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Article

Choice Overload and Height Ranking of Menus in Partially-Ordered Sets

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Abstract: When agents face incomplete information and their knowledge about the objects of choice is vague and imprecise, they tend to consider fewer choices and to process a smaller portion of the available information regarding their choices. This phenomenon is well-known as choice overload and is strictly related to the existence of a considerable amount of option-pairs that are not easily comparable. Thus, we use a finite partially-ordered set (poset) to model the subset of easily-comparable pairs within a set of options/items. The height ranking, a new ranking rule for menus, namely subposets of a finite poset, is then introduced and characterized. The height ranking rule ranks subsets of options in terms of the size of the longest chain that they include and is therefore meant to assess menus of available options in terms of the maximum number of distinct and easily-comparable alternative options that they offer.

Keywords: choice overload; poset; maximum chain; height-based ranking

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1. Introduction

Most typically, decision makers regard flexibility as a highly-desirable feature and therefore tend to appreciate menus including several distinct options: to that extent, the diversity of available alternatives is a key requirement for menus. On the other hand, several works in the behavioral and experimental economic literature suggest that the human ability to manage a diversity of options is definitely bounded. Indeed, in the face of abundant options, the observed behavior of decision makers seem to disconfirm the common assumption that the more choices they have, the better off they are. On the contrary, as agents face a great variety of plans or goods (too much choice), they tend to regard as a burden the task of identifying an optimal choice. In fact, there is growing evidence that people can easily experience difficulties in managing complex choices. Under those circumstances, decision makers experience conflict and tend to defer decisions and to search for new alternatives, choose the default option or simply opt not to choose [1–3].

Additionally, just as [4] might have predicted, consumer research suggests that as both the number of options and the information about options increases, consumers tend to consider fewer choices and to process a smaller portion of the available information concerning their choices [5]. This phenomenon is known as choice overload, and it has been observed both in inconsequential contexts (e.g., choice of snack foods) and in very consequential decision making processes, such as the choice of retirement savings plans. Thus, when agents have either too much and hard-to-process, or unreliable-information, their decisions seem to be more and more influenced by default rules, framing, anchoring, procrastination and endorsement effects.

To address this kind of situation, [6,7] suggested to implement a form of libertarian paternalism or asymmetric paternalism, respectively, *i.e.*, institutional attempts to affect individual behavior while respecting freedom of choice. In a similar vein, the present paper stresses the possible role of "regulation by transparency" and focuses on a further approach to the issue of decision support for boundedly-rational agents, namely providing a population of diverse agents with readily accessible information about the relevant and easily comparable options that are actually available in each proposed menu.

Indeed, consider a diverse population of agents whose types are defined by their characteristic

(i) diversity requirement (or diversity-subtype): the minimum number of distinct and easily comparable alternatives an agent expects to be offered from a good menu, and

(ii) focus (or focus-subtype): the subset of criteria an agent regards as relevant in order to guide her own choice, possibly including specification of an acceptable range for some of them (Conceivably, a single agent might be characterized by *several* subsets of criteria she might regard as relevant (and corresponding acceptable ranges, if any), to the effect of sharing *several focus-subtypes*. That possibility would suggest a *maxmin-height variant* of our height-ranking rule based upon maximization of minimum height across the relevant focus-subtypes. In what follows we disregard this possibility just for the sake of simplicity).

Then, for each specification of the focus subtype and for each menu, the larger the number of distinct and easily-comparable items included in the menu, the wider the set of diversity subtypes of agents whose requirements are met by the menu.

Accordingly, a regulatory agency (henceforth, the Authority) can first identify a set of most common focus subtypes and, then, for each one of such subtypes, single out of the set of "easily-comparable" pairs of options within each feasible menu (that goal can be achieved by the Authority either by forcing providers to produce that focus subtype-tailored information for customers or by producing that information by itself and making it freely available to any interested party). Then, the Authority, or indeed any other interested agency, is in a position to assess the ability of different menus to accommodate the requirements of a range of diversity subtypes. A menu can be assessed by attaching to it a rank number (or "pseudo-rank" if "rank" is to be reserved to values of a rank function that is well-defined only in graded posets, *i.e.*, posets whose maximal chains having the same extrema have also the same size; in the present paper, we chose to use "rank" as opposed to "pseudo-rank" just for the sake of convenience) as given by the size of the longest chain of (pairwise) easily-comparable options it includes.

In particular, if the relevant criteria for the focus-subtype under consideration are representable by *ordinal criteria* (either binary or not) then the subset of easily comparable pairs amounts to a *partially ordered set (poset)* namely a transitive, reflexive and antisymmetric binary relation (A binary relation $R \subseteq X \times X$ is *transitive* if xRy and yRz imply xRz for each $x, y, z \in X$, *reflexive* if xRx for each $x \in X$, and *antisymmetric* if xRy and yRx imply $x = y$ for each $x, y \in X$.) on the set of available options. The ranking method just mentioned—*the height-ranking rule*—consists in counting the size of the largest chain included in the poset attached to a menu, namely *the height of that poset*. (The present paper focuses on the case of ordinal criteria, hence of posets, for the sake of simplicity. But it should be clear that a similar approach is available if the comparability relation taken into consideration is not transitive.)

In that connection, the height ranking rule establishes a comparative assessment of alternative menus of available items in terms of the maximum number of easily-comparable options they offer to consumers: menu A is effectively richer than B if and only if the largest subsets of pairwise easily-comparable alternatives in A are larger than the largest subsets of corresponding subsets of menu B. Again, the height rank of each menu is a piece of information that the Authority might either produce by itself and make available to the public or force the relevant providers to produce and make available for free to customers. (As suggested by one of the referees, non-governmental organizations might in fact provide themselves that kind of information. In any case, the Authority would conceivably fulfil a key role by setting the format and standard of the menu rankings to be provided to the general public. That is why we think it proper to label "Authority" the agency responsible for the establishment of the menu-ranking under consideration.). In any case, the induced height ranking of menus offers an inexpensive and convenient support to boundedly-rational decision makers, helping them to assess and compare the menus of alternative options they are offered. In particular, minimum requirements on option sets and public rating of vendors might be established relying on that ranking rule.

In that connection, the present paper focuses precisely on the height ranking rule for arbitrary menus of alternative options and provides a simple axiomatic characterization of that rule when the set of easily-comparable options is a (finite) partially-ordered set.

The paper is organized as follows. Section 2 introduces and characterizes the proposed ranking rule for menus. Section 3 provides a simple illustration of the height ranking for menus of options. Section 4 includes a discussion of some related literature. Section 5 consists of some concluding remarks.

2. Coping with Choice Overload and the Height Ranking Rule

Let us consider a universal finite set X of alternative options and a population of diverse decision makers or agents sharing the following characteristics:

(i) preference for flexibility, namely each agent appreciates the opportunity to choose among many distinct alternative options (where "many" means "at least k" and k is a positive integer that may vary across agents).

(ii) severely-bounded information processing ability, hence the alternative feasible options to be considered should be easily pairwise comparable.

Moreover, each option is described by values of a universal set C^* of criteria, and each agent relies on some subset $C \subseteq C^*$ of relevant criteria. As mentioned above, we also assume for the sake of convenience that criteria are attributes that are representable by ordinal scales, but devoid of any prefixed relationship to agents' preferences: if a criterion has, say, "high, medium, low" as its possible values, then an agent's optimum choice may require any such value, including of course "medium".

Under the foregoing assumptions, for any set C of (ordinal) criteria, the set of easily-comparable pairs of available options specifies a partial order $\leqslant$ on the universal set X, namely a poset $(X, \leqslant)$ (each element of X can be conveniently regarded as the lists of values of the relevant criteria for some available item and $\leqslant$ as the dominance relation induced by the intersection of the orderings attached to the relevant ordinal criteria; however, in what follows, we shall stick for the sake of convenience to the more compact notation $(X, \leqslant)$). Then, we focus on the task of comparing distinct menus, *i.e.*, feasible subsets of X from the point of view of an Authority whose aim is to accommodate the requirements of the largest possible class of agent-types (this approach to menu ranking is akin to two-stage procedures in which, given a set of possible alternatives, first a subset with required characteristics is chosen and then a rational decision rule is applied [8–10]). As mentioned in the Introduction, such considerations lead immediately to a ranking of menus in terms of the largest chains of comparable items they include, namely to the height ranking.

Indeed, under the proposed interpretation of the underlying poset as a codification of the set of easily-comparable pairs, the height ranking of menus from X ranks the latter according to the size of the maximum number of easily-comparable options that they include. There are some different methods for finding a chain of maximum size, possibly depending on specific features of the poset. In particular, if the poset is the majorization poset (majorization is a partial order among vectors, and it is related to Schur convexity or submodularity; the majorization ordering was defined by [11] and developed in an application on symmetric means by [12]) of integer partition special algorithms to compute its height are available. (The study of posets of integer partitions has a very long tradition from Euler through Ferrers diagrams to Young diagram lattices. Integer partitions are studied in terms of lexicographic order and majorization dominance order by [13]. There are some methods for finding a chain of maximum length between two integer partitions ordered by majorization. The work in [14] provides an algorithm to compute the longest chain in the lattice of integer partitions ordered by majorization.) Other methods consider the *Maximum Entropy Principle* (The *Maximum Entropy Principle* was introduced by Jaynes [15,16] to elicit the most unbiased or the most uniform distribution among all of the possible ones as a generalization of the classical principle of insufficient reason of Laplace) and suggest some algorithms to find the longest chain inspired by that principle [17]. Computational issues, however, are beyond the scope of the present paper, which is rather focused on the quite different task of providing an independent, self-standing justification of the height ranking through a simple characterization of that ranking in terms of general properties for menu ranking rules in partially-ordered sets.

Thus, the next subsection will introduce the height ranking in its full generality for an arbitrary finite partially-ordered set and a list of properties that will enable our characterization of that ranking rule (an alternative "dual" ranking rule that, on the contrary, relies on a notion of diversity as "incomparability" is presented and characterized in [18]).

2.1. Height-Based Rankings for Menus of Alternative Options: Formal Definitions and Preliminaries

Let $\mathcal{X} = (X, \leqslant)$ be the universal (finite) partially-ordered set (henceforth, poset) of alternative options, *i.e.*, $\leqslant$ is a transitive, reflexive and antisymmetric binary relation on X, and $\mathcal{P}(X)$ the power set of X and $\mathcal{D} \subseteq \mathcal{P}(X)$. Two alternative options $x, y \in X$ are said to be $\leqslant$-incomparable, written $x||y$, if neither $x \leqslant y$ nor $y \leqslant x$ hold. A chain of $\mathcal{X}$ is a set $C \subseteq X$, such that the restriction $\leqslant_{|C} := \{(x, y) \in C \times C : x \leqslant y\})$ is a total, transitive and antisymmetric binary relation on C. The set of all chains of $\mathcal{X}$ is denoted $\mathcal{C}_{\mathcal{X}}$. An order-isomorphism of posets $(A, \leqslant')$ and $(B, \leqslant'')$ is an injective function $f : A \to B$, such that for any $x, y \in A$, $x \leqslant' y$ if and only if $f(x) \leqslant'' f(y)$. (It is easily checked that, by definition, an order-isomorphism f has to be injective, as well. An order-isomorphism from $(A, \leqslant)$ to itself is an order-automorphism of $(X, \leqslant)$.) Subsets $A, B \subseteq X$ are isomorphic in $\mathcal{X}$ if and only if there exists an order-isomorphism from $(A, \leqslant_{|A})$ to $(B, \leqslant_{|B})$.

The height function $h_{\mathcal{X}} : \mathcal{P}(X) \to \mathbb{Z}_+$ of $\mathcal{X}$ attaches to each set $Y \subseteq X$ the size of any chain $A \in \mathcal{C}_{\mathcal{X}}$ of maximum size amongst chains of $\mathcal{X}$ included in Y, namely $h_{\mathcal{X}}(Y) = \#A$, where: (i) $A \subseteq Y$, (ii) $A \in \mathcal{C}_{\mathcal{X}}$ and (iii) $\#A \geq \#B$ for any B, which also satisfies Clauses (i) and (ii) above (thus, the height function records the size of the largest totally-ordered subset of any given subpopulation).

Remark 1. *It should be noticed that, by definition, $h_{\mathcal{X}}$ is subposet-invariant, i.e., for any $A \subseteq Y \subseteq X$, $h_{\mathcal{X}}(A) = h_{\mathcal{X}|Y}(A)$ where $\mathcal{X}|Y := (Y, \leqslant_{|Y})$.*

A simple binary relational system is a pair $(V, \succcurlyeq)$ where V is a set and $\succcurlyeq \subseteq V \times V$ is a binary relation on V (while $\succ$ and $\sim$ denote the asymmetric and symmetric components of $\succcurlyeq$, respectively, and $\Delta_V = \{(x, x) : x \in V\}$ denotes the "diagonal" of V).

We are mainly interested in the height-based rankings of subpopulations as defined below:

Definition 1. *The height ranking induced by $\mathcal{X} = (X, \leqslant)$ on $\mathcal{P}(X)$ is the total and transitive binary relation $\succcurlyeq_{h_{\mathcal{X}}}$ defined by the following rule: for any $A, B \in \mathcal{P}(X)$, $A \succcurlyeq_{h_{\mathcal{X}}} B$ iff $h_{\mathcal{X}}(A) \geq h_{\mathcal{X}}(B)$.*

Clearly enough, the height ranking decrees a subset to be more diverse than another if and only if the longest chain of the former is longer than the longest chain of the latter.

We shall now provide a characterization of the height-based ranking through the following axioms for simple binary relational systems:

Definition 2. *Indifference between isomorphic sets (IIS): A simple binary relational system $(\mathcal{P}(X), \succcurlyeq)$ satisfies indifference between isomorphic sets with respect to poset $\mathcal{X} = (X, \leqslant)$ iff for any $A, B \in \mathcal{P}(X)$, if A and B are order-isomorphic in $\mathcal{X}$, then $A \succcurlyeq B$.*

In other words, IIS requires that two order-isomorphic sets to be equally ranked in terms of diversity. It amounts to a strengthened, and adapted, version of the standard notion of indifference between no choice situations, *i.e.*, between singletons (e.g., [19]). Under the present interpretation of poset $(X, \leqslant)$, IIS simply establishes that the ranking should ignore the precise identity of options of two equally-sized menus, provided that the options of those two menus exhibit the same pattern of pairwise comparability.

Definition 3. *Weak monotonicity (WMON): A simple binary relational system* $(\mathcal{P}(X), \succcurlyeq)$ *satisfies weak monotonicity iff* $A \succcurlyeq B$ *for any* $A, B \subseteq X$, *such that* $B \subseteq A$.

Definition 4. *Strict monotonicity for chains (SMONC): A simple binary relational system* $(\mathcal{P}(X), \succcurlyeq)$ *satisfies strict monotonicity for chains with respect to poset* $\mathcal{X} = (X, \leqslant)$ *iff for any* $A, B \in \mathcal{C}_{\mathcal{X}}$, $B \subset A$ *entails* $A \succ B$.

Of course, WMON amounts to requiring that the diversity preorder be set-inclusion preserving. SMONC is the restriction of the strict version of set-inclusion monotonicity to chains. Both conditions embody appropriately distinct facets of the notion that "more flexibility is better".

Definition 5. *Irrelevance of maximal-chain-disconnected units (IMDU): A simple binary relational system* $(\mathcal{P}(X), \succcurlyeq)$ *satisfies the irrelevance of maximal-chain-disconnected units with respect to poset* $\mathcal{X} = (X, \leqslant)$ *iff* $A \succcurlyeq A \cup \{x\}$ *for any* $A \in \mathcal{P}(X)$ *and any* $x \in X$, *such that* $x||y$ *for each* $y \in \bigcup\{B \subseteq A : B \in \mathcal{C}_{\mathcal{X}}$ *and* $|B| \geq |C|$ *for any* $C \in \mathcal{C}_{\mathcal{X}}, C \subseteq A\}$.

Thus, IMDU is a restricted independence condition dictating that the addition to any menu of an alternative that is disconnected from each chain of maximum size of that menu should not increase its diversity ranking. Under the suggested interpretation of $(X, \leqslant)$ as the poset of "easily-comparable pairs of options" IMDU reflects the notion that the options of a menu should be precisely "easily-comparable" to each other in order to expand the "flexibility" they offer in an effective way.

2.2. Height-Based Ranking: Characterization

We are now ready to state and prove our characterization of the height ranking of menus, namely:

Theorem 1. *Let* $\mathcal{X} = (X, \leqslant)$ *be a poset and* $\succcurlyeq$ *a preorder, i.e., a reflexive and transitive binary relation on* $\mathcal{P}(X)$. *Then,* $(\mathcal{P}(X), \succcurlyeq)$ *satisfies IIS, WMON, SMONC and IMDU if and only if* $\succcurlyeq = \succcurlyeq_{h_{\mathcal{X}}}$.

Proof. $\Longleftarrow$: It is immediately checked that $(\mathcal{P}(X), \succcurlyeq_{h_{\mathcal{X}}})$ is by definition a (totally) pre-ordered set and satisfies WMON. Furthermore, if $A, B \in \mathcal{C}_{\mathcal{X}}$ and $B \subset A$, then by definition, $h_{\mathcal{X}}(A) = \#A > \#B = h_{\mathcal{X}}(B)$, *i.e.*, $A \succ_{h_{\mathcal{X}}} B$; hence, SMONC is also satisfied.

To check that IIS holds, notice that if $A, B \subseteq X$ are order-isomorphic w.r.t. $\mathcal{X}$, then $\#A = \#B$ and for any $x, y \in A$, $x \leqslant y$ iff $f(x) \leqslant f(y)$ and $x \| y$ iff $f(x) \| f(y)$, where f is an order-automorphism of $\mathcal{X}$, such that $f[A] = B$. It follows that for any chain B' of $\mathcal{X}|B$, $f^{-1}[B']$ is a chain of $\mathcal{X}|A$. In particular, let $B' \subseteq B$ be a chain of $\mathcal{X}$ of maximum size, *i.e.*, $h_{\mathcal{X}}(B) = \#B'$. Then, $f^{-1}[B'] = A'$ is a chain of A and $\#A' = \#B'$, hence $h_{\mathcal{X}}(A) \geq h_{\mathcal{X}}(B)$, *i.e.*, $A \succcurlyeq_{h_{\mathcal{X}}} B$. Thus, IIS is satisfied.

To check that $(\mathcal{P}(X), \succcurlyeq_{h_{\mathcal{X}}})$ satisfies IMDU, as well, take any $A \subseteq X$ and any $x \in X$, such that $x||y$ for each $y \in \bigcup\{B \subseteq A : B \in \mathcal{C}_{\mathcal{X}}$ and $|B| \geq |C|$ for any $C \in \mathcal{C}_{\mathcal{X}}, C \subseteq A\}$. Clearly, by construction, $h_{\mathcal{X}}(A \cup \{x\}) = h_{\mathcal{X}}(A)$, hence, in particular, $A \succcurlyeq_{h_{\mathcal{X}}} A \cup \{x\}$.

$\Longrightarrow$: Conversely, let $(\mathcal{P}(X), \succcurlyeq)$ be a pre-ordered set that satisfies IIS, WMON, SMONC and IMDU.

First, assume $A \succcurlyeq B$. Suppose that A' is a chain of maximum size in $(A, \leqslant_{|A})$ and B' is a chain of maximum size in $(B, \leqslant_{|B})$. Now, notice that, by construction, for any $x \in A \backslash A'$, $y \in B \backslash B'$ and chains $A'' \subseteq A'$ and $B'' \subseteq B'$ of maximum size in $(A, \leqslant_{|A})$ and $(B, \leqslant_{|B})$, respectively, $x||u$ for each $u \in A''$ and $y||v$ for each $v \in B''$. The same statement holds starting from $A' \cup \{x\}$ and $B' \cup \{y\}$. Thus, by suitably-repeated applications of IMDU, it follows that $B' \succcurlyeq B$ and $A' \succcurlyeq A$, while by WMON, $A \succcurlyeq A'$ and $B \succcurlyeq B'$. Therefore, $A \sim A'$ and $B' \sim B$, whence $A' \succcurlyeq B'$. Let us now assume that $h_{\mathcal{X}}(B) > h_{\mathcal{X}}(A)$ *i.e.*, $\#B' > \#A'$. Hence, there exists $B'' \subset B'$, such that $\#B'' = \#A'$. Since, by construction, both $A' \in \mathcal{C}_{\mathcal{X}}$ and $B'' \in \mathcal{C}_{\mathcal{X}}$, it follows that A' and B'' are order-isomorphic in $\mathcal{X}$, hence by IIS $A' \sim B''$. However, $B' \succ B''$ by SMONC. Thus, by transitivity of $\succcurlyeq$, $B' \succ A'$, a contradiction. As a consequence, it must be the case that $h_{\mathcal{X}}(A) \geq h_{\mathcal{X}}(B)$, *i.e.*, $A \succcurlyeq_{h_{\mathcal{X}}} B$.

Next, assume $A \succcurlyeq_{h_{\mathcal{X}}} B$, *i.e.*, $h_{\mathcal{X}}(A) \geq h_{\mathcal{X}}(B)$. Let A', B' be chains of maximum size of $(A, \leqslant_{|A})$ and $(B, \leqslant_{|B})$, respectively, as defined in the paragraphs above: clearly $\#A' \geq \#B'$. Furthermore, notice that, again, by IMDU and WMON, $A \sim A'$ and $B \sim B'$. Then, observe that if $\#A' = \#B'$, then since both A' and B' are chains of $\mathcal{X}$, they are also order-isomorphic in $\mathcal{X}$, whence in particular $A \sim A' \succcurlyeq B' \sim B$, by IIS. Moreover, if $\#A' > \#B'$, then there exists a chain $A'' \subset A'$, such that $\#A'' = \#B'$. Again, A'' and B' are order-isomorphic in $\mathcal{X}$ (since they are both chains), hence by IIS $A'' \sim B'$. However, $A' \succ A''$ by SMONC, hence $A \sim A' \succ B' \sim B$, *i.e.*, $A \succcurlyeq B$ by transitivity. In any case, $A \succcurlyeq B$ holds, and the proof of the thesis is completed. $\square$

The foregoing characterization result is tight, *i.e.*, irredundant. Indeed, to check the independence of the axioms employed, let us consider the following list of examples showing that for each axiom of our characterization, there exists a ranking of menus that fails to satisfy that axiom, while satisfying the others.

Example 1. *The independence of IIS from the other axioms can be shown by considering the following example. First, consider a finite poset $\mathcal{X} = (X, \leqslant)$ with $\#X \geq 3$ and at least two distinct elements $y, z \in X$, such that $y \leqslant x$, $z \leqslant x$ for any $x \in X \backslash \{y, z\}$ and $y||z$. Then, take $(\mathcal{P}(X), \succcurlyeq^z_{h_{\mathcal{X}}})$ where $\succcurlyeq^z_{h_{\mathcal{X}}}$ is the "refinement" of $\succcurlyeq_{h_{\mathcal{X}}}$ defined as follows: for any $A, B \subseteq X$, $A \succcurlyeq^z_{h_{\mathcal{X}}} B$ iff either $h_{\mathcal{X}}(A) > h_{\mathcal{X}}(B)$ or $h_{\mathcal{X}}(A) = h_{\mathcal{X}}(B)$ and ($z \notin B$ or $z \in A \cap B$ or $z \notin A \cup B$). Notice that $\succcurlyeq^z_{h_{\mathcal{X}}}$ is indeed a preorder: reflexivity is obvious, and transitivity also holds (to see this, assume $A \succcurlyeq^z_{h_{\mathcal{X}}} B$ and $B \succcurlyeq^z_{h_{\mathcal{X}}} C$, then: (i) $h_{\mathcal{X}}(A) > h_{\mathcal{X}}(B)$ and $h_{\mathcal{X}}(B) > h_{\mathcal{X}}(C)$ or $h_{\mathcal{X}}(B) = h_{\mathcal{X}}(C)$ imply $h_{\mathcal{X}}(A) > h_{\mathcal{X}}(C)$, and similarly, $h_{\mathcal{X}}(A) = h_{\mathcal{X}}(B)$ and $h_{\mathcal{X}}(B) > h_{\mathcal{X}}(C)$ imply $h_{\mathcal{X}}(A) > h_{\mathcal{X}}(C)$, whence $A \succcurlyeq^z_{h_{\mathcal{X}}} C$; (ii) if $h_{\mathcal{X}}(A) = h_{\mathcal{X}}(B) = h_{\mathcal{X}}(C)$ and $z \notin B$ then $B \succcurlyeq^z_{h_{\mathcal{X}}} C$ entails $z \notin C$ (indeed $z \notin B \cup C$), whence again $A \succcurlyeq^z_{h_{\mathcal{X}}} C$; (iii) if $h_{\mathcal{X}}(A) = h_{\mathcal{X}}(B) = h_{\mathcal{X}}(C)$ and $z \notin C$ then $A \succcurlyeq^z_{h_{\mathcal{X}}} C$ by definition; (iv) if $h_{\mathcal{X}}(A) = h_{\mathcal{X}}(B) = h_{\mathcal{X}}(C)$ and $z \in A \cap B$, then it cannot be the case that $z \notin B \cup C$; thus $B \succcurlyeq^z_{h_{\mathcal{X}}} C$ entails $z \in B \cap C$, hence $z \in A \cap C$, and therefore, $A \succcurlyeq^z_{h_{\mathcal{X}}} C$). Moreover, if $B \subseteq A$, then by definition, $h_{\mathcal{X}}(A) \geq h_{\mathcal{X}}(B)$ and either $z \notin B$ or $z \in A \cap B$. Thus, $A \succcurlyeq^z_{h_{\mathcal{X}}} B$ holds in any case, and $(\mathcal{P}(X), \succcurlyeq^z_{h_{\mathcal{X}}})$ satisfies WMON. If A, B are chains and $B \subset A$, then $h_{\mathcal{X}}(A) > h_{\mathcal{X}}(B)$, whence both $A \succcurlyeq^z_{h_{\mathcal{X}}} B$ and not $B \succcurlyeq^z_{h_{\mathcal{X}}} A$: it follows that $(\mathcal{P}(X), \succcurlyeq^z_{w_{\mathcal{X}}})$ satisfies SMONC. Now, let us consider any $A \subseteq X$ and $x \in X$, such that $x||y$ for each $y \in \bigcup\{B \subseteq A : B \in \mathcal{C}_{\mathcal{X}}$ and $|B| \geq |C|$ for any $C \in \mathcal{C}_{\mathcal{X}}, C \subseteq A\}$. Clearly, $h_{\mathcal{X}}(A) = h_{\mathcal{X}}(A \cup \{x\})$ and $x \neq z$, by definition of z. It follows that either $z \notin A \cup \{x\}$ or $z \in A$, hence,* *in any case, $A \succcurlyeq^z_{w\mathcal{X}} A \cup \{x\}$, and IMDU is also satisfied by $(\mathcal{P}(X), \succcurlyeq^z_{w\mathcal{X}})$. However, it is immediately checked that $\{z\} \succ^z_{w\mathcal{X}} \{y\}$; thus, IIS is violated by $(\mathcal{P}(X), \succcurlyeq^z_{w\mathcal{X}})$.*

Example 2. *The independence of WMON from the other axioms can be shown by considering the "refinement" of $\succcurlyeq_{h_\mathcal{X}}$ defined as follows: for any $A, B \subseteq X$, $A \succcurlyeq^*_{h_\mathcal{X}} B$ iff either $h_\mathcal{X}(A) > h_\mathcal{X}(B)$ or:*

$$h_\mathcal{X}(A) = h_\mathcal{X}(B) \; and \# \{(x,y) \in A \times A : x||y\} < \# \{(x,y) \in B \times B : x||y\}$$

Indeed, it is easily checked that by construction, $(\mathcal{P}(X), \succcurlyeq^*_{h_\mathcal{X}})$ is a pre-ordered set that satisfies SMONC and IIS. Moreover, $(\mathcal{P}(X), \succcurlyeq^*_{w\mathcal{X}})$ satisfies IMDU: for any $A \subseteq X$ and any $x \in X$, such that $x||y$ for each $y \in \bigcup \{B \subseteq A : B \in \mathcal{C}_\mathcal{X}$ and $|B| \geq |C|$ for any $C \in \mathcal{C}_\mathcal{X}, C \subseteq A\}$, $A' = A \cup \{x\} \notin \mathcal{C}_\mathcal{X}$, hence, by definition, $A \succcurlyeq^*_{h_\mathcal{X}} A'$. However, in general, $(\mathcal{P}(X), \succcurlyeq^*_{w\mathcal{X}})$ does not satisfy WMON. To see this, take $\mathcal{X} = (\{x, y, z\}, \leqslant)$ with $\leqslant \smallsetminus \Delta_X = \{(x, y)\}$. Clearly, $\{x, y\} \succ^*_{h_\mathcal{X}} \{x, y, z\}$, hence WMON is violated.

Example 3. *The independence of SMONC is immediately verified by considering the universal binary relation $\succcurlyeq^U = \mathcal{P}(X) \times \mathcal{P}(X)$. Clearly enough, $(\mathcal{P}(X), \succcurlyeq^U)$ is a (totally) pre-ordered set and satisfies IIS, WMON and IMDU, but violates SMONC.*

Example 4. *The independence of IMDU can be shown by considering the binary relational system $(\mathcal{P}(X), \succcurlyeq^\#_{\|(\mathcal{X})})$ where $\succcurlyeq^\#_{\|(\mathcal{X})}$ is defined by the following rule: for any $A, B \subseteq X$, $A \succcurlyeq^\#_\mathcal{X} B$ iff:*

$$\# \{(x,y) : (x,y) \in A \times A \text{ and } x \leqslant y\} \geq \# \{(x,y) : (x,y) \in B \times B \text{ and } x \leqslant y\}$$

Clearly, $\succcurlyeq^\#_\mathcal{X}$ is a (total) preorder. IIS, WMON and SMONC are also obviously satisfied. However, if $A = \{y, z\}$ with $y \neq z$ is a chain, i.e., either $y \leqslant z$ or $z \leqslant y$, and $x||y$, $x||z$, then $\leqslant_{|A \cup \{x\}} = \leqslant_{|A} \cup \{(x,x)\}$, hence $\{x, y, z\} \succ^\#_\mathcal{X} \{y, z\}$, while $x||y$ for each $y \in \bigcup \{B \subseteq A : B \in \mathcal{C}_\mathcal{X}$ and $|B| \geq |C|$ for any $C \in \mathcal{C}_\mathcal{X}$, $C \subseteq A\}$. It follows that $(\mathcal{P}(X), \succcurlyeq^\#_\mathcal{X})$ does not satisfy IMDU.

3. A Simple Illustration

Consider the case of bundles of services supplied by phone operators (Tim, Vodafone, Telefonica, Orange, Vivendi, *etc.*) for home phone and cellular phones, indeed sms, mms, video, voice, Internet, smartphone, tablet, cost, and so on. A national authority on telecommunications (*i.e.*, AGCOM, OFCOM, ARCEP, *etc.*) might consider any subset of characteristics corresponding to the focal set of criteria for agents of the most common type. Here, menus are just the sets of options offered by distinct operators.

Consider all of the characteristics that define the supply of a phone operator. Suppose that consumers are prepared to assess options in terms of some simple ordered characterization of the values of focal attributes (such as "huge", "large", "intermediate", "moderate", "small", "very small", denoted h, l, i, m, s, v, respectively). Moreover, consumers are also typified by the minimum amount of (easily) comparable diversity they require. Finally, let us also suppose for the sake of discussion that the range of diversity subtypes is quite large and the most common focus subtype is characterized by "(up to) moderate cost" with the set of relevant criteria given by (cost, sms, voice, internet), denoted C, S, V, I respectively.

Now, consider menus $\mathcal{A}$ and $\mathcal{B}$ and take their sub-menus A_m, B_m consisting of their "(up to) moderate cost" options with:

$$\mathcal{A}_m = \left\{ \begin{array}{c} (C_m, S_i, V_i, I_m), (C_m, S_i, V_m, I_m), (C_m, S_m, V_i, I_m), \\ (C_m, S_m, V_m, I_m),\ (C_m, S_m, V_m, I_s) \end{array} \right\}$$

$$\mathcal{B}_m = \left\{ \begin{array}{c} (C_m, S_i, V_i, I_m), (C_m, S_i, V_m, I_l), (C_m, S_m, V_m, I_l), \\ (C_m, S_l, V_m, I_m), (C_m, S_m, V_l, I_m), \end{array} \right\}$$

where X_a stands for "attribute/criterion X has value a".

Clearly, the largest chain of $\mathcal{A}_m$ is:

$$\{(C_m, S_i, V_i, I_m), (C_m, S_i, V_m, I_m), (C_m, S_m, V_m, I_m),\ (C_m, S_m, V_m, I_s)\}$$

since:

$$(C_m, S_m, V_m, I_s) \leqslant (C_m, S_m, V_m, I_m) \leqslant (C_m, S_i, V_m, I_m) \leqslant (C_m, S_i, V_i, I_m)$$

On the other hand, the largest chains of $\mathcal{B}_m$ are its unit subsets, because by construction, any two distinct options of $\mathcal{B}_m$ are not easily comparable.

It follows that $h_{\mathcal{X}}(\mathcal{A}_m) = 4$, $h_{\mathcal{X}}(\mathcal{B}_m) = 1$ and $\mathcal{A}_m \succ_{h_{\mathcal{X}}} \mathcal{B}_m$.

4. Related Literature

As previously mentioned, serious difficulties in processing information when choosing among many non-comparable options are observed in different decision making contests, ranging from choice of snack foods ([5] tested choice overload hypothesis in field experiments (Studies 1 and 2) and laboratory experiment (Study 3)) to the decision to participate in the 401(k) retirement saving plans. (401(k)s are retirement saving plans, named for the section of the tax code that regulates them and appeared in the 1980s, promoted by an employer. A 401(k) is a supplement to pension in which workers invest part of their paycheck before taxes. The work in [20] tested the hypothesis that employee 401(k) participation rates fall as the number of fund options increase. They put in evidence that "if a plan offered more funds, this depressed probability of employee 401(k) participation. Other things equal, every ten funds added was associated with 1.5 percent to 2 percent drop in participation rate".) In neuroscience, there is much evidence that simple choices are made by computing and comparing the decision value of each feasible alternative option [21–23] when those comparisons are easily available. There is also the indication that if a decision making process is made under a strict time constraint (time pressure), the resulting choices are noisy [24,25]. Finally, there is no strong behavioral evidence that different attributes of the stimuli information, for example taste *versus* health attributes for foods, induce different decision values in the short period or long period conditions [26].

Consider the case of retirement saving plans. Many members of the European Union, such as Italy, Denmark, Hungary, Poland, Sweden, *etc.*, and many other states worldwide (including Mexico, Hong Kong, India, Australia, Russia, *etc.*) introduced mandatory funded pension plans where individuals bear investment risk during the accumulation phase and can opt for different types of income streams at retirement (primarily annuities or programmed withdrawals). "The acknowledgment of the weaknesses of the rationality assumption has led some to suggest a libertarian paternalistic approach to pension plan design. The key aspects of pension plan design are the range of investment options, the design of the default strategy option and the extent of freedom of choice. A review of the international experience shows different approaches to these regulatory issues" [27].

In the Swedish privatization plan of social security when designers opted for a laissez-faire approach that induced participants to choose their own portfolios, one-third of them opted for the default fund. Crucially-active participants "elected to invest nearly half their money (48.2%) in the stocks of Swedish companies... Sweden accounts for approximately 1% of the World Economy. A rational investor in the United States or Japan would invest about 1% of his assets in Swedish stocks" [28] (p. 153). Similar behavior was observed in the school choice program; in fact "when parents pick schools, status quo bias plays a big role. The neighborhood school that one knows, failing or not, may be preferable to an unknown school half an hour away" [28] (p. 207). Similar conditions and evidence were observed in insurance policies, as well.

We assume that, normally, consumers do not have enough information to perfectly discriminate objects with respect to their characteristics, but are at least able to aggregate them with respect to a few relevant aspects (that are represented by ordinal criteria). Then, all of the objects of choice can be considered as elements of a poset, and each element in the poset can be grouped with similar objects with respect to these basic attributes. As a consequence of that aggregation, a poset is obtained whose elements consist of sets of similar objects with respect to given characteristics in a sort of RECAP (record, evaluate and compare alternative prices), as suggested by Thaler and Sunstein with reference to the cell phone markets: "The government would not regulate how much issuers could charge for services, but it would rather regulate their disclosures practices. The central goal would be to inform customers of every kind of fee that currently exists. This would not be done by printing a long unintelligible document in fine print. Instead, issuers would be required to make public their fee schedule in a spreadsheet-like format that would include all relevant formulas" [28] (p. 95).

5. Concluding Remarks

A growing body of experimental literature suggests that human choices are enormously conditioned by menu design. In principle, consumers are better off whenever their option set is expanded, but too many options can induce a consumer to be highly prone to customary options (status quo bias) or default options (default bias), even when superior alternatives are available. That is what in fact typically happens when people face menus of unfamiliar objects, such as saving plans, investment plans, cell phone plans, *etc*. Thus, the diversity of options should be supplemented with the ease of comparisons among them. If the relevant criteria are ordinal, then the set of easily-comparable pairs of options defines a partial order on the set of alternative options. In that setting, a most natural index of the ability of a menu to accommodate the largest range of types of "preference for flexibility" is given by its height, namely by the size of the longest chains included in that menu. The present paper offers a supplementary justification for the height rule by providing a simple characterization of that ranking rule for menus of options in an arbitrary finite partially-ordered set.

Author Contributions

Both authors substantially contributed to the paper. Both authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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