Study on the Stability and Entropy Complexity of an Energy-Saving and Emission-Reduction Model with Two Delays

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Abstract: In this paper, we build a model of energy-savings and emission-reductions with two delays. In this model, it is assumed that the interaction between energy-savings and emission-reduction and that between carbon emissions and economic growth are delayed. We examine the local stability and the existence of a Hopf bifurcation at the equilibrium point of the system. By employing System Complexity Theory, we also analyze the impact of delays and the feedback control on stability and entropy of the system are analyzed from two aspects: single delay and double delays. In numerical simulation section, we test the theoretical analysis by using means bifurcation diagram, the largest Lyapunov exponent diagrams, attractor, time-domain plot, Poincare section plot, power spectrum, entropy diagram, 3-D surface chart and 4-D graph, the simulation results demonstrating that the inappropriate changes of delays and the feedback control will result in instability and fluctuation of carbon emissions. Finally, the bifurcation control is achieved by using the method of variable feedback control. Hence, we conclude that the greater the value of the control parameter, the better the effect of the bifurcation control. The results will provide for the development of energy-saving and emission-reduction policies.

Keywords: energy-savings; emission-reduction; entropy; stability; two delays; Hopf bifurcation

1. Introduction

Energy resources are the backbone of any national economy. Scholars worldwide have been studying energy prices, the low-carbon economy and energy-savings and emission-reduction. Lv and Zhou [1] studied the dynamic behavior of the energy price model with time delay. They explained the reasons why the energy price model was developed and maintained periodically by the bifurcation theory. Tian et al. [2] developed a nonlinear model for the development of oil, gas and other energy resources against the backdrop of the energy structure in Jiangsu, China, one that predominantly relies on the coal consumption, and evidenced empirically the feasibility of the corresponding energy countermeasures.

Despite the fact that energy promotes economic development and improves people's living standards, excessive energy consumption undoubtedly damages the environment, so it is necessary to adopt a low-carbon economy and reduce the possible environmental damage. In fact, the energy-saving and emission-reduction system is a complex nonlinear system, which involves the interactions between a variety of factors such as energy-saving and emission-reduction, carbon emissions, economic growth, energy efficiency, carbon tax and energy intensity, and so forth [3–5]. In view of this, a novel three-dimensional energy-saving and emission-reduction chaotic system is proposed [6].

The system is established in accordance with the complicated relationship among energy-savings and emission-reduction, carbon emissions and economic growth. This system displays a very complex phenomenon by including a special chaotic attractor named the energy-saving and emission-reduction attractor (the EE attractor for short), which is different from the previous chaotic attractor, such as Lorenz attractor [7], Chen attractor [8], Lü attractor [9], Energy resource attractor [10–12] and so on. Moreover, drawing up rational and effective policies and laws will ensure a better progress in energy-saving and emission-reduction [13,14]. Wang and Xu [15] reported a new four-dimensional energy-saving and emission-reduction chaotic system. The system is obtained in accordance with the complicated relationship among energy-saving and emission-reduction, carbon emission, economic growth and new energy development. The model is described as follows:

$$
\begin{cases}
\dot{x}(t) = a_1 x(t)(\frac{y(t)}{M} - 1) - a_2 y(t) + a_3 z(t), \\
\dot{y}(t) = -b_1 x(t) + b_2 y(t)(1 - \frac{y(t)}{C}) + b_3 z(t)(1 - \frac{z(t)}{E}) - b_4 u(t), \\
\dot{z}(t) = c_1 x(t)(\frac{x(t)}{N} - 1) - c_2 y(t) - c_3 z(t) + c_4 u(t)(\frac{u(t)}{L} - 1), \\
\dot{u}(t) = d_1 y(t) + d_2 z(t)(\frac{z(t)}{K} - 1) - d_3 u(t),
\end{cases}
\tag{1}
$$

where $x(t)$ is the time-dependent variable of energy-saving and emission-reduction; $y(t)$ the time-dependent variable of carbon emissions; $z(t)$ the time-dependent variable of economic growth (GDP) and $u(t)$ the time-dependent variable of new energy development. $a_i, d_i, b_j, c_j, (i = 1, 2, 3; j = 1, 2, 3, 4)$ are coefficients and M, N, L, K, C, E positive constants [15]. Model (1) stands for the development and utilization of new energy sources. Therefore, the four-dimensional model of energy-saving and emission-reduction is a step closer to the actual conditions.

In view of the current mode of economic development and technological conditions, economic development will increase the consumption of coal and oil, and in turn this will cause an increase in carbon emissions. However, there are many other approaches to promote economic growth, such as those in the modern service sector that are low energy-consuming but high value-added, so economic development may not immediately cause a substantial energy consumption load. In other words, it might not lead to an obvious increase in carbon emissions in the short term. In this essay, we have included the delay between economic development and the significant growth of carbon emission in this kind of economic model.

Moreover, energy-savings and emission-reduction reduce carbon emissions by saving or reducing energy consumption. However, due to the complexity of the economic system, the implementation of energy-saving and emission-reduction measures normally will not lead to a significant reduction, showing a certain lag. Meanwhile, energy-saving and emission-reduction will slow down the pace of economic development. This will force enterprises to change their mode of economic development, strengthen energy conservation and efficient use, and actively comply with a recycling economy and low carbon economy, so energy-savings and emission-reduction will promote the sound and fast development of economy, but it still demonstrates an obvious delay. Therefore, the impact of energy-saving and emission-reduction on carbon emission and economic development is estimated as delayed. On the basis of Model (1), we propose a two-delay model as follows:

$$
\begin{cases}
\dot{x}(t) = a_1 x(t)(\frac{y(t)}{M} - 1) - a_2 y(t) + a_3 z(t - \tau_1), \\
\dot{y}(t) = -b_1 x(t - \tau_2) + b_2 y(t)(1 - \frac{y(t)}{C}) + b_3 z(t - \tau_1)(1 - \frac{z(t - \tau_1)}{E}) - b_4 u(t), \\
\dot{z}(t) = c_1 x(t - \tau_2)(\frac{x(t - \tau_2)}{N} - 1) - c_2 y(t) - c_3 z(t) + c_4 u(t)(\frac{u(t)}{L} - 1), \\
\dot{u}(t) = d_1 y(t) + d_2 z(t)(\frac{z(t)}{K} - 1) - d_3 u(t),
\end{cases}
\tag{2}
$$

where τ_1 represents the delay time between economic development and carbon emission, and τ_2 represents the delay time among energy-saving and emission-reduction, economic development and carbon emission.

The rest of this paper is organized as follows: in Section 2, we focus on the local stability and the existence of Hopf bifurcation at the equilibrium point. In Section 3, we study the effects of delays and

the feedback control on the stability of the system. In Section 4, the effective Hopf bifurcation control of the system is examined. Finally, conclusions are drawn in Section 5.

2. Equilibrium Points and Local Stability

The calculation formulas of equilibrium points of Equation (2) are fairly complicated, so the equilibrium points can be calculated directly by using specific parameters value in numerical simulation section. We assume that Equation (2) has an equilibrium point, called hereafter by $E(x^*, y^*, z^*, u^*)$. It means that $x(t)$, $y(t)$, $z(t)$ and $u(t)$ can be in an equilibrium state through dynamic game. In this paper, we focus on the influence of τ_1, τ_2 and the feedback control on the stability of system (2) at the equilibrium point.

Next, Equation (2) is linearized at the equilibrium point $E(x^*, y^*, z^*, u^*)$ by Jacobian matrix as follows:

$$
\begin{cases}
\dot{x}(t) = (a_1 \frac{y^*}{M} - a_1)x(t) + (a_1 \frac{x^*}{M} - a_2)y(t) + a_3 z(t - \tau_1), \\
\dot{y}(t) = -b_1 x(t - \tau_2) + (b_2 - \frac{2b_2 y^*}{C})y(t) + (b_3 - \frac{2b_3 z^*}{E})z(t - \tau_1) - b_4 u(t), \\
\dot{z}(t) = (\frac{2c_1 x^*}{N} - c_1)x(t - \tau_2) - c_2 y(t) - c_3 z(t) + (\frac{2c_4 u^*}{L} - c_4)u(t), \\
\dot{u}(t) = d_1 y(t) + (\frac{2d_2 z^*}{K} - d_2)z(t) - d_3 u(t),
\end{cases}
\tag{3}
$$

The characteristic equation of Equation (3) is:

$$
\begin{vmatrix}
\lambda - J_{11} & -J_{12} & -J_{13}e^{-\lambda\tau_1} & -J_{14} \\
-J_{21}e^{-\lambda\tau_2} & \lambda - J_{22} & -J_{23}e^{-\lambda\tau_1} & -J_{24} \\
-J_{31}e^{-\lambda\tau_2} & -J_{32} & \lambda - J_{33} & -J_{34} \\
-J_{41} & -J_{42} & -J_{43} & \lambda - J_{44}
\end{vmatrix} = 0
$$

where:

$$
J_{11} = (a_1 \frac{y^*}{M} - a_1), \ J_{12} = a_1 \frac{x^*}{M} - a_2, \ J_{13} = a_3, \ J_{14} = 0,
$$

$$
J_{21} = -b_1, \ J_{22} = b_2 - \frac{2b_2 y^*}{C}, \ J_{23} = (b_3 - \frac{2b_3 z^*}{E}), \ J_{24} = -b_4,
$$

$$
J_{31} = (\frac{2c_1 x^*}{N} - c_1), \ J_{32} = -c_2, \ J_{33} = -c_3, \ J_{34} = \frac{2c_4 u^*}{L} - c_4,
$$

$$
J_{41} = 0, \ J_{42} = d_1, \ J_{43} = \frac{2d_2 z^*}{K} - d_2, \ J_{44} = -d_3.
$$

We further get:

$$
\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + (B_2\lambda^2 + B_1\lambda + B_0)e^{-\lambda\tau_1} + \\
(C_2\lambda^2 + C_1\lambda + C_0)e^{-\lambda\tau_2} + (D_2\lambda^2 + D_1\lambda + D_0)e^{-\lambda(\tau_1+\tau_2)} = 0
\tag{4}
$$

where:

$$
A_3 = -J_{11} - J_{22} - J_{33} - J_{44},
$$

$$
A_2 = J_{11}J_{22} + J_{11}J_{33} + J_{11}J_{44} - J_{14}J_{41} + J_{22}J_{33} + J_{22}J_{44} - J_{24}J_{42} + J_{33}J_{44} - J_{34}J_{43},
$$

$$
A_1 = -J_{11}J_{22}J_{33} - J_{11}J_{22}J_{44} + J_{11}J_{24}J_{42} - J_{12}J_{24}J_{41} + J_{14}J_{22}J_{41} - J_{11}J_{33}J_{44} \\
+ J_{11}J_{34}J_{43} + J_{14}J_{33}J_{41} - J_{22}J_{33}J_{44} + J_{22}J_{34}J_{43} - J_{24}J_{32}J_{43} + J_{24}J_{33}J_{42},
$$

$$
A_0 = J_{11}J_{22}J_{33}J_{44} - J_{11}J_{22}J_{34}J_{43} + J_{11}J_{24}J_{32}J_{43} - J_{11}J_{24}J_{33}J_{42} + J_{12}J_{24}J_{33}J_{41} - J_{14}J_{22}J_{33}J_{41},
$$

$$
B_2 = J_{23}J_{32}, \ B_1 = J_{11}J_{23}J_{32} - J_{13}J_{34}J_{41} + J_{23}J_{32}J_{44} - J_{23}J_{34}J_{42},
$$

$$
B_0 = -J_{11}J_{23}J_{32}J_{44} + J_{11}J_{23}J_{34}J_{42} - J_{12}J_{23}J_{34}J_{41} + J_{13}J_{22}J_{34}J_{41} - J_{13}J_{24}J_{32}J_{41} + J_{14}J_{23}J_{32}J_{41},
$$

$$
C_2 = J_{12}^2, \ C_1 = J_{12}^2 J_{33} + J_{12}^2 J_{44} - J_{12}J_{14}J_{42} - J_{14}J_{31}J_{43},
$$

$$
C_0 = J_{12}^2 J_{33}J_{44} + J_{12}^2 J_{34}J_{43} - J_{12}J_{14}J_{32}J_{43} + J_{12}J_{14}J_{33}J_{42} - J_{12}J_{24}J_{31}J_{43} + J_{14}J_{22}J_{31}J_{43},
$$

$$D_2 = J_{13}J_{31}, \ D_1 = J_{12}J_{13}J_{32} - J_{12}J_{23}J_{31} + J_{13}J_{22}J_{31} + J_{13}J_{31}J_{44},$$

$$D_0 = J_{12}J_{13}J_{32}J_{44} - J_{12}J_{13}J_{34}J_{42} + J_{12}J_{23}J_{31}J_{44} - J_{13}J_{22}J_{31}J_{44} + J_{13}J_{24}J_{31}J_{42} - J_{14}J_{23}J_{31}J_{42}.$$

2.1. Case 1 $\tau_1 > 0, \tau_2 = 0$

For $\tau_2 = 0$, Equation (4) can be simplified as follows:

$$\lambda^4 + M_3\lambda^3 + M_2\lambda^2 + M_1\lambda + M_0 + (N_2\lambda^2 + N_1\lambda + N_0)e^{-\lambda\tau_1} = 0, \tag{5}$$

where:

$$M_3 = A_3, \ M_2 = A_2 + C_2, \ M_1 = A_1 + C_1, \ M_0 = A_0 + C_0,$$

$$N_2 = B_2 + D_2, \ N_1 = B_1 + D_1, \ N_0 = B_0 + D_0,$$

Let $\lambda = i\omega_1 \ (\omega_1 > 0)$ be the root of Equation (5). Separating the real and imaginary parts, we obtain the following:

$$\begin{cases} N_1\omega_1\cos\omega_1\tau_1 + (N_2\omega_1^2 - N_0)\sin\omega_1\tau_1 = M_3\omega_1^3 - M_1\omega_1 \\ N_1\omega_1\sin\omega_1\tau_1 - (N_2\omega_1^2 - N_0)\cos\omega_1\tau_1 = M_2\omega_1^2 - \omega_1^4 - M_0 \end{cases}. \tag{6}$$

From Equation (6), we can get:

$$\cos\omega_1\tau_1 = \frac{N_2\omega_1^6 + (N_1M_3 - N_2M_2 - N_0)\omega_1^4 + (N_2M_0 + N_0M_2 - N_1M_1)\omega_1^2 - N_0M_0}{N_1^2\omega_1^2 + (N_2\omega_1^2 - N_0)^2}. \tag{7}$$

Squaring both sides, adding both equations and regrouping by powers of ω_1, we obtain that ω_1 ($\omega_1 > 0$) satisfies the following polynomial:

$$\omega_1^8 + (M_3^6 - 2M_2)\omega_1^6 + (M_2^2 - 2M_1M_3 + 2M_0 - N_2^2)\omega_1^4 + (M_1^2 - 2M_0M_2 - N_1^2 + 2N_0N_2)\omega_1^2 + M_0^2 - N_0^2 = 0. \tag{8}$$

Let $r_1 = \omega_1^2$, Equation (8) transformed into:

$$r_1^4 + q_3r_1^3 + q_2r_1^2 + q_1r_1 + q_0 = 0, \tag{9}$$

where:

$$q_3 = M_3^6 - 2M_2, q_2 = M_2^2 - 2M_1M_3 + 2M_0 - N_2^2,$$

$$q_1 = M_1^2 - 2M_0M_2 - N_1^2 + 2N_0N_2, q_0 = M_0^2 - N_0^2.$$

In the following, we need to seek conditions under which Equation (9) has at least one positive root. Denote:

$$h(r_1) = r_1^4 + q_3r_1^3 + q_2r_1^2 + q_1r_1 + q_0.$$

Since $\lim_{r_1 \to \infty} h(r_1) = \infty$, we conclude that if $q_0 < 0$, then Equation (9) has at least one positive root [16].

Suppose that Equation (9) has positive roots. Without loss of generality, we assume that it has four positive roots, defined by r_{11}, r_{12}, r_{13} and r_{14}, respectively. Then Equation (8) has four positive roots:

$$\omega_{11} = \sqrt{r_{11}}, \omega_{12} = \sqrt{r_{12}}, \omega_{13} = \sqrt{r_{13}}, \omega_{14} = \sqrt{r_{14}}.$$

For each fixed ω_{1k} $(k = 1, 2, 3, 4)$, there exists a sequence $\{\tau_{1k}^{(j)} | k = 1, 2, 3, 4; j = 0, 1, 2, ...\}$ that satisfies Equation (6). According to Equation (7), we can get:

$$\tau_{1k}^{(j)} = \frac{1}{w_{1k}}\arccos\frac{N_2\omega_{1k}^6 + (N_1M_3 - N_2M_2 - N_0)\omega_{1k}^4 + (N_2M_0 + N_0M_2 - N_1M_1)\omega_{1k}^2 - N_0M_0}{N_1^2\omega_{1k}^2 + (N_2\omega_{1k}^2 - N_0)^2} + \frac{2j\pi}{\omega_{1k}}$$
$$k = 1, 2, 3, 4; j = 0, 1, 2, 3, ... \tag{10}$$

Let $\tau_{10} = \min\{\tau_{1k}^{(j)} | k = 1, 2, 3, 4; j = 0, 1, 2, ...\} = \min_{k \in \{1,2,3,4\}}\{\tau_{1k}^{(0)}\} = \tau_{1k_0}$, $\omega_{10} = \omega_{1k_0}$. So the τ_{10} is:

$$\tau_{10} = \frac{1}{\omega_{10}}\arccos\frac{N_2\omega_{10}^6 + (N_1M_3 - N_2M_2 - N_0)\omega_{10}^4 + (N_2M_0 + N_0M_2 - N_1M_1)\omega_{10}^2 - N_0M_0}{N_1^2\omega_{10}^2 + (N_2\omega_{10}^2 - N_0)^2} \quad (11)$$

Based on the above analysis, the main results are presented as below:

Lemma 1. *If $q_0 < 0$, Equation (5) has a pair of pure imaginary roots $\pm i\omega_{10}$ when $\tau_1 = \tau_{10}$. Next, take the derivative with respect to τ_1 in Equation (5) and we can obtain:*

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \frac{(4\lambda^3 + 3M_3\lambda^2 + 2M_2\lambda + M_1)e^{\lambda\tau_1} + (2N_2\lambda + N_1)}{(N_2\lambda^3 + N_1\lambda^2 + N_0\lambda)} - \frac{\tau_1}{\lambda}$$

$$\mathrm{Re}\left[\frac{d\lambda(\tau_{10})}{d\tau_1}\right]^{-1}_{\lambda=i\omega_{10}} = \frac{P_1P_3 + P_2P_4}{P_1^2 + P_2^2}$$

where:

$$P_1 = -N_1\omega_{10}^2, P_2 = N_0\omega_{10} - N_2\omega_{10}^3,$$

$$P_3 = 4\omega_{10}^3\sin\omega_{10}\tau_{10} - 3M_3\omega_{10}^2\cos\omega_{10}\tau_{10} - 2M_2\omega_{10}\sin\omega_{10}\tau_{10} + M_1\cos\omega_{10}\tau_{10},$$

$$P_4 = -4\omega_{10}^3\cos\omega_{10}\tau_{10} - 3M_3\omega_{10}^2\sin\omega_{10}\tau_{10} + 2M_2\omega_{10}\cos\omega_{10}\tau_{10} + M_1\sin\omega_{10}\tau_{10} + 2N_2\omega_{10} + N_1,$$

Lemma 2. *Suppose that $P_1P_3 + P_2P_4 \neq 0$, then $\frac{d\mathrm{Re}\lambda(\tau_{10})}{d\tau_1}\Big|_{\lambda=i\omega_{10}} = \mathrm{Re}\left[\frac{d\lambda(\tau_{10})}{d\tau_1}\right]^{-1}_{\lambda=i\omega_{10}} \neq 0$. So it satisfies the transversality condition.*

According to the Lemmas 1–2 and the Hopf bifurcation theorem in [17], we obtain the following results:

Theorem 1: *The equilibrium point $E(x^*, y^*, z^*, u^*)$ of Equation (2) is asymptotically stable for $\tau_1 \in [0, \tau_{10})$ and unstable for $\tau_1 > \tau_{10}$; Equation (2) undergoes a Hopf bifurcation when $\tau_1 = \tau_{10}$.*

2.2. Case 2 $\tau_1 > 0, \tau_2 > 0$

For $\tau_1 > 0, \tau_2 > 0$, we consider the characteristic Equation (4) with τ_1 in its stable intervals (the range of τ_1 when the stability of the system is not affected), that is to say, $\tau_1 \in [0, \tau_{10})$ [18]. We study the influence of τ_2 on the stability of the system when τ_1 is fixed.

Proposition 1. *Suppose that $L_0 < 0$, then the characteristic Equation (4) has a pair of pure imaginary roots $\pm i\omega_{20}$ for $\tau_2 = \tau_{20}$. where:*

$$\tau_{20} = \frac{1}{\omega_{20}}\arccos\frac{h_6\omega_{20}^6 + h_5\omega_{20}^5 + h_4\omega_{20}^4 + h_3\omega_{20}^3 + h_2\omega_{20}^2 + h_1\omega_{20} + h_0}{f_4\omega_{20}^4 + f_3\omega_{20}^3 + f_2\omega_{20}^2 + f_1\omega_{20} + f_0} \quad (12)$$

Proof. Let $\lambda = i\omega_2$ $(\omega_2 > 0)$ be a root of Equation (4). Then we get:

$$\begin{cases} (C_2\omega_2^2 - C_0 + D_2\omega_2^2\cos\omega_2\tau_1 - D_1\omega_2\sin\omega_2\tau_1 - D_0\cos\omega_2\tau_1)\sin\omega_2\tau_2 \\ +(C_1\omega_2 + D_2\omega_2^2\sin\omega_2\tau_1 + D_1\omega_2\cos\omega_2\tau_1 - D_0\sin\omega_2\tau_1)\cos\omega_2\tau_2 \\ = A_3\omega_2^3 - A_1\omega_2 - B_2\omega_2^2\sin\omega_2\tau_1 - B_1\omega_2\cos\omega_2\tau_1 + B_0\sin\omega_2\tau_1, \\ -(C_2\omega_2^2 - C_0 + D_2\omega_2^2\cos\omega_2\tau_1 - D_1\omega_2\sin\omega_2\tau_1 - D_0\cos\omega_2\tau_1)\cos\omega_2\tau_2 \\ +(C_1\omega_2 + D_2\omega_2^2\sin\omega_2\tau_1 + D_1\omega_2\cos\omega_2\tau_1 - D_0\sin\omega_2\tau_1)\sin\omega_2\tau_2 \\ = A_2\omega_2^2 - \omega_2^4 - A_0 + B_2\omega_2^2\cos\omega_2\tau_1 - B_1\omega_2\sin\omega_2\tau_1 - B_0\cos\omega_2\tau_1 \end{cases} \quad (13)$$

From Equation (13), we can obtain:

$$\cos\omega_2\tau_2 = \frac{h_6\omega_2^6 + h_5\omega_2^5 + h_4\omega_2^4 + h_3\omega_2^3 + h_2\omega_2^2 + h_1\omega_2 + h_0}{f_4\omega_2^4 + f_3\omega_2^3 + f_2\omega_2^2 + f_1\omega_2 + f_0} \tag{14}$$

where:

$$h_6 = C_2 + D_2\cos\omega_2\tau_1,$$

$$h_5 = D_2A_3\sin\omega_2\tau_1 - D_1\sin\omega_2\tau_1$$

$$h_4 = C_1A_3 - D_2B_2 + D_1A_3\cos\omega_2\tau_1 - C_2A_2 - C_2B_2\cos\omega_2\tau_1 - C_0 - D_2A_2\cos\omega_2\tau_1 - D_0\cos\omega_2\tau_1,$$

$$h_3 = C_2B_1\sin\omega_2\tau_1 - C_1B_2\sin\omega_2\tau_1 - D_2A_1\sin\omega_2\tau_1 - D_0A_3\sin\omega_2\tau_1 + D_1A_2\sin\omega_2\tau_1,$$

$$\begin{aligned} h_2 = -C_1A_1 - C_1B_1\cos\omega_2\tau_1 + D_2B_0 - D_1A_1\cos\omega_2\tau_1 - D_1B_1 + D_0B_2 + C_2A_0 \\ +C_2B_0\cos\omega_2\tau_1 + C_0A_2 + C_0B_2\cos\omega_2\tau_1 + D_2A_0\cos\omega_2\tau_1 + D_0A_2\cos\omega_2\tau_1 \end{aligned},$$

$$\begin{aligned} h_1 = C_1B_0\sin\omega_2\tau_1 + D_1B_0\cos\omega_2\tau_1\sin\omega_2\tau_1 + D_0A_1\sin\omega_2\tau_1 \\ -C_0B_1\sin\omega_2\tau_1 - D_1A_0\sin\omega_2\tau_1 - D_1B_0\sin\omega_2\tau_1\cos\omega_2\tau_1 \end{aligned},$$

$$h_0 = -D_0B_0 - C_0A_0 - C_0B_0\cos\omega_2\tau_1 - D_0A_0\cos\omega_2\tau_1,$$

$$f_4 = C_2^2 + D_2^2 + 2D_2C_2\cos\omega_2\tau_1,$$

$$f_3 = -2D_1C_2\sin\omega_2\tau_1 + 2C_1D_2\sin\omega_2\tau_1,$$

$$f_2 = 2D_2C_0\cos\omega_2\tau_1 - 2C_0C_2 + D_1^2 - 2D_0C_2\cos\omega_2\tau_1 - 2D_0D_2 + C_1^2 + 2D_1C_1\cos\omega_2\tau_1,$$

$$f_1 = 2D_1C_0\sin\omega_2\tau_1 - 2D_0C_1\sin\omega_2\tau_1,$$

$$f_0 = C_0^2 + D_0^2 + 2C_0D_0\cos\omega_2\tau_1.$$

Similar to Case 1, from Equation (13), we have:

$$\omega_2^8 + L_7\omega_2^7 + L_6\omega_2^6 + L_5\omega_2^5 + L_4\omega_2^4 + L_3\omega_2^3 + L_2\omega_2^2 + L_1\omega_2 + L_0 = 0 \tag{15}$$

where

$$L_7 = 0, L_6 = A_3^2 - 2A_2 - 2B_2\cos\omega_2\tau_1, L_5 = 2B_2A_3\sin\omega_2\tau_1 - 2B_1\sin\omega_2\tau_1,$$

$$L_4 = B_2^2 - 2A_1A_3 - 2A_3B_1\cos\omega_2\tau_1 + A_2^2 + 2A_0 + 2A_2B_2\cos\omega_2\tau_1 + 2B_0\cos\omega_2\tau_1 - C_2^2 - D_2^2 - 2D_2C_2\cos\omega_2\tau_1,$$

$$L_3 = 2B_2A_1\sin\omega_2\tau_1 + 2B_0A_3\sin\omega_2\tau_1 - 2B_1A_2\sin\omega_2\tau_1 + 2D_1C_2\sin\omega_2\tau_1 - 2C_1D_2\sin\omega_2\tau_1,$$

$$\begin{aligned} L_2 = A_1^2 + B_1^2 + 2A_1B_1\cos\omega_2\tau_1 - 2B_0B_2 - 2A_0A_2 - 2B_2A_0\cos\omega_2\tau_1 - 2B_0A_2\cos\omega_2\tau_1 \\ -2D_2C_0\cos\omega_2\tau_1 + 2C_0C_2 + D_1^2 + 2D_0C_2\cos\omega_2\tau_1 + 2D_0D_2 - C_1^2 - 2D_1C_1\cos\omega_2\tau_1 \end{aligned},$$

$$L_1 = -2B_0A_1\sin\omega_2\tau_1 + 2B_1A_0\sin\omega_2\tau_1 - 2D_1C_0\sin\omega_2\tau_1 + 2D_0C_1\sin\omega_2\tau_1,$$

$$L_0 = A_0^2 + 2B_0A_0\cos\omega_2\tau_1 - C_0^2 - D_0^2 - 2D_0C_0\cos\omega_2\tau_1,$$

We turn Equation (15) into the following form:

$$f(\omega_2) = \omega_2^8 + L_7\omega_2^7 + L_6\omega_2^6 + L_5\omega_2^5 + L_4\omega_2^4 + L_3\omega_2^3 + L_2\omega_2^2 + L_1\omega_2 + L_0 \tag{16}$$

Since $\lim_{\omega_2\to\infty} f(\omega_2) = \infty$, we conclude that if $L_0 < 0$, then Equation (15) has at least one positive root. Without loss of generality, we assume that Equation (15) has a finite number of positive roots defined by $\{\omega_{21}, \omega_{22}, ..., \omega_{2s}\}$.

From Equation (14), we denote:

$$\tau_{2k}^{(j)} = \frac{1}{\omega_{2k}}\arccos\frac{h_6\omega_{2k}^6 + h_5\omega_{2k}^5 + h_4\omega_{2k}^4 + h_3\omega_{2k}^3 + h_2\omega_{2k}^2 + h_1\omega_{2k} + h_0}{f_4\omega_{2k}^4 + f_3\omega_{2k}^3 + f_2\omega_{2k}^2 + f_1\omega_{2k} + f_0} + \frac{2j\pi}{\omega_{2k}}, k = 1, 2, ..., s; j = 0, 1, 2, ...$$

Then $(\tau_{2k}^{(j)}, \omega_{2k})$ be a root of Equation (13). Therefore, when $\tau_2 = \tau_{2k}^{(j)}$, the characteristic Equation (4) has a pair of pure imaginary roots $\pm i\omega_{2k}$. Define:

$$\tau_{20} = \min\{\tau_{2k}^{(j)}|k = 1,2,...,s; j = 0,1,2,...\} = \min_{k\in\{1,2,...,s\}}\{\tau_{2k}^{(0)}\} = \tau_{2k_0}, \omega_{20} = \omega_{2k_0}.$$

Let $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ be the root of the characteristic Equation (4) near $\tau_2 = \tau_{2k}^{(j)}$ satisfying:

$$\alpha\left(\tau_{2k}^{(j)}\right) = 0, \ \omega\left(\tau_{2k}^{(j)}\right) = \omega_{2k}.$$

Then on the basis of above analysis, we can get the following conclusion:

$$\tau_{20} = \frac{1}{\omega_{20}}\arccos\frac{h_6\omega_{20}^6 + h_5\omega_{20}^5 + h_4\omega_{20}^4 + h_3\omega_{20}^3 + h_2\omega_{20}^2 + h_1\omega_{20} + h_0}{f_4\omega_{20}^4 + f_3\omega_{20}^3 + f_2\omega_{20}^2 + f_1\omega_{20} + f_0} \tag{17}$$

The characteristic Equation (4) has a pair of pure imaginary roots $\pm i\omega_{20}$ when $\tau_2 = \tau_{20}$. □

Proposition 2. *Suppose that* $\Delta \neq 0$, *then* $\mathrm{Re}\left[\frac{d\lambda(\tau_{20})}{d\tau_2}\right]^{-1}_{\lambda=i\omega_{20}} \neq 0$, $\mathrm{Re}\left[\frac{d\lambda(\tau_{20})}{d\tau_2}\right]^{-1}_{\lambda=i\omega_{20}}$ *and* Δ *have the same sign, where:*

$$\Delta = Q_1 Q_3 + Q_2 Q_4$$

Proof. Substituting $\lambda(\tau_2)$ (the root of the characteristic Equation (4), which is defined above) into Equation (4) and taking the derivative with respect to τ_2, we obtain:

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{Q_{10} + Q_{20} + Q_{30} + Q_{40}}{\lambda e^{-\lambda(\tau_1+\tau_2)}(D_2\lambda^2 + D_1\lambda + D_0) + \lambda e^{-\lambda\tau_2}(C_2\lambda^2 + C_1\lambda + C_0)} - \frac{\tau_2}{\lambda} \tag{18}$$

$$Q_{10} = 4\lambda^3 + 3A_3\lambda^2 + 2A_2\lambda + A_1, Q_{20} = (2B_2\lambda + B_1 - \tau_1 B_2\lambda^2 - \tau_1 B_1\lambda - \tau_1 B_0)e^{-\lambda\tau_1},$$

$$Q_{30} = (2C_2\lambda + C_1)e^{-\lambda\tau_2}, Q_{40} = (2D_2\lambda + D_1 - \tau_1 D_2\lambda^2 - \tau_1 D_1\lambda - \tau_1 D_0)e^{-\lambda(\tau_1+\tau_2)},$$

From Equation (18), we have:

$$\mathrm{Re}\left[\frac{d\lambda(\tau_{20})}{d\tau_2}\right]^{-1}_{\lambda=i\omega_{20}} = \frac{Q_1 Q_3 + Q_2 Q_4}{Q_1^2 + Q_2^2} = \frac{\Delta}{Q_1^2 + Q_2^2}$$

where:

$$Q_1 = -D_1\omega_{20}^2\cos(\tau_1 + \tau_{20}) - D_2\omega_{20}^3\sin(\tau_1 + \tau_{20}) + D_0\omega_{20}\sin(\tau_1 + \tau_{20})$$
$$-C_1\omega_{20}^2\cos\omega_{20}\tau_{20} - C_2\omega_{20}^3\sin\omega_{20}\tau_{20} + C_0\omega_{20}\sin\omega_{20}\tau_{20}$$

$$Q_2 = -D_2\omega_{20}^3\cos(\tau_1 + \tau_{20}) + D_0\omega_{20}\cos(\tau_1 + \tau_{20}) + D_1\omega_{20}^2\sin(\tau_1 + \tau_{20})$$
$$-C_2\omega_{20}^3\cos\omega_{20}\tau_{20} + C_0\omega_{20}\cos\omega_{20}\tau_{20} + C_1\omega_{20}^2\sin\omega_{20}\tau_{20}$$

$$Q_3 = -3A_3\omega_{20}^2 + A_1 + 2B_2\omega_{20}\sin\omega_{20}\tau_1 + B_1\cos\omega_{20}\tau_1 + \tau_1 B_2\omega_{20}^2\cos\omega_{20}\tau_1 - \tau_1 B_1\omega_{20}\sin\omega_{20}\tau_1$$
$$-\tau_1 B_0\cos\omega_{20}\tau_1 + 2C_2\omega_{20}\sin\omega_{20}\tau_{20} + C_1\cos\omega_{20}\tau_{20} + 2D_2\omega_{20}\sin(\tau_1 + \tau_{20})$$
$$+D_1\cos(\tau_1 + \tau_{20}) + \tau_1 D_2\omega_{20}^2\cos(\tau_1 + \tau_{20}) - \tau_1 D_1\omega_{20}\sin(\tau_1 + \tau_{20}) - \tau_1 D_0\cos(\tau_1 + \tau_{20})$$

$$Q_4 = -4\omega_{20}^3 + 2A_2\omega_{20} + 2B_2\omega_{20}\cos\omega_{20}\tau_1 - B_1\sin\omega_{20}\tau_1 - \tau_1 B_2\omega_{20}^2\sin\omega_{20}\tau_1 - \tau_1 B_1\omega_{20}\cos\omega_{20}\tau_1$$
$$+\tau_1 B_0\sin\omega_{20}\tau_1 + 2C_2\omega_{20}\cos\omega_{20}\tau_{20} - C_1\sin\omega_{20}\tau_{20} + 2D_2\omega_{20}\cos(\tau_1 + \tau_{20})$$
$$-D_1\sin(\tau_1 + \tau_{20}) - \tau_1 D_2\omega_{20}^2\sin(\tau_1 + \tau_{20}) - \tau_1 D_1\omega_{20}\cos(\tau_1 + \tau_{20}) + \tau_1 D_0\sin(\tau_1 + \tau_{20})$$

Since $Q_1^2 + Q_2^2 > 0$, thus $\mathrm{Re}\left[\frac{d\lambda(\tau_{20})}{d\tau_2}\right]^{-1}_{\lambda=i\omega_{20}} \neq 0$. We conclude that $\mathrm{Re}\left[\frac{d\lambda(\tau_{20})}{d\tau_2}\right]^{-1}_{\lambda=i\omega_{20}}$ and Δ have the same sign. □

Thus, according to propositions 1 and 2 and the Hopf bifurcation theorem in [17], we have the following theorem.

Theorem 2: *For $\tau_1 \in [0, \tau_{10})$, τ_{10} is defined by Equation (11). The equilibrium point $E(x^*, y^*, z^*, u^*)$ of Equation (2) is asymptotically stable for $\tau_2 \in [0, \tau_{20})$ and unstable when $\tau_2 > \tau_{20}$. Equation (2) has a Hopf bifurcation at $\tau_2 = \tau_{20}$.*

3. Numerical Simulation and Analysis

In this section, the numerical simulation is carried out in order to support the theoretical analysis. Let $x(0) = 0.3$; $y(0) = 0.5$; $z(0) = 0.7$; $u(0) = 0.9$; $a_1 = 4.5$; $a_2 = 1.3$; $a_3 = 20$; $b_1 = 0.25$; $b_2 = 0.85$; $b_3 = 0.35$; $b_4 = 0.04$; $c_1 = 0.5$; $c_2 = 0.3$; $c_3 = 0.2$; $c_4 = 0.1$; $d_1 = 0.01$; $d_2 = 0.02$; $d_3 = 0.06$; $M = 10$; $C = 1$; $E = 8$; $N = 12$; $L = 2$; $K = 2$. We consider the following system with given parameter values:

$$\begin{cases} \dot{x}(t) = 4.5x(t)(\frac{y(t)}{10} - 1) - 1.3y(t) + 20z(t - \tau_1), \\ \dot{y}(t) = -0.25x(t - \tau_2) + 0.85y(t)(1 - y(t)) + 0.35z(t - \tau_1)(1 - \frac{z(t-\tau_1)}{8}) - 0.04u(t), \\ \dot{z}(t) = 0.5x(t - \tau_2)(\frac{x(t-\tau_2)}{12} - 1) - 0.3y(t) - 0.2z(t) + 0.1u(t)(\frac{u(t)}{2} - 1), \\ \dot{u}(t) = 0.01y(t) + 0.02z(t)(\frac{z(t)}{2} - 1) - 0.06u(t), \end{cases} \tag{19}$$

The following equilibrium points can be obtained by simple calculation:

$$E_0(0,0,0,0), E_1(-0.6625, 1.1410, -0.0578, 0.2100),$$

$$E_2(-5.2219 \pm 54.1330i, -3.8795 \pm 0.7677i, -0.9477 \pm 17.0453i, -48.6048 \pm 10.9389i),$$

$$E_3(13.3977 \pm 0.7680i, 0.4248 \pm 1.7180i, 2.9437 \pm 0.2407i, 0.5241 \pm 0.1303i),$$

$$E_4(-251.5640 \pm 371.2492i, 14.7184 \pm 1.4202i, 15.8006 \pm 47.5443i, -337.9472 \pm 234.7986i),$$

$$E_5(-48.6048 \pm 10.9389i, 6.9068 \pm 6.8940i, -28.3090 \pm 13.8694i, 112.0947 \pm 134.3508i),$$

$$E_6(131.6126 \pm 3.3325i, 3.3618 \pm 6.3095i, 19.4028 \pm 18.7720i, -1.8941 \pm 114.1013i),$$

According to actual economic meaning, only E_1 is Nash Equilibrium point. So we examine the system stability of the system at the equilibrium point $E_1(-0.6625, 1.1410, -0.0578, 0.2100)$.

For Case 1, from Equations (8) and (11), we can obtain $\omega_{10} = 4.5329$, $\tau_{10} = 0.4618$, so Equation (5) has a pair of pure imaginary roots $\pm i\omega_{10}$ when $\tau_1 = \tau_{10}$ and $\tau_2 = 0$. We also get $P_1 P_3 + P_2 P_4 = 37.4226 \neq 0$. By theorem 1, the equilibrium point E_1 of Equation (2) is asymptotically stable when $\tau_1 \in [0, 0.4618)$ and unstable when $\tau_1 > 0.4618$. It has a Hopf bifurcation at $\tau_1 = 0.4618$.

For Case 2, Let $\tau_1 = 0.4 \in [0, \tau_{10})$, from Equations (16) and (17), we can obtain $\omega_{20} = 39.6736$, $\tau_{20} = 0.0622$, so Equation (4) has a pair of pure imaginary roots $\pm i\omega_{20}$ when $\tau_1 = 0.4$ and $\tau_2 = \tau_{20}$. At this point, $Q_1 Q_3 + Q_2 Q_4 = 23.8153 \neq 0$. By theorem 2, the equilibrium point E_1 of Equation (2) is asymptotically stable when $\tau_2 \in [0, 0.0622)$ for $\tau_1 = 0.4$ and unstable when $\tau_2 > 0.0622$ for $\tau_1 = 0.4$. Equation (2) undergoes a Hopf bifurcation when $\tau_2 = 0.0622$ for $\tau_1 = 0.4$.

3.1. The Influence of τ_1 on the Stability of Equation (19)

Equation (19) moves from stable to unstable with the increase in τ_1 and undergoes Hopf bifurcation at $\tau_1 = 0.4618$ when $\tau_2 = 0$. Figure 1a shows the dynamic evolution process of Equation (19). In this paper, we calculate the largest Lyapunov exponent (LLE) by the method of Wolf reconstruction [19]. We judge whether the system is stable according to the exponent value. If it is less than 0, the system is stable. If it is greater than 0, the system is unstable. If it is equal to 0, the system appears bifurcation. In Figure 1b, we can know that the system has a bifurcation at $\tau_1 = 0.4618$. This is consistent with the conclusion of theoretical analysis.
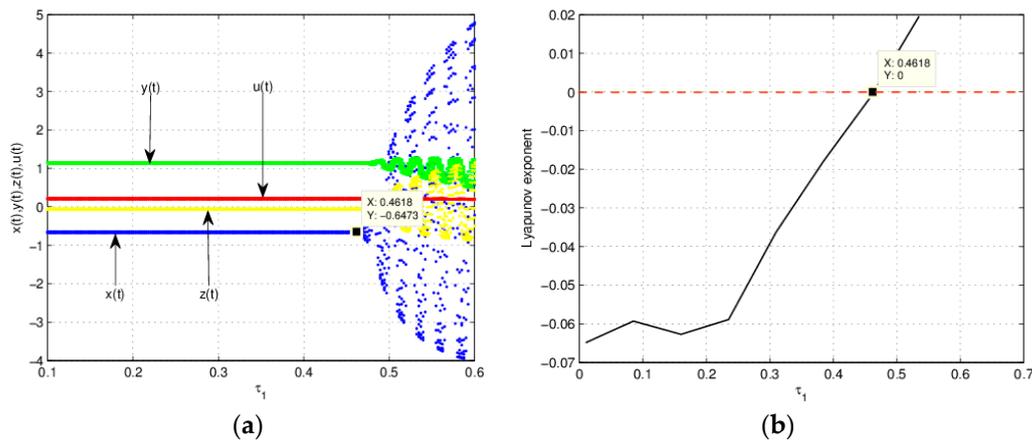
Figure 1. The influence of τ_1 on the stability of Equation (19) when $\tau_2 = 0$. (**a**) bifurcation diagram; (**b**) the largest Lyapunov exponent plot.

According to Theorem 1, Equation (19) is stable when $\tau_1 = 0.4 < \tau_{10} = 0.4618$ and $\tau_2 = 0$. In this case, $x(t)$, $y(t)$, $z(t)$ and $u(t)$ will converge to the equilibrium point E_1 through the game. Figures 2 and 3 show the above properties.

Equation (19) is unstable when $\tau_1 = 0.5 > \tau_{10} = 0.4618$ and $\tau_2 = 0$ by Theorem 1. At this point, the frequency spectrum plot is discrete, which means that the system has a periodic solution. The $x(t)$, $y(t)$, $z(t)$ and $u(t)$ will lie on the basin of attraction [20] through the game. These analyses can be illustrated by Figures 4 and 5.

We discover that economic growth (slowdown) will lead to an increase (reduction) in carbon emissions when other parameters are fixed. The two variables share a consistent pattern, but with a certain delay in time. Based on the above simulation, we find out that when the delay is greater than the bifurcation value, the carbon emissions pattern will not be consistent with the economic growth trend and volatility surfaces. Therefore, we should guarantee the timing of achieving the goal of reducing emissions and make it less than the bifurcation value. Only in such a stable environment, the implementation of energy-saving and emission-reduction policies will intend a favorable effect.
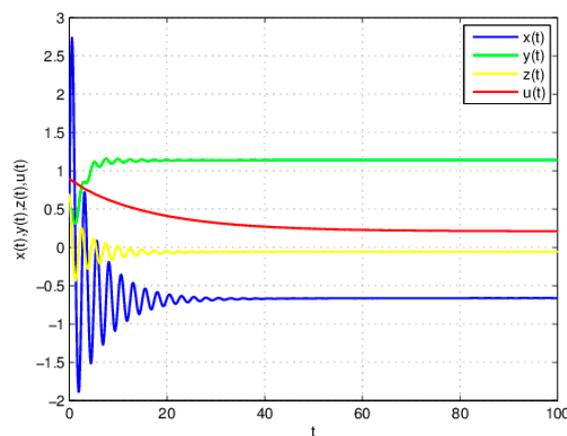
Figure 2. Time-domain plot when $\tau_1 = 0.4 < \tau_{10} = 0.4618$, $\tau_2 = 0$.
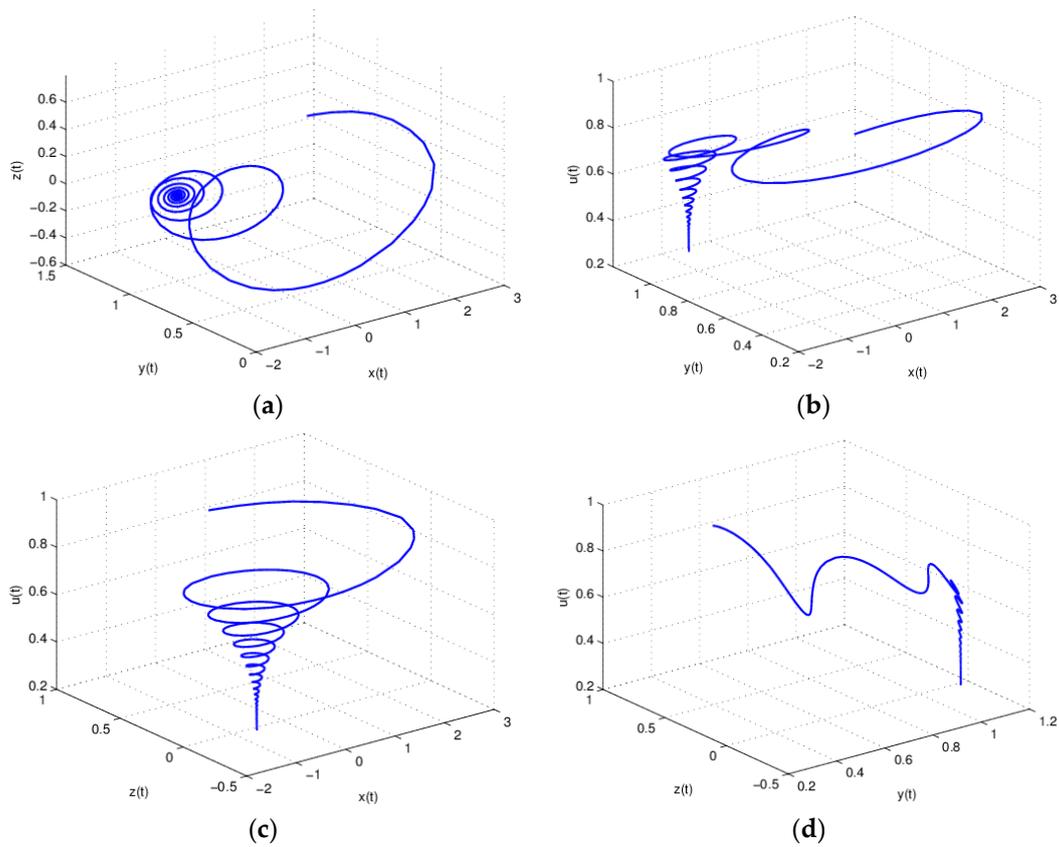
Figure 3. The EE attractor when $\tau_1 = 0.4 < \tau_{10} = 0.4618$, $\tau_2 = 0$ and $(x(0), y(0), z(0), u(0)) = (0.3, 0.5, 0.7, 0.9)$. (**a**) $x(t), y(t), z(t)$; (**b**) $x(t), y(t), u(t)$; (**c**) $x(t), z(t), u(t)$; (**d**) $y(t), z(t), u(t)$.
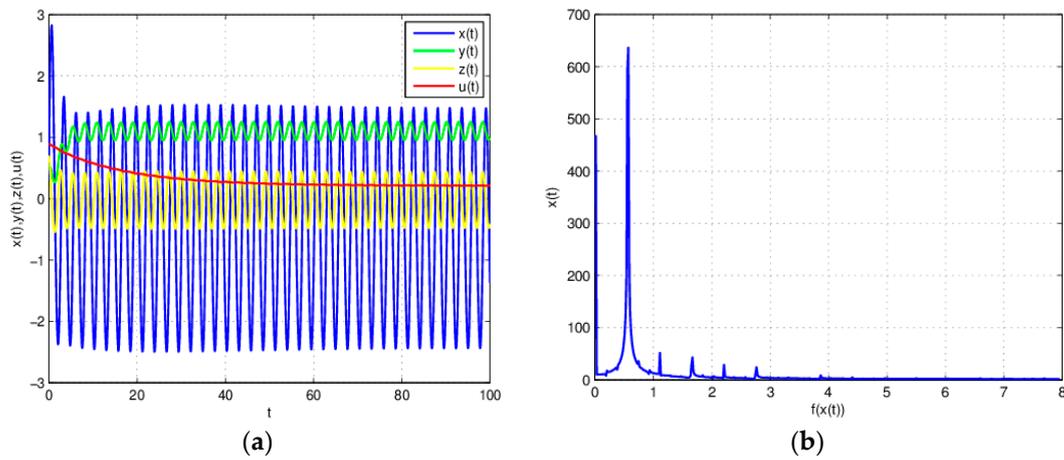

Figure 4. Equation (19) is unstable when $\tau_1 = 0.5 > \tau_{10} = 0.4618$, $\tau_2 = 0$. (**a**) time-domain plot; (**b**) frequency spectrum plot.
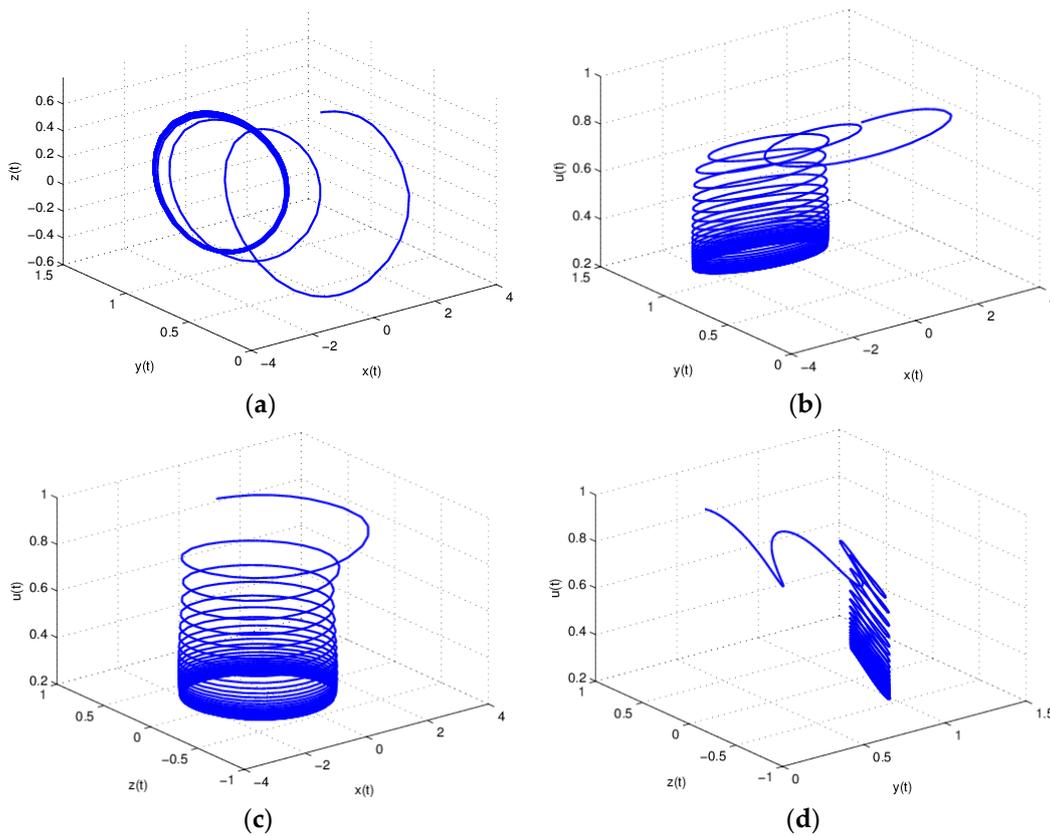
Figure 5. The EE attractor when $\tau_1 = 0.5 > \tau_{10} = 0.4618, \tau_2 = 0$ and $(x(0), y(0), z(0), u(0)) = (0.3, 0.5, 0.7, 0.9)$. (**a**) $x(t), y(t), z(t)$; (**b**) $x(t), y(t), u(t)$; (**c**) $x(t), z(t), u(t)$; (**d**) $y(t), z(t), u(t)$.

3.2. The Influence of τ_1 on the Entropy of Equation (19)

Kolmogorov entropy can be used to measure the degree of complexity of the system. Let k be the value of Kolmogorov entropy. If the system is stable and in regular motion, then $k = 0$, otherwise $k > 0$. The greater the value of k is, the more complex the system is. We have already found that Equation (19) displays bifurcation at $\tau_1 = 0.4618$, as is shown in Figure 1. Based on the above analysis, we can get that if $\tau_1 < 0.4618$, then $k = 0$, otherwise $k > 0$. The more complex the system is, the longer and the more difficult it will be for the system to return to stability. The entropy properties are displayed in Figure 6.
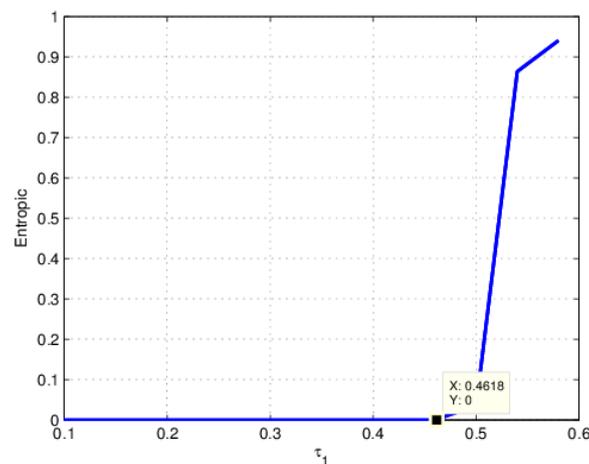
Figure 6. The entropy plot respect to τ_1 when $\tau_2 = 0$.

3.3. The Influence of τ_1, b_3, b_4 on the Stability of Equation (19)

The influence of τ_1 and b_3 on y is shown in Figure 7. The system shows a sign of instability at $\tau_1 = 0.4618$. However, b_3 has no obvious impact on y when $\tau_1 < 0.55$. With the increase of b_3, y becomes larger and then decreases when $\tau_1 > 0.55$. The minimum value of y is -1.448 for $(\tau_1, b_3) = (0.5, 0.85)$ while the maximum value of y is 5.124 for $(\tau_1, b_3) = (0.6, 0.55)$. The value of y is -0.66 approximately when $\tau_1 < 0.4618$. Thus, the impact of b_3 on y can be ignored when $\tau_1 < 0.4618$.
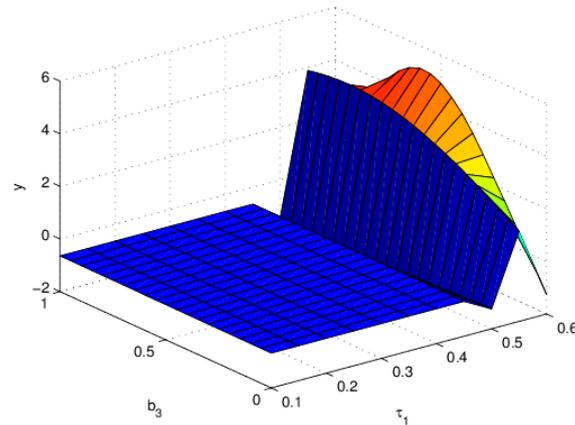
Figure 7. The influence of τ_1 and b_3 on y when $\tau_2 = 0$.

Figure 8 shows that b_4 has a greater impact on y. With an increase in b_4, the system shifts from unstable to stable. When $b_4 = 0.2$, the system is beginning to become stable. The minimum value of y is -1.838 for $(\tau_1, b_4) = (0.5, 0.15)$ and the maximum value of y is 4.207 for $(\tau_1, b_4) = (0.6, 0)$. The value of y is stable at 2.889, but τ_1 has little effect on y. Therefore, we should consider the effect of b_4 on the stability of the system.
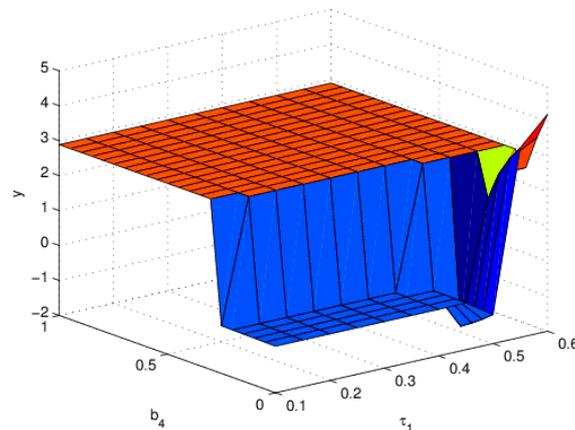
Figure 8. The influence of τ_1 and b_4 on y when $\tau_2 = 0$.

We focus on the influence of τ_1, b_3 and b_4 on y in Figure 9. The system is stable when $\tau_1 < 0.4618$, and unstable when $\tau_1 > 0.4618$. The change in b_3 and b_4 leads only to a larger fluctuation but when $\tau_1 > 0.4618$, the fluctuation trend is basically symmetrical. However, b_3 and b_4 have no effect on y when $\tau_1 < 0.4618$. We conclude that the stability of energy-saving and emission-reduction system can be maintained, only controlling for the value of delay parameter τ_1.
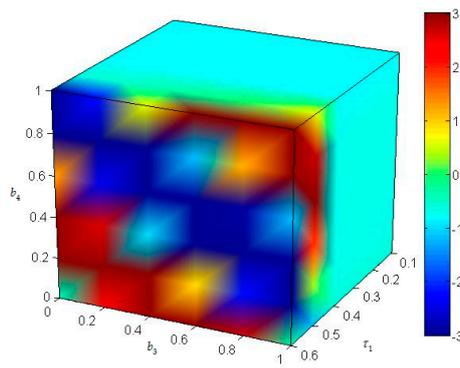
Figure 9. The influence of τ_1, b_3 and b_4 on y when $\tau_2 = 0$.

3.4. The Influence of τ_2 on the Stability of Equation (19)

In this part, we study the effect of τ_2 on the stability of the system when τ_1 is fixed. With an increase in τ_2, Equation (19) will lose stability. Figure 10 displays that Equation (19) undergoes bifurcation at $\tau_2 = 0.0622$ for $\tau_1 = 0.4$. This is consistent with the theoretical analysis.
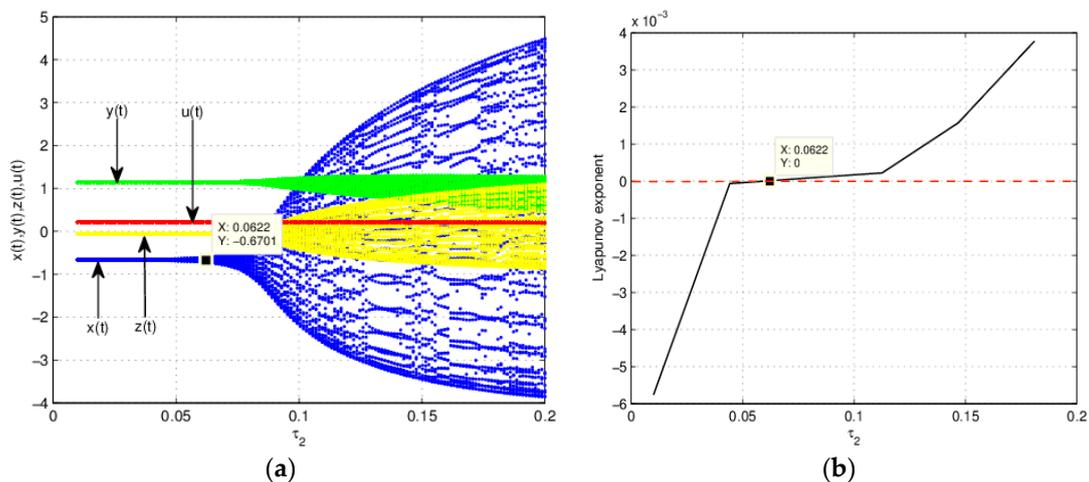

Figure 10. The influence of τ_2 on the stability of Equation (19) when $\tau_1 = 0$. (**a**) bifurcation diagram; (**b**) the largest Lyapunov exponent plot.

Equation (19) is stable when $\tau_1 = 0.4, \tau_2 = 0.04 < \tau_{20} = 0.0622$ by Theorem 2. $x(t), y(t), z(t)$ and $u(t)$ will converge to the equilibrium point E_1 through the game. The properties are shown in Figures 11 and 12.
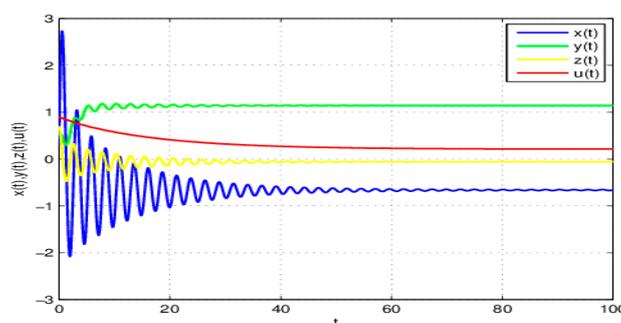
**Figure 11.** Time-domain plot when $\tau_1 = 0.4, \tau_2 = 0.04 < \tau_{20} = 0.0622$.
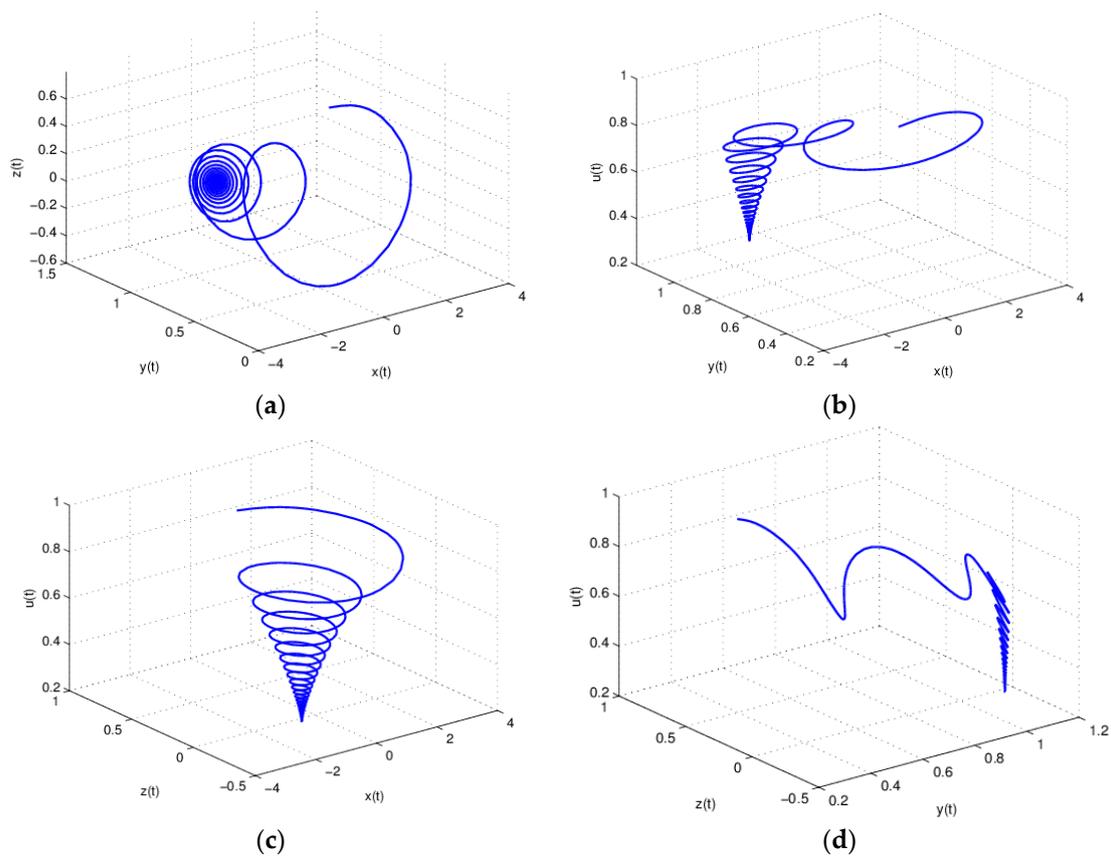
Figure 12. The EE attractor when $\tau_1 = 0.4$, $\tau_2 = 0.04 < \tau_{20} = 0.0622$ and $(x(0), y(0), z(0), u(0)) = (0.3, 0.5, 0.7, 0.9)$. (**a**) $x(t), y(t), z(t)$; (**b**) $x(t), y(t), u(t)$; (**c**) $x(t), z(t), u(t)$; (**d**) $y(t), z(t), u(t)$.

On the basis of theorem 2, Equation (19) is unstable when $\tau_1 = 0.4$, $\tau_2 = 0.08 > \tau_{20} = 0.0622$. Meanwhile, the Poincare plot has five discrete points, which means that the system has a periodic solution. $x(t)$, $y(t)$, $z(t)$ and $u(t)$ will lie on the basin of attraction through the game. Figures 13 and 14 show these characteristics.
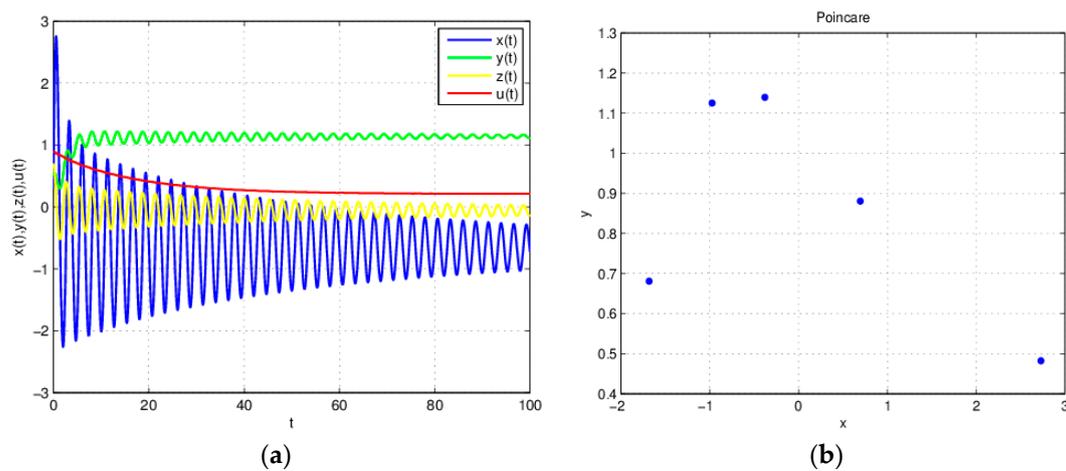
Figure 13. Equation (19) is unstable when $\tau_1 = 0.4$, $\tau_2 = 0.08 > \tau_{20} = 0.0622$. (**a**) Time-domain plot; (**b**) Poincare plot.

(a)

(b)

(c)

(d)
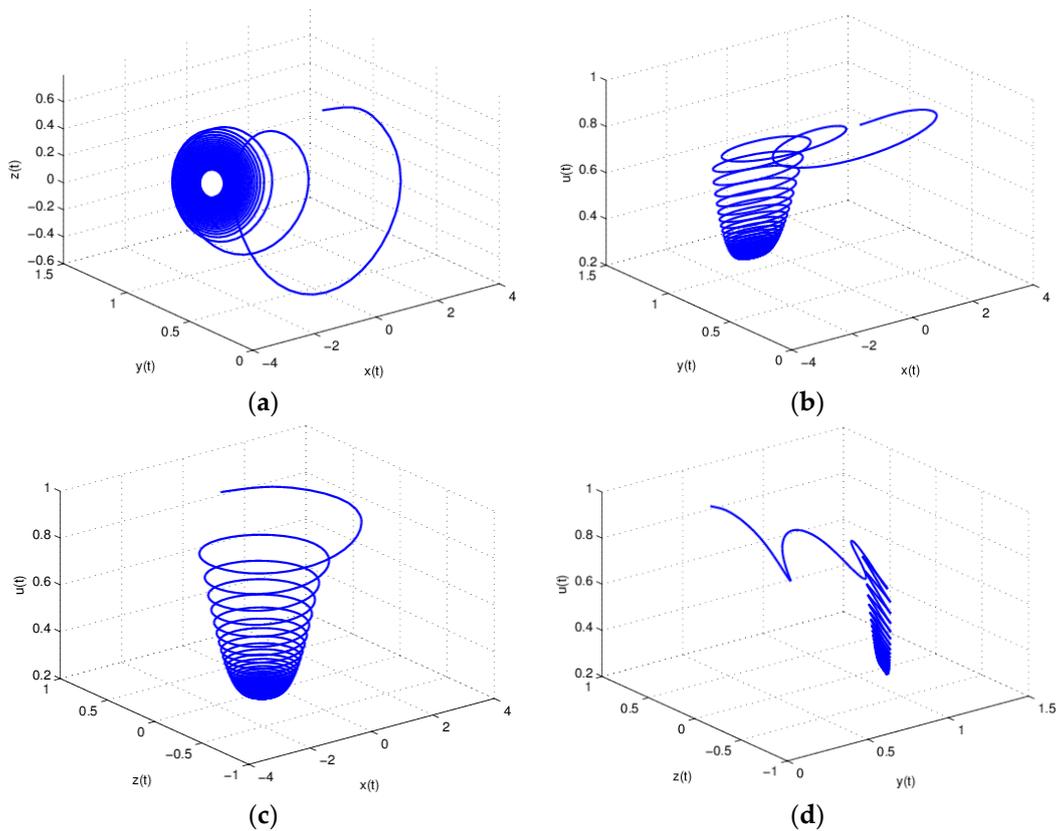
Figure 14. The EE attractor when $\tau_1 = 0.4, \tau_2 = 0.08 > \tau_{20} = 0.0622$ and $(x(0), y(0), z(0), u(0)) = (0.3, 0.5, 0.7, 0.9)$. (**a**) $x(t), y(t), z(t)$; (**b**) $x(t), y(t), u(t)$; (**c**) $x(t), z(t), u(t)$; (**d**) $y(t), z(t), u(t)$.

3.5. The Influence of τ_2 on the Entropy of Equation (19)

From Figure 10, we know that Equation (19) has a bifurcation when $\tau_1 = 0.4, \tau_2 = 0.0622$. Equation (19) is unstable for $\tau_2 > 0.0622$, then the entropy value is more than 0. As in Section 3.2, the growth in the value of entropy shares the same pattern with that in τ_2. The change of entropy is shown in Figure 15. With the increase of entropy value, the system will be more unstable. In this case, the implementation effect of energy-saving and emission-reduction is poorly maintained.
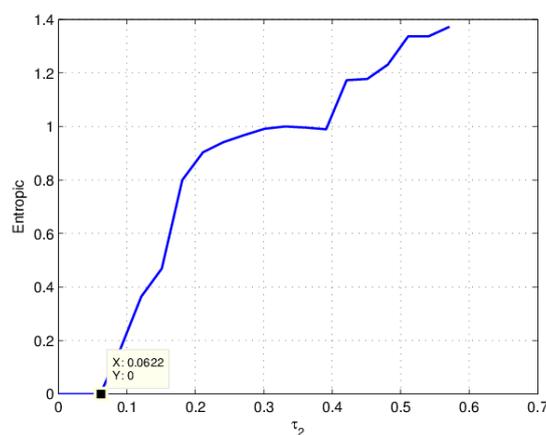

Figure 15. The entropy plot respect to τ_2 when $\tau_1 = 0.4$.

3.6. The Influence of τ_2, b_3, b_4 on the Stability of Equation (19)

Figure 16 shows that the influence of τ_2 on y is greater. With an increase in τ_2, the system moves from stable to unstable. The system is becoming unstable at $\tau_2 = 0.06$. The minimum value of y is 1.048 for $(\tau_2, b_3) = (0.1, 0.1)$ and the maximum value of y is 1.212 for $(\tau_2, b_3) = (0.09, 0.1)$. The value of y is stable in the vicinity of 1.1. But b_3 has little effect on y. Only when $\tau_2 > 0.06$, the value of y will turn from large to small with the increase of b_3. Therefore, we have to make $\tau_2 < 0.06$, so as to guarantee the effective implementation of energy-savings and emission-reduction.
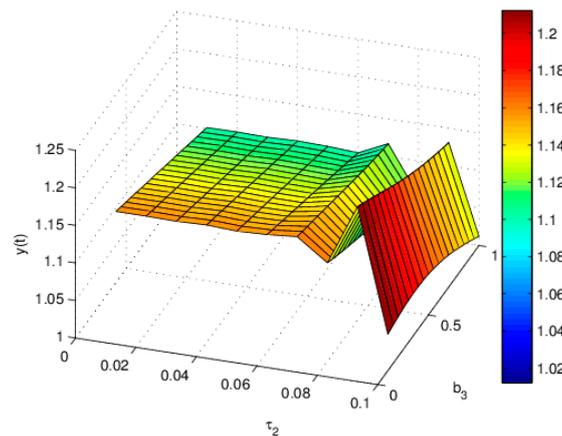

Figure 16. The influence of τ_2 and b_3 on y when $\tau_1 = 0.4$.

When τ_2 and b_3 are larger and b_4 is smaller, the value of y is closer to 1. Otherwise, y approximates -1. The change in the value of the parameters will cause the transformation of y between two states. These properties are shown in Figure 17. Therefore, we have to ensure that τ_2 and b_3 are as small as possible, so as to reduce carbon emissions.


(a)

(b)
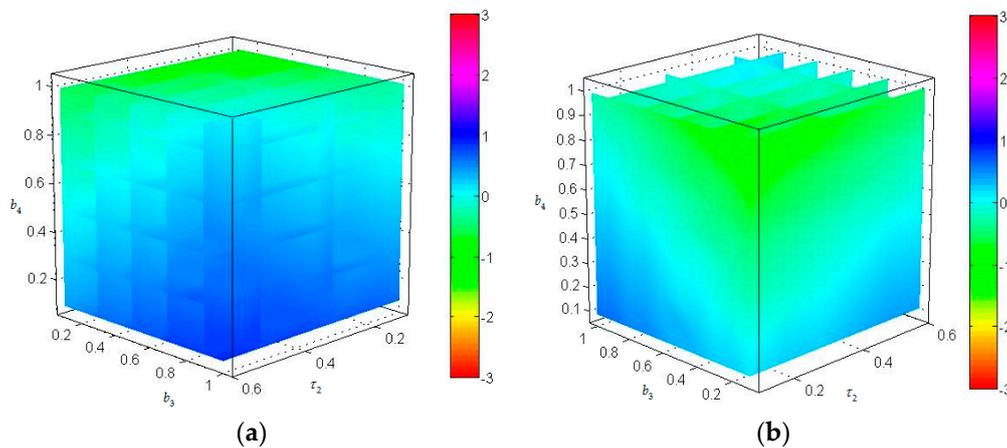
Figure 17. The influence of τ_2, b_3 and b_4 on y when $\tau_1 = 0.4$. (**a**,**b**) shown from different angles.

3.7. The Influence of τ_1, τ_2 on the Stability of Equation (19)

It can be seen from Figure 18 that the system loses stability if τ_1 shows an increase. In that case, x demonstrates a large range of ups and downs. On the other hand, τ_2 has no impact on x.
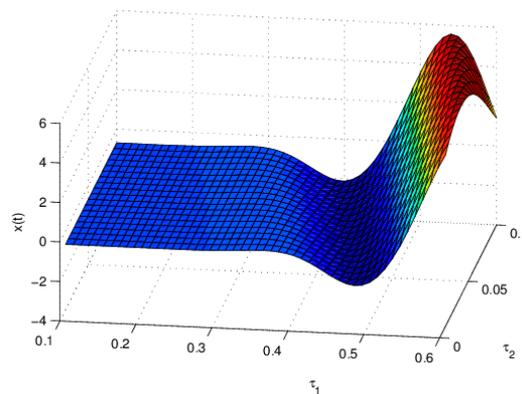
Figure 18. The influence of τ_1 and τ_2 on x when $\tau_1 \in [0.1, 0.6]$, $\tau_2 \in [0.01, 0.1]$.

3.8. The Influence of τ_1, τ_2 on the Entropy of Equation (19)

Figure 19 shows that the entropy becomes greater than 0 with an increase in τ_1. Meanwhile, the system is unstable. According to Section 3.4, $(\tau_1, \tau_2) = (0.4, 0.0622)$ is a bifurcation point, so we can infer that the point $(0.4, 0.0622)$ is on the boundary of the entropy change. τ_2 still has no effect on entropy. So keeping the other parameters constant, the system is stable when $\tau_1 < 0.4$.
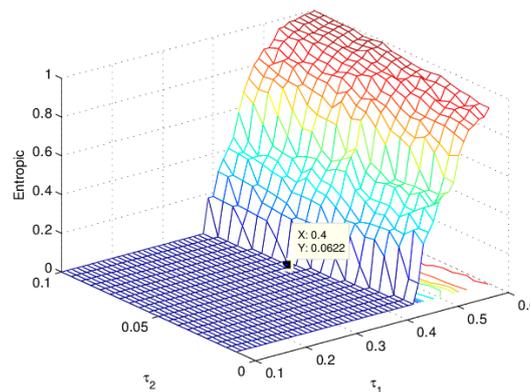

Figure 19. The influence of τ_1 and τ_2 on entropy of Equation (19) when $\tau_1 \in [0.1, 0.6]$, $\tau_2 \in [0.01, 0.1]$.

4. Bifurcation Control

The fact that the system loses stability at the bifurcation will seriously impact the effectiveness of the implementation of the policy. Thus we have to take measures to ensure the stability of the system. The system can be controlled by some approaches, for example, modified straight-line stabilization method [21], pole placement method [22], the OGY method (a control method of chaos proposed by Ott, Grebogi and Yorke) [23], time-delayed feedback method [24], and so forth. Here, we adopt the method of variable feedback control (see, e.g., [25], and references therein) to control for the Hopf bifurcation. The controlled system is given as follows:

$$
\begin{cases}
\dot{x}(t) = a_1 x(t)(\frac{y(t)}{M} - 1) - a_2 y(t) + a_3 z(t - \tau_1) - kx(t), \\
\dot{y}(t) = -b_1 x(t - \tau_2) + b_2 y(t)(1 - \frac{y(t)}{C}) + b_3 z(t - \tau_1)(1 - \frac{z(t-\tau_1)}{E}) - b_4 u(t) - ky(t), \\
\dot{z}(t) = c_1 x(t - \tau_2)(\frac{x(t-\tau_2)}{N} - 1) - c_2 y(t) - c_3 z(t) + c_4 u(t)(\frac{u(t)}{L} - 1), \\
\dot{u}(t) = d_1 y(t) + d_2 z(t)(\frac{z(t)}{K} - 1) - d_3 u(t),
\end{cases} \tag{20}
$$

where k is a feedback control parameter. The state of Equation (20) can be changed by adjusting the value of k. If k is greater than the critical value, then Equation (20) returns to the stable state, otherwise the system is still in an unstable state.

4.1. Bifurcation Value of Equation (20) to k

In Case 1, we know that Equation (2) demonstrates the bifurcation when $\tau_1 = 0.4618, \tau_2 = 0$ and it is unstable for $\tau_1 = 0.5, \tau_2 = 0$. It can be seen from Figures 4 and 5.

Figure 20 shows the dynamic evolution process of Equation (20) to k for $(\tau_1, \tau_2) = (0.5, 0)$. We find out that Equation (20) undergoes bifurcation at $k = 0.1689$. In other words, Equation (20) is unstable when $k < 0.1689$ and is stable when $k > 0.1689$.
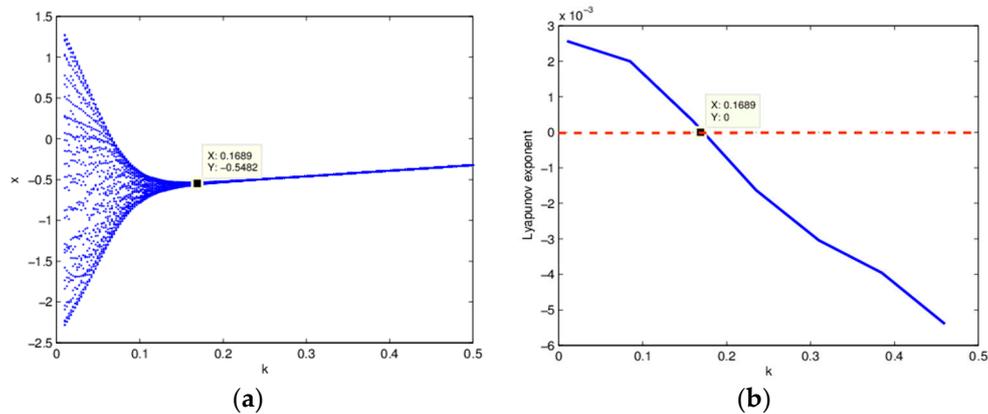

Figure 20. The influence of k on the stability of Equation (20) when $(\tau_1, \tau_2) = (0.5, 0)$. (**a**) bifurcation diagram; (**b**) the largest Lyapunov exponent plot.

4.2. Equation (20) is Unstable When k < 0.1689

Keep the values of other parameters unchanged and let $k = 0.05 < 0.1689$, and the time-domain plot and the EE attractor of Equation (20) are shown in Figure 21. We find out that the system (20) is unstable and has a basin of attraction. It fails to achieve bifurcation control.
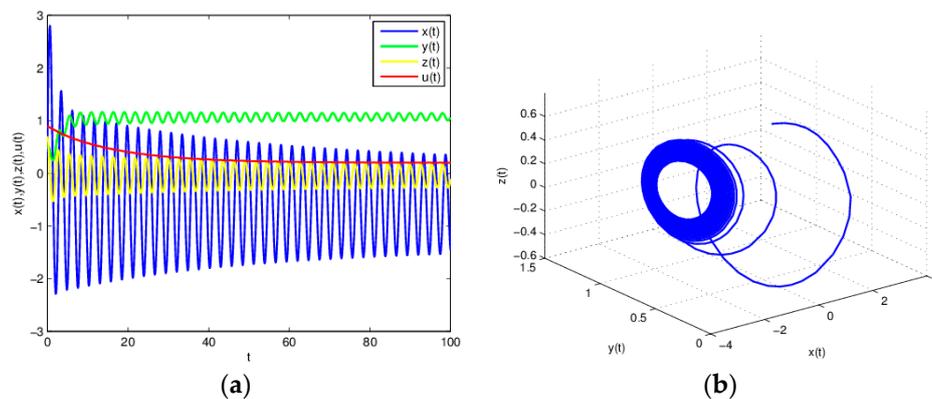

Figure 21. Equation (20) is unstable when $k = 0.05 < 0.1689$ for $(\tau_1, \tau_2) = (0.5, 0)$. (**a**) time-domain plot; (**b**) the EE attractor.

4.3. Equation (20) is Stable When k > 0.1689

When the other parameters are consistent with the original, let $k = 0.2 > 0.1689$ and the time-domain plot and the EE attractor of Equation (20) are displayed in Figure 22. We can see that Equation (20) maintains stable and x, y and z tend to equilibrium point. Therefore, bifurcation control is successful as $k > 0.1689$. We can conclude that the control effect will be larger if the control parameter k stands higher.
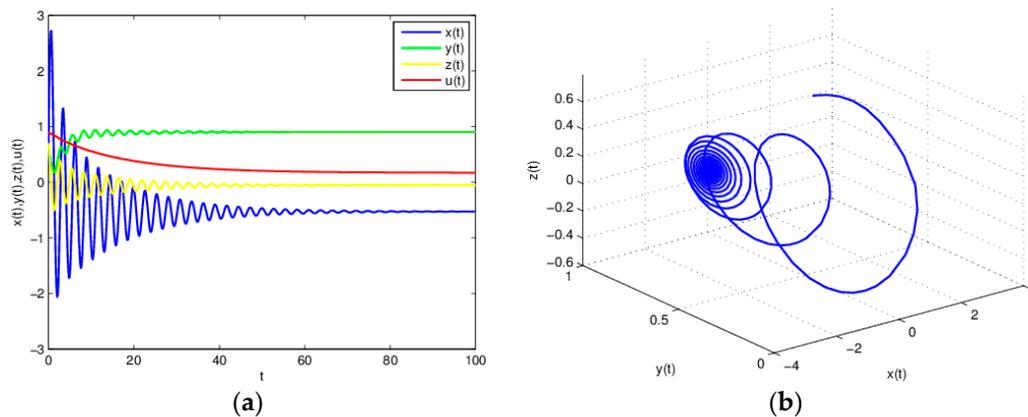
Figure 22. Equation (20) is stable when $k = 0.2 > 0.1689$ for $(\tau_1, \tau_2) = (0.5, 0)$. (**a**) time-domain plot; (**b**) the EE attractor.

Based on the above analysis, we assume that the unstable system can return to stability if effective control is exerted. This means that when problems arise from energy-saving and emission-reduction policies and lead to market volatility, we can take control measures to ensure that the system returns to a stable state.

5. Conclusions

In this paper, we examine the stability of a four dimensional energy-saving and emission-reduction model with two delays. We focus on the impacts of delays and the feedback control on the stability of the system. The system shifts from stable to unstable when the delays are larger than the bifurcation values. In this case, a large fluctuation shows up in the system and affects the implementation of energy-saving and emission reduction policies. However, we can control the bifurcation of the system by adopting the variable feedback control approach and the system will return to a stable state when the control parameters are set.

The results show that time delays play an important role in the stability of the system. Only when the energy-savings and emission-reduction system is stable, can the purpose of reducing carbon emissions be achieved. Therefore, we must ensure that the delay parameters are in a stable region and then add other parameters to maintain a long-term stable system.

According to the above analysis, if we want to keep the energy-saving and emission-reduction system running in a stable state, we can take the following strategies when other things are equal. First, when $\tau_2 = 0$, we must make sure that $\tau_1 < 0.4618$ or $b_4 > 0.5$. Second, when $\tau_1 = 0.4$, we must keep $\tau_2 < 0.0622$.

Author Contributions: Jing Wang performed mathematical derivation and numerical simulation; Yuling Wang built the economic model and provided economic interpretation of the conclusions; they wrote this research article together. Both authors have read and approved the final manuscript.

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