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New Derivatives on the Fractal Subset of Real-Line

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Abstract: In this manuscript we introduced the generalized fractional Riemann-Liouville and Caputo like derivative for functions defined on fractal sets. The Gamma, Mittag-Leffler and Beta functions were defined on the fractal sets. The non-local Laplace transformation is given and applied for solving linear and non-linear fractal equations. The advantage of using these new nonlocal derivatives on the fractals subset of real-line lies in the fact that they are better at modeling processes with memory effect.

Keywords: fractal calculus; triadic Cantor set; non-local Laplace transformation; memory processes; generalized Mittag-Leffler function; generalized gamma function; generalized beta function

1. Introduction

The calculus involving arbitrary orders of derivatives and integrals is called fractional calculus. Recently, fractional calculus has found many applications in several areas of science and engineering [1–6]. The nonlocal property of the fractional derivatives and integrals is used to model the processes with memory effect [1,2]. For example, the fractional derivatives are used to model more appropriately the dynamics of the non-conservative systems in Hamilton, Lagrange and Nambu mechanics [7–10]. The continuous but non-differentiable functions admit the local fractional derivatives [11]. The local fractional derivative gives a measurement of fractal sets. Consequently, recently, the F^{α}-calculus on the fractal subset of real line and fractal curves is built as a framework [12,13]. Fractal analysis has been established by many researchers by using different methods [14–17]. Using F^{α}-calculus the Newton, Lagrange and Hamilton mechanics were built on fractal sets [18,19]. Also, Schrödinger's equation on a fractal curve was derived in [20–22]. Motivated by the above-mentioned interesting results, in this work, we define the non-local derivative on fractal sets. These new derivatives can be successfully used to derive new mathematical models on fractal sets involving processes with memory.

We organize our manuscript as follows:

In Section 2, we give a brief exposition of F^{α}-calculus and defined fractal Gamma and Beta functions. In Section 3 we define the non-local derivative on fractals as generalized Riemann-Liouville and Caputo fractional derivatives. In Section 4, Mittag-Leffler function and non-local Laplace fractional on fractal sets are introduced. We solve the non-local differential equations on fractal using the suggested methods. Section 5 is devoted to our conclusion.

2. A Review of Fractional Local Derivatives

In this section, we review the F^α-calculus [12,13].

Calculus on Fractal Subset of Real-Line

Fractal geometry is the geometry of the real world [1]. Fractal shape is an object with fractional dimension and the self similarity property [9,10]. In a seminal paper, Parvate and Gangal established a calculus on fractals which is similar to Riemann integration. The suggested framework became a mathematical model for many phenomena in fractal media [12,13]. We recall that the triadic Cantor set is a fractal that can be obtained by an iterative process. In Figure 1 we show the Triadic Cantor set [14].

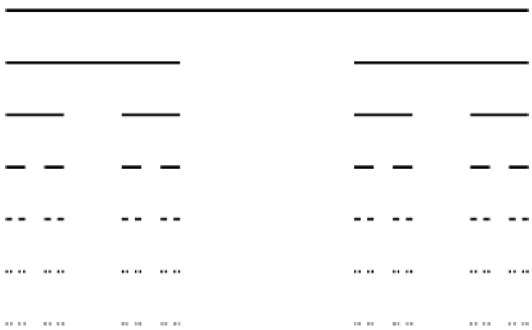

Figure 1. The finite iteration for constructing the triadic Cantor set.

The integral staircase function for the triadic Cantor set is defined as [12,13].

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F,a,b), & \text{if } x \geqslant a_0, \\ -\gamma^\alpha(F,a,b), & \text{othewise.} \end{cases} \quad (1)$$

where α is the γ-dimension of triadic Cantor set. In Figure 2 we plot the integral staircase function for a triadic Cantor set.

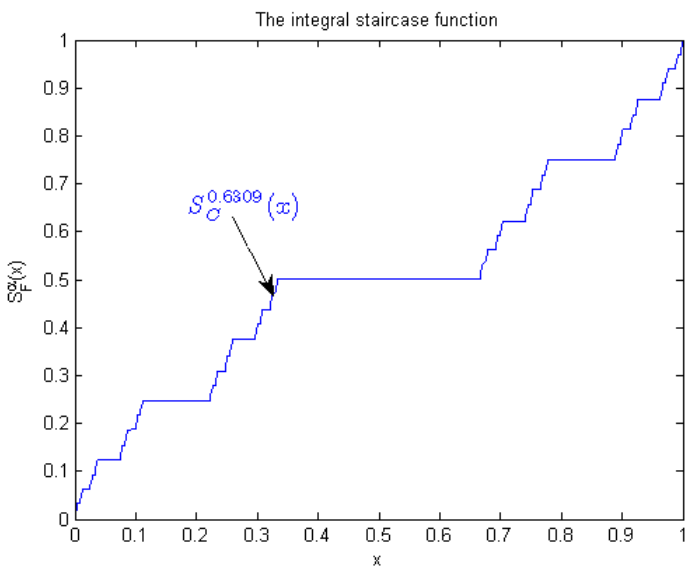

Figure 2. We plot the integral staircase function for triadic Cantor.

The definitions of F^α-limit, F^α-continuity and F^α-integration are given in the ref. [12,13]. The F^α-differentiation is denoted by D_F^α and it is defined as

$$D_F^\alpha f(x) = \begin{cases} F - \lim_{y\to x} \dfrac{f(y)-f(x)}{S_F^\alpha(y)-S_F^\alpha(x)}, & \text{if } x \in F, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

if the limit exists [12,13].

Definition 1. The Gamma function with the fractal support is defined as

$$\Gamma_F^\alpha(x) = \int_{S_F^\alpha(0)}^{S_F^\alpha(\infty)} e^{-S_F^\alpha(t)} S_F^\alpha(t)^{S_F^\alpha(x)-1} d_F^\alpha t, \tag{3}$$

where

$$e^{-S_F^\alpha(t)} = F - \lim_{n\to\infty}(1 - \frac{S_F^\alpha(t)}{n})^n. \tag{4}$$

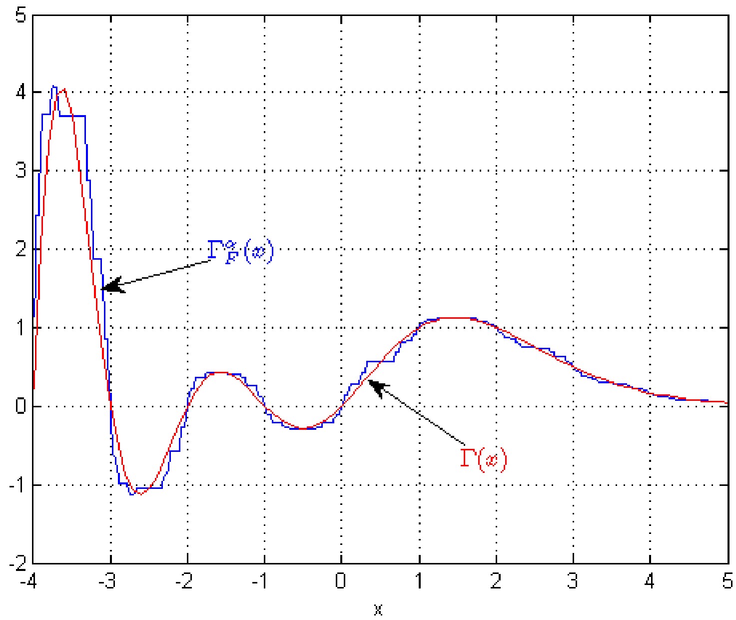

Figure 3. We sketch the fractal Gamma function which is compared with the standard case.

Definition 2. The fractal Beta function on the fractal set is defined as follows

$$B_F^\alpha(r,w) = \int_0^1 S_F^\alpha(\zeta)^{r-1}(1 - S_F^\alpha(\zeta))^{w-1} d_F^\alpha \zeta, \tag{5}$$

which is called two-parameter (r,w) fractal integral, where $\Re(r) > 0$ and $\Re(w) > 0$.

In the following we present some properties of fractal Beta function.

(1) The fractal Beta function has a symmetry $B_F^\alpha(w,r) = B_F^\alpha(r,w)$. Since, we have

$$B_F^\alpha(r,w) = \int_{S_F^\alpha(0)}^{S_F^\alpha(1)} (S_F^\alpha(x))^{r-1}(1 - S_F^\alpha(x))^{w-1} d_F^\alpha x, \tag{6}$$

using the transformation $S_F^\alpha(x) = 1 - S_F^\alpha(y)$, we conclude that

$$B_F^\alpha(r,w) = \int_{S_F^\alpha(0)}^{S_F^\alpha(1)} (1 - S_F^\alpha(y))^{r-1} (S_F^\alpha(y))^{w-1} d_F^\alpha y = B_F^\alpha(w,r). \tag{7}$$

(2) Using the transformation $S_F^\alpha(x) = \sin^2(S_F^\alpha(\theta))$, we get following form for the fractal the Beta function

$$B_F^\alpha(r,w) = \int_{S_F^\alpha(0)}^{S_F^\alpha(\pi/2)} \sin^2(S_F^\alpha(\theta))^{r-1} \cos^2(S_F^\alpha(\theta))^{w-1} (2\sin(S_F^\alpha(\theta))\cos(S_F^\alpha(\theta))) d_F^\alpha x, \tag{8}$$

$$= 2\int_{S_F^\alpha(0)}^{S_F^\alpha(\pi/2)} \sin^{2r-1}(S_F^\alpha(\theta)) \cos^{2w-1}(S_F^\alpha(\theta)) d_F^\alpha x. \tag{9}$$

(3) The Beta fractal function is related to the fractal Gamma function as

$$B_F^\alpha(r,w) = \frac{\Gamma_F^\alpha(r)\Gamma_F^\alpha(w)}{\Gamma_F^\alpha(r+w)}. \tag{10}$$

Proof. We have

$$\Gamma_F^\alpha(r)\Gamma_F^\alpha(w) = 4\int_{S_F^\alpha(0)}^{S_F^\alpha(\infty)} (S_F^\alpha(x))^{2r-1} (S_F^\alpha(y))^{2w-1} e^{S_F^\alpha(x)^2 + S_F^\alpha(y)^2} d_F^\alpha x d_F^\alpha y. \tag{11}$$

Transforming to the polar coordinates $S_F^\alpha(x) = S_F^\alpha(\rho)\cos(S_F^\alpha(\phi))$, $\quad S_F^\alpha(x) = S_F^\alpha(\rho)\sin(S_F^\alpha(\phi))$ we obtain

$$\Gamma_F^\alpha(r)\Gamma_F^\alpha(w) = 4\int_{S_F^\alpha(0)}^{S_F^\alpha(\infty)} (S_F^\alpha(\rho))^{2(r+w)-1} e^{-S_F^\alpha(\rho)^2} d_F^\alpha \rho \int_{S_F^\alpha(0)}^{S_F^\alpha(\pi/2)} \cos^{2r-1}(S_F^\alpha(\phi)) \sin^{2w-1}(S_F^\alpha(\phi)) d_F^\alpha \phi,$$

$$= B_F^\alpha(r,w)\Gamma_F^\alpha(r+w). \tag{12}$$

Thus, the proof is completed. □

3. Non-Local Fractal Derivative and Integral

In this section, we define the non-local derivative for the functions with fractal support.

Definition 3. If $f(x) \in C_F^\alpha[a,b]$ (α-order differentiable function on $[a,b]$) and $\beta > 0$ then we have

$$_a\mathcal{I}_x^\beta f(x) := \frac{1}{\Gamma_F^\alpha(\beta)} \int_{S_F^\alpha(a)}^{S_F^\alpha(x)} \frac{f(t)}{(S_F^\alpha(x) - S_F^\alpha(t))^{\alpha-\beta}} d_F^\alpha t, \quad S_F^\alpha(x) > S_F^\alpha(a), \tag{13}$$

where if $\beta = \alpha$ then we have fractal integral whose order is equal the dimension of the fractal, and

$$_x\mathcal{I}_b^\beta f(x) := \frac{1}{\Gamma_F^\alpha(\beta)} \int_{S_F^\alpha(x)}^{S_F^\alpha(b)} \frac{f(t)}{(S_F^\alpha(x) - S_F^\alpha(t))^{\alpha-\beta}} d_F^\alpha t, \quad S_F^\alpha(x) < S_F^\alpha(b), \tag{14}$$

are called the analogous left sided and the right sided Riemann-Liouville fractal integral of order β.

Definition 4. Let $n - \alpha \leqslant \beta < n$, then the analogous left and right Riemann-Liouville fractal derivative are defined as follows:

$$_a\mathcal{D}_x^\beta f(x) := \frac{1}{\Gamma_F^\alpha(n-\beta)} (D_F^\alpha)^n \int_{S_F^\alpha(a)}^{S_F^\alpha(x)} \frac{f(t)}{(S_F^\alpha(x) - S_F^\alpha(t))^{-n+\beta+\alpha}} d_F^\alpha t, \tag{15}$$

$$_x\mathcal{D}_b^\beta f(x) := \frac{1}{\Gamma_F^\alpha(n-\beta)} (-D_F^\alpha)^n \int_{S_F^\alpha(x)}^{S_F^\alpha(b)} \frac{f(t)}{(S_F^\alpha(t) - S_F^\alpha(x))^{-n+\beta+\alpha}} d_F^\alpha t. \tag{16}$$

Definition 5. Let $f(x) \in C^{\alpha n}[a,b]$, then the analogous left sided Caputo fractal derivative is defined by

$$ {}_a^C\mathcal{D}_x^{\beta} f(x) := \frac{1}{\Gamma_F^{\alpha}(n-\beta)} \int_{S_F^{\alpha}(a)}^{S_F^{\alpha}(x)} (S_F^{\alpha}(x) - S_F^{\alpha}(t))^{n-\beta-\alpha} (D_F^{\alpha})^n f(t) d_F^{\alpha} t, \quad n = \max(0, -[-\beta]). \tag{17} $$

Also, the analogous right sided Caputo fractal derivative has the form

$$ {}_x^C\mathcal{D}_b^{\beta} f(x) := \frac{1}{\Gamma_F^{\alpha}(n-\beta)} \int_{S_F^{\alpha}(x)}^{S_F^{\alpha}(b)} (S_F^{\alpha}(t) - S_F^{\alpha}(x))^{n-\beta-\alpha} (-D_F^{\alpha})^n f(t) d_F^{\alpha} t. \tag{18} $$

Now, we give some important relations, namely

$$ {}_a\mathcal{I}_x^{\beta} (S_F^{\alpha}(x) - S_F^{\alpha}(a))^{\eta} = \frac{\Gamma_F^{\alpha}(\eta+1)}{\Gamma_F^{\alpha}(\eta+\beta+1)} (S_F^{\alpha}(x) - S_F^{\alpha}(a))^{\eta+\beta}, \quad \eta > -1. \tag{19} $$

Proof. Using the Equation (13) we conclude

$$ {}_a\mathcal{I}_x^{\beta} (S_F^{\alpha}(x) - S_F^{\alpha}(a))^{\eta} = \frac{1}{\Gamma_F^{\alpha}(\beta)} \int_{S_F^{\alpha}(a)}^{S_F^{\alpha}(x)} (S_F^{\alpha}(x) - S_F^{\alpha}(t))^{\beta-1} (S_F^{\alpha}(t) - S_F^{\alpha}(a))^{\eta} d_F^{\alpha} t. \tag{20} $$

Let us consider

$$ S_F^{\alpha}(\xi) = \frac{S_F^{\alpha}(t) - S_F^{\alpha}(a)}{S_F^{\alpha}(x) - S_F^{\alpha}(a)}, \qquad d_F^{\alpha} t = (S_F^{\alpha}(x) - S_F^{\alpha}(a)) d_F^{\alpha} \xi. \tag{21} $$

Therefore, we have $S_F^{\alpha}(\xi) : S_F^{\alpha}(0) \to S_F^{\alpha}(1)$ while $S_F^{\alpha}(t) : S_F^{\alpha}(a) \to S_F^{\alpha}(x)$. As a result we obtain

$$ S_F^{\alpha}(x) - S_F^{\alpha}(t) = \frac{S_F^{\alpha}(1) - S_F^{\alpha}(\xi)}{S_F^{\alpha}(\xi)} (S_F^{\alpha}(t) - S_F^{\alpha}(0)). \tag{22} $$

Substituting Equations (21) and (22) in Equation (20) we conclude that

$$ \begin{aligned} {}_a\mathcal{I}_x^{\beta} (S_F^{\alpha}(x) - S_F^{\alpha}(a))^{\eta} &= \frac{1}{\Gamma_F^{\alpha}(\beta)} \int_{S_F^{\alpha}(0)}^{S_F^{\alpha}(1)} (1 - S_F^{\alpha}(\xi))^{\beta-1} S_F^{\alpha}(\xi)^{1-\beta} (S_F^{\alpha}(t) - S_F^{\alpha}(a))^{\beta+\eta-1} (S_F^{\alpha}(x) - S_F^{\alpha}(a)) d_F^{\alpha} \xi, \\ &= \frac{1}{\Gamma_F^{\alpha}(\beta)} \int_{S_F^{\alpha}(0)}^{S_F^{\alpha}(1)} (1 - S_F^{\alpha}(\xi))^{\beta-1} \left(\frac{S_F^{\alpha}(t) - S_F^{\alpha}(a)}{S_F^{\alpha}(x) - S_F^{\alpha}(a)} \right)^{1-\beta} (S_F^{\alpha}(t) - S_F^{\alpha}(a))^{\beta+\eta-1} (S_F^{\alpha}(x) - S_F^{\alpha}(a)) d_F^{\alpha} \xi. \end{aligned} \tag{23} $$

Then, we have

$$ {}_a\mathcal{I}_x^{\beta} (S_F^{\alpha}(x) - S_F^{\alpha}(a))^{\eta} = \frac{(S_F^{\alpha}(x) - S_F^{\alpha}(a))^{\beta+\eta}}{\Gamma_F^{\alpha}(\beta)} \int_{S_F^{\alpha}(0)}^{S_F^{\alpha}(1)} (1 - S_F^{\alpha}(\xi))^{\beta-1} (S_F^{\alpha}(\xi))^{\eta} d_F^{\alpha} \xi. \tag{24} $$

In view of Equation (5) we derive

$$ {}_a\mathcal{I}_x^{\beta} (S_F^{\alpha}(x) - S_F^{\alpha}(a))^{\eta} = \frac{(S_F^{\alpha}(x) - S_F^{\alpha}(a))^{\beta+\eta}}{\Gamma_F^{\alpha}(\beta)} B_F^{\alpha}(\beta, \eta+1). \tag{25} $$

Applying Equation (10) we get

$$ \begin{aligned} {}_a\mathcal{I}_x^{\beta} (S_F^{\alpha}(x) - S_F^{\alpha}(a))^{\eta} &= \frac{(S_F^{\alpha}(x) - S_F^{\alpha}(a))^{\beta+\eta}}{\Gamma_F^{\alpha}(\beta)} \frac{\Gamma_F^{\alpha}(\beta)\Gamma_F^{\alpha}(\eta+1)}{\Gamma_F^{\alpha}(\beta+\eta+1)}, \\ &= \frac{\Gamma_F^{\alpha}(\eta+1)}{\Gamma_F^{\alpha}(\beta+\eta+1)} (S_F^{\alpha}(x) - S_F^{\alpha}(a))^{\beta+\eta}. \end{aligned} \tag{26} $$

□

Now, we consider following formula

$$ {}_a\mathcal{D}_x^{\beta}(S_F^{\alpha}(x)-S_F^{\alpha}(a))^{\eta} = \frac{\Gamma_F^{\alpha}(\eta+1)}{\Gamma_F^{\alpha}(\eta+1-\beta)}(S_F^{\alpha}(x)-S_F^{\alpha}(a))^{\eta-\beta}. \quad (27) $$

Proof. By rewriting the Equation (27) we get

$$ {}_a\mathcal{D}_x^{\beta}(S_F^{\alpha}(x)-S_F^{\alpha}(a))^{\eta} = (D_F^{\alpha})^n \, {}_a\mathcal{I}_x^{n-\beta}(S_F^{\alpha}(x)-S_F^{\alpha}(a))^{\eta}. \quad (28) $$

Utilizing the Equation (19) we conclude

$$ {}_a\mathcal{D}_x^{\beta}(S_F^{\alpha}(x)-S_F^{\alpha}(a))^{\eta} = \frac{\Gamma_F^{\alpha}(\eta+1)}{\Gamma_F^{\alpha}(\eta+n-\beta+1)}(D_F^{\alpha})^n(S_F^{\alpha}(x)-S_F^{\alpha}(a))^{\eta+n-\beta}, \quad (29) $$

$$ = \frac{\Gamma_F^{\alpha}(\eta+1)}{\Gamma_F^{\alpha}(\eta-\beta+1)}(D_F^{\alpha})^n(S_F^{\alpha}(x)-S_F^{\alpha}(a))^{\eta-\beta}, \quad \eta > -1. \quad (30) $$

□

Now, we write some important composition relations, namely

$$ {}_a\mathcal{I}_x^{\beta} \, {}_a\mathcal{D}_x^{\beta} f(x) = f(x) - \sum_{j=1}^{n} \frac{({}_a\mathcal{D}_x^{\beta-j} f(x))|_{(S_F^{\alpha}(a))}}{\Gamma_F^{\alpha}(\beta+1-j)}(S_F^{\alpha}(x)-S_F^{\alpha}(a))^{\beta-j}. \quad (31) $$

Proof. Using the definitions we get

$$ {}_a\mathcal{I}_x^{\beta} \, {}_a\mathcal{D}_x^{\beta} f(x) = \frac{1}{\Gamma_F^{\alpha}(\beta)} \int_{S_F^{\alpha}(a)}^{S_F^{\alpha}(x)} (S_F^{\alpha}(x)-S_F^{\alpha}(t))^{\beta-1} \mathcal{D}_x^{\beta} f(t) d_F^{\alpha} t \quad (32) $$

$$ = \frac{1}{\Gamma_F^{\alpha}(\beta+1)} D_F^{\alpha} \int_{S_F^{\alpha}(a)}^{S_F^{\alpha}(x)} (S_F^{\alpha}(x)-S_F^{\alpha}(t))^{\beta} (D_F^{\alpha})^n \, {}_a\mathcal{I}_x^{n-\beta} f(t) d_F^{\alpha} t f(t) d_F^{\alpha} t. \quad (33) $$

Applying, n-times integration by part it leads to

$$ {}_a\mathcal{I}_x^{\beta} \, {}_a\mathcal{D}_x^{\beta} f(x) = D_F^{\alpha} \, {}_a\mathcal{I}_x^{\beta+1-n}({}_a\mathcal{I}_x^{n-\beta} f(x)) - \sum_{k=1}^{n} \frac{(D_F^{\alpha})^{n-k} \, {}_a\mathcal{D}_x^{\beta-n} f(x)|_{S_F^{\alpha}(a)}}{\Gamma_F^{\alpha}(\beta-k+1)}(S_F^{\alpha}(x)-S_F^{\alpha}(a))^{\beta-k}, $$

$$ = f(x) - \sum_{k=1}^{n} \frac{{}_a\mathcal{D}_x^{\beta-k} f(x)|_{S_F^{\alpha}(a)}}{\Gamma_F^{\alpha}(\beta-k+1)}(S_F^{\alpha}(x)-S_F^{\alpha}(a))^{\beta-k}. \quad (34) $$

□

The similar proof works for the following formulas

$$ {}_x\mathcal{I}_b^{\beta} \, {}_x\mathcal{D}_b^{\beta} f(x) = f(x) - \sum_{j=1}^{n} \frac{({}_x\mathcal{D}_b^{\beta-j} f(x))|_{(S_F^{\alpha}(b))}}{\Gamma_F^{\alpha}(\beta+1-j)}(S_F^{\alpha}(b)-S_F^{\alpha}(x))^{\beta-j}, \quad (35) $$

$$ {}_a\mathcal{I}_x^{\beta} \, {}_a^{C}\mathcal{D}_x^{\beta} f(x) = f(x) - \sum_{j=1}^{n} \frac{((D_F^{\alpha})^j f(x))|_{(S_F^{\alpha}(a))}}{\Gamma_F^{\alpha}(j+1)}(S_F^{\alpha}(x)-S_F^{\alpha}(a))^{j}, \quad (36) $$

$$ {}_x\mathcal{I}_b^{\beta} \, {}_x^{C}\mathcal{D}_b^{\beta} f(x) = f(x) - \sum_{j=1}^{n} \frac{((D_F^{\alpha})^j f(x))|_{(S_F^{\alpha}(b))}}{\Gamma_F^{\alpha}(j+1)}(S_F^{\alpha}(b)-S_F^{\alpha}(x))^{j}. \quad (37) $$

In Figures 4 and 5 we compared the non-local standard derivative versus non-local fractal derivative and the generalized fractal integral.

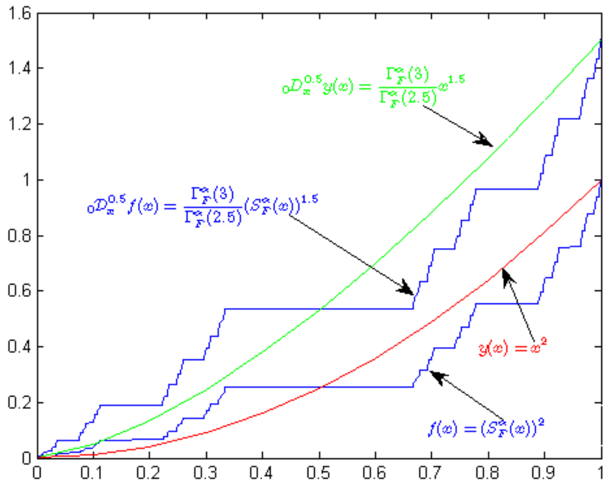

Figure 4. We plot $y(x) = x^2$ and $f(x) = S_F^\alpha(x)^2$ and their non-local derivative ${}_0D_x^{0.5}y(x)$ and ${}_0\mathcal{D}_x^{0.5}f(x)$, respectively.

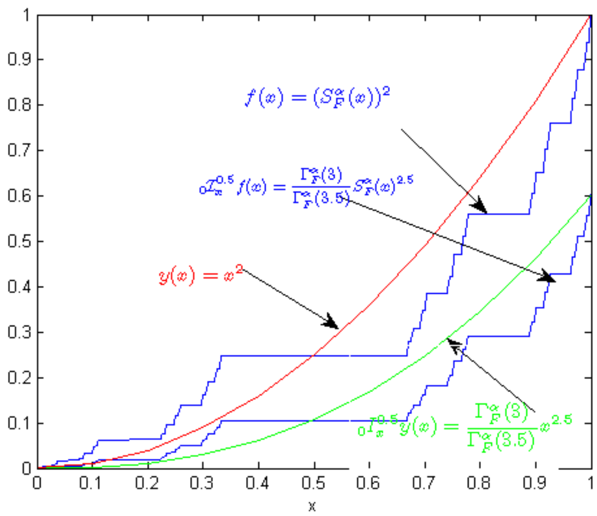

Figure 5. We show the graph of $g(x) = x^2$ and $f(x) = S_F^\alpha(x)^2$ and their non-local integral ${}_0I_x^{0.5}g(x)$ and ${}_0\mathcal{I}_x^{0.5}f(x)$, respectively.

4. Generalized Functions in the Non-Local Calculus on the Fractal Subset of Real-Line

In this section, we suggest the mathematical tools for solving the non-local fractal differential equations.

4.1. Gamma Function on Fractal Subset of Real Line

Now, we define the Gamma function for the fractal calculus that will be used in non-local calculus on fractals.

4.2. Mittag-Leffler Function on Fractal Subset of Real-Line

It is well known that the exponential function has an important role in the theory of standard differential equation. The generalized exponential function is called the Mittag-Leffer function and plays an important role in fractional differential equations [1].

Definition 6. The generalized two parameter η, ν Mittag-Liffler function on fractal F with α-dimension is defined as

$$E^{\alpha}_{F,\eta,\nu}(x) = \sum_{k=0}^{\infty} \frac{S^{\alpha}_{F}(x)^{k}}{\Gamma^{\alpha}_{F}(\eta k + \nu)}, \quad \eta > 0, \;\; \nu \in \Re. \tag{38}$$

In some special cases we have the following results, namely

$$E^{\alpha}_{F,1,1}(x) = e^{S^{\alpha}_{F}(x)}, \tag{39}$$

$$E^{\alpha}_{F,1,2}(x) = \frac{e^{S^{\alpha}_{F}(x)-1}}{S^{\alpha}_{F}(x)}, \tag{40}$$

$$E^{\alpha}_{F,2,1}(x) = \cosh(S^{\alpha}_{F}(x)), \tag{41}$$

$$E^{\alpha}_{F,2,2}(x) = \frac{\sinh(S^{\alpha}_{F}(x))}{S^{\alpha}_{F}(x)}. \tag{42}$$

4.3. Non-Local Laplace Transformation on Fractal Subset of Real-Line

The Laplace transformation is a very useful tool for solving a standard linear differential equation with constant coefficients. The generalized Laplace transformation is applied to solve the fractional differential equations. Thus, in this section, we generalized the Laplace transformation for the function with fractal support which is utilized to solve the non-local differential equation on the fractal set [1].

Definition 7. Laplace transformation for the function $f(x)$ is denoted by $F(s)$ and it is defined as

$$\mathcal{F}^{\alpha}_{F}(S^{\alpha}_{F}(s)) = \mathcal{L}^{\alpha}_{F}[f(x)] = \int_{S^{\alpha}_{F}(0)}^{S^{\alpha}_{F}(\infty)} f(x) e^{-S^{\alpha}_{F}(s) S^{\alpha}_{F}(x)} d^{\alpha}_{F}x. \tag{43}$$

Now, we give the fractal Laplace transformation of some functions. If we define the fractal convolution of two function $f(x)$ and $g(x)$ as follows:

$$f(x)g(x) = \int_{S^{\alpha}_{F}(0)}^{S^{\alpha}_{F}(x)} f(S^{\alpha}_{F}(x) - S^{\alpha}_{F}(\tau)) g(S^{\alpha}_{F}(\tau)) d^{\alpha}_{F}\tau, \tag{44}$$

the fractal Laplace transformation of power function of $S^{\alpha}_{F}(x)$ is

$$\mathcal{L}^{\alpha}_{F}[S^{\alpha}_{F}(x)] \int_{S^{\alpha}_{F}(0)}^{S^{\alpha}_{F}(\infty)} S^{\alpha}_{F}(x)^{\beta} e^{-S^{\alpha}_{F}(s) S^{\alpha}_{F}(x)} d^{\alpha}_{F}x = \frac{\Gamma^{\alpha}_{F}(1+\beta)}{s^{\beta+1}}. \tag{45}$$

Lemma 1. *The Laplace transformation of the non-local fractal Riemann-Liouville integral is given by*

$$\mathcal{L}^{\alpha}_{F}[{}_{0}\mathcal{I}^{\beta}_{x} f(x)] = \frac{\mathcal{F}^{\alpha}_{F}(S^{\alpha}_{F}(s))}{S^{\alpha}_{F}(s)^{\beta}}. \tag{46}$$

Proof. The Laplace transform of the fractal Riemann-Liouville integral is

$$\mathcal{L}^{\alpha}_{F}[{}_{0}\mathcal{I}^{\beta}_{x} f(x)] = \mathcal{L}^{\alpha}_{F}\left[\frac{1}{\Gamma^{\alpha}_{F}(\beta)} \int_{S^{\alpha}_{F}(0)}^{S^{\alpha}_{F}(x)} \frac{f(t)}{(S^{\alpha}_{F}(x) - S^{\alpha}_{F}(t))^{\alpha-\beta}} d^{\alpha}_{F}t\right]. \tag{47}$$

Using the Equations (44) and (45) we arrive at

$$\begin{aligned} \mathcal{L}_F^{\alpha}[{}_0\mathcal{I}_x^{\beta} f(x)] &= \frac{1}{\Gamma_F^{\alpha}(\beta)} \mathcal{F}_F^{\alpha}(S_F^{\alpha}(s)) \mathcal{L}_F^{\alpha}[S_F^{\alpha}(x)^{\beta-1}], \\ &= \frac{1}{\Gamma_F^{\alpha}(\beta)} \mathcal{F}_F^{\alpha}(S_F^{\alpha}(s)) \frac{\Gamma_F^{\alpha}(\beta)}{S_F^{\alpha}(s)^{\beta}}, \\ &= \frac{\mathcal{F}_F^{\alpha}(S_F^{\alpha}(s))}{S_F^{\alpha}(s)^{\beta}}. \end{aligned} \tag{48}$$

□

The fractal Laplace transform of the non-local fractal Riemann-Liouville derivative of order $\beta \in [0,1)$ is given by

$$\mathcal{L}_F^{\alpha}\{{}_0\mathcal{D}_x^{\beta} f(x), x, s\} = S_F^{\alpha}(s)^{\beta} \mathcal{F}_F^{\alpha}(s) - \sum_{k=1}^{n} S_F^{\alpha}(s)^{n-k}\, {}_0\mathcal{D}_x^{\beta-n+k-1} f(x)|_{S_F^{\alpha}(0)}, \tag{49}$$

where $n = [\beta] + 1$. The fractal Laplace transform of the non-local fractal Caputo derivative of order $\beta \in [0,1)$ is given by

$$\mathcal{L}_F^{\alpha}\{{}_0^C\mathcal{D}_x^{\beta} f(x), x, s\} = (S_F^{\alpha}(s))^{\beta} \mathcal{F}_F^{\alpha}(s) - \sum_{k=1}^{n} S_F^{\alpha}(s)^{\beta-k}\, {}_0\mathcal{D}_x^{k-1} f(x)|_{S_F^{\alpha}(0)}. \tag{50}$$

where $n = \max(0, -[-\beta])$.

5. Non-Local Fractal Differential Equations

In this section, we solve some illustrative examples.

Example 1. Consider the following linear fractal equation

$${}_0^C\mathcal{D}_x^{\frac{1}{2}} y(x) = 2, \tag{51}$$

with the initial condition

$$D_F^{\alpha} y(x)|_{S_F^{\alpha}(0)=0} = 1, \tag{52}$$

where $\alpha = 0.6309$ is Cantor set dimension. By applying ${}_0\mathcal{I}_x^{\frac{1}{2}}$ on the both sides of the Equation (52) we obtain

$$y(x) = \frac{1}{\Gamma_F^{\alpha}(1+\frac{1}{2})} S_F^{\alpha}(x) + \frac{2}{\Gamma_F^{\alpha}(1-\frac{1}{2})} S_F^{\alpha}(x)^{-\frac{1}{2}}. \tag{53}$$

Example 2. Consider a linear fractal differential equation

$${}_0^C\mathcal{D}_x^{\frac{1}{2}} y(x) = 1 - S_F^{\alpha}(x), \quad S_F^{\alpha}(x) \geqslant 1, \tag{54}$$

with initial condition as

$$D_F^{\alpha} y(x)|_{S_F^{\alpha}(1)} = 0. \tag{55}$$

By applying ${}_0\mathcal{I}_x^{\frac{1}{2}}$ on the both sides of the Equation (55) we arrive at

$$y(x) = -\frac{\Gamma_F^\alpha(2)}{\Gamma_F^\alpha(2+\frac{1}{2})}(S_F^\alpha(x)-1)^{1+\frac{1}{2}}, \quad S_F^\alpha(x) \geqslant 1. \tag{56}$$

In Figures 6 and 7, we plot the solutions of Equations (51) and (54) , respectively.

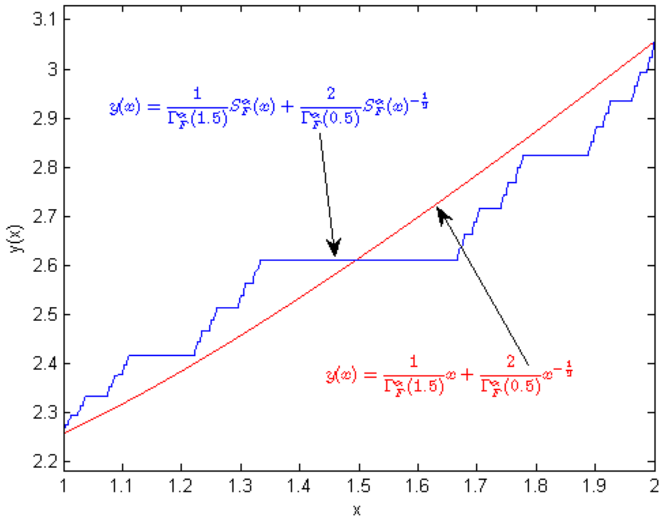

Figure 6. We present the solution of Equation (51) on the real-line and Cantor set.

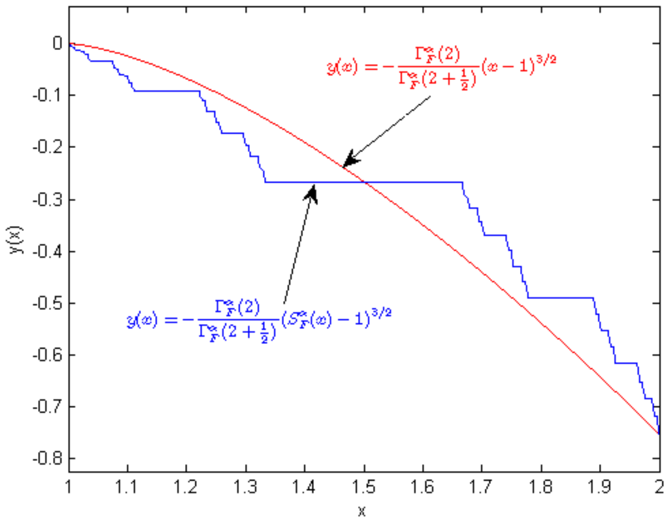

Figure 7. We give the graph of the solution of Equation (54) on the real-line and Cantor set.

Example 3. Consider a linear differential equation

$$_0\mathcal{D}_x^{\frac{1}{2}} y(x) = y(x), \tag{57}$$

with the following initial condition, namely

$$_0\mathcal{D}_x^{-\frac{1}{2}} y(x)|_{S_F^\alpha(0)} = 1. \tag{58}$$

By inspection, the solution for the Equation (57) becomes

$$y(x) = S_F^{\alpha}(x)^{-\frac{1}{2}} E^{\alpha}_{F,1/2,1/2}(-\sqrt{S_F^{\alpha}(x)}). \tag{59}$$

In Figure 8 we sketched the solution of Equation (57) on the Cantor set and real-line.

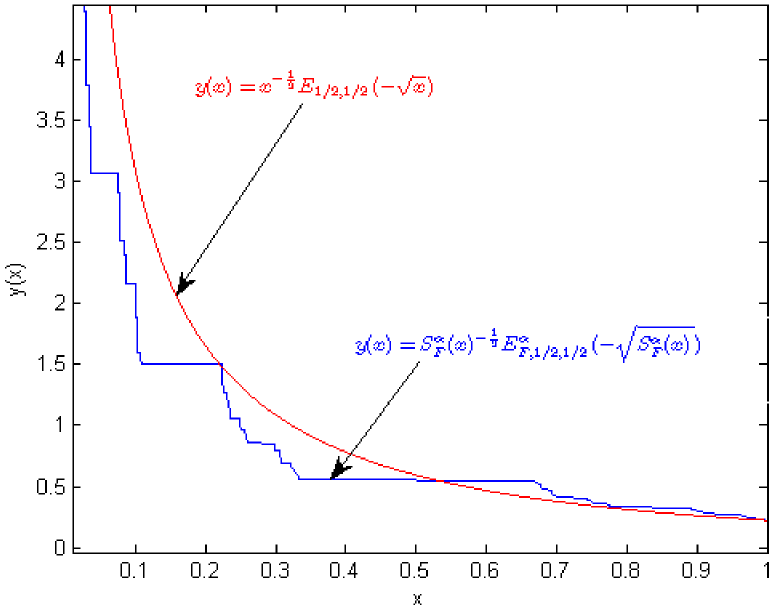

Figure 8. We plot the solution of Equation (57) on the real-line and Cantor set.

Example 4. We examine the following non-local differential equation on a fractal subset of real-line, namely with the following initial condition

$${}_0\mathcal{D}_F^{\frac{4}{3}} y(x) - \lambda y(x) = (S_F^{\alpha}(x))^2, \tag{60}$$

$${}_0\mathcal{D}_F^{\frac{1}{3}} y(x)|_{S_F^{\alpha}(0)} = 1, \qquad {}_0\mathcal{D}_F^{\frac{-1}{6}} y(x)|_{S_F^{\alpha}(0)} = 2. \tag{61}$$

For solving Equation (60) we apply the fractal Laplace transformation on both side of it and we get

$$S_F^{\alpha}(s)^{\frac{4}{3}} \mathcal{F}_F^{\alpha}(s) - 1 - 2(S_F^{\alpha}(s))^{\frac{1}{2}} - \lambda \mathcal{F}_F^{\alpha}(s) = \frac{2}{S_F^{\alpha}(s)^3}. \tag{62}$$

After some calculations we obtain

$$\mathcal{F}_F^{\alpha}(s) = \frac{1}{S_F^{\alpha}(s)^{\frac{4}{3}} - \lambda} + \frac{2S_F^{\alpha}(s)^{\frac{1}{2}}}{S_F^{\alpha}(s)^{\frac{4}{3}} - \lambda} + \frac{2S_F^{\alpha}(s)^{-3}}{S_F^{\alpha}(s)^{\frac{4}{3}} - \lambda}. \tag{63}$$

By computing the inverse fractal Laplace transform we conclude

$$y(x) = S_F^\alpha(x)^{\frac{4}{3}} E^\alpha_{F,4/3,4/3}(\lambda S_F^\alpha(x)^{\frac{4}{3}}) + 2S_F^\alpha(x)^{\frac{-1}{6}} E^\alpha_{F,4/3,5/6}(\lambda S_F^\alpha(x)^{\frac{4}{3}})$$
$$+ 2S_F^\alpha(x)^{\frac{10}{3}} E^\alpha_{F,4/3,13/3}(\lambda S_F^\alpha(x))^{\frac{4}{3}}. \quad (64)$$

Remark 1. *The Figures 6–8 show that the solution of Equations (51), (54) and (57) leads to the standard non-local fractional cases when* $\alpha = 1$*, respectively.*

6. Conclusions

In this work, we defined new non-local derivatives on fractal sets. These new types of non-local derivatives can describe better the dynamics of complex systems which possess memory effect on a fractal set. Four illustrative examples were solved in detail. Finally, one can recover the standard non-local fractional cases when assigning $\alpha = 1$.

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