Entropy **2015**, *17*, 6519-6533; doi:10.3390/e17096519

OPEN ACCESS
entropy
ISSN 1099-4300
www.mdpi.com/journal/entropy

Article

Approximate Analytical Solutions of Time Fractional Whitham–Broer–Kaup Equations by a Residual Power Series Method

Linjun Wang and Xumei Chen *

Department of Mathematics, Faculty of Science, Jiangsu University, Zhenjiang 212013, China;
E-Mail: wanglinjun@ujs.edu.cn

* Author to whom correspondence should be addressed; E-Mail: chenxumei@ujs.edu.cn;
Tel.: +86-511-88780161.

Academic Editor: J. A. Tenreiro Machado

Received: 11 August 2015 / Accepted: 18 September 2015 / Published: 23 September 2015

Abstract: In this paper, a new analytic iterative technique, called the residual power series method (RPSM), is applied to time fractional Whitham–Broer–Kaup equations. The explicit approximate traveling solutions are obtained by using this method. The efficiency and accuracy of the present method is demonstrated by two aspects. One is analyzing the approximate solutions graphically. The other is comparing the results with those of the Adomian decomposition method (ADM), the variational iteration method (VIM) and the optimal homotopy asymptotic method (OHAM). Illustrative examples reveal that the present technique outperforms the aforementioned methods and can be used as an alternative for solving fractional equations.

Keywords: fractional power series; fractional Whitham–Broer–Kaup equations; residual power series method

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1. Introduction

Fractional calculus, including integrals and derivatives of arbitrary order, is a generalization of classical integer-order differentiation and integration [1]. In the past few decades, fractional calculus theory has played an important role in the fields of fluid mechanics, physics, entropy and engineering [2–5]. By using fractional calculus, some physical models and engineering processes can be described more reasonably and applicably. For example, entropies based on fractional calculus could be used more widely than traditional Shannon entropy [6]. Due to its wide application, fractional entropy has become a hot research field [7]. Another example is fractional differential equations, which are powerful for modeling various phenomena [8]. The reason is that the next state of a system depends not only on its current state, but also on all of its historical states. Such equations, to a certain extent, may reflect the physical reality better than the integer-order differential equations. For example, the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumptions of continuum traffic flow [9]. It is of interest to note that the fractional calculus theory and its application have been studied in great detail in the literature ([10–14] and the references therein).

As is known to all, the Whitham–Broer–Kaup (WBK) equations were originally introduced to describe the propagation of shallow water waves [9], with different dispersion relations. The WBK equations have the following form:

$$\begin{aligned} &P_t + PP_x + Q_x + bP_{xx} = 0, \\ &Q_t + (PQ)_x + aP_{xxx} - bQ_{xx} = 0, \end{aligned} \tag{1}$$

where $P(x,t)$ denotes the horizontal velocity, $Q(x,t)$ is the height that deviates from the equilibrium position of liquid and a and b are constants that are represented in different diffusion powers.

So far, a great deal of effort has focused on the exact or approximate solutions for the WBK equations. Xie *et al.* [15] obtained some new solitary wave solutions by the hyperbolic function method. Sayed and Kaya [16] used the Adomian decomposition method (ADM) to get the approximate solutions. Rafei and Daniali [17] applied the variational iteration method (VIM) to construct the analytical solutions. Recently, Hap and Ishap [18] made use of the optimal homotopy asymptotic method (OHAM) to solve the WBK equations and acquired the numerical solutions.

Here, we consider the WBK equations in fractional case, which are called time fractional Whitham–Broer–Kaup equations:

$$\begin{aligned} &D_t^\alpha P + PP_x + Q_x + bP_{xx} = 0, \\ &D_t^\alpha Q + (PQ)_x + aP_{xxx} - bQ_{xx} = 0, \end{aligned} \tag{2}$$

where $0 < \alpha \leq 1$ and a, b are real constants. Note that $\alpha = 1$; System (2) becomes the standard WBK equations. It is also necessary to point out that when $a = 1$ and $b = 0$, we have fractional modified Boussinesq (MB) equations, and when $a = 0$ and $b = \frac{1}{2}$, approximate long wave (ALW) equations are obtained.

Nowadays, information theory is generalized in view of fractional calculus. By using fractional calculus, Machado introduced a novel formula for entropy [19]. Finding the exact or numerical solutions of a given fractional differential equation is still a challenging task in the field of fractional calculus. Therefore, an immediate and natural question arises if we can get the explicit approximate solutions of time fractional WBK equations. In recent years, many powerful techniques have been extended and developed to obtain numerical and analytical solutions of fractional differential equations, such as the tau spectral method [20], the spectral collocation method [21–24], the Jacobi–Gauss–Lobatto collocation method [25], the operational matrices and spectral techniques [26] and the mesh-less boundary collocation methods [27,28].

The residual power series method (RPSM) was initially developed to compute the numerical solutions of the first-order and the second-order fuzzy differential equations [29]. This method provides a power series solution with rapid convergence. It has been advantageously implemented for the nonlinear fractional Korteweg–de Vries–Burgers equation [30], for the fractional foam drainage equation [31], for the time-fractional two-component evolutionary system of order two [32] and for other equations [33]. It has been proven that the RPSM is a convenient and effective method in its application.

The main purpose of this paper is to use the RPSM to study various properties of time fractional WBK equations. After a few steps, high accuracy analytical traveling solutions for System (2) can be given in the form of the truncated power series.

The remainder of the paper is organized as follows. In the next section, we review some fundamental definitions and theorems of fractional calculus theory. In Section 3, the procedure of the RPSM is described, and then, the residual power series solution to System (2) is derived. Three examples are discussed, and the results are compared to various methods in Section 4. Finally, a short conclusion is presented in Section 5.

2. Preliminaries

In this section, some fundamental definitions and preliminary results of fractional calculus are presented [32,33]. There are different definitions of fractional integration and differentiation, such as Grunwald–Letnikov's definition, Riemann–Liouville's definition and Caputo's definition. In this context, the fractional derivative is in the Caputo sense, which is defined as:

$$D^{\alpha}P(x,t)=\frac{\partial^{\alpha}P(x,t)}{\partial t^{\alpha}}=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}\frac{\partial^m P(x,\tau)}{\partial\tau^m}d\tau, \ m-1<\alpha<m\\ \frac{\partial^m P(x,t)}{\partial t^m}, \ \alpha=m\in N,\end{cases} \tag{3}$$

where m is the smallest integer that exceeds α. Some properties of the Caputo fractional derivatives are stated here:

$$\begin{aligned}&D^{\alpha}C=0, \ C \text{ is a constant},\\ &D^{\alpha}(\gamma f(t)+\delta g(t))=\gamma D^{\alpha}f(t)+\delta D^{\alpha}g(t).\end{aligned} \tag{4}$$

Next, we will collect some important definitions and theorems of fractional power series. For a more detailed discussion, the reader is referred to [30,34].

Definition 1. A power series (PS) of the form:

$$\sum_{m=0}^{\infty}c_m(t-t_0)^{m\alpha}=c_0+c_1(t-t_0)^{\alpha}+c_2(t-t_0)^{2\alpha}+\cdots \ 0\leq n-1<\alpha\leq n, \ t\leq t_0$$

is called the fractional power series about $t=t_0$.

Theorem 1. Suppose that f has a fractional PS representation at $t=t_0$ of the form:

$$f(t)=\sum_{m=0}^{\infty}c_m(t-t_0)^{m\alpha}, \ t_0\leq t<t_0+R.$$

If $f(t)$ is continuous on $[t_0,t_0+R)$ and $D^{m\alpha}f(t), \ m=0,1,2,\cdots$ are continuous on (t_0,t_0+R), then $c_m=\frac{D^{m\alpha}f(t_0)}{\Gamma(1+m\alpha)}$. Here, $D^{m\alpha}=D^{\alpha}\cdot D^{\alpha}\cdot\cdots\cdot D^{\alpha}$ (m-times).

Remark 1. The number R in Theorem 2 is called the radius of convergence of fractional PS.

Definition 2. A power series of the form:

$$\sum_{m=0}^{\infty} f_m(x)(t-t_0)^{m\alpha}$$

is called the multiple fractional power series (PS) about $t = t_0$.

Theorem 2. Suppose that $P(x,t)$ has a multiple fractional PS representation at $t = t_0$ of the form:

$$P(x,t) = \sum_{m=0}^{\infty} f_m(x)(t-t_0)^{m\alpha}, x \in I,\ t_0 \leq t < t_0 + R.$$

If $D_t^{m\alpha}P(x,t),\ m = 0,1,2,\cdots$ are continuous on $I \times (t_0, t_0+R)$, then $f_m(x) = \frac{D_t^{m\alpha}P(x,t_0)}{\Gamma(1+m\alpha)}$.

3. Residual Power Series for Time Fractional WBK

We consider time fractional WBK equations:

$$\begin{aligned} &D_t^{\alpha}P + PP_x + Q_x + bP_{xx} = 0, \\ &D_t^{\alpha}Q + (PQ)_x + aP_{xxx} - bQ_{xx} = 0, \end{aligned} \tag{5}$$

subject to the initial conditions

$$\begin{aligned} P(x,0) &= f(x), \\ Q(x,0) &= g(x). \end{aligned} \tag{6}$$

We aim to construct a power series solution to the above system by its power series expansion among its truncated residual function.

The procedure of the RPSM for System (5) and (6) is summarized as follows.

Step 1. Suppose that the solutions to System (5) and (6) as a fractional PS about $t = 0$ can be written as:

$$\begin{aligned} P(x,t) &= \sum_{n=0}^{\infty} f_n(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \\ Q(x,t) &= \sum_{n=0}^{\infty} g_n(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \end{aligned} \tag{7}$$

Then, the k-th truncated series of $P(x,t)$, $Q(x,t)$ could be represented as:

$$\begin{aligned} P_k(x,t) &= \sum_{n=0}^{k} f_n(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \\ Q_k(x,t) &= \sum_{n=0}^{k} g_n(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)},\ 0 < \alpha \leq 1, x \in I,\ 0 \leq t < R. \end{aligned} \tag{8}$$

If we take $k = 0$, by the initial conditions (6), it is easy to check that the zeroth RPS truncated solutions of $P(x,t)$, $Q(x,t)$ are:

$$\begin{aligned} P_0(x,t) &= f_0(x) = P(x,0) = f(x), \\ Q_0(x,t) &= g_0(x) = Q(x,0) = g(x). \end{aligned} \tag{9}$$

Therefore, the *k*-th truncated series of $P(x,t)$, $Q(x,t)$ could be rewritten as:

$$\begin{aligned} &P_k(x,t) = f(x) + \sum_{n=1}^{k} f_n(x)\tfrac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \\ &Q_k(x,t) = g(x) + \sum_{n=1}^{k} g_n(x)\tfrac{t^{n\alpha}}{\Gamma(1+n\alpha)},\ 0<\alpha\le 1, x\in I,\ 0\le t<R. \end{aligned} \tag{10}$$

By the representations of $P_k(x,t)$ and $Q_k(x,t)$, the *k*-th RPS approximate solution will be obtained after $f_i(x)$ and $g_i(x),\ i=1,2,\cdots,k$ are available.

Step 2. The residual functions for System (5) and (6) are defined respectively:

$$\begin{aligned} &Res_P(x,t) = D_t^{\alpha}P + P\tfrac{\partial P}{\partial x} + \tfrac{\partial Q}{\partial x} + b\tfrac{\partial^2 P}{\partial x^2}, \\ &Res_Q(x,t) = D_t^{\alpha}Q + \tfrac{\partial (PQ)}{\partial x} + a\tfrac{\partial^3 P}{\partial x^3} - b\tfrac{\partial^2 Q}{\partial x^2}. \end{aligned} \tag{11}$$

Moreover, the *k*-th residual functions take the form:

$$\begin{aligned} &Res_{P,k}(x,t) = D_t^{\alpha}P_k + P_k\tfrac{\partial P_k}{\partial x} + \tfrac{\partial Q_k}{\partial x} + b\tfrac{\partial^2 P_k}{\partial x^2}, \\ &Res_{Q,k}(x,t) = D_t^{\alpha}Q_k + \tfrac{\partial (P_kQ_k)}{\partial x} + a\tfrac{\partial^3 P_k}{\partial x^3} - b\tfrac{\partial^2 Q_k}{\partial x^2}. \end{aligned} \tag{12}$$

We state some useful results of $Res_P(x,t)$ and $Res_Q(x,t)$ from [29,31–33], which are essential in the RPSM.

$$\begin{aligned} &Res_P(x,t) = 0,\ Res_Q(x,t) = 0, \\ &\lim_{k\to\infty} Res_{P,k}(x,t) = Res_P(x,t),\ \lim_{k\to\infty} Res_{Q,k}(x,t) = Res_Q(x,t),\ x\in I,\ t\ge 0, \\ &D_t^{r\alpha}Res_{P,k}(x,0) = 0,\ D_t^{r\alpha}Res_{Q,k}(x,0) = 0,\ r=0,1,2,\cdots,k. \end{aligned} \tag{13}$$

Step 3. By substituting (10) into (12) and calculating the fractional derivative $D_t^{(k-1)\alpha}Res_{P,k}(x,t)$ and $D_t^{(k-1)\alpha}Res_{Q,k}(x,t)$ together with (13), we get following algebraic system:

$$\begin{aligned} &D_t^{(k-1)\alpha}Res_{P,k}(x,0) = 0, \\ &D_t^{(k-1)\alpha}Res_{Q,k}(x,0) = 0,\ k=0,1,2,\cdots,. \end{aligned} \tag{14}$$

Step 4. After solving algebraic System (14), we have $f_i(x)$ and $g_i(x),\ i=1,2,\cdots,k$. Therefore, the *k*-th RPS approximate solution is derived.

Next, we will deduce the first approximate solution in detail. In fact, it is very convenient to perform computations by using the Maple 13 software package.

For $k=1$, the first RPS approximate solution could be written as:

$$\begin{aligned} &P_1(x,t) = f(x) + f_1(x)\tfrac{t^{\alpha}}{\Gamma(1+\alpha)}, \\ &Q_1(x,t) = g(x) + g_1(x)\tfrac{t^{\alpha}}{\Gamma(1+\alpha)}. \end{aligned} \tag{15}$$

Following the above steps, we get the first residual functions:

$$\begin{aligned} Res_{P,1}(x,t) &= D_t^{\alpha}P_1 + P_1\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial x} + b\frac{\partial^2 P_1}{\partial x^2} \\ &= f_1 + (f + f_1\frac{t^{\alpha}}{\Gamma(1+\alpha)})(f' + f_1'\frac{t^{\alpha}}{\Gamma(1+\alpha)}) + g' + g_1'\frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ &\quad + b(f'' + f_1''\frac{t^{\alpha}}{\Gamma(1+\alpha)}) \end{aligned}$$

and:

$$\begin{aligned} Res_{Q,1}(x,t) &= D_t^{\alpha} Q_1 + \frac{\partial (P_1 Q_1)}{\partial x} + a\frac{\partial^3 P_1}{\partial x^3} - b\frac{\partial^2 Q_1}{\partial x^2} \\ &= g_1 + (f' + f_1' \frac{t^{\alpha}}{\Gamma(1+\alpha)})(g + g_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}) + (g' + g_1' \frac{t^{\alpha}}{\Gamma(1+\alpha)})(f + f_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}) \\ & \quad + a(f''' + f_1''' \frac{t^{\alpha}}{\Gamma(1+\alpha)}) - b(g'' + g_1'' \frac{t^{\alpha}}{\Gamma(1+\alpha)}). \end{aligned}$$

According to $Res_{P,1}(x,0) = Res_{Q,1}(x,0) = 0$, we have following algebraic system:

$$\begin{aligned} & f_1(x) + f(x)f'(x) + g'(x) + bf''(x) = 0, \\ & g_1(x) + f'(x)g(x) + g'(x)f(x) + af'''(x) - bg''(x) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} & f_1(x) = -f(x)f'(x) - g'(x) - bf''(x), \\ & g_1(x) = -f'(x)g(x) - g'(x)f(x) - af'''(x) + bg''(x). \end{aligned} \tag{16}$$

The higher degree of approximate solutions can be derived in a similar way. When $k = 2$, we express the second RPS approximate solution as:

$$\begin{aligned} & P_2(x,t) = f(x) + f_1(x)\tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + f_2(x)\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ & Q_2(x,t) = g(x) + g_1(x)\tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + g_2(x)\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \end{aligned} \tag{17}$$

According to the procedure of the RPSM, we obtain:

$$\begin{aligned} & f_2(x) = -f(x)f_1'(x) - f'(x)f_1(x) - g_1'(x) - bf_1''(x), \\ & g_2(x) = -f'(x)g_1(x) - g(x)f_1'(x) - f_1(x)g'(x) - g_1'(x)f(x) - af_1'''(x) + bg_1''(x). \end{aligned} \tag{18}$$

By the Maple 13 software package, we deduce the following result:

$$\begin{aligned} f_3(x) &= -f(x)f_2'(x) - 2f_1(x)f_1'(x) - f'(x)f_1(x) - g_2'(x) - bf_2''(x), \qquad (19) \\ g_3(x) &= -f'(x)g_2(x) - 2g_1(x)f_1'(x) - g(x)f_2'(x) - f(x)g_2'(x) - g'(x)f_2(x) \\ & \quad -2g_1'(x)f_1(x) - af_2'''(x) + bg_2''(x), \end{aligned}$$

and:

$$\begin{aligned} f_4(x) &= -f(x)f_3'(x) - f_1(x)f_2'(x) - 2f_2(x)f_1'(x) - 2f_2'(x)f_1(x) - f'(x)f_3(x) - f_2(x)f_1'(x) \\ & \quad -g_3'(x) - bf_3''(x), \qquad (20) \\ g_4(x) &= -f'(x)g_3(x) - 2g_2(x)f_1'(x) - 2g_1(x)f_2'(x) - 2g_1'(x)f_2(x) - g(x)f_3'(x) - g_2(x)f_1'(x) \\ & \quad -g_1(x)f_2'(x) - g'(x)f_3(x) - g_1'(x)f_2(x) - g_3'(x)f(x) - f_1(x)g_2'(x) - af_3'''(x) + bg_3''(x). \end{aligned}$$

After the above discussion, we can represent the *k*-th RPS ($k = 1,2,3,4$) approximate solution of Systems (5) and (6). If we repeat the process of RPSM, we will get a higher degree of approximate solution.

4. Applications

The purpose of this section is to present some examples to show the efficiency and accuracy of the method proposed in Section 3.

Application 1. Consider the following time fractional WBK equations:

$$\begin{aligned} &D_t^\alpha P + PP_x + Q_x + bP_{xx} = 0, \\ &D_t^\alpha Q + (PQ)_x + aP_{xxx} - bQ_{xx} = 0, \end{aligned} \tag{21}$$

subject to the initial conditions:

$$\begin{aligned} &P(x,0) = \lambda - 2Bk\coth(k\xi), \\ &Q(x,0) = -2B(B+b)k^2 \csc h^2(k\xi), \end{aligned} \tag{22}$$

where $B = \sqrt{a+b^2}$, $\xi = x + c$ and c,k, λ are arbitrary constants.

For $\alpha = 1$, the exact solutions of Systems (21) and (22) are:

$$\begin{aligned} &P(x,t) = \lambda - 2Bk\coth(k(\xi - \lambda t)), \\ &Q(x,t) = -2B(B+b)k^2 \csc h^2(k(\xi - \lambda t)). \end{aligned} \tag{23}$$

According to the process of the RPSM described in Section 3, we get the first RPS approximate solutions:

$$\begin{aligned} &P_1(x,t) = f(x) + f_1(x)\tfrac{t^\alpha}{\Gamma(1+\alpha)}, \\ &Q_1(x,t) = g(x) + g_1(x)\tfrac{t^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \tag{24}$$

and the second, third and fourth RPS approximate solutions are obtained respectively:

$$\begin{aligned} &P_2(x,t) = f(x) + f_1(x)\tfrac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ &Q_2(x,t) = g(x) + g_1(x)\tfrac{t^\alpha}{\Gamma(1+\alpha)} + g_2(x)\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ &P_3(x,t) = f(x) + f_1(x)\tfrac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3(x)\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ &Q_3(x,t) = g(x) + g_1(x)\tfrac{t^\alpha}{\Gamma(1+\alpha)} + g_2(x)\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + g_3(x)\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ &P_4(x,t) = f(x) + f_1(x)\tfrac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3(x)\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)} + f_4(x)\tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)}, \\ &Q_4(x,t) = g(x) + g_1(x)\tfrac{t^\alpha}{\Gamma(1+\alpha)} + g_2(x)\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + g_3(x)\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)} + f_4(x)\tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)}, \end{aligned} \tag{25}$$

where f_i and g_i, $(i = 1, 2, \cdots, 4)$ satisfied (16,18–20).

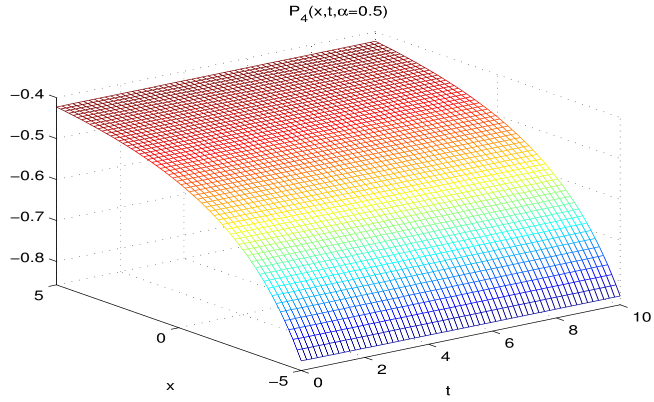
To obtain the first RPS solutions for Application 1, we substitute (16) into (24), *i.e.*,

$$\begin{aligned} &P_1(x,t) = f(x) + (-f(x)f'(x) - g'(x) - bf''(x))\tfrac{t^\alpha}{\Gamma(1+\alpha)}, \\ &Q_1(x,t) = g(x) + (-f'(x)g(x) - g'(x)f(x) - af'''(x) + bg''(x))\tfrac{t^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{26}$$

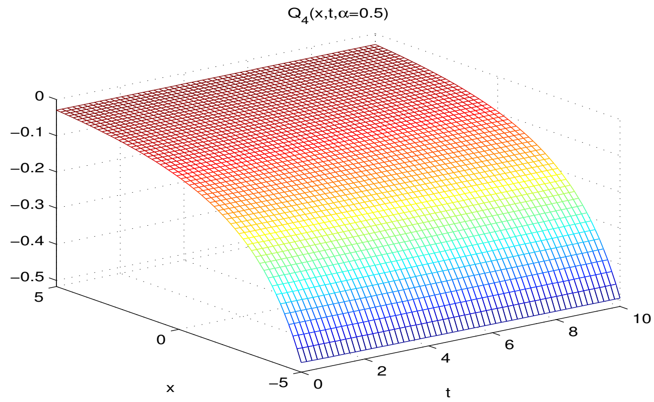
The second, third and fourth RPS approximate solutions can be derived by the same manner as above.

We take $k = 0.1$, $\lambda = 0.005$, $a = 1.5$, $b = 1.5$ and $c = 10$. Figure 1 explores the fourth RPS approximate solutions of $P(x,t)$ and $Q(x,t)$ for $\alpha = 0.5$.

The comparison results of the absolute errors by the RPSM and other methods [16–18] for Application 1 are shown in Tables 1 and 2.



(**a**)



(**b**)

Figure 1. The figures of the numerical solutions for Application 1. RPS, residual power series. (**a**) The fourth RPS solution of $P(x,t)$ for $\alpha = 0.5$; (**b**) The fourth RPS solution of $Q(x,t)$ for $\alpha = 0.5$.

Table 1. The absolute errors of $P(x,t)$ obtained by the various methods for Application 1.

(x,t)	$\lvert P_{Exact} - P_{ADM}\rvert$	$\lvert P_{Exact} - P_{VIM}\rvert$	$\lvert P_{Exact} - P_{OHAM}\rvert$	$\lvert P_{Exact} - P_2\rvert$	$\lvert P_{Exact} - P_4\rvert$
(0.1,0.1)	1.04892×10^{-4}	1.23033×10^{-4}	1.07078×10^{-4}	4.69624×10^{-14}	1.11022×10^{-16}
(0.1,0.3)	9.64474×10^{-5}	3.69597×10^{-4}	3.04565×10^{-4}	1.26521×10^{-12}	1.11022×10^{-16}
(0.1,0.5)	8.88312×10^{-5}	6.16873×10^{-4}	4.81303×10^{-4}	5.85787×10^{-12}	1.11022×10^{-16}
(0.2,0.1)	4.25408×10^{-4}	1.19869×10^{-4}	1.04388×10^{-4}	4.49640×10^{-14}	0
(0.2,0.3)	3.91098×10^{-4}	3.60098×10^{-4}	2.97260×10^{-4}	1.21614×10^{-12}	1.11022×10^{-16}
(0.2,0.5)	3.60161×10^{-4}	6.01006×10^{-4}	4.70138×10^{-4}	5.63161×10^{-12}	1.11022×10^{-16}
(0.3,0.1)	9.71922×10^{-4}	1.16789×10^{-4}	1.01776×10^{-4}	4.34097×10^{-14}	1.11022×10^{-16}
(0.3,0.3)	8.93309×10^{-4}	3.50866×10^{-4}	2.90150×10^{-4}	1.16984×10^{-12}	0
(0.3,0.5)	8.22452×10^{-4}	5.85610×10^{-4}	4.59590×10^{-4}	5.41645×10^{-12}	1.11022×10^{-16}
(0.4,0.1)	1.75596×10^{-3}	1.13829×10^{-4}	9.92418×10^{-5}	4.18554×10^{-14}	1.66533×10^{-16}
(0.4,0.3)	1.61430×10^{-3}	3.41948×10^{-4}	2.83229×10^{-4}	1.12560×10^{-12}	5.55111×10^{-17}
(0.4,0.5)	1.48578×10^{-3}	5.70710×10^{-4}	4.49118×10^{-4}	5.21133×10^{-12}	5.55111×10^{-17}
(0.5,0.1)	2.79519×10^{-3}	1.10936×10^{-4}	9.67808×10^{-4}	4.00791×10^{-14}	5.55111×10^{-17}
(0.5,0.3)	2.56714×10^{-3}	3.33274×10^{-4}	2.76492×10^{-4}	1.08330×10^{-12}	0
(0.5,0.5)	2.36184×10^{-3}	5.56235×10^{-4}	4.38895×10^{-4}	5.01577×10^{-12}	1.11022×10^{-16}

Table 2. The absolute errors of $Q(x, t)$ obtained by the various methods for Application 1.

(x, t)	$\vert Q_{Exact} - Q_{ADM}\vert$	$\vert Q_{Exact} - Q_{VIM}\vert$	$\vert Q_{Exact} - Q_{OHAM}\vert$	$\vert Q_{Exact} - Q_2\vert$	$\vert Q_{Exact} - Q_4\vert$
(0.1,0.1)	6.41419×10^{-3}	1.10430×10^{-4}	5.86860×10^{-5}	6.37142×10^{-14}	4.29274×10^{-17}
(0.1,0.3)	5.99783×10^{-3}	3.31865×10^{-4}	3.04565×10^{-4}	1.71937×10^{-12}	2.55237×10^{-17}
(0.1,0.5)	5.61507×10^{-3}	5.54071×10^{-4}	3.08812×10^{-4}	7.96097×10^{-12}	3.23499×10^{-17}
(0.2,0.1)	1.33181×10^{-2}	1.07016×10^{-4}	5.56884×10^{-5}	6.06144×10^{-14}	3.78170×10^{-18}
(0.2,0.3)	1.24441×10^{-2}	3.21601×10^{-4}	2.97260×10^{-4}	1.63713×10^{-12}	7.59809×10^{-18}
(0.2,0.5)	1.16416×10^{-2}	5.36927×10^{-4}	2.92626×10^{-4}	7.58015×10^{-12}	1.12757×10^{-17}
(0.3,0.1)	2.07641×10^{-2}	1.03737×10^{-4}	5.28609×10^{-5}	5.77301×10^{-14}	1.53762×10^{-17}
(0.3,0.3)	1.93852×10^{-2}	3.11737×10^{-4}	2.90150×10^{-4}	1.55956×10^{-12}	1.34853×10^{-17}
(0.3,0.5)	1.81209×10^{-2}	5.20447×10^{-4}	2.77382×10^{-4}	7.22117×10^{-12}	3.19298×10^{-17}
(0.4,0.1)	2.88100×10^{-2}	1.00579×10^{-4}	5.01929×10^{-5}	5.50600×10^{-14}	2.06725×10^{-17}
(0.4,0.3)	2.68724×10^{-2}	3.02245×10^{-4}	2.83229×10^{-4}	1.48644×10^{-12}	1.95189×10^{-17}
(0.4,0.5)	2.50985×10^{-2}	5.04593×10^{-4}	2.63019×10^{-4}	6.88241×10^{-12}	1.72117×10^{-17}
(0.5,0.1)	3.75193×10^{-2}	9.75385×10^{-5}	4.76741×10^{-5}	5.25135×10^{-14}	2.76894×10^{-17}
(0.5,0.3)	3.49617×10^{-2}	2.93107×10^{-4}	2.76492×10^{-4}	1.41738×10^{-12}	1.32826×10^{-17}
(0.5,0.5)	3.26239×10^{-2}	4.89335×10^{-4}	2.49480×10^{-4}	6.56265×10^{-12}	1.22244×10^{-17}

Application 2. We consider the special case of the time fractional WBK equations, namely the time fractional MB equations:

$$\begin{aligned} &D_t^\alpha P + PP_x + Q_x = 0, \\ &D_t^\alpha Q + (PQ)_x + P_{xxx} = 0, \end{aligned} \tag{27}$$

with the initial conditions:

$$\begin{aligned} P(x, 0) &= \lambda - 2k\coth(k\xi), \\ Q(x, 0) &= -2k^2 \csc h^2(k\xi). \end{aligned} \tag{28}$$

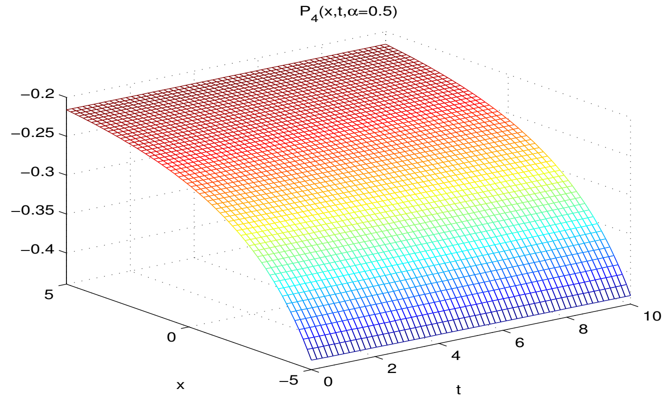
For the special case where $\alpha = 1$, we have the exact solutions of Systems (27) and (28):

$$\begin{aligned} P(x, t) &= \lambda - 2k\coth(k(\xi - \lambda t)), \\ Q(x, t) &= -2k^2 \csc h^2(k(\xi - \lambda t)). \end{aligned} \tag{29}$$

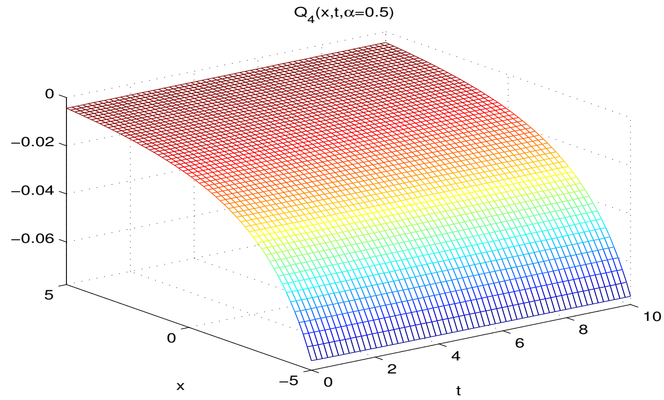
The parameters are taken to be the same as in Application 1. Taking $a = 1$ and $b = 0$ from (26), we can write the first RPS approximate solutions for the time fractional MB equations as:

$$\begin{aligned} P_1(x, t) &= f(x) + (-f(x)f'(x) - g'(x))\tfrac{t^\alpha}{\Gamma(1+\alpha)}, \\ Q_1(x, t) &= g(x) + (-f'(x)g(x) - g'(x)f(x) - f'''(x))\tfrac{t^\alpha}{\Gamma(1+\alpha)}. \end{aligned}$$

The other higher degree of approximate solutions can be obtained similarly. The fourth RPS approximate solutions of $P(x, t)$ and $Q(x, t)$ for $\alpha = 0.5$ are in Figure 2. The results of absolute errors by various methods [16–18] for Application 2 are given in Tables 3 and 4.



(**a**)



(**b**)

Figure 2. The figures of the numerical solutions for Application 2. (**a**) The fourth RPS solution of $P(x,t)$ for $\alpha = 0.5$; (**b**) The fourth RPS solution of $Q(x,t)$ for $\alpha = 0.5$.

Table 3. The absolute errors of $P(x,t)$ obtained by the various methods for Application 2.

(x,t)	$\lvert P_{Exact} - P_{ADM}\rvert$	$\lvert P_{Exact} - P_{VIM}\rvert$	$\lvert P_{Exact} - P_{OHAM}\rvert$	$\lvert P_{Exact} - P_2\rvert$	$\lvert P_{Exact} - P_4\rvert$
(0.1,0.1)	8.16297×10^{-7}	6.35269×10^{-5}	6.35267×10^{-5}	2.42082×10^{-14}	1.34780×10^{-17}
(0.1,0.3)	7.64245×10^{-7}	1.90854×10^{-4}	1.90854×10^{-4}	6.53345×10^{-13}	2.17823×10^{-17}
(0.1,0.5)	7.16083×10^{-7}	3.18549×10^{-4}	3.18548×10^{-4}	3.02494×10^{-12}	2.24972×10^{-18}
(0.2,0.1)	3.26243×10^{-6}	6.18930×10^{-5}	6.18931×10^{-5}	2.32705×10^{-14}	9.38089×10^{-18}
(0.2,0.3)	3.05458×10^{-6}	1.85945×10^{-4}	1.85945×10^{-4}	6.28096×10^{-13}	1.43996×10^{-17}
(0.2,0.5)	2.86226×10^{-6}	3.10352×10^{-4}	3.10352×10^{-4}	2.90822×10^{-12}	1.97731×10^{-17}
(0.3,0.1)	7.33445×10^{-6}	6.03095×10^{-5}	6.03098×10^{-5}	2.24143×10^{-14}	4.24677×10^{-17}
(0.3,0.3)	6.86758×10^{-6}	1.81187×10^{-4}	1.81187×10^{-4}	6.04127×10^{-13}	2.87822×10^{-17}
(0.3,0.5)	6.43557×10^{-6}	3.02408×10^{-4}	3.02408×10^{-4}	2.79704×10^{-12}	2.11487×10^{-17}
(0.4,0.1)	1.30286×10^{-5}	5.87746×10^{-5}	5.87749×10^{-5}	2.16117×10^{-14}	8.72342×10^{-17}
(0.4,0.3)	1.22000×10^{-5}	1.76574×10^{-4}	1.76574×10^{-4}	5.81296×10^{-13}	7.90451×10^{-17}
(0.4,0.5)	1.14333×10^{-5}	2.94707×10^{-4}	2.94708×10^{-4}	2.69115×10^{-12}	6.79998×10^{-17}
(0.5,0.1)	2.03415×10^{-5}	5.72867×10^{-5}	5.72865×10^{-4}	2.07126×10^{-14}	4.07931×10^{-18}
(0.5,0.3)	1.90489×10^{-5}	1.72102×10^{-4}	1.72102×10^{-4}	5.59417×10^{-13}	1.19703×10^{-17}
(0.5,0.5)	1.78528×10^{-5}	2.87241×10^{-4}	2.87240×10^{-4}	2.59014×10^{-12}	6.07085×10^{-17}

Table 4. The absolute errors of $Q(x,t)$ obtained by the various methods for Application 2.

(x,t)	$\lvert Q_{Exact}-Q_{ADM}\rvert$	$\lvert Q_{Exact}-Q_{VIM}\rvert$	$\lvert Q_{Exact}-Q_{OHAM}\rvert$	$\lvert Q_{Exact}-Q_2\rvert$	$\lvert Q_{Exact}-Q_4\rvert$
(0.1,0.1)	5.88676×10^{-5}	1.65942×10^{-5}	1.65942×10^{-5}	9.57047×10^{-15}	3.17215×10^{-18}
(0.1,0.3)	5.56914×10^{-5}	4.98691×10^{-5}	4.98691×10^{-5}	2.58365×10^{-13}	1.08813×10^{-18}
(0.1,0.5)	5.27169×10^{-5}	8.32598×10^{-5}	8.26491×10^{-4}	1.19628×10^{-12}	2.73648×10^{-18}
(0.2,0.1)	1.18213×10^{-4}	1.60813×10^{-5}	1.60812×10^{-5}	9.10730×10^{-15}	2.42893×10^{-18}
(0.2,0.3)	1.11833×10^{-4}	4.83269×10^{-5}	4.83269×10^{-5}	2.46008×10^{-13}	1.39638×10^{-18}
(0.2,0.5)	1.05858×10^{-4}	8.06837×10^{-5}	7.94290×10^{-4}	1.13906×10^{-12}	3.50926×10^{-18}
(0.3,0.1)	1.78041×10^{-4}	1.55880×10^{-5}	1.55880×10^{-5}	8.67709×10^{-15}	1.18742×10^{-18}
(0.3,0.3)	1.68429×10^{-4}	4.68440×10^{-5}	4.68439×10^{-5}	2.34354×10^{-13}	8.35230×10^{-19}
(0.3,0.5)	1.59428×10^{-4}	7.82068×10^{-5}	7.63646×10^{-4}	1.08512×10^{-12}	4.22825×10^{-18}
(0.4,0.1)	2.38356×10^{-4}	1.51135×10^{-5}	1.51135×10^{-5}	8.27463×10^{-15}	3.41750×10^{-18}
(0.4,0.3)	2.25483×10^{-4}	4.54174×10^{-5}	4.54174×10^{-5}	2.23366×10^{-13}	3.80156×10^{-18}
(0.4,0.5)	2.13430×10^{-4}	7.58243×10^{-5}	7.34471×10^{-4}	1.03421×10^{-12}	2.18005×10^{-18}
(0.5,0.1)	2.99162×10^{-4}	1.46569×10^{-5}	1.46569×10^{-5}	7.89473×10^{-15}	7.77731×10^{-18}
(0.5,0.3)	2.83001×10^{-4}	4.40448×10^{-5}	4.40448×10^{-5}	2.12988×10^{-13}	1.82067×10^{-18}
(0.5,0.5)	2.67868×10^{-4}	7.35317×10^{-5}	7.06678×10^{-4}	9.86164×10^{-13}	4.18858×10^{-18}

Application 3. Finally, we consider the time fractional ALW equations:

$$\begin{aligned} &D_t^{\alpha}P + PP_x + \tfrac{1}{2}P_{xx} + Q_x = 0,\\ &D_t^{\alpha}Q + (PQ)_x + \tfrac{1}{2}Q_{xx} = 0, \end{aligned} \tag{30}$$

and the initial conditions:

$$\begin{aligned} P(x,0) &= \lambda - k\coth(k\xi),\\ Q(x,0) &= -k^2\csc h^2(k\xi). \end{aligned} \tag{31}$$

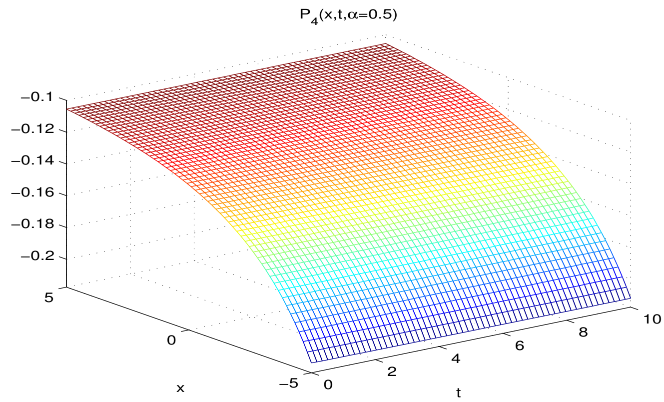
If $\alpha = 1$, the exact solutions of Systems (30) and (31) are:

$$\begin{aligned} P(x,t) &= \lambda - k\coth(k(\xi-\lambda t)),\\ Q(x,t) &= -k^2\csc h^2(k(\xi-\lambda t)). \end{aligned} \tag{32}$$

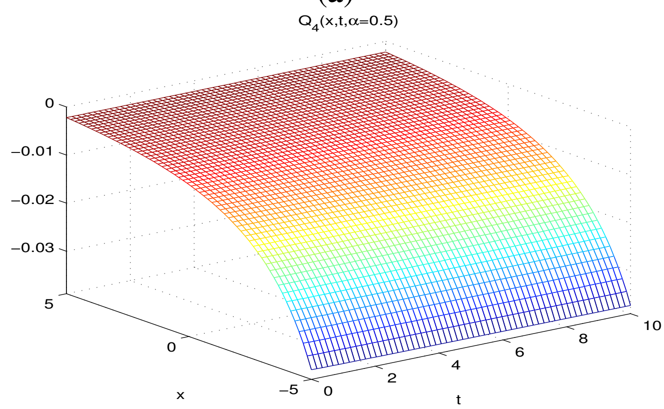
We set the parameters the same as in Application 1. By (26) together with the constants $a = 0$, $b = \frac{1}{2}$, it is easy to get the first RPS approximate solutions for the time fractional ALW equations as:

$$\begin{aligned} P_1(x,t) &= f(x) + (-f(x)f'(x) - g'(x) - \tfrac{1}{2}f''(x))\tfrac{t^{\alpha}}{\Gamma(1+\alpha)},\\ Q_1(x,t) &= g(x) + (-f'(x)g(x) - g'(x)f(x) + \tfrac{1}{2}g''(x))\tfrac{t^{\alpha}}{\Gamma(1+\alpha)}. \end{aligned}$$

Similarly, we have the higher degree of approximate solutions. Like the above two applications, the graphical results and numerical descriptions are presented in Figure 3 and Tables 5 and 6.



(a)



(b)

Figure 3. The figures of the numerical solutions for Application 3. (**a**) The fourth RPS solution of $P(x,t)$ for $\alpha = 0.5$; (**b**) The fourth RPS solution of $Q(x,t)$ for $\alpha = 0.5$.

Table 5. The absolute errors of $P(x,t)$ obtained by the various methods for Application 3.

(x,t)	$\lvert P_{Exact} - P_{ADM} \rvert$	$\lvert P_{Exact} - P_{VIM} \rvert$	$\lvert P_{Exact} - P_{OHAM} \rvert$	$\lvert P_{Exact} - P_2 \rvert$	$\lvert P_{Exact} - P_4 \rvert$
(0.1,0.1)	8.02989×10^{-6}	3.17634×10^{-5}	3.17634×10^{-5}	1.21041×10^{-14}	6.73913×10^{-18}
(0.1,0.3)	7.38281×10^{-6}	9.54273×10^{-5}	9.54269×10^{-5}	3.26673×10^{-13}	1.08892×10^{-17}
(0.1,0.5)	6.79923×10^{-6}	1.59274×10^{-4}	1.59274×10^{-4}	1.51247×10^{-12}	1.12040×10^{-18}
(0.2,0.1)	3.23228×10^{-5}	3.09466×10^{-5}	3.09465×10^{-5}	1.16352×10^{-14}	4.72239×10^{-18}
(0.2,0.3)	2.97172×10^{-5}	9.29725×10^{-5}	9.29723×10^{-5}	3.14048×10^{-13}	7.10655×10^{-18}
(0.2,0.5)	2.73673×10^{-5}	1.55176×10^{-4}	1.55176×10^{-4}	1.45411×10^{-12}	1.00389×10^{-17}
(0.3,0.1)	7.32051×10^{-5}	3.01549×10^{-5}	3.01549×10^{-5}	1.12072×10^{-14}	2.12448×10^{-17}
(0.3,0.3)	6.73006×10^{-5}	9.05935×10^{-5}	9.05932×10^{-5}	3.02063×10^{-13}	1.44235×10^{-17}
(0.3,0.5)	6.19760×10^{-5}	1.51204×10^{-4}	1.51204×10^{-4}	1.39852×10^{-12}	1.06299×10^{-17}
(0.4,0.1)	1.31032×10^{-4}	2.93874×10^{-5}	2.93874×10^{-5}	1.08059×10^{-14}	4.36379×10^{-17}
(0.4,0.3)	1.20455×10^{-4}	8.82871×10^{-5}	8.82870×10^{-5}	2.90648×10^{-13}	3.95777×10^{-17}
(0.4,0.5)	1.10919×10^{-4}	1.47354×10^{-4}	1.47354×10^{-4}	1.34557×10^{-12}	3.40839×10^{-17}
(0.5,0.1)	2.06186×10^{-4}	2.86433×10^{-5}	2.86432×10^{-5}	1.03563×10^{-14}	2.05053×10^{-18}
(0.5,0.3)	1.89528×10^{-4}	8.60509×10^{-5}	8.60506×10^{-5}	2.79708×10^{-13}	5.94932×10^{-17}
(0.5,0.5)	1.74510×10^{-4}	1.43620×10^{-4}	1.43620×10^{-4}	1.29507×10^{-12}	3.02966×10^{-17}

Table 6. The absolute errors of $Q(x,t)$ obtained by the various methods for Application 3.

(x,t)	$\|Q_{Exact}-Q_{ADM}\|$	$\|Q_{Exact}-Q_{VIM}\|$	$\|Q_{Exact}-Q_{OHAM}\|$	$\|Q_{Exact}-Q_2\|$	$\|Q_{Exact}-Q_4\|$
(0.1,0.1)	4.81902×10^{-4}	8.29712×10^{-6}	8.29711×10^{-6}	4.78523×10^{-15}	1.58608×10^{-18}
(0.1,0.3)	4.50818×10^{-4}	2.49346×10^{-5}	2.49345×10^{-5}	1.29182×10^{-13}	5.43944×10^{-19}
(0.1,0.5)	4.22221×10^{-4}	4.16299×10^{-5}	4.16298×10^{-5}	5.98141×10^{-13}	1.36881×10^{-18}
(0.2,0.1)	9.76644×10^{-4}	8.04063×10^{-6}	8.04063×10^{-6}	4.55365×10^{-15}	1.21449×10^{-18}
(0.2,0.3)	9.13502×10^{-4}	2.41634×10^{-5}	2.41634×10^{-5}	1.23004×10^{-13}	1.68513×10^{-19}
(0.2,0.5)	8.55426×10^{-4}	4.03419×10^{-5}	4.03418×10^{-5}	5.69530×10^{-13}	1.75768×10^{-18}
(0.3,0.1)	1.48482×10^{-3}	7.79401×10^{-6}	7.79400×10^{-6}	4.33854×10^{-15}	5.93717×10^{-19}
(0.3,0.3)	1.38858×10^{-3}	2.34220×10^{-5}	2.34219×10^{-5}	1.17177×10^{-13}	4.17842×10^{-19}
(0.3,0.5)	1.30009×10^{-3}	3.91034×10^{-5}	3.91034×10^{-5}	5.42558×10^{-13}	2.11323×10^{-18}
(0.4,0.1)	2.00705×10^{-3}	7.55675×10^{-6}	7.55675×10^{-6}	4.13732×10^{-15}	1.70872×10^{-18}
(0.4,0.3)	1.87661×10^{-3}	2.27087×10^{-5}	2.27087×10^{-5}	1.11683×10^{-13}	1.89994×10^{-18}
(0.4,0.5)	1.75670×10^{-3}	3.79121×10^{-5}	3.79121×10^{-5}	5.17105×10^{-13}	1.08612×10^{-18}
(0.5,0.1)	2.54396×10^{-3}	7.32847×10^{-6}	7.32846×10^{-6}	3.94736×10^{-15}	3.88866×10^{-18}
(0.5,0.3)	2.37815×10^{-3}	2.20224×10^{-5}	2.20224×10^{-5}	1.06494×10^{-13}	9.10368×10^{-18}
(0.5,0.5)	2.22578×10^{-3}	3.67658×10^{-5}	3.67658×10^{-5}	4.93082×10^{-13}	2.09445×10^{-18}

Due to the high accuracy of the present method, there are nearly no differences between the graphs of numerical solutions and exact solutions.

From Tables 1–6, there is no doubt that the absolute errors obtained through the RPSM are better than ADM, VIM and OHAM. It is also clear that we can get very good approximation solutions from a few iterations. That is the pertinent feature of the proposed method for solving time fractional WBK equations.

5. Conclusions

In this paper, we have demonstrated the feasibility of the RPSM for solving time fractional WBK equations. The steps of this method are summarized, and the relevant applications are developed. All of the given examples reveal that the RPSM can be used as an alternative to obtain analytical solutions of time fractional nonlinear differential equations. The numerical results also show that the RPSM yields a very effective and accurate approach to the approximate solution of time fractional WBK equations.

Acknowledgments

The research leading to these results has received funding from the Natural Science Foundation of Jiangsu Province (Grant No. BK20140522) and the Startup Fund for Advanced Talents of Jiangsu University (Grant No. 10JDG124).

Author Contributions

Both authors designed and performed the methods. Linjun Wang analyzed the results. Linjun Wang wrote the paper, and Xumei Chen edited it. Both authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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