(k))\xi_l(k) \le 0 \tag{21}$$

which is equivalent to

$$2\lambda_{il}\phi_l^2(\xi_l(k)) - 2\lambda_{il}h_l\phi_l(\xi_l(k))\xi_l(k) \le 0 \tag{22}$$

where $\lambda_{il} > 0$, $l = 1, \cdots, L$.

The above inequality can also be written in the matrix form as follows:

$$
\begin{aligned}
&-2\phi^T(\xi(k))\Lambda_i\phi(\xi(k)) + 2x^T(k)\bar{C}_{qi}^T H\Lambda_i\phi(\xi(k)) \\
&+2x^T(k-\tau)C_{qdi}^T H\Lambda_i\phi(\xi(k)) + 2\phi^T(\xi(k))D_{pi}^T H\Lambda_i\phi(\xi(k)) \\
&+2n_1 w^T(k)D_{wi}^T H\Lambda_i\phi(\xi(k)) \ge 0
\end{aligned} \tag{23}
$$

where $\Lambda_i = \{\lambda_{i1}, \cdots, \lambda_{iL}\}$, $H = \{h_1, \cdots, h_L\}$.

Based on (20) and (23), we have

$$E\{\tilde{V}(k)\} \le \Psi^T G_i \Psi \tag{24}$$

where $\Psi = [x^T(k)\ x^T(k-1)\ \cdots\ x^T(k-\tau+1)\ x^T(k-\tau)\ \phi^T(\xi(k))\ w^T(k)]^T$.

Since $G_i < 0$, we have

$$E\{V(k+1)\} < \alpha E\{V(k)\} + dE\{w^T(k)Q_i w(k)\} \tag{25}$$

which implies that

$$E\{x^T(k)P_ix(k)\} < E\{\alpha^N V(0) + d\sum_{\eta=0}^{N-1}\alpha^{N-\eta}w^T(\eta)Q_iw(\eta)\} \tag{26}$$

Since $\alpha \geq 1$, we have

$$\begin{aligned} E\{x^T(k)P_ix(k)\} &< E\{\alpha^N V(0) + d\alpha^N\sum_{\eta=0}^{N-1}w^T(\eta)Q_iw(\eta)\} \\ &= \alpha^N E\{x^T(0)P_ix(0) + \sum_{\theta=-\tau}^{-1}x^T(\theta)\Gamma x^T(\theta) + d\sum_{\eta=0}^{N-1}w^T(\eta)Q_iw(\eta)\} \end{aligned} \tag{27}$$

Let $P_i = R_i^{1/2}R_i^{-1/2}P_iR_i^{-1/2}R_i^{1/2}$, we have

$$\begin{aligned} E\{x^T(k)P_ix(k)\} &= E\{x^T(k)R_i^{-1/2}R_i^{1/2}P_iR_i^{-1/2}R_i^{1/2}x(k)\} \\ &\geq E\{\lambda_{min}(R_i^{-1/2}P_iR_i^{-1/2})x^T(k)R_ix(k)\} \end{aligned} \tag{28}$$

and let $\Gamma = R_i^{1/2}R_i^{-1/2}\Gamma R_i^{-1/2}R_i^{1/2}$, according to the preconditions of **Definition 1**,

$$\begin{aligned} &\alpha^N E\{x^T(0)P_ix(0) + \sum_{\theta=-\tau}^{-1}x^T(\theta)\Gamma x^T(\theta) + d\sum_{\eta=0}^{N-1}w^T(\eta)Q_iw(\eta)\} \\ &\leq \lambda_{max}(R_i^{-1/2}P_iR_i^{-1/2})\alpha^N c_1^2 + \tau\lambda_{max}(R_i^{-1/2}\Gamma R_i^{-1/2})\alpha^N c_1^2 + d\alpha^N\lambda_{max}(Q_i)c_2^2 \end{aligned} \tag{29}$$

Based on (28) and (29), we derive the following inequality:

$$\begin{aligned} &E\{\lambda_{min}(R_i^{-1/2}P_iR_i^{-1/2})x^T(k)R_ix(k)\} \\ &< \lambda_{max}(R_i^{-1/2}P_iR_i^{-1/2})\alpha^N c_1^2 + \tau\lambda_{max}(R_i^{-1/2}\Gamma R_i^{-1/2})\alpha^N c_1^2 + d\alpha^N\lambda_{max}(Q_i)c_2^2 \end{aligned}$$

i.e.,

$$\begin{aligned} &E\{x^T(k)R_ix(k)\} \\ &< \lambda_{min}^{-1}(R_i^{-1/2}P_iR_i^{-1/2})\alpha^N\left(\lambda_{max}(R_i^{-1/2}P_iR_i^{-1/2})c_1^2 + \tau\lambda_{max}(R_i^{-1/2}\Gamma R_i^{-1/2})c_1^2 + d\lambda_{max}(Q_i)c_2^2\right) \end{aligned} \tag{30}$$

According to (11), we have

$$\lambda_{min}(R_i^{-1/2}P_iR_i^{-1/2}) \geq 1,\ \lambda_{max}(R_i^{-1/2}P_iR_i^{-1/2}) \leq \sigma_1^{-1} \tag{31}$$

Similarly, (12) and (13) yield that

$$\lambda_{max}(Q_i) \leq \sigma_2 \tag{32}$$

$$\lambda_{max}(R_i^{-1/2}\Gamma R_i^{-1/2}) < \sigma_3^{-1} \tag{33}$$

Based on (31), (32), and (33), (30) yields that

$$E\{x^T(k)R_ix(k)\} < \alpha^N\left(\sigma_1^{-1}c_1^2 + \tau\sigma_3^{-1}c_1^2 + d\sigma_2c_2^2\right) \tag{34}$$

By virtue of (14), we obtain

$$E\{x^T(k)R_ix(k)\} < \beta^2 \tag{35}$$

According to **Definition 1**, system (9) is FTB with respect to $\{c_1, c_2, \beta, R_i, N\}$. Thus the proof is completed. □

Theorem 2. *Given* $0 \le c_1 \le \beta$, $c_2 \ge 0$, $R_i > 0$, $N \in N_0$, *and* $d > 0$, *if there exist* $\sigma_1^{-1} > 0$, $\sigma_3^{-1} > 0$, $\alpha \ge 1$, *symmetric positive definite matrices* P_i, Γ, *a set of diagonal positive definite matrices* Λ_i, *and matrices* K_i *such that the following LMIs hold:*

$$\bar{G}_i = \begin{bmatrix} \bar{G}_{i11} & 0 & G_{i13} & G_{i14} & \bar{G}_{i15} \\ * & G_{i22} & 0 & 0 & 0 \\ * & * & G_{i33} & G_{i34} & G_{i35} \\ * & * & * & G_{i44} & G_{i45} \\ * & * & * & * & \bar{G}_{i55} \end{bmatrix} < 0 \tag{36}$$

$$R_i < P_i < \sigma_1^{-1} R_i \tag{37}$$

$$0 < \Gamma < \sigma_3^{-1} R_i \tag{38}$$

$$dc_2^2 + \tau\sigma_3^{-1}c_1^2 - \beta^2\alpha^{-N} + c_1^2\sigma_1^{-1} < 0 \tag{39}$$

where

$$\bar{G}_{i11} = \sum_{j=1}^{s} \mu_{ij}\bar{A}_i^T P_j \bar{A}_i - \alpha P_i + \Gamma + C_{zi}^T C_{zi},\ \bar{G}_{i15} = n_1 \sum_{j=1}^{s} \mu_{ij}\bar{A}_i^T P_j B_{wi} + n_2 C_{zi}^T D_{zwi}$$

$$\bar{G}_{i55} = (n_1^2 + \bar{n}_1) \sum_{j=1}^{s} \mu_{ij} B_{wi}^T P_j B_{wi} - dI + D_{zwi}^T (n_2^2 + \bar{n}_2) D_{zwi}$$

$$n_2 = 1 - b_1(1-b),\ \bar{n}_2 = (\frac{1-b}{v})^2 m_1 + \frac{mm_1}{v^2} + mb_1^2,\ i = 1,2,\cdots,s$$

$$m = E\left\{(\alpha(k) - b)^2\right\} = b(1-b),\ m_1 = E\left\{(\alpha_1(k) - vb_1)^2\right\} = vb_1(1-b_1)$$

then system (9) is said to be FTB with H_∞ *performances,* $\gamma = \sqrt{d}$ *is called the disturbance attenuation rate.*

Proof. According to the Schur Complement [26], $\bar{G}_i < 0$ is equivalent to

$$\begin{bmatrix} G_{i11} & 0 & G_{i13} & G_{i14} & G_{i15} & C_{zi} & 0 \\ * & G_{i22} & 0 & 0 & 0 & 0 & 0 \\ * & * & G_{i33} & G_{i34} & G_{i35} & 0 & 0 \\ * & * & * & G_{i44} & G_{i45} & 0 & 0 \\ * & * & * & * & (n_1^2 + \bar{n}_1)\sum_{j=1}^{s} \mu_{ij} B_{wi}^T P_j B_{wi} - dI & n_2 D_{zwi} & \sqrt{\bar{n}_2} D_{zwi}^T \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{40}$$

In **Theorem 1**, let $Q_i = I$, then G_i becomes the principle minor of the left side of (40). Thus $G_i < 0$ is derived according to (40). Together with the conditions (37)–(39), it can be concluded that system (9) is FTB based on **Theorem 1**. On the other hand, consider the following function:

$$\bar{V}(k) = V(k+1) - \alpha V(k) + z^T(k)z(k) - dw^T(k)w(k) \tag{41}$$

Since

$$\begin{aligned}
&E\{\bar{V}(k)\}\\
&=E\{V(k+1)\}-\alpha E\{V(k)\}+E\{z^T(k)z(k)\}-dE\{w^T(k)w(k)\}\\
&=x^T(k)(\sum_{j=1}^{s}\mu_{ij}\bar{A}_i^T P_j\bar{A}_i-\alpha P_i+\Gamma+C_{zi}^T C_{zi})x(k)+x^T(k-\tau)(\sum_{j=1}^{s}A_{di}^T P_j A_{di}-\alpha\Gamma)x(k-\tau)\\
&\quad+\phi^T(\xi(k))\sum_{j=1}^{s}\mu_{ij}B_{pi}^T P_j B_{pi}\phi(\xi(k))+w^T(k)((\bar{n}_1+n_1^2)\sum_{j=1}^{s}\mu_{ij}B_{wi}^T P_j B_{wi}-dI+D_{zwi}^T\\
&\quad\times(n_2^2+\bar{n}_2)D_{zwi})w(k)+2x^T(k)\sum_{j=1}^{s}\mu_{ij}\bar{A}_i^T P_j A_{di}x(k-\tau)+2x^T(k)\sum_{j=1}^{s}\mu_{ij}\bar{A}_i^T P_j B_{pi}\phi(\xi(k))\\
&\quad+2x^T(k)(n_1\sum_{j=1}^{s}\mu_{ij}\bar{A}_i^T P_j B_{wi}+n_2 C_{zi}^T D_{zwi})w(k)+2x^T(k-\tau)\sum_{j=1}^{s}\mu_{ij}A_{di}^T P_j B_{pi}\phi(\xi(k))\\
&\quad+2n_1 x^T(k-\tau)\sum_{j=1}^{s}\mu_{ij}A_{di}^T P_j B_{wi}w(k)+2n_1\phi^T(\xi(k))\sum_{j=1}^{s}\mu_{ij}B_{pi}^T P_j B_{wi}w(k)\\
&\quad+\sum_{\theta=-1}^{-\tau+1}x^T(k+\theta)(1-\alpha)\Gamma x(k+\theta)
\end{aligned} \tag{42}$$

Consider the sector condition (23), we derive

$$E\{\bar{V}(k)\}<\Psi^T\bar{G}_i\Psi \tag{43}$$

Since $\bar{G}_i<0$, we have

$$E\{\bar{V}(k)\}<0 \tag{44}$$

i.e.,

$$E\{V(k+1)\}<\alpha E\{V(k)\}-E\{z^T(k)z(k)-dw^T(k)w(k)\} \tag{45}$$

which indicates that

$$\begin{aligned}
E\{V(k)\}&<\alpha^k E\{V(0)\}-E\left\{\sum_{\eta=0}^{k-1}\alpha^{k-1-\eta}\left(z^T(\eta)z(\eta)-dw^T(\eta)w(\eta)\right)\right\}\\
&<\alpha^k E\{V(0)\}-E\left\{\sum_{\eta=0}^{N}\alpha^{N-\eta}\left(z^T(\eta)z(\eta)-dw^T(\eta)w(\eta)\right)\right\}
\end{aligned} \tag{46}$$

Due to $E\{V(k)\}>0$ and $V(0)=0$, we derive

$$0<-E\left\{\sum_{\eta=0}^{N}\left(z^T(\eta)z(\eta)-dw^T(\eta)w(\eta)\right)\right\} \tag{47}$$

i.e.,

$$E\{\sum_{\eta=0}^{N}z^T(\eta)z(\eta)\}<E\{\sum_{\eta=0}^{N}dw^T(\eta)w(\eta)\} \tag{48}$$

Let $\gamma=\sqrt{d}$. According to **Definition 2**, it concludes that system (9) has H_∞ performances with the disturbance attenuation rate $\gamma=\sqrt{d}$. Thus the proof is completed. □

Theorem 3. *Given $0 \le c_1 \le \beta$, $c_2 \ge 0$, $R_i > 0$, $N \in N_0$, and $d > 0$, if there exist $\sigma_1 > 0$, $\sigma_3 > 0$, $\alpha \ge 1$, symmetric positive definite matrices X_i, Y, a set of diagonal positive definite matrices S_i, and matrices W_i such that the following LMIs hold:*

$$M_i = \begin{bmatrix} -\alpha X_i & 0 & 0 & X_iC_{qi}^TH + W_i^TD_{ui}^TH & 0 & X_iL_{1i}^T & 0 & X_i & X_iC_{zi}^T & 0 \\ * & M_{i22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\alpha Y & YC_{qdi}^TH & 0 & YL_{2i}^T & 0 & 0 & 0 & 0 \\ * & * & * & \begin{matrix} -2S_i + HD_{pi}S_i \\ +S_iD_{pi}^TH \end{matrix} & n_1HD_{wi} & S_iL_{3i}^T & 0 & 0 & 0 & 0 \\ * & * & * & * & -dI & L_{4i}^T & L_{5i}^T & 0 & n_2D_{zwi}^T & \sqrt{\bar{n}_2}D_{zwi}^T \\ * & * & * & * & * & -X & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -X & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -Y & 0 & 0 \\ * & * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{49}$$

$$\sigma_1 R_i^{-1} < X_i > R_i^{-1} \tag{50}$$

$$Y > \sigma_3 R_i^{-1} \tag{51}$$

$$\begin{bmatrix} dc_2^2 - \beta^2\alpha^{-N} & c_1 & \sqrt{\tau}c_1 \\ * & -\sigma_1 & 0 \\ * & * & -\sigma_3 \end{bmatrix} < 0 \tag{52}$$

where

$$\begin{aligned} M_{i22} &= diag\{\underbrace{(1-\alpha)\Gamma, \cdots, (1-\alpha)\Gamma}_{\tau-1}\} \\ X_iL_{1i}^T &= \left[\sqrt{\mu_{i1}}X_iA_i^T + \sqrt{\mu_{i1}}W_i^TB_{ui}, \quad \cdots, \quad \sqrt{\mu_{is}}X_iA_i^T + \sqrt{\mu_{is}}W_i^TB_{ui}\right] \\ Y_iL_{2i}^T &= \left[\sqrt{\mu_{i1}}YA_{di}^T, \quad \cdots, \quad \sqrt{\mu_{is}}YA_{di}^T\right] \\ S_iL_{3i}^T &= \left[\sqrt{\mu_{i1}}S_iB_{pi}^T, \quad \cdots, \quad \sqrt{\mu_{is}}S_iB_{pi}^T\right] \\ L_{4i}^T &= \left[\sqrt{\mu_{i1}}n_1B_{wi}^T, \quad \cdots, \quad \sqrt{\mu_{is}}n_1B_{wi}^T\right] \\ L_{5i}^T &= \left[\sqrt{\mu_{i1}}\bar{n}_1B_{wi}^T, \quad \cdots, \quad \sqrt{\mu_{is}}\bar{n}_1B_{wi}^T\right] \end{aligned}$$

then system (9) is FTB with disturbance attenuation rate $\gamma = \sqrt{d}$. Furthermore, the feedback controller gains are determined by $K_i = W_iX_i^{-1}$.

Proof.

$$\bar{G}_i = \tilde{G}_i + L_i^T\bar{P}L_i < 0 \tag{53}$$

where

$$\tilde{G}_i = \begin{bmatrix} -\alpha P_i + \Gamma + C_{zi}^T C_{zi} & 0 & 0 & \bar{C}_{qi}^T H \Lambda_i & n_2 C_{zi}^T D_{zwi} \\ * & G_{i22} & 0 & 0 & 0 \\ * & * & -\alpha\Gamma & C_{qdi}^T H \Lambda_i & 0 \\ * & * & * & \begin{matrix} -2\Lambda_i + \Lambda_i H D_{pi} \\ +D_{pi}^T H \Lambda_i \end{matrix} & n_1 H \Lambda_i D_{wi} \\ * & * & * & * & -dI + (n_2^2 + \bar{n}_2) D_{zwi}^T D_{zwi} \end{bmatrix} < 0$$

$$L_i = \begin{bmatrix} L_{1i} & 0 & \cdots & 0 & L_{2i} & L_{3i} & L_{4i} \\ 0 & 0 & \cdots & 0 & 0 & 0 & L_{5i} \end{bmatrix}$$

$$\bar{P} = diag\{P, P\},\ P = diag\{P_1, P_2, \cdots, P_s\}$$

According to the Schur Complement [26], (53) is equivalent to

$$\begin{bmatrix} \tilde{G}_i & L_i^T \\ * & -\bar{P}^{-1} \end{bmatrix} < 0 \tag{54}$$

i.e.,

$$\begin{bmatrix} -\alpha P_i & 0 & 0 & \bar{C}_{qi}^T H \Lambda_i & 0 & L_{1i}^T & 0 & I & C_{zi}^T & 0 \\ * & G_{i22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\alpha\Gamma & C_{qdi}^T H \Lambda_i & 0 & L_{2i}^T & 0 & 0 & 0 & 0 \\ * & * & * & \begin{matrix} -2\Lambda_i + \Lambda_i H D_{pi} \\ +D_{pi}^T H \Lambda_i \end{matrix} & n_1 H \Lambda_i D_{wi} & L_{3i}^T & 0 & 0 & 0 & 0 \\ * & * & * & * & -dI & L_{4i}^T & L_{5i}^T & 0 & n_2 D_{zwi}^T & \sqrt{\bar{n}_2} D_{zwi}^T \\ * & * & * & * & * & -P^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -P^{-1} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\Gamma^{-1} & 0 & 0 \\ * & * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{55}$$

Pre-and-post multiply $diag\{P_i^{-1}, \underbrace{\Gamma^{-1}, \cdots, \Gamma^{-1}}_{\tau}, \Lambda_i, I, I, I, I, I\}$ on both sides of (55), and let

$$X_i = P_i^{-1}, Y = \Gamma^{-1}, S_i = \Lambda_i^{-1}, W_i = K_i X_i \tag{56}$$

we have $M_i < 0$ in (49).

Similarly, we can derive (50), (51), and (52) from (37), (38) and (39), respectively. Thus the proof is completed. □

Remark 5. *If $\tau = 0$ or $A_{di} = C_{qdi} = 0$, system (9) becomes a non-delayed system denoted as follows*

$$\begin{cases} x(k+1) = \bar{A}_i x(k) + B_{pi}\phi(\xi(k)) + \left(\alpha_0(k) + (1-\alpha_0(k))\dfrac{\alpha_1(k)}{v}\right) B_{wi} w(k) \\ \xi(k) = \bar{C}_{qi} x(k) + D_{pi}\phi(\xi(k)) + \left(\alpha_0(k) + (1-\alpha_0(k))\dfrac{\alpha_1(k)}{v}\right) D_{wi} w(k) \\ z(k) = C_{zi} x(k) + (\alpha_0(k) + \left(1-\alpha_0(k))(1-\dfrac{\alpha_1(k)}{v})\right) D_{zwi} w(k) \end{cases} \tag{57}$$

In this case, H_∞ FTB controller can still be designed by virtue of the following corollary.

Corollary 1. *Given $0 \le c_1 \le \beta$, $c_2 \ge 0$, $R_i > 0$, $N \in N_0$, and $d > 0$, if there exist $\sigma_1 > 0$, $\alpha \ge 1$, symmetric positive definite matrices X_i, a set of diagonal positive definite matrices S_i, and matrices W_i such that the following LMIs hold:*

$$\bar{M}_i = \begin{bmatrix} -\alpha X_i & X_i C_{qi}^T H + W_i^T D_{ui}^T H & 0 & X_i L_{1i}^T & 0 & X_i C_{zi}^T & 0 \\ * & \begin{matrix} -2S_i + HD_{pi}S_i \\ +S_i D_{pi}^T H \end{matrix} & n_1 H D_{wi} & S_i L_{3i} & 0 & 0 & 0 \\ * & * & -dI & L_{4i}^T & L_{5i}^T & n_2 D_{zwi}^T & \sqrt{\bar{n}_2} D_{zwi}^T \\ * & * & * & -X & 0 & 0 & 0 \\ * & * & * & * & -X & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{58}$$

$$\sigma_1 R_i^{-1} < X_i > R_i^{-1} \tag{59}$$

$$\begin{bmatrix} dc_2^2 - \beta^2 \alpha^{-N} & c_1 \\ * & -\sigma_1 \end{bmatrix} < 0 \tag{60}$$

then system (57) is FTB with disturbance attenuation rate $\gamma = \sqrt{d}$. Furthermore, the feedback controller gains can still be determined by $K_i = W_i X_i^{-1}$.

4. Numerical Example

In this section, a numerical example is provided to demonstrate the effectiveness of the proposed design method.

Example. *Consider system (9) with the following parameters:*

$$A_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix}, A_{d1} = 0_{2\times 2}, B_{p1} = \begin{bmatrix} 0.3 & 0.2 & -0.3 & 0.2 \\ -0.1 & 0.4 & 0.15 & 0.05 \end{bmatrix}, B_{u1} = B_{w1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C_{q1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T, C_{qd1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T, D_{p1} = 0_{4\times 4}, D_{u1} = D_{w1} = 0_{4\times 1}$$

$$C_{z1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, D_{zw1} = 0_{2\times 1}$$

$$A_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, A_{d2} = 0_{2\times 2}, B_{p2} = \begin{bmatrix} 0.3 & 0.25 & -0.2 & 0.15 \\ -0.15 & 0.2 & 0.18 & 0.1 \end{bmatrix}, B_{u2} = B_{w2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C_{q2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T, C_{qd2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T, D_{p2} = 0_{4\times 4}, D_{u2} = D_{w2} = 0_{4\times 1}$$

$$C_{z2} = \begin{bmatrix} 0.15 & 0 \\ 0 & 0.15 \end{bmatrix}, \ D_{zw2} = 0_{2\times 1}$$

$$c_1 = 2, \ c_2 = 1, \ \beta = 10, \ N = 20, \ \tau = 2, \ b = 0.8, \ b_1 = 0.4, \ v = 10, \ d = 6, \ R_1 = R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the disturbance $w(t)$ *is of uniform distribution over the interval* $[0, 0.1]$*, and the mode transition matrix is*

$$\begin{bmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \\ 0.45 & 0.55 \end{bmatrix}$$

According to **Theorem 3***, we can design the feedback controller as follows:*

$$\alpha_{max} = 1.1377, \ X_1 = \begin{bmatrix} 0.8756 & 0.0625 \\ 0.0625 & 0.9464 \end{bmatrix}, \ X_2 = \begin{bmatrix} 0.9783 & -0.0193 \\ -0.0193 & 0.9806 \end{bmatrix}, \ Y = \begin{bmatrix} 3.0747 & 0.9991 \\ 0.9991 & 8.1905 \end{bmatrix}$$

$$S_1 = diag\{2.0097, 3.4877, 2.4420, 4.7930\}, \ S_2 = diag\{3.4612, 4.4578, 4.1899, 14.6321\}$$

$$W_1 = \begin{bmatrix} -0.3067 & -0.3638 \end{bmatrix}, \ W_2 = \begin{bmatrix} -0.3820 & -0.2099 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} -0.3244 & -0.3629 \end{bmatrix}, \ K_2 = \begin{bmatrix} -0.3949 & -0.2218 \end{bmatrix}$$

The simulation results are demonstrated in Figures 2–9. Figure 2 shows the Markov jumping signals subject to the mode-to-mode transition probabilities. The signal which determines whether the distribution of the disturbance will occur is displayed by the Bernoulli sequence in Figure 3. Particularly, the distribution ratio q *is assumed to be described by the Binomial sequence in Figure 4. With these signals and the developed controller, system (9) is FTB since* $E\{x^T(k)R_i x(k)\}$ $(i = 1, 2)$ *in Figure 5 doesn't exceed the prescribed level* $\beta = 10$*. The trajectories of system states are depicted in Figures 6 and 7, respectively. And the system outputs perform as shown in Figures 8 and 9. From Figures 6 and 7, it also conclude that the FTB performance is achieved as the trajectories converge to zeros asymptotically in the finite time interval* $[0, \ 20]$.

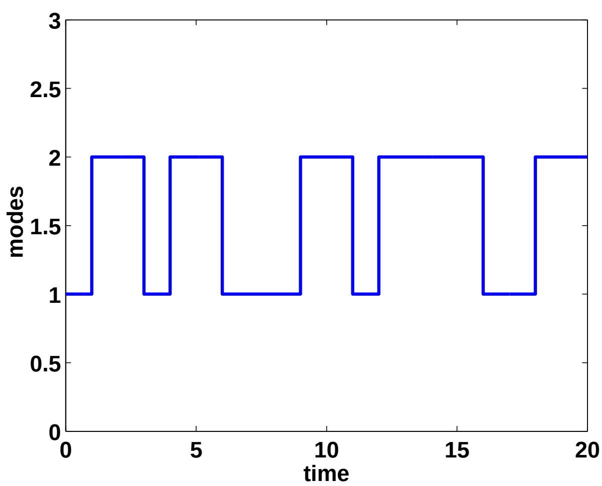

Figure 2. Markovian jumping modes.

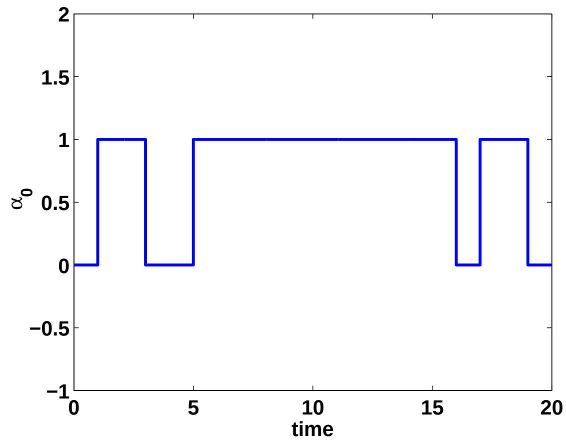

Figure 3. Bernoulli distribution of the disturbance.

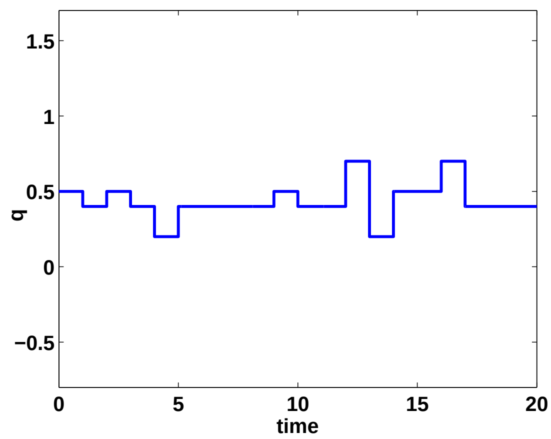

Figure 4. Binomial distribution of the distribution ratio.

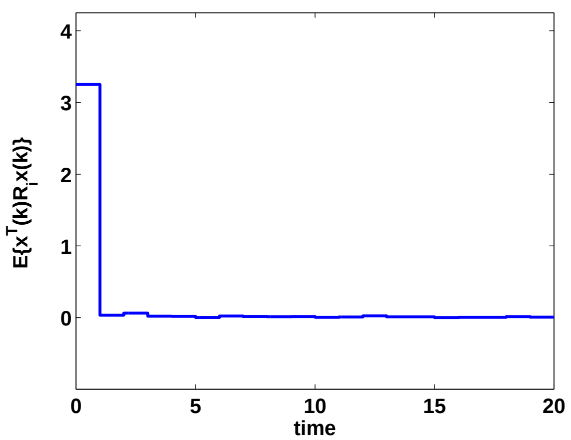

Figure 5. Mathematical expectation of $x^T(k)R_ix(k)$.

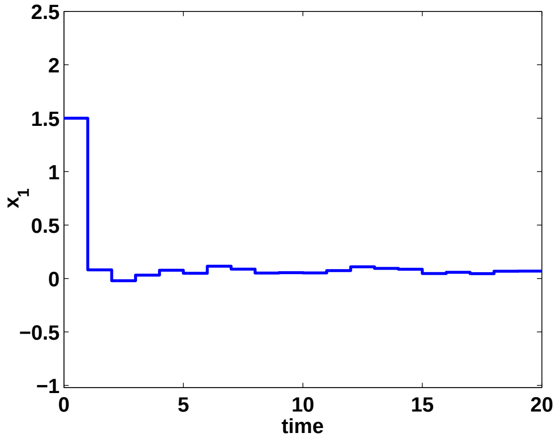

Figure 6. Trajectory for system state $x_1(k)$.

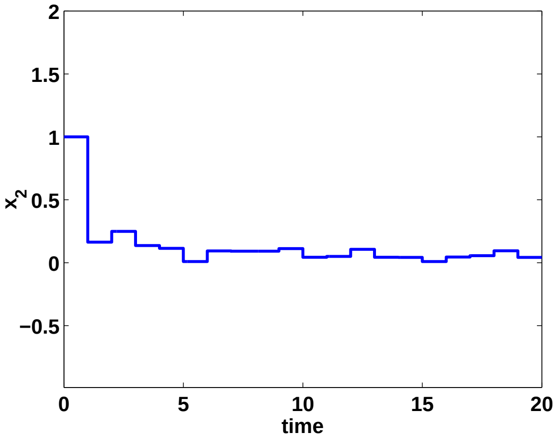

Figure 7. Trajectory for system state $x_2(k)$.

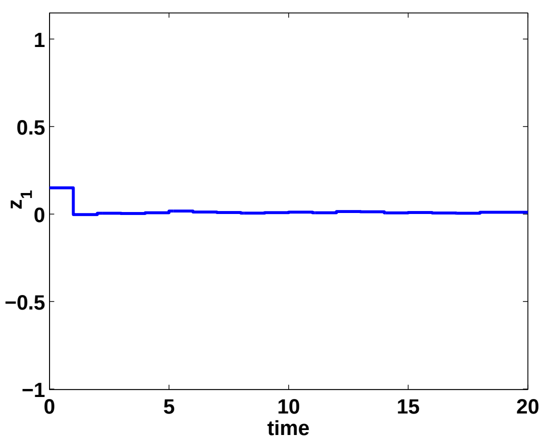

Figure 8. Output performance for $z_1(k)$.

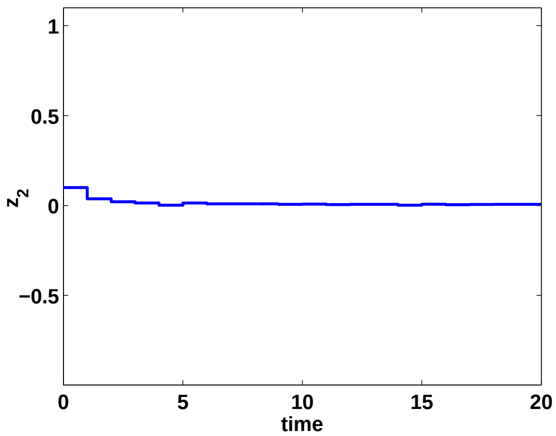

Figure 9. Output performance for $z_2(k)$.

Tables 1 and 2 show the relations between the disturbance influence degree and the disturbance attenuate rate. It can be seen from Table 1 that with the same b_1 values, d_{min} becomes larger with the increase of b, which demonstrates that with the same distribution ratios, a larger disturbance attenuation rate is required for a lower occurrence probability of disturbance distribution. On the other hand, Table 2 presents that with the same b values, larger values of b_1 demands higher d_{min}, which means with the same occurrence probability of disturbance distribution, if the disturbance affects the system states more than the outputs, the disturbance attenuation performances deserve more strict criteria. In a word, H_∞ performances demands higher levels if the disturbance affects the system states more than the outputs and is distributed with a lower occurrence probability.

Table 1. Relations between the disturbance distribution and the attenuation rate: Part I ($\alpha = 1.1364$).

b	b_1	d_{min}
0.2	0.4	1.1400
0.4	0.4	1.6900
0.6	0.4	2.2900
0.8	0.4	2.9300
1	Null	3.6200

*Compared with [4], the novelty of proposed results in our work mainly lies in the introduction of disturbance distribution, where the disturbance enters the system states and system outputs with different percentages of the total influence degrees. In addition, the simulations have proved that the disturbance distribution does have an effect on control performances presented above. Besides, from Table 1, we can see that when no distribution occurs in the disturbance (*i.e.*, $b = 1$, $b_1 = Null$), d_{min} is larger than the counterparts in the case of distributed disturbances. And Table 2 shows the same conclusion.*

To this regard, the conservatism of the proposed criteria is much reduced with the idea of disturbance distribution. Here we note that since no distribution occurs when $b = 1$ holds, it makes no sense for the existence of b_1 any more. Actually whatever b_1 takes over the interval $[0, 1]$, the simulation results for d_{min} make no difference. Therefore, the expression "Null" is used to denote the value of b_1 in the case of $b = 1$.

Table 2. Relations between the disturbance distribution and the attenuation rate: Part II ($\alpha = 1.1338$).

b	b_1	d_{min}
0.8	0.2	2.4400
0.8	0.4	2.5700
0.8	0.6	2.7300
0.8	0.8	2.9300
1	Null	3.1500

5. Conclusions

This paper investigates the FTB and H_∞ FTB issues for discrete delayed nonlinear systems with Markovian jumps and disturbances of probabilistic distributions. Concepts of FTB and H_∞ FTB are proposed for the discussed system. Then new criteria are derived which guarantee the FTB and H_∞ FTB performances for discrete delayed nonlinear systems with Markovian jumps and disturbances of probabilistic distributions. And finally a numerical example is provided to validate the designed controller. The system we considered here contains no uncertainties and linear feedback control is used. Thus in our future work, output feedback control method will be considered for the FTB and H_∞ FTB of discrete delayed nonlinear uncertain systems with Markovian jumps and disturbances of probabilistic distributions.

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Author Contributions

Haiyang Chen, Meiqin Liu, and Senlin Zhang put forward original ideas of this research after many discussions. Haiyang Chen performed the research, designed the simulation experiment, wrote and revised the paper. Meiqin Liu reviewed and modified the paper. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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