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Positive Sofic Entropy Implies Finite Stabilizer

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Abstract: We prove that, for a measure preserving action of a sofic group with positive sofic entropy, the stabilizer is finite on a set of positive measures. This extends the results of Weiss and Seward for amenable groups and free groups, respectively. It follows that the action of a sofic group on its subgroups by inner automorphisms has zero topological sofic entropy, and that a faithful action that has completely positive sofic entropy must be free.

Keywords: sofic entropy; stabilizers; invariant random subgroups

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1. Introduction

The last decade brought a number of important developments in dynamics of non-amenable group actions. Among these we note the various extensions of classical entropy theory. For actions of free groups, Bowen introduced a numerical invariant known as f-invariant entropy [1]. Some time later Bowen defined a new invariant for actions of *sofic groups*, called *sofic entropy* [2]. Kerr and Li further developed sofic entropy theory and also adapted it to groups actions on topological spaces by homeomorphisms [3]. The classical notion of entropy for amenable groups and Bowen's f-invariant both turned out to be special cases of sofic entropy [4,5].

The study of *non-free measure-preserving group actions* is another fruitful and active trend in dynamics. These are closely related to the notion of *invariant random subgroups*, that is, probability measures on the space of subgroups whose law is invariant under conjugation. Any such law can be realized as the law of the stabilizer for a random point for some probability preserving action [6]. In this note we prove the following:

Theorem 1. *Suppose $G \curvearrowright (X, \mathcal{B}, \mu)$ is an action of a countable sofic group G that has positive sofic entropy (with respect to some sofic approximation). Then the set of points in X with finite stabilizer has positive measure. In particular, if the action is ergodic, almost every point has finite stabilizer.*

The amenable case of Theorem 1 appears as a remark in the last section of Weiss's survey paper on actions of amenable groups [7]. To be precise, Weiss stated the amenable case of Corollary 2 below.

Another interesting case of Theorem 1 for free groups is due to Seward [8]. The result proved in [8] applies to the *random sofic approximation*. By a non-trivial result of Bowen, this coincides with the f-invariant for free groups. Sewards's proof in [8] is based on a specific formula for f-entropy, which does not seem to be available for sofic entropy in general. Our proof below proceeds essentially by proving a combinatorial statement about finite objects. In personal communication, Seward informed me of another proof of Theorem 1 that is expected to appear in a forthcoming paper of Alpeev and Seward as a byproduct of their study of an entropy theory for general countable groups.

Theorem 1 confirms the point of view that the "usual" notions of sofic entropy for sofic groups (or mean-entropy in the amenable case) are not very useful as invariants for non-free actions. A version of sofic entropy for some non-free actions of sofic groups was developed by Bowen [9] as a particular instance of a more general framework called "entropy theory of sofic groupoids". It seems likely that both the statement of Theorem 1 and our proof should have a generalization to "sofic class bijective extensions of groupoids" (see [9] for definition of this term) . We will not pursue this direction. Thanks to Yair Glasner, Guy Salomon, Brandon Seward and Benjy Weiss for interesting discussions, and the referees for valuable remarks and suggestions.

2. Notation and Definitions

2.1. Sofic Groups

Sofic groups were introduced by Gromov [10] (under a different name) towards the end of the millennium. The name "sofic groups" is due to Weiss [11]. Sofic groups retain some properties of finite groups. They are a common generalization of amenable and residually finite groups. We include a definition below. There are several other interesting equivalent definitions. For further background, motivation and discussions on sofic groups, we refer the reader, for instance to [12].

Throughout G, we will denote a countable discrete group with identity denoted by 1. We will write $F \Subset G$ to indicate that F is a finite subset of G. For a finite set V, let S_V denote the group of permutations over V. We will consider maps from a group G to S_V. These maps are not necessarily homomorphisms. Given a map $\xi : G \to S_V$, $g \in G$ and $v \in V$, we write $\xi_g \in S_V$ for the image of g under ξ and $\xi_g(v) \in V$ for the image of v under the permutation ξ_g.

Let V be a finite set, $F \Subset G$ and $\epsilon > 0$. A map $\xi : G \to S_V$ is called an *(F, ϵ)-approximation of G* if it satisfies the following properties:

$$\frac{1}{|V|}\#\{v \in V \ : \ \xi_g(\xi_h(v)) \neq \xi_{gh}(v)\} < \epsilon \text{ for every choice of } g, h \in F, \tag{1}$$

and

$$\frac{1}{|V|}|\#\{v \in V \ : \ \xi_g(v) = v\} < \epsilon \text{ for every choice of } g \in F \setminus \{1\}. \tag{2}$$

A *sofic group* is a group G that admits an (F, ϵ)-approximation for any $F \Subset G$ and any $\epsilon > 0$. A *symmetric (F, ϵ)-approximation of* G is $\xi : G \to S_V$ that in addition to (1) and (2) also satisfies

$$\xi_g(\xi_{g^{-1}}(v)) = v \text{ for every choice of } g \in F \setminus \{1\} \text{ and } v \in V. \tag{3}$$

Standard arguments show that a sofic group admits a symmetric (F, ϵ)-approximation for any $F \Subset G$ and any $\epsilon > 0$, so from now assume our (F, ϵ)-approximations also satisfy (3).

Let $(V_n)_{n=1}^{\infty}$ be a sequence of finite sets. A sequence $(\xi_n)_{n=1}^{\infty}$ of maps $\xi_n : G \to V_n$ is called a *sofic approximation for* G if

$$\{n \in \mathbb{N} \ : \ \xi_n \text{ is an } (F, \epsilon) - \text{approximation}\}$$

is co-finite in $\mathbb{N}$, for every $F \Subset G$ and every $\epsilon > 0$. It follows directly from the definition that G is sofic if and only if there exists a sofic approximation for G.

2.2. Sofic Entropy

Roughly speaking, the sofic entropy of an action is h if there are "approximately" $e^{h|V|}$ "sufficiently distinct good approximations" for the action that "factor through" a finite "approximate action" $\xi : G \to S_V$. Various definitions have been introduced in the literature, that have been shown to lead to an equivalent notion. Most definitions involve some auxiliary structure. Here, we follow a recent presentation of sofic entropy by Austin [13]. Ultimately, this presentation is equivalent to Bowen's original definition and also to definitions given by Kerr and Li. We briefly describe Austin's definition and refer to [13] for details.

From now on we denote

$$\mathbb{X} := \{0,1\}^{\mathbb{N}}. \tag{4}$$

The space $\mathbb{X}$ will be equipped with the metric d defined by

$$d(\omega,\omega') := 2^{-\min\{n\in\mathbb{N}\,:\,\omega_n\neq\omega'_n\}} \text{ for } \omega,\omega'\in\mathbb{X}. \tag{5}$$

The above metric induces the product topology on $\mathbb{X}$ with respect to the discrete topology on $\{0,1\}$, making $\mathbb{X}$ into a compact topological space.

Let $G \curvearrowright (X,\mu)$ be a probability preserving action on a standard probability space. As explained in [13], by passing to an isomorphic action we can assume without loss of generality that:

1. The space X is equal to $\mathbb{X}^G$, equipped the product topology;
2. The action of G on $\mathbb{X}^G$ is the shift action: $(gx)_h := x_{g^{-1}h}$;
3. $\mu \in Prob(\mathbb{X}^G)$ is a Borel probability measure on $\mathbb{X}^G$.

Note that X is a compact topological space (in fact X is homeomorphic to $\mathbb{X}$).

To see that the above assumptions are no loss of generality, start with an arbitrary measure preserving G-action on a standard Borel probability space (X,μ). Choose a countable sequence $(A_n)_{n=1}^{\infty}$ of Borel subsets $A_n \subset X$ so that the smallest G-invariant σ-algebra containing $\{A_n\}_{n=1}^{\infty}$ is the Borel σ-algebra. There is a G-equivariant Borel embedding of $x \in X$ to $\hat{x} \in \mathbb{X}^G$ defined by

$$(\hat{x}_g)_n := 1_{g^{-1}A_n}(x) \text{ for } n\in\mathbb{N} \text{ and } g\in G.$$

Let $\hat{\mu} \in Prob(\mathbb{X}^G)$ denote the push-forward measure of μ, it follows that the G-action on $(\mathbb{X}^G,\hat{\mu})$ is measure-theoretically isomorphic to the G-action on (X,μ).

It follows from the choice of metric in (5) that

$$d(\omega,\omega') \in \{0\}\cup\{2^{-n} \;:\; n\in\mathbb{Z}_+\} \text{ for every } \omega,\omega'\in\mathbb{X}. \tag{6}$$

We recall some definitions and notation that Austin introduced in [13]:

Definition 1. Given a finite or countable set V, $\omega\in\mathbb{X}^V$, $\xi : G \to S_V$ and $v \in V$, the *pullback name of* ω *at* v, denoted by $\Pi_v^{\xi}(\omega) \in \mathbb{X}^G$ is defined to be:

$$(\Pi_v^{\xi}(\omega))_{g^{-1}} := \omega_{\xi_g(v)}. \tag{7}$$

The *empirical distribution* of ω with respect to ξ is defined by:

$$P_{\omega}^{\xi} := \frac{1}{|V|}\sum_{v\in V}\delta_{\Pi_v^{\xi}(\omega)}. \tag{8}$$

Given a weak-$*$ neighborhood $\mathcal{O} \subset Prob(\mathbb{X}^G)$ of $\mu \in Prob(\mathbb{X}^G)$, the set of $(\mathcal{O},\xi)$*-approximations* for the action $G \curvearrowright (\mathbb{X}^G,\mu)$ is given by

$$Map(\mathcal{O},\xi) := \{\omega\in\mathbb{X}^V \;:\; P_{\omega}^{\xi}\in\mathcal{O}\}.$$

In [13] elements of $Map(\mathcal{O},\xi)$ are called "good models".

The space $Map(\mathcal{O},\xi) \subset \mathbb{X}^V$, if it is non-empty, is considered as a metric space with respect to the following metric

$$d^V(\omega,\omega') := \frac{1}{|V|}\sum_{v\in V} d(\omega_v,\omega'_v) \text{ for } \omega,\omega'\in Map(\mathcal{O},\xi).$$

Given a compact metric space (Y,ρ) and $\delta>0$ we denote by $sep_{\delta}(Y,\rho)$ the maximal cardinality of a δ-separated set in (Y,ρ), and by $cov_{\delta}(Y,\rho)$ the minimal number of ρ-balls of radius δ needed to cover $5Y$. Let us recall a couple of classical relations between these quantities. Because distinct 2δ-separated points cannot be in the same δ-ball the following holds:

$$sep_{2\delta}(Y,\rho) \leq cov_{\delta}(Y,\rho).$$

Consider a maximal δ-separated set $Y_0 \subset Y$. The collection of δ-balls with centers in Y_0 covers Y. Thus:

$$cov_{\delta}(Y,\rho) \leq sep_{\delta}(Y,\rho).$$

Definition 2. Let $\Sigma = (\xi_n)_{n=1}^{\infty}$ be a sofic approximation of G, with $\xi_n : G \to S_{V_n}$. The Σ-*entropy* (or *sofic entropy* with respect to Σ) of $G \curvearrowright (\mathbb{X}^G, \mu)$ is defined by:

$$h_{\Sigma}(\mu) := \sup_{\delta>0} \inf_{\mathcal{O}\ni\mu} \limsup_{n\to\infty} \frac{1}{|V_n|} \log sep_{\delta}\left(Map(\mathcal{O},\xi_n), d^{V_n}\right), \tag{9}$$

where the infimum is over weak-$*$ neighborhoods $\mathcal{O}$ of μ in $Prob(X)$. If $Map(\mathcal{O},\xi_n) = \emptyset$ for all large n's, define $h_{\Sigma}(\mu) := -\infty$.

A curial property of the quantity $h_{\Sigma}(\mu)$ is that it does not depend on the way we choose the topological model $\mathbb{X}^G$ or on choice of metric d, and thus defines an isomorphism-invariant for probability preserving G-actions.

Remark 1. We recall a slight generalization of Σ-entropy: A *random sofic approximation* is $\Sigma = (P_n)_{n=1}^{\infty}$ where $P_n \in Prob((S_{V_n})^G)$ so that the conditions (1) and (2) hold "on average" with respect to P_n for any $\epsilon > 0$ and $F \Subset G$, if n is large enough.

In this case Σ-entropy is defined by

$$h_{\Sigma}(\mu) := \sup_{\delta>0} \inf_{\mathcal{O}\ni\mu} \limsup_{n\to\infty} \frac{1}{|V_n|} \log\left(\int sep_{\delta}\left(Map(\mathcal{O},\xi), d^{V_n}\right) dP_n(\xi)\right). \tag{10}$$

For the special case where G is a free group on d generators and P_n is chosen uniformly among the homomorphisms from G to the group of permutations of $\{1,\ldots,n\}$, Bowen proved that Σ-entropy coincides with the so called f-invariant [4].

Our proof of Theorem 1 applies directly with no changes to random sofic approximations, in particular to f-entropy.

2.3. Stabilizers and the Space of Subgroups

Let $Sub_G \subset 2^G$ denote the space of subgroups of G. The space Sub_G comes with a compact topology, inherited from the product topology on 2^G. The group G acts on Sub_G by inner automorphisms. Now let $G \curvearrowright X$ be an action of G on a standard Borel space X. For $x \in X$ let

$$stab(x) := \{g \in G \,:\, g(x) = x\}\,. \tag{11}$$

It is routine to check that the map $stab : X \to Sub_G$ is Borel and G-equivariant. The following observation appears implicitly for instance in [7]:

Lemma 1. *Let $G \curvearrowright (X,\mu)$ be an ergodic action of a countable group. If the action has finite stabilizers, the map $stab : X \to Sub_G$ induces a finite factor $G \curvearrowright (Sub_G, \mu\circ stab^{-1})$.*

Proof. Suppose $stab(x)$ is finite on a set of positive measure. By ergodicity $|stab(x)| < \infty$ on a set of full measure. Since there are only countably many finite subgroups, the measure $\mu \circ stab^{-1} \in Prob(Sub_G)$ must be purely atomic. To finish the proof, note that a purely atomic invariant probability measure must be supported on a single finite orbit, if it is ergodic. $\square$

Here is a quick corollary of Theorem 1 that concerns the action $G \curvearrowright Sub_G$:

Corollary 1. *Let G be an infinite sofic group and Σ a sofic approximation sequence. The topological Σ-entropy of the action $G \curvearrowright Sub_G$ by conjugation is zero.*

Proof. The variational principle for Σ-entropy states that the topological Σ-entropy of an action $G \curvearrowright X$ is equal to the supremum of the measure-theoretic Σ-entropy over all G-invariant measures [3]. Sub_G always admits at least two trivial fixed points G and $\{1\}$. The delta measures δ_G and $\delta_{\{1\}}$ $Prob(Sub_G)$ are thus G-invariant and have $h_\Sigma(\delta_G) = h_\Sigma(\delta_{\{1\}}) = 0$. It thus suffices to prove that any G-invariant measure $\mu \in Prob(Sub_G)$ has $h_\Sigma(\mu) \leq 0$. By Theorem 1, it is enough to show that the set $A = \{H \in Sub_G \ : \ |stab(H)| < \infty\}$ is null. Indeed, for any $H \in Sub_G$, $H \subset stab(H)$, because any subgroup is contained in its normalizer. Thus, groups $H \in Sub_G$ with finite stabilizer must be finite, so A is a countable set. Suppose $\mu(A) > 0$. Then A has positive measure for some ergodic measure and as in Lemma 1 this measure must be supported on a finite set. An action of an infinite group on a finite set cannot have finite stabilizers. This shows that $\mu(A) = 0$. $\square$

3. Sampling from Finite Graphs

In this section we prove an auxiliary result on finite labeled graphs. We begin with some terminology:

Definition 3. A *finite , simple and directed graph* is a pair $\mathcal{G} = (V, E)$ where V is a finite set and $E \subset V^2$ (we allow self-loops but no parallel edges).

- The *out-degree* and *in-degree* of $v \in V$ are given by
$$\deg_{out}(v) := |\{w \in V \ : \ (v,w) \in E\}|\,,$$
$$\deg_{in}(v) := |\{w \in V \ : \ (w,v) \in E\}|\,.$$
- We say that $\mathcal{G}$ is (ϵ, k, M)*-regular* if at most $\epsilon|V|$ vertices have out-degree less than k, and all vertices have in-degree at most M.
- A set $W \subset V$ is ϵ*-dominating* if the number of vertices in $v \in V$ so that $\{w \in W \ : \ (v,w) \in E\} = \emptyset$ is at most $\epsilon|V|$.
- A p*-Bernoulli* set $W \subset V$ for $p \in (0,1)$ is a random subset of V such that for each $v \in V$ the probability that $v \in W$ is p, independently of the other vertices.

Lemma 2. *Fix any $\kappa \in (0,1)$. Suppose $k \leq M \leq N$ satisfy*

$$(1 - \frac{1}{\sqrt{k}})^k < \kappa \text{ and } N > 2M^2\kappa^{-3}. \tag{12}$$

For any (κ, k, M)-regular graph $G = (V, E)$ with $|V| > N$, a $\frac{1}{\sqrt{k}}$-Bernoulli subset is 3κ-dominating and has size at most $\frac{2}{\sqrt{k}}|V|$ with probability at least $1 - \kappa$.

Proof. Suppose (12) holds. Let $\mathcal{G} = (V, E)$ be a graph satisfying the assumptions in the statement of the lemma, and let $W \subset V$ be $\frac{1}{\sqrt{k}}$-Bernoulli.

For $v \in V$, let $n(v)$ be number of edges $(v, w) \in E$ with $w \in W$. The random variable $n(v)$ is Binomial $B(\frac{1}{\sqrt{k}}, \deg_{out}(v))$. Let

$$\Phi := \sum_{v \in V} 1_{[n(v)=0]}.$$

It follows that

$$E(\Phi) = \sum_{v \in V} P\left(n(v) = 0\right) = \sum_{v \in V\ ,\ \deg_{out}(v) < k} P\left(n(v) = 0\right) + \sum_{v \in V\ ,\ \deg_{out}(v) \geq k} P\left(n(v) = 0\right).$$

Thus

$$E(\Phi) \le \kappa|V| + \left(1 - \frac{1}{\sqrt{k}}\right)^k |V| < 2\kappa|V|.$$

For $v, w \in V$, the random variables $n(v)$ and $n(w)$ are independent, unless there is a vertex $u \in V$ so that $(v, u) \in E$ and $(w, u) \in E$. Because the maximal in-degree is at most M, each $u \in V$ can account for at most M^2 such pairs, so there are at most $M^2|V|$ pairs which are not independent. Also note that $Var(1_{[n(v)=0]}) \le 1$ for every $v \in V$ so $Cov(1_{[n(v)=0]}, 1_{[n(w)=0]}) \le 1$. It follows that

$$Var(\Phi) = \sum_{v,w \in V} Cov(1_{[n(v)=0]}, 1_{[n(w)=0]}) \le M^2|V|.$$

By Chebyshev's inequality, the probability that W is not 3κ-dominating is at most

$$P(\Phi > 3\kappa|V|) \le P(|\Phi - E(Y)| > \kappa|V|) \le \frac{Var(\Phi)}{\kappa^2|V|^2} \le \frac{M^2}{\kappa^2|V|} < \frac{\kappa}{2}.$$

Also $E(|W|) = \frac{1}{\sqrt{k}}|V|$ and $Var(|W|) < \frac{1}{\sqrt{k}}|V|$, so again by Chebyshev's inequality

$$P(|W| > \frac{2}{\sqrt{k}}|V|) \le \frac{kVar(|W|)}{|V|^2} \le \frac{\sqrt{k}}{|V|} \le \frac{\kappa}{2}.$$

It follows that with probability at least $1 - \kappa$, W is 3κ-dominating and $|W| \le \frac{2}{\sqrt{k}}|V|$. □

4. Proof of Theorem 1

From now on we assume that $\mathbb{X}^G$ with the shift action described above is a fixed topological model for an action $G \curvearrowright (X, \mu)$. Suppose $stab(x)$ is infinite μ-almost-surely. Our goal is to prove that the sofic entropy of this G-action is non-positive with respect to any sofic approximation (in the case of a deterministic approximation sequence this means it is either 0 or $-\infty$). By a direct inspection of the definition of sofic entropy in (9), our goal is to show that for any $\eta > 0$ there exists a neighborhood $\mathcal{O} \subset Prob(\mathbb{X}^G)$ of μ so that for any sufficiently good approximation $\xi : G \to S_V$,

$$\frac{1}{|V|} \log sep_\eta(Map(\mathcal{O}, \xi), d^V) < \eta.$$

We will show that we can choose the neighborhood $\mathcal{O} \subset Prob(\mathbb{X}^G)$ to be of the form $\mathcal{O} = \mathcal{O}[M, \delta, \epsilon, F_1, F_2, \psi]$ (see Definition 5 below), for some parameters $F_1, F_2 \Subset G$ and $\epsilon, \delta > 0$.

Definition 4. (*Approximate stabilizer*) For $F \Subset G$ and $\delta > 0$ and $x \in \mathbb{X}^G$ let

$$stab_{\delta,F}(x) := \bigcap_{h \in F} \{g \in G \,:\, d(x_h, g(x)_h) < \delta\}. \tag{13}$$

Lemma 3. *For any measure $\mu \in Prob(\mathbb{X}^G)$, $\epsilon > 0$, $F_1 \Subset G$ there exists $F_2 \Subset G$ and a continuous function $\psi : \mathbb{X}^{F_2} \to 2^{F_1}$ so that*

$$\mu\left(\left\{x \in \mathbb{X}^G \,:\, \psi(x\,|_{F_2}) \ne (stab(x) \cap F_1)\right\}\right) < \epsilon. \tag{14}$$

Proof. By Lusin's theorem there exists a compact set $E \subset \mathbb{X}^G$ with $\mu(X \setminus E) < \epsilon$ so that the function $x \mapsto (stab(x) \cap F_1)$ is continuous on E, where $(stab(x) \cap F_1)$ is considered as a discrete set. Let $\tilde{\psi} : E \to (stab(x) \cap F_1)$ denote the restriction of $x \mapsto (stab(x) \cap F_1)$ to E, then $\tilde{\psi}^{-1}(s)$ is clopen for every $s \in 2^{F_1}$. Thus there exists $F_2 \subset G$ and $\psi : \mathbb{X}_2^F \to G$ so that $\tilde{\psi}(x) = \psi(x\,|_{F_2})$. □

Definition 5. Let $\epsilon, \delta, M > 0$, $F_1, F_2 \Subset G$ and $\psi : \mathbb{X}^{F_2} \to 2^{F_1}$. Define

$$\mathcal{O}[M, \delta, \epsilon, F_1, F_2, \psi] \subset Prob(\mathbb{X}^G)$$

to be the set of probability measures $\nu \in Prob(\mathbb{X}^G)$ satisfying the following conditions:

$$\nu(\{x : |\psi(x\mid_{F_2})| < M\}) < \epsilon, \tag{15}$$

$$\nu\left(\left\{x : \psi(x\mid_{F_2}) \neq (stab_{\delta, F_2^2}(x) \cap F_1)\right\}\right) < \epsilon. \tag{16}$$

Lemma 4. *If δ^{-1} is not an integer power of 2, then the set $\mathcal{O}[M, \delta, \epsilon, F_1, F_2, \psi] \subset Prob(\mathbb{X}^G)$ is open.*

Proof. Suppose δ^{-1} is not an integer power of 2. By (6) it follows that for $\omega, \omega' \in \mathbb{X}$, $d(\omega, \omega') < \delta$ if and only if $d(\omega, \omega') \leq \delta$. So for every $F \Subset G$,

$$stab_{\delta,F}(x) = \bigcap_{h \in F} \{g \in G : d(x_h, g(x)_h) < \delta\} = \bigcap_{h \in F} \{g \in G : d(x_h, g(x)_h) \leq \delta\}.$$

It follows that for any $g \in G$ and $F \Subset G$ the set $\{x \in \mathbb{X}^G : g \in stab_{\delta,F}(x)\}$ is a *clopen* set: It is both open and closed in $\mathbb{X}^G$.

Because F_1 and F_2 are both finite and $\psi : \mathbb{X}^{F_2} \to 2^{F_1}$ is continuous,

$$A := \left\{x \in \mathbb{X}^G : \psi(x\mid_{F_2}) \neq (stab_{\delta, F_2^2}(x) \cap F_1)\right\}$$

and

$$B := \{x \in X : |\psi(x\mid_{F_2})| < M\}$$

are also clopen in X. So the indicator functions $1_A, 1_B : X \to \mathbb{R}$ are continuous. Now

$$\mathcal{O}[M, \delta, \epsilon, F_1, F_2, \psi] = \left\{\nu \in Prob(\mathbb{X}^G) : \int 1_A(x) d\nu(x) < \epsilon \text{ and } \int 1_B(x) d\nu(x) < \epsilon\right\},$$

so $\mathcal{O}[M, \delta, \epsilon, F_1, F_2, \psi] \subset Prob(\mathbb{X}^G)$ is an open set.
□

Fix $\eta > 0$. We now specify how to choose the parameters $\epsilon > 0$, $\delta > 0$, $M > 0$, $F_1, F_2 \Subset G$, and $\psi : \mathbb{X}_2^F \to 2^{F_1}$.

- Choose ϵ so that
$$0 < \epsilon < \min\{\frac{\eta}{100}, \frac{1}{3}\}. \tag{17}$$
- Choose $\delta > 0$ so that δ^{-1} is not an integer power of 2 and so that
$$3\delta < \eta - 100\epsilon. \tag{18}$$
- Choose $M > 0$ depending on ϵ and δ, and big enough so that
$$\sup_{n \geq M} (1 - \frac{1}{\sqrt{n}})^n < \epsilon/3 \text{ and } \frac{4}{\sqrt{M}} \log sep_{\delta/2}(\mathbb{X}, d) < \frac{\eta}{2}. \tag{19}$$
It is clear that the left hand side in both expressions tends to 0 as $M \to \infty$, so such choice of M is indeed possible.
- Choose a finite subset $F_1 \Subset G$ depending on M, ϵ, and the measure μ so that $1 \in F_1$ and
$$\mu\left(\left\{x \in \mathbb{X}^G : |stab(x) \cap F_1| \leq M\right\}\right) < \epsilon/2. \tag{20}$$

We prove the existence of such a set F_1 in Lemma 5 below.

- By Lemma 3 choose $F_2 \Subset G$ and $\psi : \mathbb{X}^{F_2} \to 2^{F_1}$ so that (14) holds. Furthermore, by making F_2 bigger, assume that $F_1 \subset F_2$, that $F_2 = F_2^{-1}$ and that

$$\frac{2}{\sqrt{|F_2|}}|F_1|\log(2) < \frac{\eta}{2}. \tag{21}$$

- Choose a finite set V big enough so that

$$|V| > 2|F_2|^2(\epsilon/3)^{-3}. \tag{22}$$

- Choose $\xi : G \to S_V$ to be a symmetric $(F_2^6, \epsilon/6)$-approximation of G.

Lemma 5. *Under the assumption that $\mu(|stab(x)| < \infty) = 0$, for every $M > 0$ and $\epsilon > 0$ there exists $F_1 \Subset G$ so that $1 \in F$ and (20) holds.*

Proof. Because $stab(x) = \infty$ μ-a.e, it follows that for any $M > 0$,

$$\mu\left(\{x \in X \ : \ |stab(x)| \leq M\}\right) = 0.$$

Note that

$$\{x \in X \ : \ |stab(x)| \leq M\} = \bigcap_{F \Subset G} \{x \in X \ : \ |stab(x) \cap F| \leq M\}.$$

So by σ-additivity of μ, it follows that (20) holds for some $F_1 \Subset G$. Furthermore, we can assume that $1 \in F_1$ by further increasing F_1. □

Lemma 6. *For $M > 0$, $\epsilon, \delta > 0$ and $F_1, F_2 \Subset G$ as above, $\mu \in \mathcal{O}[M, \delta, \epsilon, F_1, F_2, \psi]$.*

Proof. Because $F_1 \subseteq F_2^2$, it follows that

$$stab(x) \subseteq stab_{\delta, F_2^2}(x) \subseteq stab_{\delta, F_1}(x).$$

So by (14) it follows that (16) also holds with ν replaced by μ. Using (14) and (20) we see that (15) holds with ν replaced by μ. Thus $\mu \in \mathcal{O}[M, \delta, \epsilon, F_1, F_2, \psi]$. □

The following lemma shows that approximate stabilizers behave well under conjugation:

Lemma 7. *If $F_1 \subset F_2 = F_2^{-1}$ and $x \in \mathbb{X}^G$ satisfies*

$$\psi(x\mid_{F_2}) = stab_{\delta, F_2^2}(x) \cap F_1, \tag{23}$$

then

$$g\psi(x\mid_{F_2})g^{-1} \subseteq stab_{\delta, F_1}(g(x)) \text{ for every } g \in F_2. \tag{24}$$

Proof. Suppose (23) holds. Choose any $f \in \psi(x\mid_{F_2})$. By (23),

$$d(x_h, x_{f^{-1}h}) < \delta \text{ for every } h \in F_2^2. \tag{25}$$

Now choose any $g \in F_2$. For any $h \in F_1$ we have $g^{-1}h \in F_2^{-1}F_1 \subset F_2^2$ so we can substitute $g^{-1}h$ instead of h in (25) to obtain

$$d(x_{g^{-1}h}, x_{f^{-1}g^{-1}h}) < \delta.$$

Now $(g(x))_h = x_{g^{-1}h}$ and

$$(gfg^{-1}g(x))_h = x_{f^{-1}g^{-1}h}.$$

So we have

$$d((g(x))_h, (gfg^{-1}g(x))_h) < \delta.$$

This means that $(gfg^{-1}) \in stab_{\delta,F_1}(g(x))$. We conclude that (23) implies (24). □

Definition 6. Call $v \in V$ *good for* $\omega \in \mathbb{X}^V$ if the following conditions are satisfied:

$$|\psi(\Pi_v^\xi(\omega) \mid_{F_2})| \geq M, \tag{26}$$

$$\psi(\Pi_v^\xi(\omega) \mid_{F_2}) = stab_{\delta,F_2^2}(\Pi_v^\xi(\omega)) \cap F_1, \tag{27}$$

and

$$\xi_{g_1} \circ \xi_{g_2} \circ \xi_{g_3}(v) = \xi_{g_1 g_2 g_3}(v) \text{ for every } g_1, g_2, g_3 \in F_2^4. \tag{28}$$

Otherwise, say that $v \in V$ is *bad for* $\omega \in \mathbb{X}^V$.

Lemma 8. *Let $\Omega \subset Map(\mathcal{O}[M,\delta,\epsilon,F_1,F_2,\psi],\xi)$ with $2 \leq |\Omega| < \infty$. Then there exists a set $C \subset V$ and a function $\tau : V \to F_2$ with the following properties:*

(I) $|C| < \frac{2}{\sqrt{|F_2|}}|V|$,

(II) $\frac{1}{|V|}|\{v \in V \ : \ \xi_\tau(v) \notin C\}| \leq \epsilon$,

(III) $\frac{1}{|\Omega|}\,|\{x \in \Omega \ : \ |\{v \in V \ : \ \xi_\tau(v) \text{ is bad for } x\}| < 8\epsilon|V|\}| \geq \frac{1}{2}$,

where $\xi_\tau : V \to V$ is defined by

$$\xi_\tau(v) := \xi_{\tau(v)}(v). \tag{29}$$

Proof. Consider the directed graph $\mathcal{G}_{\xi,F_2} = (V,E)$ with

$$E = \left\{(u,v) \in V \times V \ : \ \exists g \in F_2 \text{ s.t. } \xi_g(u) = v\right\}.$$

We aim to apply Lemma 2 to find a small ϵ-dominating set $C \subset V$ in $\mathcal{G}_{\xi,F_2}$. Let us check that $\mathcal{G}_{\xi,F_2}$ satisfies the assumptions of Lemma 2:

Because ξ_g is a permutation of V for every $g \in G$, the maximal out-degree in $\mathcal{G}_{\xi,F_2}$ is at most $|F_2|$. Because the approximation $\xi : G \to S_V$ is symmetric and $F_2 = F_2^{-1}$, the maximal in-degree in $\mathcal{G}_{\xi,F_2}$ is also at most $|F_2|$. Let $V' \subset V$ denote the set of v's for which the mapping $g \mapsto \xi_g(v)$ is injective on F_2. Because $\xi : G \to S_V$ is an $(F_2^6, \epsilon/6)$-good approximation of G it follows that $|V \setminus V'| \leq \frac{\epsilon}{3}|V|$, so $\mathcal{G}_{\xi,F_2}$ is $(\epsilon/3, |F_2|, |F_2|)$-regular.

By Lemma 2, a $\frac{1}{\sqrt{|F_2|}}$-Bernoulli set $C \subset V$ is ϵ-dominating and has size at most $\frac{2}{\sqrt{|F_2|}}|V|$ with probability at least $1 - \frac{2}{3}\epsilon > \frac{1}{2}$. To see that Lemma 2 applies, we used the left inequality in (19) (keeping in mind that $|V| > M$), and (22) to deduce that (12) is satisfied with $k = M = |F_2|$ and κ replaced with $\epsilon/3$ and $N = |V|$. In this case $C \subset V$ satisfies (I). Choose the value of $\tau : V \to F_2$ at $v \in V$ randomly as follows: Whenever the set $N_v := \{g \in F_2 \ : \ \xi_g(v) \in C\}$ is non-empty, choose $\tau(v)$ uniformly at random from $N_v \subset F_2$. If $N_v = \emptyset$, then $\tau(v)$ be chosen uniformly at random from F_2. We see that if C is ϵ-dominating, then (II) is satisfied.

To conclude the proof we will show that (III) is satisfied with probability at least $1/2$.

For $\omega \in \Omega$ and $v \in V$ denote:

$$\Psi_{\omega,v} := \begin{cases} 0 & \text{if } v \text{ is good for } \omega; \\ 1 & \text{if } v \text{ is bad for } \omega. \end{cases} \tag{30}$$

Because $\Omega \subset \mathcal{O}[M,\delta,\epsilon,F_1,F_2,\psi]$, it follows that for every $\omega \in \Omega$, all but an ϵ-fraction of the v's are good so

$$\frac{1}{|V|}\sum_{v \in V} \Psi_{\omega,v} < \epsilon \text{ for every } \omega \in \Omega. \tag{31}$$

Now let $Z_{\omega,v}$ denote the indicator of the event "$\xi_\tau(v)$ is bad for ω", that is,

$$Z_{\omega,v} := \sum_{g\in F_2} 1_{[\tau(v)=g]}\Psi_{\omega,\xi_g(v)}.$$

Then $Z_{\omega,v}$ is a random variable, because $\tau : V \to F_2$ is a random function.

Because $g \mapsto \xi_g(v)$ is injective for every $v \in V'$, each $v \in V'$ has $|F_2|$ outgoing edges. So for $v \in V'$, $\tau(v)$ is uniform in F_2 in case N_v is empty and uniform in $\{g \in F_2 \ : \ \xi_g(v) \in C\}$ otherwise. It follows that

$$\mathbb{P}\left(\tau(v) = g\right) = |F_2|^{-1} \text{ for every } v \in V' \text{ and } g \in F_2. \tag{32}$$

It follows that for $v \in V'$,

$$\mathbb{E}(Z_{\omega,v}) = \frac{1}{|F_2|}\sum_{g\in F_2}\Psi_{\omega,\xi_g(v)}.$$

Because $|V \setminus V'| < \epsilon|V|$ it follows that

$$\mathbb{E}\left(\frac{1}{|V|}\sum_{v\in V} Z_{\omega,v}\right) \le \frac{1}{|V|}\sum_{v\in V}\frac{1}{|F_2|}\sum_{g\in F_2}\Psi_{\omega,\xi_g(v)} + \epsilon. \tag{33}$$

Because $\xi_g \in S_V$ is a permutation:

$$\sum_{v\in V}\Psi_{\omega,\xi_g(v)} = \sum_{v\in V}\Psi_{x,v}.$$

So from (33) and (31) we get that for every $x \in \Omega$

$$\mathbb{E}\left(\frac{1}{|V|}\sum_{v\in V} Z_{\omega,v}\right) \le \frac{1}{|V|}\sum_{v\in V}\Psi_{\omega,v} + \epsilon \le 2\epsilon.$$

Averaging over $\omega \in \Omega$ we obtain:

$$\mathbb{E}\left(\frac{1}{|\Omega|}\sum_{\omega\in\Omega}\frac{1}{|V|}\sum_{v\in V} Z_{\omega,v}\right) = \frac{1}{|\Omega|}\sum_{\omega\in\Omega}\mathbb{E}\left[\frac{1}{|V|}\sum_{v\in V} Z_{\omega,v}\right] \le 2\epsilon.$$

Using Markov's inequality, we get that

$$\mathbb{P}\left[\left(\frac{1}{|\Omega|}\sum_{x\in\Omega}\frac{1}{|V|}\sum_{v\in V} Z_{\omega,v}\right) > 4\epsilon\right] \le \frac{1}{2}.$$

So (III) holds with probability at least $\frac{1}{2}$. $\square$

Given a finite set V, the following "Hamming-like" metric is defined on $\mathbb{X}^V$:

$$d^V(\omega,\omega') := \frac{1}{|V|}\sum_{v\in V} d(\omega_v,\omega'_v) \text{ for } \omega,\omega' \in \mathbb{X}^V. \tag{34}$$

We also have the following "uniform" metric d^D_∞ on $\mathbb{X}^D$, where D is a finite set:

$$d^D_\infty(\omega,\omega') := \max_{v\in D} d(\omega_v,\omega'_v) \text{ for } \omega,\omega' \in \mathbb{X}^V. \tag{35}$$

We will use the following estimate:

Lemma 9. *For any finite set D and $\delta > 0$ we have*

$$\log sep_{2\delta}(\mathbb{X}^D, d_\infty^D) \leq |D| \log sep_\delta(\mathbb{X}, d).$$

Proof. If $S \subset \mathbb{X}$ is such that $\mathbb{X} = \bigcup_{\omega \in S} B_\delta(\omega)$ and $|S| = cov_\delta(\mathbb{X}, d)$ then the union of δ-balls in $(\mathbb{X}^D, d_\infty^D)$ with centers in S^V covers $\mathbb{X}^V$. It follows that

$$\log cov_\delta(\mathbb{X}^D, d_\infty^D) \leq |D| \log cov_\delta(\mathbb{X}, d).$$

The claim now follows because $sep_{2\delta}(\mathbb{X}^D, d_\infty^D) \leq cov_\delta(\mathbb{X}^D, d_\infty^D)$ and $cov_\delta(\mathbb{X}, d) \leq sep_\delta(\mathbb{X}, d)$. □

We record the following Lemma (see Lemma 3.1 in [13], and recall that we use a left-action):

Lemma 10. *Suppose $v \in V$ is good for $\omega \in \mathbb{X}^V$ and $g \in F_2^3$. Then*

$$g^{-1}(\Pi^\xi_{\xi_g(v)}(\omega)) \mid_{F_2^3} = \Pi^\xi_v(\omega) \mid_{F_2^3}.$$

Proof. Because v is good for ω it follows that

$$\xi_{h^{-1}g^{-1}}(\xi_g(v)) = \xi_{h^{-1}}(v) \text{for every } h \in F_2^3,$$

so for every $h \in F_2^3$ we have

$$g^{-1}(\Pi^\xi_{\xi_g(v)}(\omega))_h = (\Pi^\xi_{\xi_g(v)}(\omega))_{gh} = \omega_{\xi_{h^{-1}g^{-1}}(\xi_g(v))} = \omega_{\xi_{h^{-1}}(v)} = (\Pi^\xi_v(x))_h. \quad \square$$

The following lemma is the heart of our proof of Theorem 1:

Lemma 11. *The following holds:*

$$\frac{1}{|V|} \log sep_\eta(Map(\mathcal{O}[M, \delta, \epsilon, F_1, F_2, \psi], \xi), d^V) < \frac{4}{\sqrt{|F_2|}} \cdot |F_1| \log(2) + \frac{4}{\sqrt{M}} \log sep_{\delta/2}(\mathbb{X}, d). \tag{36}$$

Proof. Fix any subset $\Omega \subset Map(\mathcal{O}[M, \delta, \epsilon, F_1, F_2, \psi], \xi)$ that is η-separated with respect to the metric d^V. Let $\tau : V \to F_2$ and $C \subset V$ be as in the conclusion of Lemma 8. By condition (III) there exists $\Omega' \subset \Omega$ with $2|\Omega'| \geq |\Omega|$ such that

$$\frac{1}{|V|} |\{v \in V \ : \ \xi_\tau(v) \text{ is bad for } \omega\}| < 8\epsilon \text{ for every } \omega \in \Omega'. \tag{37}$$

Denote by S the set of functions from C to subsets of F_1, that is $S := (2^{F_1})^C$. For each $s \in S$, let

$$\Omega_s := \left\{\omega \in \Omega' \ : \ \psi(\Pi^\xi_v(\omega) \mid_{F_2}) = s_v \text{ for every } v \in C\right\}.$$

Then $\Omega' = \bigcup_{s \in S} \Omega_s$, so

$$|\Omega| \leq 2 \cdot 2^{|F_1| \cdot |C|} \cdot \max_{s \in S} |\Omega_s|. \tag{38}$$

By (I), $|C| \leq \frac{2}{\sqrt{|F_2|}}|V|$. It follows that

$$\frac{1}{|V|} \log |\Omega| \leq \frac{4}{\sqrt{|F_2|}} |F_1| \log(2) + \max_{s \in S} \frac{1}{|V|} \log |\Omega_s|. \tag{39}$$

So our next goal is to bound $|\Omega_s|$, for $s \in S$. For this s and $v \in V$ define:

$$stab(v,s) := \begin{cases} (\tau(v))^{-1} s_{\xi_\tau(v)} \tau(v) & \text{if } \xi_\tau(v) \in C, \\ \emptyset & \text{otherwise.} \end{cases} \tag{40}$$

We claim that if $\omega \in \Omega_s$ and $v, \xi_\tau(v)$ are both good for ω then

$$stab(v,s) \subset stab_{\delta,F_1}(\Pi_v^\xi(\omega)). \tag{41}$$

Indeed, we can assume that $\xi_\tau(v) \in C$ otherwise $stab(v,s) = \emptyset$ and (41) holds trivially. Then

$$s_{\xi_\tau(v)} = stab_{\delta,F_1}(\Pi^\xi_{\xi_\tau(v)}(\omega)) \cap F_1.$$

Denote $g_v := \tau(v)$. Because v is good for ω and $g_v \in F_2$, by Lemma 10,

$$g_v^{-1}(\Pi^\xi_{\xi_{g_v}(v)}(\omega)) \mid_{F_2^3} = \Pi_v^\xi(\omega) \mid_{F_2^3}.$$

So

$$stab_{\delta,F_1}\left(g_v^{-1}(\Pi^\xi_{\xi_{g_v}(v)}(\omega))\right) \cap F_2^3 = stab_{\delta,F_1}\left(\Pi_v^\xi(\omega)\right) \cap F_2^3.$$

Because $\xi_\tau(v)$ is good for ω (27) holds with v replaced by $\xi_\tau(v)$. So by Lemma 7 applied with $g = \tau(v)^{-1}$,

$$\tau(v)^{-1} s_{\xi_\tau(v)} \tau(v) \subset stab_{\delta,F_1}(g_v^{-1}(\Pi^\xi_{\xi_\tau(v)}(\omega))) \cap F_2^3 = stab_{\delta,F_1}(\Pi_v^\xi(\omega)) \cap F_2^3.$$

This proves that (41) holds.

Consider the graph $\mathcal{G}_s = (V, E_s)$ where

$$E_s := \{(v, \xi_g(v)) \;:\; g \in stab(v,s)\}.$$

Claim A: If (v,u) is an edge in $\mathcal{G}_s$ and $\omega \in \Omega_s$ and $v, \xi_\tau(v)$ are both good for ω then $d(\omega_v, \omega_u) < \delta$.

Proof of Claim A: By definition of $\mathcal{G}_s$ there exists $g \in stab(v,s)$ so that $\xi_g(v) = u$. By the argument above $g \in stab_{\delta,F_1}(\Pi_v^\xi(\omega))$, so $d((\Pi_v^\xi(\omega))_1, g(\Pi_v^\xi(\omega))_1) < \delta$. Now $x_v = (\Pi_v^\xi(\omega))_1$ and

$$x_u = \omega_{\xi_g(v)} = (\Pi_v^\xi(\omega))_{g^{-1}} = g((\Pi_v^\xi(\omega))_1,$$

so indeed $d(\omega_v, \omega_u) < \delta$.

Claim B: The graph $\mathcal{G}_s$ is $(11\epsilon, M, |F_2|^3)$-regular.

Proof of Claim B: Note that by definition $stab(v,s) \subset F_2^{-1} F_1 F_2 \subset F_2^3$, so $(u,v) \in E_s$ implies that $v = \xi_g(u)$ for some $g \in F_2^3$. This shows that $\mathcal{G}_s$ has maximal in-degree at most $|F_2|^3$.

The properties of C, τ and Ω' assure that

$$|\{v \in V \;:\; \xi_\tau(v) \notin C\}| < 2\epsilon |V|$$

and

$$|\{v \in V \;:\; \xi_\tau(v) \text{ is bad for } \omega\}| < 8\epsilon |V| \text{ for every } \omega \in \Omega'.$$

It follows that $|stab(v,x)| < M$ for at most $10\epsilon|V|$ v's. Also, as in the proof of Lemma 8, because $\xi : G \to S_V$ is a sufficiently good sofic approximation the map $g \mapsto \xi_g(v)$ is injective on

F_2^3 for all but at most $\epsilon|V|$ v's. It follows that at most $11\epsilon|V|$ of the vertices in $\mathcal{G}_s$ have degree smaller than M. This completes the proof of Claim B.

By (19) and (22), the condition (12) is satisfied with M replaced by $|F_2|^3$, k replaced by M and κ replaced by 11ϵ. So using Claim B we can apply Lemma 2 to deduce that there is a set $D \subset V$ of size at most $\frac{2}{\sqrt{M}}|V|$ which is 33ϵ-dominating in $\mathcal{G}_s$. As in the proof of Lemma 8, there exists a function $\tau' : V \to F_2^3$ so that for all but 33ϵ v's the pair $(v, \xi_{\tau'(v)}(v))$ is an edge in $\mathcal{G}_s$ and $\xi_{\tau'(v)}(v) \in D$.

Claim C: If $\omega, \omega' \in \Omega_s$ and $d(\omega_v, \omega'_v) < \delta$ for all $v \in D$ then $d^V(\omega, \omega') < \eta$.

Proof of Claim C: Suppose $\omega, \omega' \in \Omega_s$ and $d(\omega_v, \omega'_v) < \delta$ for all $v \in D$. Fix $u \in V$. Denote $v = \xi_{\tau'(u)}(u)$. If u and $\xi_\tau(u)$ are both good with respect to ω and with respect to ω', and (u, v) is an edge in $\mathcal{G}_s$, it follows from Claim A that $d(\omega_u, \omega'_v) < \delta$ and $d(\omega'_u, \omega'_v) < \delta$. Furthermore, if $v \in D$, then $d(\omega_v, \omega'_v) < \delta$ so in that case $d(\omega_u, \omega'_u) < 3\delta$. So if $d(\omega_u, \omega'_u) > 3\delta$, either $v \notin D$ or one of $u, \xi_\tau(u)$ is not good for ω or for ω'. Thus

$$\left| \left\{ u \in V \ : \ d(\omega_u, \omega'_u) > 3\delta \right\} \right| < 2(33+8)\epsilon|V| < 100\epsilon|V|.$$

Thus, because the diameter of $\mathbb{X}$ is bounded by 1,

$$\frac{1}{|V|} \sum_{v \in V} d(\omega_v, \omega'_v) \leq 3\delta + 100\epsilon < \eta,$$

where in the last inequality we used (18). This completes the proof of Claim C.

Because Ω_s is η-separated, Claim C implies that the restriction map $\pi_D : \mathbb{X}^V \to \mathbb{X}^D$ is injective on Ω_s, and that $\pi_D(\Omega_s)$ is δ-separated with respect to the metric d_∞^D. Thus by Lemma 9,

$$\log|\Omega_s| = \log|\pi_D(\Omega_s)| \leq \log sep_\delta(\mathbb{X}^d, d_\infty^D) \leq |D| \log sep_{\delta/2}(\mathbb{X}, d).$$

We conclude that

$$\log|\Omega_s| \leq \frac{4}{\sqrt{M}}|V| \log sep_{\delta/2}(\mathbb{X}, d). \tag{42}$$

Together with (39) this shows that

$$\frac{1}{|V|}\log|\Omega| \leq \frac{4}{\sqrt{|F_2|}} \cdot |F_1|\log(2) + \frac{4}{\sqrt{M}} \log sep_{\delta/2}(\mathbb{X}, d).$$

Since Ω was an arbitrary η-separated subset of $\mathcal{O}[M, \delta, \epsilon, F_1, F_2, \psi]$, this completes the proof. □

To conclude the proof of Theorem 1, observe that the right hand side of (36) is bounded above by η because of (21) and the right inequality in (19).

5. Finite Stabilizers and Completely Positive Entropy

We conclude with a corollary regarding actions with completely positive Σ-entropy, due to Weiss [7] in the amenable case. Recall that an action $G \curvearrowright (X, \mu)$ is *faithful* if $\mu(\{x \in X \ : \ g(x) \neq x\}) > 0$ for all $g \in G$, and *free* if $\mu(\{x \in X \ : \ stab(x) \neq \{1\}\}) = 0$. Also, recall that $G \curvearrowright (Y, \nu)$ is a *factor* of $G \curvearrowright (X, \mu)$, that is there is a G-equivariant map $\pi : X \to Y$ with $\nu = \mu \circ \pi^{-1}$. An action $G \curvearrowright (X, \mu)$ of a sofic group has *completely positive Σ-entropy* if any non-trivial factor has positive Σ-entropy.

Corollary 2. *Let G be an infinite countable sofic group. If an ergodic action $G \curvearrowright (X, \mu)$ is faithful and has completely positive entropy with respect to some sofic approximation Σ, it is free.*

Proof. By Theorem 1, the stabilizers must be finite, thus by Lemma 1 the map $x \mapsto stab(x)$ induces a finite factor. But an action of an infinite group on finite probability space must have infinite stabilizers.

In particular by Theorem 1 this factor has zero entropy. Because $G \curvearrowright (X, \mu)$ has completely positive sofic entropy it follows that $stab(x)$ is constant, and because the action is faithful it must be trivial, so the action is free. $\square$

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