Proofs and Additional Tables and Figures on "Reproducibility Probability Estimation and RP-Testing for Some Nonparametric Tests"

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1. Proof of Lemma 1

Proof. Consider first point 1. Note that $\hat{\tilde{\pi}}_a > 1/2$ if and only if $t_{n,1-\alpha} - e(\hat{\theta}) < 0$, that is, if and only if $t_{n,1-\alpha} < T_n$. Consequently, the identity in point 1 is proved and the proof for point 2 is exactly the same. $\square$

2. Proof of Lemma 2 and Lemma 3

Proof. To prove point 1, consider a realization $(x_1, ..., x_n)$ of $\mathbf{X}_n$ such that $\sum_{l=1}^n x_l = s$. The number of all the possible re-samples $(x_1^i, ..., x_n^i)$ that can be drawn with replacement from $(x_1, ..., x_n)$ such that $\sum_{l=1}^n x_l^i = k$ is given by $\binom{n}{k} s^k (n-s)^{n-k}$. Consequently, the number of possible re-samples with $\sum_{l=1}^n x_l^i \geq (nc_\alpha + 1)$ (i.e. the number of re-samples leading to a rejection of H_0 using the exact test Ψ_α) is given by $\sum_{k=nc_\alpha+1}^n \binom{n}{k} s^k (n-s)^{n-k}$. Then, the value of $\hat{\pi}_e$ associated to a realization of $(x_1, ..., x_n)$ with $\sum_{l=1}^n x_l = s$ is given by $\hat{\pi}_e^{PI}(s) = \sum_{k=nc_\alpha+1}^n \binom{n}{k} \left(\frac{s}{n}\right)^k \left(1 - \frac{s}{n}\right)^{n-k} = 1 - B\left(nc_\alpha; n, \frac{s}{n}\right) = 1 - B\left(nc_\alpha; n, \hat{p}\right)$ and $P(\hat{\pi}_e^{PI} = \hat{\pi}_e^{PI}(s)) = \binom{n}{s} p^s (1-p)^{n-s}$. This proves points 1 and 2. Concerning point 3, note that the value of the estimators $\hat{\pi}_e^{PI}$ corresponding to the sample realizations leading to the critical value $\hat{p} = c_\alpha$ is $\hat{\pi}_e^{PI}(nc_\alpha) = \sum_{k=nc_\alpha+1}^n \binom{n}{k} \left(\frac{nc_\alpha}{n}\right)^k \left(1 - \frac{nc_\alpha}{n}\right)^{n-k} = 1 - B(nc_\alpha; n, c_\alpha)$. Since nc_α is an integer, the median of the binomial distribution with parameters n and c_α coincides with nc_α. Consequently, $\hat{\pi}_e^{PI}(nc_\alpha) \leq 1/2$. Analogously, the value of $\hat{\pi}_e^{PI}$ corresponding to $nc_\alpha + 1$ is given by $\hat{\pi}_e^{PI}(nc_\alpha + 1) = \sum_{k=nc_\alpha+1}^n \binom{n}{k} \left(\frac{nc_\alpha+1}{n}\right)^k \left(1 - \frac{nc_\alpha+1}{n}\right)^{n-k} = 1 - B(nc_\alpha; n, (nc_\alpha+1)/n)$.

Again, since $nc_\alpha + 1$ is an integer, it coincides with the median of the binomial distribution with parameters n and $(nc_\alpha+1)/n$. Then $1 - B(nc_\alpha; n, (nc_\alpha+1)/n) > 1/2$, which coincides with $\hat{\pi}_e^{PI}(nc_\alpha + 1) > 1/2$. This demonstrates that, if the null hypotheses is rejected (accepted) by the classical exact test (or by the test based on the parametric exact RP-estimator), also the RP-testing rule defined on the basis of $\hat{\pi}_e^{PI}$ reject (accept) H_0. The converse implication is straightforward since a binomial random variable is strictly increasing in p_t with respect to the usual stochastic ordering. Then, the RP-testing rule based on $\hat{\pi}_e^{PI}$ is equivalent to the exact one. The proof for Lemma 3.2 is analogous. $\square$

3. RP-Estimation and Testing for the Sign Test

Let X be a continuous random variable with distribution function ${}_tF$ and median θ_t. Let $\mathbf{X}_n = (X_1, ..., X_n)$ be random sample drawn from ${}_tF$ in order to test the statistical hypotheses $H_0 : \theta_t \leq \theta_0$ vs $H_1 : \theta_t > \theta_0$. It is well known that the previous hypotheses can be tested by using the Sing test which is based on the test statistics

$$B = \sum_{i=1}^n I_i$$

with

$$I_i = \begin{cases} 1 & \text{if} \quad X_i > \theta_0 \\ 0 & \text{if} \quad X_i \leq \theta_0 \end{cases} .$$

The exact and asymptotic distribution of B is known both under H_0 and under H_1 and, consequently, this test falls under case (**A**). Putting $p_t = 1 -_t F(\theta_t)$ results that $B \sim \mathit{Binomial}\,(n, p_t)$ and $\frac{B - np_t}{\sqrt{np_t(1-p_t)}} \xrightarrow{d} \mathcal{N}(0,1)$. It is now clear that all the results obtained for the Binomial test can be applied also for the Sing test. Note that, under H_0, $p_t = 0.5$ and, consequently, the performances of the semi-parametric and non-parametric RP-estimators for the Sign test are reported in Figure 2 of the Supplementary Material, which is related to the binomial test with $p_0 = 0.5$.

4. Proof of Corollary 4

Proof. Remember that $W = \sum_{i=1}^{n} \sum_{j=i}^{n} I_{ij}$ where I_{ij} is defined as in formula (8) of the main document. Observe that $E[I_{i_j}] = p_1$ if $i \neq j$ and $E[I_{i_j}] = p$ if $i = j$. Moreover, recall that $E_{F_{\theta_t}}[W] = e(p, p_1) = \frac{n(n-1)}{2} p_1 + np$.

Thanks to Lemma 1, in order to prove Corollary 4 it is sufficient to show that $W = e(\hat{p}, \hat{p}_1) = \hat{E}$. Observe that

$$W = \sum_{i=1}^{n} \sum_{j=i}^{n} I_{ij} = \sum_{i=1}^{n} \sum_{j=i+1}^{n} I_{ij} + \sum_{i=1}^{n} I_{ii} \, . \tag{1}$$

Remembering that $\hat{p} = \frac{1}{n} \sum_{i=1}^{n} I_{ii}$ and $\hat{p}_1 = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} I_{ij}$, expression (1) can be rewritten as follows:

$$W = \sum_{i=1}^{n} \sum_{j=i}^{n} I_{ij} = \frac{n(n-1)}{2} \left(\frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} I_{ij} \right) + n \left(\frac{1}{n} \sum_{i=1}^{n} I_{ii} \right) = \frac{n(n-1)}{2} \hat{p}_1 + n\hat{p} = \hat{E} \, .$$

□

5. Tables and Figures

Table S1. Probability of disagreement $D(p_t, n, \alpha, p_0)$ between the tests $\Psi_\alpha(\mathbf{X}_n)$ and $\widetilde{\Psi}_\alpha(\mathbf{X}_n)$ evaluated for $\alpha = 0.05$, $n = (5, 15, 30)$, $p_0 = (0.2, 0.5)$ and $p_t = (0.1, 0.2, ..., 0.9)$.

	$p_0 = 0.2$			$p_0 = 0.5$		
p_t	$n = 5$	$n = 15$	$n = 30$	$n = 5$	$n = 15$	$n = 30$
0.1	0.0081	0.0019390	0.0003653	0	0.0000000	0
0.2	0.0512	0.0429926	0.0354709	0	0.0000115	0
0.3	0.1323	0.1472360	0.1415617	0	0.0005806	0
0.4	0.2304	0.2065976	0.1151854	0	0.0074199	0
0.5	0.3125	0.1527405	0.0279816	0	0.0416565	0
0.6	0.3456	0.0612141	0.0019975	0	0.1267758	0
0.7	0.3087	0.0115900	0.0000296	0	0.2186231	0
0.8	0.2048	0.0006718	0.0000000	0	0.1876042	0
0.9	0.0729	0.0000027	0.0000000	0	0.0428352	0

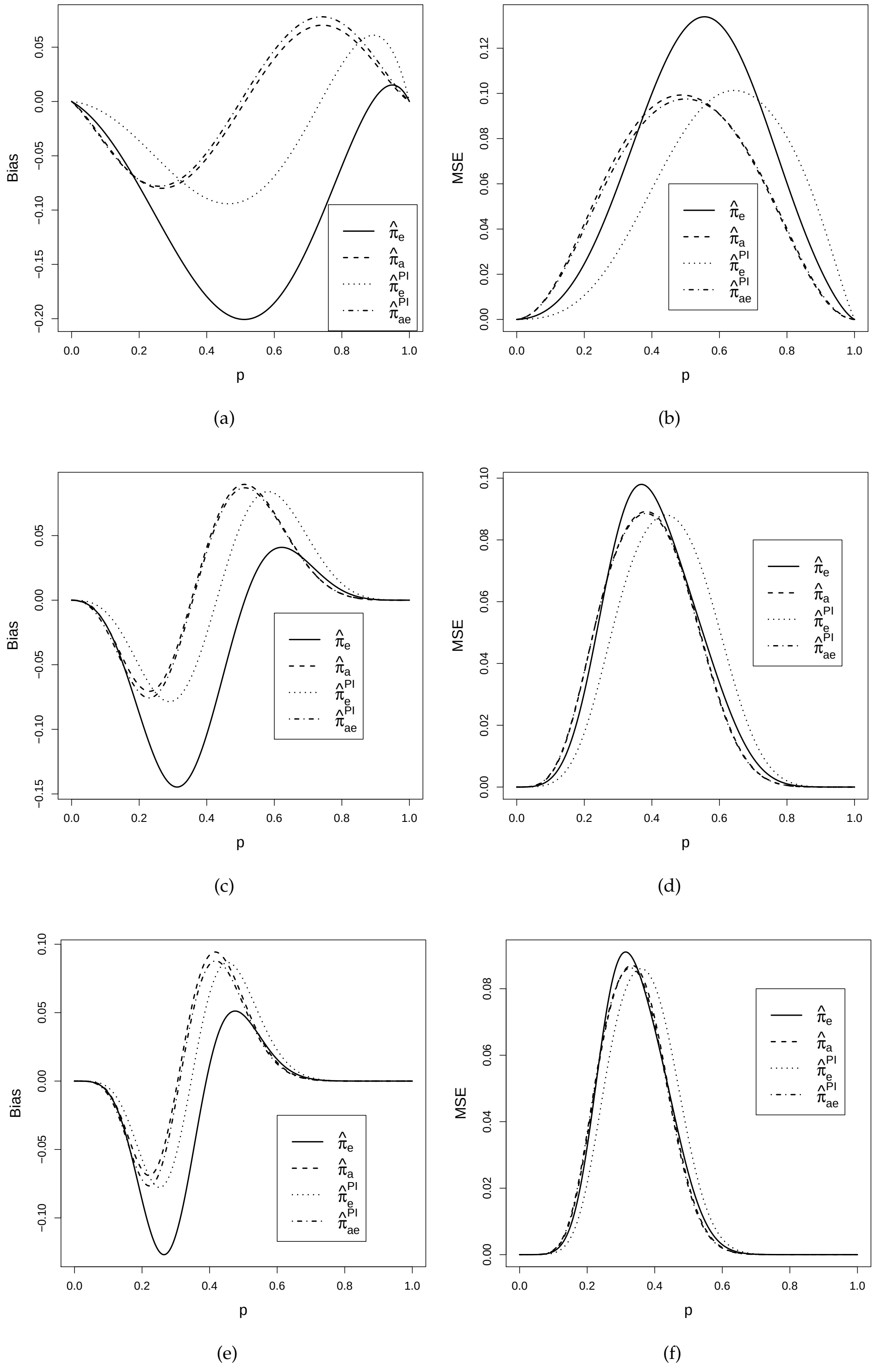

Figure S1. Bias (left) and MSE (right) of the RP estimators $\hat{\pi}_e$ (solid), $\hat{\pi}_a$ (dashed), $\hat{\pi}_e^{PI}$ (dotted), and $\hat{\pi}_{a,e}^{PI}$ (dot-dashed). The MSE and the Bias are computed setting $\alpha = 0.05$ and considering the testing problem (3.1) by setting $\alpha = 0.05$, $p_0 = 0.2$ and $n = (5, 10, 15)$. (**a**) $p_0 = 0.2$, $n = 5$; (**b**) $p_0 = 0.2$, $n = 5$; (**c**) $p_0 = 0.2$, $n = 15$; (**d**) $p_0 = 0.2$, $n = 15$; (**e**) $p_0 = 0.2$, $n = 30$; (**f**) $p_0 = 0.2$, $n = 30$.

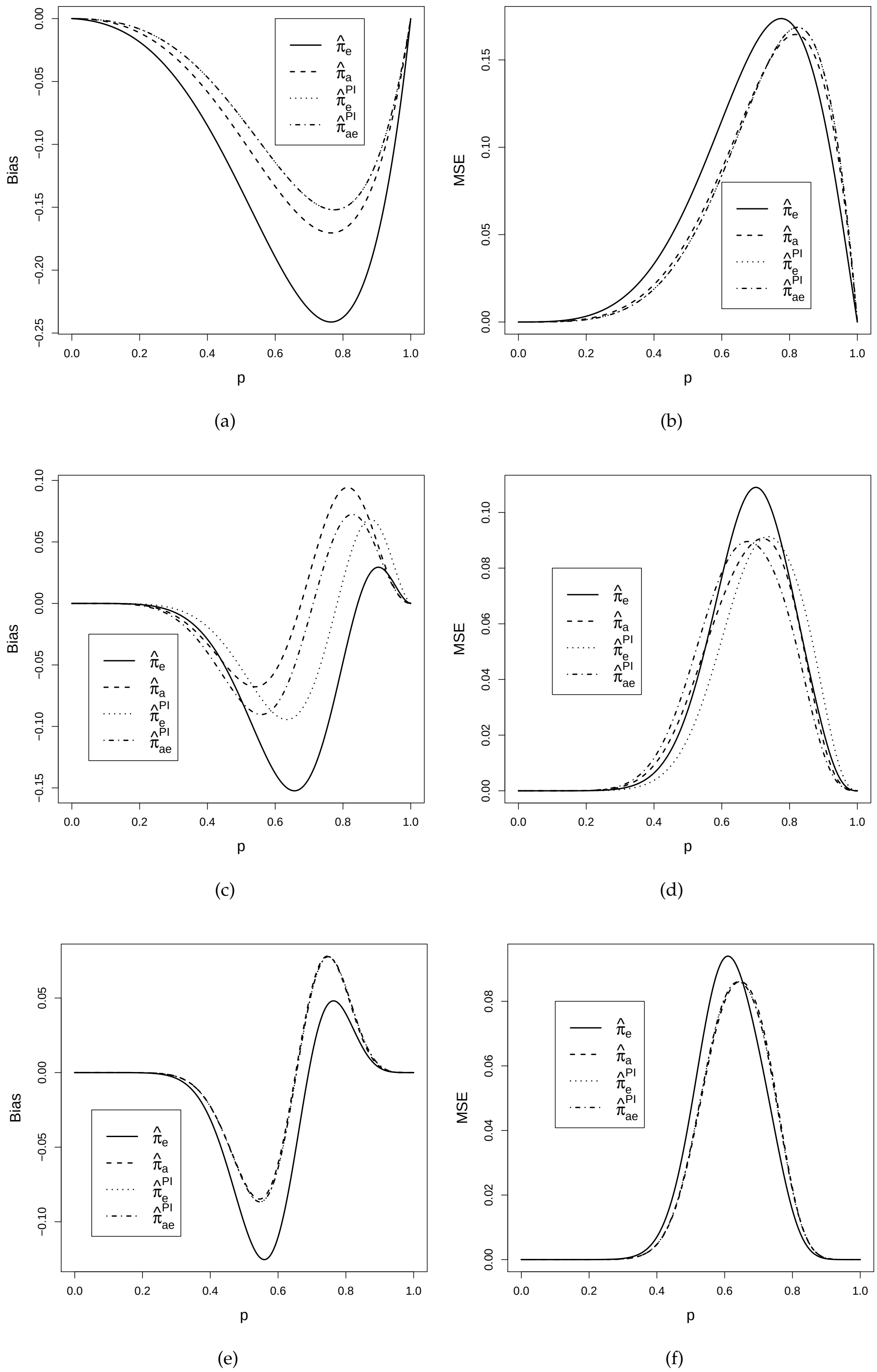

Figure S2. Bias (left) and MSE (right) of the RP estimators $\hat{\pi}_e$ (solid), $\hat{\pi}_a$ (dashed), $\hat{\pi}_e^{PI}$ (dotted), and $\hat{\pi}_{a,e}^{PI}$ (dot-dashed). The MSE and the Bias are computed setting $\alpha = 0.05$ and considering the testing problem (3.1) by setting $\alpha = 0.05$, $p_0 = 0.5$ and $n = (5, 10, 15)$. (**a**) $p_0 = 0.5$, $n = 5$; (**b**) $p_0 = 0.5$, $n = 5$; (**c**) $p_0 = 0.5$, $n = 15$; (**d**) $p_0 = 0.5$, $n = 15$; (**e**) $p_0 = 0.5$, $n = 30$; (**f**) $p_0 = 0.5$, $n = 30$.

Table S2. Probability of disagreement $D(\alpha, n, F_{\theta_t}, \theta_0)$ between the tests $\Psi_\alpha(\mathbf{X}_n)$ and $\widetilde{\Psi}_\alpha(\mathbf{X}_n)$ evaluated for $\alpha = 0.05$, $n = (15, 30, 60, 120, 240)$, $\theta_0 = 0$, and assuming that $F_{\theta_t} \equiv \mathcal{N}(\theta_t, 1)$ or $F_{\theta_t} \equiv \text{Cauchy}(\theta_t)$ with $\theta_t = (0.0, 0.1, 0.2, ..., 0.9)$.

		Gaussian Distribution				
		$n = 15$	$n = 30$	$n = 60$	$n = 120$	$n = 240$
	w_α	89	313	1139	4258	16231
	$\tilde{w}_\alpha$	88.9606	312.4702	1138.4369	4258.0778	16230.9582
θ_t						
0.0		0.0065	0.0023	0.0009	0.0000	0.0001
0.1		0.0111	0.0044	0.0020	0.0000	0.0004
0.2		0.0165	0.0070	0.0028	0.0000	0.0001
0.3		0.0210	0.0082	0.0025	0.0000	0.0000
0.4		0.0252	0.0080	0.0012	0.0000	0.0000
0.5		0.0249	0.0053	0.0003	0.0000	0.0000
0.6		0.0229	0.0031	0.0001	0.0000	0.0000
0.7		0.0180	0.0013	0.0000	0.0000	0.0000
0.8		0.0129	0.0003	0.0000	0.0000	0.0000
0.9		0.0076	0.0001	0.0000	0.0000	0.0000
		Cauchy Distribution				
		$n = 15$	$n = 30$	$n = 60$	$n = 120$	$n = 240$
	w_α	89	313	1139	4258	16231
	$\tilde{w}_\alpha$	88.961	312.470	1138.437	4258.078	16230.958
θ_t						
0.0		0.0060	0.0020	0.0008	0.0000	0.0001
0.1		0.0092	0.0031	0.0013	0.0000	0.0003
0.2		0.0113	0.0048	0.0021	0.0000	0.0004
0.3		0.0142	0.0064	0.0030	0.0000	0.0002
0.4		0.0171	0.0079	0.0030	0.0000	0.0001
0.5		0.0195	0.0085	0.0025	0.0000	0.0000
0.6		0.0216	0.0086	0.0025	0.0000	0.0000
0.7		0.0223	0.0080	0.0019	0.0000	0.0000
0.8		0.0239	0.0079	0.0011	0.0000	0.0000
0.9		0.0241	0.0070	0.0009	0.0000	0.0000

Table S3. Averaged MSE, Bias and Disagreement rate for the asymptotic and exact WSR test when sampling from the Normal distribution. The averages are computed over the 19 different values of θ considered in the simulation study. The smallest values for the averaged MSE, Bias and disagreement are highlighted in bold.

RP-estimation and Testing for the Asymptotic WSR Test															
	$n = 15$			$n = 30$			$n = 60$			$n = 120$			$n = 240$		
RP-est.	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**
$\hat{\pi}_{N1}$	−0.0197	0.0618	0.0143	−0.0044	0.0652	0.0105	**−0.0013**	0.0671	0.0039	**−0.0007**	0.0681	0.0024	0.0005	0.0685	0.0009
$\hat{\pi}_{N2}$	−0.0278	**0.0614**	0.0074	−0.0084	0.0649	0.0032	−0.0032	0.0669	0.0018	−0.0016	0.0680	0.0009	0.0000	0.0685	0.0004
$\hat{\pi}_{aS1}$	−0.0278	0.0618	**0.0000**	−0.0087	0.0650	**0.0000**	−0.0035	0.0670	**0.0000**	−0.0018	0.0681	**0.0000**	**0.0000**	0.0685	**0.0000**
$\hat{\pi}_{aS2}$	−0.0350	0.0617	0.0219	−0.0125	**0.0648**	0.0057	−0.0054	**0.0668**	0.0026	−0.0027	**0.0680**	0.0011	−0.0005	**0.0685**	0.0007
$\hat{\pi}_{a1}$	0.0051	0.0781	**0.0000**	0.0067	0.0732	**0.0000**	0.0039	0.0710	**0.0000**	0.0018	0.0701	**0.0000**	0.0018	0.0695	**0.0000**
$\hat{\pi}_{a2}$	**−0.0029**	0.0727	0.0219	**0.0026**	0.0704	0.0057	0.0019	0.0696	0.0026	0.0008	0.0693	0.0011	0.0013	0.0692	0.0007
$\hat{\pi}_{a5}^{PI}$	0.0134	0.0714	0.0171	0.0061	0.0698	0.0142	0.0035	0.0694	0.0139	0.0014	0.0694	0.0136	0.0018	0.0694	0.0138
$\hat{\pi}_{a10}^{PI}$	0.0134	0.0713	0.0152	0.0061	0.0696	0.0112	0.0035	0.0692	0.0101	0.0015	0.0692	0.0095	0.0018	0.0692	0.0096
$\hat{\pi}_{a20}^{PI}$	0.0134	0.0712	0.0137	0.0061	0.0696	0.0091	0.0035	0.0691	0.0073	0.0014	0.0692	0.0068	0.0018	0.0691	0.0067
RP-estimation and Testing for the Exact WSR Test															
	$n = 15$			$n = 30$			$n = 60$			$n = 120$			$n = 240$		
RP-est.	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**
$\hat{\pi}_{N1}$	**−0.0006**	0.0614	0.0202	0.0020	0.0652	0.0156	**0.0009**	0.0671	0.0049	**−0.0007**	0.0681	0.0024	0.0008	0.0685	0.0011
$\hat{\pi}_{N2}$	−0.0086	**0.0607**	0.0133	**−0.0019**	0.0648	0.0059	−0.0010	0.0669	0.0029	−0.0016	0.0680	0.0009	0.0003	0.0685	0.0005
$\hat{\pi}_{S1}$	−0.0092	0.0611	**0.0000**	−0.0050	0.0649	**0.0000**	−0.0022	0.0669	**0.0000**	−0.0017	0.0681	**0.0000**	0.0003	0.0685	**0.0000**
$\hat{\pi}_{S2}$	−0.0164	0.0607	0.0048	−0.0088	**0.0647**	0.0035	−0.0041	**0.0668**	0.0018	−0.0027	**0.0680**	0.0011	**−0.0002**	**0.0685**	0.0005
$\hat{\pi}_{1}$	0.0236	0.0786	**0.0000**	0.0103	0.0733	**0.0000**	0.0051	0.0710	**0.0000**	0.0019	0.0701	**0.0000**	0.0020	0.0695	**0.0000**
$\hat{\pi}_{2}$	0.0156	0.0729	0.0048	0.0063	0.0705	0.0035	0.0032	0.0696	0.0018	0.0009	0.0693	0.0011	0.0016	0.0692	0.0005
$\hat{\pi}_{5}^{PI}$	0.0189	0.0720	0.0188	0.0074	0.0698	0.0145	0.0040	0.0694	0.0140	0.0014	0.0694	0.0136	0.0019	0.0694	0.0139
$\hat{\pi}_{10}^{PI}$	0.0189	0.0718	0.0167	0.0074	0.0697	0.0112	0.0039	0.0692	0.0099	0.0015	0.0692	0.0095	0.0019	0.0692	0.0096
$\hat{\pi}_{20}^{PI}$	0.0189	0.0717	0.0161	0.0074	0.0696	0.0091	0.0039	0.0692	0.0073	0.0014	0.0692	0.0068	0.0019	0.0691	0.0067

Table S4. Averaged MSE, Bias and Disagreement rate for the asymptotic and exact WSR test when sampling from the Chaucy distribution. The averages are computed over the 19 different values of θ considered in the simulation study. The smallest values for the averaged MSE, Bias and disagreement are highlighted in bold.

RP-estimation and Testing for the Asymptotic WSR Test

	$n = 15$			$n = 30$			$n = 60$			$n = 120$			$n = 240$		
RP-est.	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**
$\hat{\pi}_{N1}$	−0.0308	0.0684	0.0286	−0.0082	0.0678	0.0073	−0.0050	0.0686	0.0045	−0.0025	0.0688	0.0018	−0.0024	0.0683	0.0012
$\hat{\pi}_{N2}$	−0.0312	0.0664	0.0131	−0.0082	0.0669	0.0057	−0.0050	0.0682	0.0025	−0.0025	0.0686	0.0010	−0.0024	0.0682	0.0008
$\hat{\pi}_{aS1}$	−0.0247	0.0652	**0.0000**	**−0.0049**	0.0664	**0.0000**	−0.0034	0.0680	**0.0000**	−0.0017	0.0685	**0.0000**	−0.0020	0.0681	**0.0000**
$\hat{\pi}_{aS2}$	−0.0250	**0.0636**	0.0122	−0.0050	**0.0656**	0.0034	−0.0034	**0.0676**	0.0019	−0.0017	**0.0683**	0.0009	−0.0020	**0.0680**	0.0004
$\hat{\pi}_{a1}$	**0.0009**	0.0793	**0.0000**	0.0068	0.0734	**0.0000**	**0.0022**	0.0713	**0.0000**	**0.0010**	0.0702	**0.0000**	−0.0006	0.0690	**0.0000**
$\hat{\pi}_{a2}$	0.0021	0.0735	0.0122	0.0071	0.0702	0.0034	0.0023	0.0697	0.0019	0.0010	0.0693	0.0009	−0.0006	0.0685	0.0004
$\hat{\pi}_{a5}^{PI}$	0.0192	0.0718	0.0180	0.0091	0.0698	0.0142	0.0031	0.0696	0.0136	0.0013	0.0695	0.0133	**−0.0003**	0.0687	0.0129
$\hat{\pi}_{a10}^{PI}$	0.0192	0.0717	0.0164	0.0091	0.0696	0.0111	0.0031	0.0694	0.0097	0.0012	0.0693	0.0093	−0.0003	0.0686	0.0091
$\hat{\pi}_{a20}^{PI}$	0.0192	0.0716	0.0156	0.0091	0.0695	0.0091	0.0031	0.0694	0.0072	0.0012	0.0692	0.0068	−0.0003	0.0685	0.0065

RP-estimation and Testing for the Exact WSR Test

	$n = 15$			$n = 30$			$n = 60$			$n = 120$			$n = 240$		
RP-est.	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**
$\hat{\pi}_{N1}$	−0.0132	0.0677	0.0200	−0.0019	0.0678	0.0082	−0.0028	0.0686	0.0040	−0.0025	0.0688	0.0018	−0.0021	0.0683	0.0011
$\hat{\pi}_{N2}$	−0.0136	0.0657	0.0045	−0.0019	0.0668	0.0033	−0.0028	0.0682	0.0020	−0.0025	0.0686	0.0010	−0.0021	0.0682	0.0006
$\hat{\pi}_{S1}$	**−0.0076**	0.0647	**0.0000**	**−0.0013**	0.0664	**0.0000**	**−0.0021**	0.0680	**0.0000**	−0.0016	0.0685	**0.0000**	−0.0017	0.0681	**0.0000**
$\hat{\pi}_{S2}$	−0.0079	**0.0631**	0.0066	−0.0013	**0.0655**	0.0036	−0.0021	**0.0675**	0.0018	−0.0016	**0.0683**	0.0009	−0.0017	**0.0680**	0.0005
$\hat{\pi}_{1}$	0.0179	0.0797	**0.0000**	0.0104	0.0734	**0.0000**	0.0034	0.0713	**0.0000**	**0.0010**	0.0702	**0.0000**	−0.0004	0.0690	**0.0000**
$\hat{\pi}_{2}$	0.0192	0.0739	0.0066	0.0108	0.0703	0.0036	0.0035	0.0697	0.0018	0.0011	0.0693	0.0009	−0.0004	0.0685	0.0005
$\hat{\pi}_{5}^{PI}$	0.0264	0.0722	0.0211	0.0105	0.0698	0.0143	0.0036	0.0696	0.0137	0.0013	0.0695	0.0133	**−0.0002**	0.0687	0.0130
$\hat{\pi}_{10}^{PI}$	0.0264	0.0721	0.0200	0.0105	0.0697	0.0113	0.0036	0.0694	0.0099	0.0012	0.0693	0.0093	-0.0003	0.0686	0.0092
$\hat{\pi}_{20}^{PI}$	0.0264	0.0720	0.0193	0.0105	0.0696	0.0091	0.0036	0.0694	0.0073	0.0012	0.0692	0.0068	−0.0002	0.0685	0.0066

Table S5. Probability of disagreement between $\widetilde{\Psi}_\alpha((\mathbf{X},\mathbf{Y})_n)$ and $\Psi_\alpha((\mathbf{X},\mathbf{Y})_n)$ when $\alpha = 0.05$, $n = (15, 30, 60, 120)$, and assuming that $\mathbf{F}_t$ is the Bivariate Normal or Bivariate Student's t (3 df) with correlation coefficient $\rho = (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6)$. The values of τ_t corresponding to the considered values of ρ are reported in the second column of the table. They have been obtained by using the relation $\tau_t = \frac{2}{\pi}\arcsin(\rho)$ which holds for all absolutely continuous elliptical distributions.

		Bivariate Gaussian Distribution			
		$\boldsymbol{n = 15}$	$\boldsymbol{n = 30}$	$\boldsymbol{n = 60}$	$\boldsymbol{n = 120}$
$t_{1-\alpha}$		0.3143	0.2138	0.1458	0.1017
$\tilde{t}_{1-\alpha}$		0.3772	0.2525	0.1736	0.1210
ρ	τ_t				
0.0	0.0000	0.0234	0.0222	0.0245	0.0251
0.1	0.0638	0.0401	0.0487	0.0682	0.0932
0.2	0.1282	0.0636	0.0839	0.1170	0.1205
0.3	0.1940	0.0920	0.1138	0.1142	0.0489
0.4	0.2620	0.1180	0.1156	0.0582	0.0051
0.5	0.3333	0.1324	0.0810	0.0134	0.0001
0.6	0.4097	0.1229	0.0350	0.0010	0.0000
		Bivariate *t* Distribution with 3 df			
		$\boldsymbol{n = 15}$	$\boldsymbol{n = 30}$	$\boldsymbol{n = 60}$	$\boldsymbol{n = 120}$
$t_{1-\alpha}$		0.3143	0.2138	0.1458	0.1017
$\tilde{t}_{1-\alpha}$		0.3772	0.2525	0.1736	0.1210
ρ	τ_t				
0.0	0.0000	0.0287	0.0282	0.0312	0.0313
0.1	0.0638	0.0453	0.0516	0.0692	0.0874
0.2	0.1282	0.0661	0.0805	0.1039	0.1073
0.3	0.1940	0.0874	0.1025	0.1029	0.0529
0.4	0.2620	0.1063	0.1026	0.0611	0.0091
0.5	0.3333	0.1172	0.0783	0.0195	0.0004
0.6	0.4097	0.1102	0.0411	0.0025	0.0000

Table S6. Averaged MSE, Bias and Disagreement rate for the asymptotic and exact Kendall's test when sampling from the Gaussian copula. The averages are computed over the 19 different values of ρ considered in the simulation study. The least values for the averaged MSE, Bias and disagreement are highlighted in bold.

RP-estimation and Testing for the Asymptotic Kendall's Test												
	$n = 15$			$n = 30$			$n = 60$			$n = 120$		
RP-est.	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**
$\hat{\pi}_N$	**−0.0001**	0.0748	0.0070	0.0187	0.0681	0.0314	0.0017	0.0676	0.0149	−0.0028	0.0678	0.0078
$\hat{\pi}_{aS}$	−0.0059	0.0741	**0.0000**	−0.0085	0.0653	**0.0000**	−0.0103	0.0664	**0.0000**	−0.0091	0.0672	**0.0000**
$\hat{\pi}_{a1}$	−0.0047	0.0693	**0.0000**	**0.0032**	0.0729	**0.0000**	−0.0046	0.0700	**0.0000**	−0.0063	0.0690	**0.0000**
$\hat{\pi}_{a2}$	−0.0088	**0.0570**	**0.0000**	−0.0063	**0.0545**	**0.0000**	−0.0115	**0.0541**	**0.0000**	−0.0119	**0.0541**	**0.0000**
$\hat{\pi}_{C1}$	0.0008	0.0697	0.0070	0.0312	0.0773	0.0314	0.0076	0.0715	0.0149	**0.0000**	0.0696	0.0078
$\hat{\pi}_{C2}$	−0.0038	0.0571	0.0070	0.0168	0.0553	0.0314	**−0.0012**	0.0543	0.0149	−0.0065	0.0542	0.0078
$\hat{\pi}_{aL}$	−0.0048	0.0745	**0.0000**	−0.0034	0.0671	**0.0000**	−0.0080	0.0672	**0.0000**	−0.0080	0.0676	**0.0000**
$\hat{\pi}_{a5}^{PI}$	0.0203	0.0680	0.0329	0.0471	0.0669	0.0654	0.0318	0.0668	0.0498	0.0218	0.0672	0.0365
$\hat{\pi}_{a10}^{PI}$	0.0203	0.0679	0.0327	0.0472	0.0668	0.0658	0.0318	0.0666	0.0501	0.0218	0.0671	0.0370
$\hat{\pi}_{a20}^{PI}$	0.0203	0.0678	0.0329	0.0472	0.0667	0.0658	0.0319	0.0665	0.0504	0.0218	0.0670	0.0373
RP-estimation and Testing for the Exact Kendall's Test												
	$n = 15$			$n = 30$			$n = 60$			$n = 120$		
RP-est.	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**
$\hat{\pi}_N$	0.0818	0.0819	0.0882	0.1061	0.0798	0.1190	0.0931	0.0769	0.1060	0.0896	0.0764	0.0995
$\hat{\pi}_S$	**0.0069**	0.0739	**0.0000**	**0.0050**	0.0645	**0.0000**	**0.0053**	0.0660	**0.0000**	**0.0084**	0.0670	**0.0000**
$\hat{\pi}_1$	0.0105	0.0693	**0.0000**	0.0147	0.0732	**0.0000**	0.0099	0.0702	**0.0000**	0.0106	0.0691	**0.0000**
$\hat{\pi}_2$	0.0134	**0.0570**	**0.0000**	0.0163	**0.0544**	**0.0000**	0.0137	**0.0541**	**0.0000**	0.0151	**0.0542**	**0.0000**
$\hat{\pi}_{C1}$	0.0827	0.0769	0.0882	0.1187	0.0914	0.1190	0.0990	0.0819	0.1060	0.0924	0.0787	0.0995
$\hat{\pi}_{C2}$	0.0780	0.0634	0.0882	0.1043	0.0664	0.1190	0.0902	0.0628	0.1060	0.0859	0.0618	0.0995
$\hat{\pi}_L$	0.0080	0.0745	**0.0000**	0.0097	0.0668	**0.0000**	0.0074	0.0671	**0.0000**	0.0094	0.0676	**0.0000**
$\hat{\pi}_5^{PI}$	0.0361	0.0696	0.0324	0.0655	0.0705	0.0679	0.0501	0.0696	0.0503	0.0396	0.0693	0.0362
$\hat{\pi}_{10}^{PI}$	0.0361	0.0695	0.0326	0.0655	0.0704	0.0684	0.0501	0.0694	0.0505	0.0396	0.0692	0.0369
$\hat{\pi}_{20}^{PI}$	0.0361	0.0694	0.0323	0.0655	0.0703	0.0685	0.0501	0.0694	0.0509	0.0396	0.0691	0.0369

Table S7. Averaged MSE, Bias and Disagreement rate for the asymptotic and exact Kendall's test when sampling from the t copula with 3 degrees of freedom. The averages are computed over the 19 different values of ρ considered in the simulation study. The least values for the averaged MSE, Bias and disagreement are highlighted in bold.

RP-estimation and Testing for the Asymptotic Kendall's Test												
	$n = 15$			$n = 30$			$n = 60$			$n = 120$		
RP-est.	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**
$\hat{\pi}_N$	0.0360	0.0759	0.0551	0.0186	0.0751	0.0281	0.0038	0.0752	0.0133	**−0.0001**	0.0748	0.0070
$\hat{\pi}_{aS}$	−0.0060	0.0689	**0.0000**	−0.0065	0.0721	**0.0000**	−0.0072	0.0738	**0.0000**	−0.0059	0.0741	**0.0000**
$\hat{\pi}_{a1}$	0.0131	0.0778	**0.0000**	0.0033	0.0735	**0.0000**	−0.0033	0.0709	**0.0000**	−0.0047	0.0693	**0.0000**
$\hat{\pi}_{a2}$	**0.0036**	**0.0591**	**0.0000**	−0.0037	**0.0581**	**0.0000**	−0.0081	**0.0577**	**0.0000**	−0.0088	**0.0570**	**0.0000**
$\hat{\pi}_{C1}$	0.0553	0.0881	0.0551	0.0278	0.0765	0.0281	0.0073	0.0720	0.0133	0.0008	0.0697	0.0070
$\hat{\pi}_{C2}$	0.0379	0.0617	0.0551	0.0175	0.0589	0.0281	**0.0012**	0.0579	0.0133	−0.0038	0.0571	0.0070
$\hat{\pi}_{aL}$	0.0062	0.0736	**0.0000**	**−0.0015**	0.0741	**0.0000**	−0.0049	0.0747	**0.0000**	−0.0048	0.0745	**0.0000**
$\hat{\pi}_{a5}^{PI}$	0.0587	0.0695	0.0769	0.0435	0.0687	0.0595	0.0295	0.0685	0.0448	0.0203	0.0680	0.0329
$\hat{\pi}_{a10}^{PI}$	0.0587	0.0694	0.0775	0.0434	0.0685	0.0597	0.0295	0.0683	0.0451	0.0203	0.0679	0.0327
$\hat{\pi}_{a20}^{PI}$	0.0587	0.0693	0.0779	0.0434	0.0684	0.0596	0.0295	0.0683	0.0455	0.0203	0.0678	0.0329
RP-estimation and Testing for the Exact Kendall's Test												
	$n = 15$			$n = 30$			$n = 60$			$n = 120$		
RP-est.	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**	**Bias**	**MSE**	**D**
$\hat{\pi}_N$	0.1224	0.0906	0.1411	0.0964	0.0847	0.1063	0.0851	0.0829	0.0955	0.0818	0.0819	0.0882
$\hat{\pi}_S$	**0.0018**	0.0672	**0.0000**	**0.0027**	0.0713	**0.0000**	**0.0040**	0.0734	**0.0000**	**0.0069**	0.0739	**0.0000**
$\hat{\pi}_1$	0.0196	0.0780	**0.0000**	0.0138	0.0738	**0.0000**	0.0099	0.0711	**0.0000**	0.0105	0.0693	**0.0000**
$\hat{\pi}_2$	0.0214	**0.0590**	**0.0000**	0.0145	**0.0580**	**0.0000**	0.0124	**0.0577**	**0.0000**	0.0134	**0.0570**	**0.0000**
$\hat{\pi}_{C1}$	0.1417	0.1064	0.1411	0.1056	0.0876	0.1063	0.0886	0.0803	0.0955	0.0827	0.0769	0.0882
$\hat{\pi}_{C2}$	0.1243	0.0764	0.1411	0.0953	0.0680	0.1063	0.0825	0.0651	0.0955	0.0780	0.0634	0.0882
$\hat{\pi}_L$	0.0138	0.0732	**0.0000**	0.0076	0.0739	**0.0000**	0.0062	0.0746	**0.0000**	0.0080	0.0745	**0.0000**
$\hat{\pi}_5^{PI}$	0.0775	0.0738	0.0781	0.0593	0.0715	0.0601	0.0456	0.0707	0.0457	0.0361	0.0696	0.0324
$\hat{\pi}_{10}^{PI}$	0.0775	0.0737	0.0784	0.0593	0.0714	0.0604	0.0456	0.0706	0.0461	0.0361	0.0695	0.0326
$\hat{\pi}_{20}^{PI}$	0.0775	0.0736	0.0791	0.0593	0.0713	0.0607	0.0456	0.0705	0.0463	0.0361	0.0694	0.0323