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Hawking-Like Radiation from the Trapping Horizon of Both Homogeneous and Inhomogeneous Spherically Symmetric Spacetime Model of the Universe

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Abstract: The present work deals with the semi-classical tunnelling approach and the Hamilton–Jacobi method to study Hawking radiation from the dynamical horizon of both the homogeneous Friedmann–Robertson–Walker (FRW) model and the inhomogeneous Lemaitre–Tolman–Bondi (LTB) model of the Universe. In the tunnelling prescription, radial null geodesics are used to visualize particles from behind the trapping horizon and the Hawking-like temperature has been calculated. On the other hand, in the Hamilton–Jacobi formulation, quantum corrections have been incorporated by solving the Klein–Gordon wave equation. In both the approaches, the temperature agrees at the semiclassical level.

Keywords: Hawking-like temperature; Hamilton–Jacobi method; tunnelling of particles

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1. Introduction

The quantum description of a black hole (BH), namely the Hawking radiation (HR) is closely related to the existence of an event horizon to the BH. The derivation of Hawking that "BH evaporates particles" [1–3] was based on quantum field theory. Hartle and Hawking [4] subsequently derived the BH temperature at the semiclassical level using the Feynmann path integral. The mathematical complexity of the above procedures forces to develop semi-classical approaches [5–11] for studying BH radiation. However, these semi-classical techniques were classified into two approaches—the tunnelling approach of Parikh and Wilczek [5,6,9–16] and the standard Hamilton–Jacobi (HJ) method (known as complex path integral formalism) by Padmanabhan et al. [7,8].

The energy conservation in tunnelling of a thin shell from the hole is the main ingredient for the first approach and is often referred to as the radial null geodesic method. The imaginary part of the action from the s-wave emission is connected to the Boltzmann factor for emission to relate with Hawking temperature (HT). In the complex paths method, the action $S(r,t)$ for a single scalar particle is obtained by solving the Klein–Gordon (KG) equation in gravitational background. Then, HT is obtained using "the principle of detailed balance" [7,8]. A remarkable behavior of the energy conservation in tunnelling is that the final spectrum is not strictly thermal [9–11], and this has important consequences on the black hole information paradox [17].

On the other hand, the discovery of HR [1,2] completes the cycle to describe BH as a thermodynamical object. The black body nature of BH reveals emission of thermal radiation with temperature proportional to its surface gravity. Before the discovery of HR, entropy was formulated by Bekenstein [18] as proportional to the horizon area of the BH. This new perspective leads to viewing general relativity (GR) from a completely different angle. The nice interrelationship between gravity and thermodynamics was first enlightened by Jacobson [19] replacing the usual definition of heat by flow of energy across a horizon. In fact, gravity might be due to the thermodynamics of the microstructure of spacetime. Then, this idea is extended to modified gravity theories [20–23]. At present, there is a general consensus that gravity might be originated by the thermodynamics of the unknown microstructure of spacetime. Thus, it is speculated that BH thermodynamics is playing the role of a bridge to put GR and quantum mechanics on the same platform—the challenging issue of quantum gravity. It is also an intuitive but general conviction that BHs result in highly excited states representing both the "hydrogen atom" and the "quasi-thermal emission" in quantum gravity. A recent approach [24–26] has shown that such an intuitive picture is more than a picture, discussing a model of quantum BH somewhat similar to the historical semi-classical model of the structure of a hydrogen atom introduced by Bohr in 1913 [27].

Normally, the global concept of event horizon is used to define the HT. However, difficulty arises in dynamical spacetime, where there is no existence of an event horizon even locally. In the recent past, Hayward et al. [28] formulated a locally defined HT for dynamical BH using the tunnelling idea of Parikh–Wilczek [5]. Then, Cai et al. [29] have shown HR from the locally defined apparent horizon of the FRW Universe, where HT is measured by an observer using the Kodama vector [30–32] inside the horizon. In the present work, we shall address this important issue in GR, namely the formulation of thermodynamics of dynamical spacetime. We shall derive HR and the corresponding HT using both the HJ method and the tunnelling approach. A comparison between these two formalism will be done for homogeneous (FRW) as well as an inhomogeneous (LTB) spherically symmetric spacetime model. The paper is organized as follows: Section 2 gives an overview of radial null geodesic method to derive HT. In Section 3, the HJ method with quantum prescription for the FRW model has been presented. Quantum corrected entropy has been formulated in Section 4. Sections 5 and 6 deal with inhomogeneous LTB model, where HT has been evaluated using both the approaches. In the last Section we write a brief summary of the whole work. Finally, for the sake of completeness the paper ends with an Appendix concerning the calculation of Kodama Vector and Surface Gravity.

2. Radial Null Geodesic Approach: Hawking-Like Temperature

In semiclassical tunnelling analysis [5,6,9–13,33], the radial null geodesic method is a common way of evaluating Hawking radiation. This method is simple as compared to the HJ method, but it has some limitations, namely:

- The method is applicable only for massless particles;
- One has to use only Painleve type coordinates to avoid singularity at the horizon;
- There is a discrepancy of factor two in this method;
- There is no general method to include quantum effects.

The basic idea in this semiclassical approximation (Wentzel–Kramers–Brillouin (WKB) approximation) is that the emission rate for the *s*-wave emission of a massless particle can be related to the imaginary part of the action of a system. In the *s*-wave approximation, particles can be viewed as a massless shell moving along radial null geodesic. Now, compared to static BH cases, there is one basic difference, as in the present case, the metric coefficients depend on both radius and time. Hence, there is no longer time translation Killing vector field. We shall have to use the Kodama [30], which is time like inside the horizon and the associate energy.

The homogeneous and isotropic model of the Universe is described by the FRW metric as:

$$ds^2 = -dt^2 + \frac{a^2(t)}{1-\kappa r^2}dr^2 + a^2(t)r^2 d\Omega_2{}^2, \tag{1}$$

where the FRW coordinates (t, r, θ, ϕ) are orthogonal comoving coordinates with "t" the comoving time corresponding to a comoving observer, and κ is the intrinsic spatial curvature.

For the above FRW model of the Universe, if we make a change of the radial coordinate $r \to R$, where $R = ar$ is known as the area radius, then the above standard FRW metric becomes a Painleve–Gullstrand like metric [34,35] as follows:

$$ds^2 = -\frac{1-\frac{R^2}{R_A^2}}{1-\frac{\kappa R^2}{a^2}}dt^2 - \frac{2HR}{1-\frac{\kappa R^2}{a^2}}dtdR + \frac{dR^2}{1-\frac{\kappa R^2}{a^2}} + R^2 d\Omega_2{}^2, \tag{2}$$

where $R_A = \frac{1}{\sqrt{H^2+\frac{\kappa}{a^2}}}$ is the radius of the apparent horizon. In the present case, for the metric (2), the Kodama vector and the corresponding energy are:

$$\kappa^\mu = \left(\sqrt{1-\frac{\kappa R^2}{a^2}}, 0, 0, 0\right) \quad and \quad \omega = -\sqrt{1-\frac{\kappa R^2}{a^2}}\frac{\partial S}{\partial t}, \tag{3}$$

respectively. Thus, $\frac{\omega}{\sqrt{1-\frac{\kappa R^2}{a^2}}}$ is the energy of the particle as measured by an observer with the Kodama vector. The differential equation for the radial null geodesic (i.e., $ds^2 = 0 = d\Omega_2{}^2$) has the form

$$\frac{dR}{dt} = HR \pm \sqrt{H^2R^2 + 1 - \frac{R^2}{R_A^2}}, \tag{4}$$

where $\pm$ sign are associated with the outgoing/ingoing null geodesic with the assumption that "t" increases towards future. As we are interested in the imaginary part of the action corresponding to the tunnelling process through a barrier (the classically forbidden region) as according to Parikh-Wilczek [5]:

$$Im\ S = Im \int_{R_{in}}^{R_{out}} p_R dR = Im \int_{R_{in}}^{R_{out}} \int_0^{p_R} dp'_R dR = Im \int_{R_{in}}^{R_{out}} \int_0^{E} \frac{dH'}{\dot{R}} dR, \tag{5}$$

where we have used the Hamiltonian equation

$$\dot{R} = \frac{\partial H}{\partial p_R} = \frac{dH}{dp_R}|_R.$$

Here, p_R is the radial momentum, R_{in} and R_{out} are positions very close to the horizon with R_{in} as the initial position and R_{out}, a classical turning point. Using the value of R from Equation (4) into Equation (5), we get

$$\begin{aligned} Im\ S &= Im \int_{R_{in}}^{R_{out}} dR \int \frac{dH'}{\dot{R}} \\ &= Im \int_{R_{in}}^{R_{out}} \frac{dR}{\dot{R}} \frac{\omega}{\sqrt{1+\frac{\kappa R^2}{a^2}}} \\ &= -\omega \int_{R_{in}}^{R_{out}} \frac{dR}{\sqrt{1+\frac{\kappa R^2}{a^2}}\left\{\sqrt{H^2R^2+1-\frac{R^2}{R_A^2}} - HR\right\}} \\ &= \pi R_A \omega, \end{aligned} \tag{6}$$

where integration over the Hamiltonian H gives the energy of the particle as $\frac{\omega}{\sqrt{1-\frac{\kappa^2}{a^2}}}$ as measured by an observer with Kodama vector. Now, comparing the tunnelling probability

$$\Gamma \sim \left\{-\frac{2}{\hbar} Im\ S\right\} \tag{7}$$

with the Boltzmann factor $exp\left\{-\frac{\omega}{T}\right\}$, we have the temperature associated with the apparent horizon of the FRW Universe as [35]

$$T_A = \frac{\hbar}{2\pi R_A}. \tag{8}$$

This is the semiclassical Hawking-like temperature of the FRW Universe for tunnelling of massless particles across the apparent horizon.

It should be noted that Im $\int_{r_{in}}^{r_{out}} p_r dr$ is not canonically invariant and hence it is not a proper observable. The canonically invariant quantity Im $\oint p_r dr$ over a closed path across the horizon cannot coincide with 2Im $\int_{r_{in}}^{r_{out}} p_r dr$ for the Painleve coordinates as a particle experiences barrier only from inside the horizon to outside, not the other way. However, for the invariant definition $\Gamma \sim e^{-\frac{i}{\hbar}\mathrm{Im}\oint p_r dr}$, the Hawking temperature turns out to be twice the original temperature. This ambiguity in the factor of two has been discussed in [36–42].

3. Hamilton–Jacobi Method: Quantum Prescription

In this section, we shall deal with the tunnelling of massless particles beyond the semiclassical approximation by Hamilton–Jacobi (HJ) method. We shall start with a KG equation for a scalar field ϕ describing a massless scalar particle of the form

$$\frac{\hbar^2}{\sqrt{-g}}\partial(g^{\mu\nu}\sqrt{-g}\partial_\nu)\psi = 0. \tag{9}$$

The explicit form of the KG equation in the background of the FRW metric (1) is given by

$$\frac{\partial^2\psi}{\partial t^2} - \frac{(1-\kappa r^2)}{a^2}\frac{\partial^2\psi}{\partial r^2} + H\frac{\partial\psi}{\partial t} + \frac{\kappa r}{a^2}\frac{\partial\psi}{\partial r} = 0. \tag{10}$$

It should be mentioned that here, due to the spherical symmetry of the FRW spacetime and consideration of the radial trajectories only, we have considered $(t-r)$—sector in the spacetime given by Equation (1), i.e., 2D hyperplane (t,r). Now, substituting the standard ansatz for the semiclassical wave function

$$\psi(r,t) = exp\left\{-\frac{i}{\hbar}S(r,t)\right\} \tag{11}$$

into the wave Equation (10), the differential equation for the action S becomes

$$\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{1-\kappa r^2}{a^2}\right)\left(\frac{\partial S}{\partial r}\right)^2 - i\hbar\left[\frac{\partial^2 S}{\partial t^2} - \left(\frac{1-\kappa r^2}{a^2}\right)\frac{\partial^2 S}{\partial r^2} + H\frac{\partial S}{\partial t} + \frac{\kappa r}{a^2}\frac{\partial S}{\partial r}\right] = 0. \tag{12}$$

As a first step to solve this partial differential equation (PDE), we expand the action in powers of Planck constant $\hbar$ as [36]

$$S(r,t) = S_0(r,t) + \Sigma_k \hbar^k S_k(r,t) \tag{13}$$

with k, a positive integer. Here, terms of the order of Planck's constant, and its higher powers, are considered as quantum corrections over the semiclassical action S_0. Now, substituting this ansatz for

S in the differential Equation (12) and equating different powers of $\hbar$ on both sides, we obtain the following set of PDEs:

$$\hbar^0 : \left(\frac{\partial S_0}{\partial t}\right)^2 - \left(\frac{1-\kappa r^2}{a^2}\right)\left(\frac{\partial S_0}{\partial r}\right)^2 = 0, \tag{14}$$

$$\hbar^1 : 2\frac{\partial S_0}{\partial t}\frac{\partial S_1}{\partial t} - 2\left(\frac{1-\kappa r^2}{a^2}\right)\frac{\partial S_0}{\partial r}\frac{\partial S_1}{\partial t} - i\hbar\left[\frac{\partial^2 S_0}{\partial t^2} - \left(\frac{1-\kappa r^2}{a^2}\right)\frac{\partial^2 S_0}{\partial r^2} + H\frac{\partial S_0}{\partial t} + \frac{\kappa r}{a^2}\frac{\partial S_0}{\partial r}\right] = 0, \tag{15}$$

$$\begin{aligned}\hbar^2 \quad : \quad & \left(\frac{\partial S_1}{\partial t}\right)^2 + 2\frac{\partial S_0}{\partial t}\frac{\partial S_2}{\partial t} - \left(\frac{1-\kappa r^2}{a^2}\right)\left\{\left(\frac{\partial S_1}{\partial r}\right)^2 + 2\frac{\partial S_0}{\partial r}\frac{\partial S_2}{\partial r}\right\} - i\hbar\left[\frac{\partial^2 S_1}{\partial t^2} - \left(\frac{1-\kappa r^2}{a^2}\right)\frac{\partial^2 S_1}{\partial r^2}\right. \\ + \quad & \left. H\frac{\partial S_1}{\partial t} + \frac{\kappa r}{a^2}\frac{\partial S_1}{\partial r}\right] = 0,\end{aligned} \tag{16}$$

and so on. The above expansion of the action (see Equation (13)) to all orders in $\hbar$ is due to [36]. However, Yale [43] criticized this expansion of the action in $\hbar$ as [36] does not take into account the higher order corrections to the energy—only zero order energy is considered. Subsequently, Singleton et al. [44] has made an attempt to justify the above expansion of the action in [36] using some ancillary assumption, namely, self-similarity. Anyway, one can say that the expansion in Equation (13) may be conditionally valid.

Apparently, different order PDEs are very complicated, but, fortunately, there will be lot of simplifications if, in the PDE corresponding to $\hbar^k$, all previous PDEs are used and finally we obtain identical PDE, namely

$$\hbar^k : \left(\frac{\partial S_k}{\partial t}\right)^2 - \left(\frac{1-\kappa r^2}{a^2}\right)\left(\frac{\partial S_k}{\partial r}\right)^2 \qquad for\;\; k = 0, 1, 2.... \tag{17}$$

We see that different order quantum corrections satisfy identical differential equations as the semiclassical action S_0, thus the correction terms are not independent, rather proportional to S_0 (i.e., $S_k \propto S_0$, for all k). To determine these proportionality constants, we shall use dimension analysis. Since S_0 has the dimension of $\hbar$, the proportionality constant d_k for S_k has the dimension of $\hbar^{-k}$. However, in standard units, namely $G = c = k_B = 1$, the Planck's constant $\hbar$ is of the order M_p^2 (M_p is the Planck mass) and hence d_k has the dimension of M^{-2k}, where M is identified as the mass of the Universe. Thus, the series expansion (13) can be written in terms of S_0 as

$$S(r,t) = S_0(r,t)\left[1 + \Sigma_k \alpha_k \left(\frac{\hbar}{M^2}\right)^k\right], \tag{18}$$

with α_k as dimensionless constant parameters.

Now, a complete solution for S requires the solution of S_0 satisfying the PDE (14). Since the metric (1) is non-static, there is no time like Killing vector in the dynamical FRW spacetime. However, Kodama vector [28] has a similar role in FRW spacetime as the time like Killing vector does in the stationary BH spacetime. Inside the apparent horizon, Kodama vector is a time like vector, and hence there is a conserved energy of a particle moving in the time like Killing vector in the stationary BH spacetime.

The Kodama vector for the FRW metric is given by

$$K^\mu = (\sqrt{1-\kappa r^2}, -Hr\sqrt{1-\kappa r^2}, 0, 0), \tag{19}$$

with associated conserved energy of the particle as

$$\omega = -\left(\sqrt{1-\kappa r^2}\right)\frac{\partial S_0}{\partial t} + \left(Hr\sqrt{1-\kappa r^2}\right)\frac{\partial S_0}{\partial r}. \tag{20}$$

Solving for $\frac{\partial S_0}{\partial t}$ and $\frac{\partial S_0}{\partial r}$, using Equations (14) and (20), we get

$$S_0 = -\int \frac{\omega dt}{\{\sqrt{1-\kappa r^2} - Har\}} \mp a\omega \int \frac{dr}{\sqrt{1-\kappa r^2}\{\sqrt{1-\kappa r^2} - Har\}}, \tag{21}$$

where the $\mp$ sign corresponds to ingoing and outgoing scalar particle, respectively, for which the wave functions have the expressions [36]

$$\psi_{in} = exp\left[-\frac{i}{\hbar}\left\{1+\Sigma_k \alpha_k \left(\frac{\hbar}{M^2}\right)^k\right\}\left\{\int \frac{\omega dt}{\{\sqrt{1-\kappa r^2} - Har\}} + a\omega \int \frac{dr}{\sqrt{1-\kappa r^2}\{\sqrt{1-\kappa r^2} - Har\}}\right\}\right] \tag{22}$$

and

$$\psi_{out} = exp\left[-\frac{i}{\hbar}\left\{1+\Sigma_k \alpha_k \left(\frac{\hbar}{M^2}\right)^k\right\}\left\{\int \frac{\omega dt}{\{\sqrt{1-\kappa r^2} - Har\}} - a\omega \int \frac{dr}{\sqrt{1-\kappa r^2}\{\sqrt{1-\kappa r^2} - Har\}}\right\}\right]. \tag{23}$$

Now, across the horizon, the metric coefficients in the (r,t) sector alter their sign. Thus, the above time integration might have imaginary parts and make contributions to the probabilities for the ingoing and the outgoing particles. As a result, the probabilities are given by

$$\begin{aligned} P_{in} &= |\psi_{in}|^2 = exp\Big[\frac{2}{\hbar}\left\{1+\Sigma_k \alpha_k \left(\frac{\hbar}{M^2}\right)^k\right\} Im\{\int \frac{\omega dt}{\{\sqrt{1-\kappa r^2} - Har\}} \\ &+ Im\, a\omega \int \frac{dr}{\sqrt{1-\kappa r^2}\{\sqrt{1-\kappa r^2} - Har\}}\}\Big], \end{aligned} \tag{24}$$

and

$$\begin{aligned} P_{out} &= |\psi_{out}|^2 = exp\Big[\frac{2}{\hbar}\left\{1+\Sigma_k \alpha_k \left(\frac{\hbar}{M^2}\right)^k\right\} Im\{\int \frac{\omega dt}{\{\sqrt{1-\kappa r^2} - Har\}} \\ &- Im\, a\omega \int \frac{dr}{\sqrt{1-\kappa r^2}\{\sqrt{1-\kappa r^2} - Har\}}\}\Big]. \end{aligned} \tag{25}$$

In the present context for the cosmological spacetime, the outgoing probability has to be unity in the classical limit $\hbar \to 0$, as there will be no observer and everything will go out [45]. Note that the situation is the opposite of what one has for the BH case, where $P_{in} = 1$ in the limit $\hbar \to 0$. Hence, from Equation (25), in the limit $\hbar \to 0$, we get,

$$Im \int \frac{\omega dt}{\{\sqrt{1-\kappa r^2} - Har\}} = Im\, a\omega \int \frac{dr}{\sqrt{1-\kappa r^2}\{\sqrt{1-\kappa r^2} - Har\}}. \tag{26}$$

The above two integrals over t and r, respectively, are explicitly shown to be equal for the FRW model in [35]. Note that, in most tunnelling problems, one not only has a spatial integral contribution to the tunnelling rate but also a time integral contribution, which is critical to getting the correct thermodynamic properties (namely, temperature and entropy) for these spacetimes in the tunnelling picture (either null geodesic or HJ). This time contribution was first pointed out in [46–48]. Furthermore, this time contribution to the tunnelling amplitude resolves the "factor of 2" puzzle mentioned in Section 2. Moreover, in [46–48], it has been shown that Painleve types of coordinates are not the only coordinates to avoid singularity at the horizon, one may use Schwarzschild or Kruskal–Szekeres coordinates in dealing with tunnelling formulation, i.e., singularity at the horizon is not an issue if treated properly. Thus, P_{in} will have the simplified form

$$\begin{aligned} P_{in} &= exp\left[\frac{4a\omega}{\hbar}\left\{1+\Sigma_k\alpha_k\left(\frac{\hbar}{M^2}\right)^k\right\} Im\int\frac{dr}{\sqrt{1-\kappa r^2}\{\sqrt{1-\kappa r^2}-Har\}}\right] \\ &= exp\left[\frac{2\omega}{\hbar}\left\{1+\Sigma_k\alpha_k\left(\frac{\hbar}{M^2}\right)^k\right\}\pi R_a\right]. \end{aligned} \tag{27}$$

Hence, from "the principle of detailed balance", i.e., [7,8]

$$P_{out} = exp\left\{-\frac{\omega}{T_h}\right\}P_{in}, \tag{28}$$

we have

$$T_k = \left\{1+\Sigma_k\alpha_k\left(\frac{\hbar}{M^2}\right)^k\right\}^{-1}\frac{1}{2\pi R_A} = \left\{1+\Sigma_k\alpha_k\left(\frac{\hbar}{M^2}\right)^k\right\}^{-1}T_c. \tag{29}$$

This is the horizon temperature of the FRW model of the Universe after quantum corrections are taken into account. The quantum corrections to this modified temperature is very similar to that of Banerjee et al. [36] for general static BHs. In addition, the modified temperature has arbitrariness due to the choice of the parameters α_k in the expression. Banerjee et al. [36] has shown that, for static BHs, different choices of α_k's will lead to different physical interpretation. For future work, we shall attempt to study particles with non-zero mass and examine whether quantum corrections as well as Hawking-like temperature depend on the mass term.

4. Hamilton–Jacobi Method in the Lemaitre–Tolman–Bondi Model

The inhomogeneous spherically symmetric LTB spacetime model is given by the matric ansatz in a comoving frame as

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)}dr^2 + R^2(d\theta^2 + sin^2\theta d\phi^2), \tag{30}$$

where $R = R(r,t)$ is the area radius of the spherical surface and the curvature scalar $f(r)$ classifies the spacetime as follows:

(a) bounded: $-1 < f(r) < 0$,
(b) marginally bounded: $f(r) = 0$,
(c) unbounded: $f(r) > 0$.

The Einstein field equations for the spacetime model can be written as: [49–52]

$$8\pi G\rho = \frac{F'(r,t)}{R^2R'} \quad and \quad 8\pi Gp = -\frac{\dot{F}(r,t)}{R^2\dot{R}}, \tag{31}$$

and the evolution equation for R is given by

$$2R\ddot{R} + \dot{R}^2 + 8\pi GpR^2 = f(r). \tag{32}$$

Hence, the mass function

$$F(r,t) = R(\dot{R}^2 - f(r)) \tag{33}$$

is related to the mass contained within the comoving radius "r". The Universe is assumed to be filled with perfect fluid having energy-momentum tensor

$$T_{\mu\nu} = (\rho+p)u_\mu u_\nu + pg_{\mu\nu}, \tag{34}$$

where the fluid 4-velocity u^μ is normalized by $u_\mu u^\nu = -1$ and ρ, p are the energy density and pressure of the fluid, respectively. The energy-momentum conservation relation $T_{\mu;\nu}^{\ \nu} = 0$ gives

$$\dot{\rho} + 3H(\rho + p) = 0\,, \quad p' = 0, \tag{35}$$

with $H = \frac{1}{3}\left(\frac{\dot{R}'}{R'} + 2\frac{\dot{R}}{R}\right)$ as the Hubble parameter; "dot" and "dash" over any quantity stand for differentiation with respect to "t" and "r", respectively. Now, splitting the metric ansatz (30) on the surface of the two-sphere and on the 2D hypersurface normal to the two-sphere as

$$ds^2 = h_{ab}dx^a dx^b + R^2 d\Omega_2{}^2, \tag{36}$$

the dynamical apparent horizon is characterized by [25,26]

$$h^{ab}\partial_a R \partial_b R = 0, \tag{37}$$

where $h_{ab} = diag[-1, \frac{R'^2}{1+f(r)}]$ is the 2D metric normal to the two-sphere. Thus, the spherical surface of radius $R = R_A$ corresponding to the apparent horizon (i.e., marginally trapped surface) satisfies

$$R_A = F(r,t) \quad and \quad \dot{R}_A^2 = 1 + f(r). \tag{38}$$

For the sake of completeness, we recall that, in principle, the energy density, see Equation (35), could be phantom dark energy. In fact, the current observational data does not rule out this possibility [53,54]. For example, in the model in [53], which is founded on a brane–antibrane system, the equation of state parameter of the 4D Universe can change due to owing energy from an extra dimension and decrease from higher values of -1 (non-phantom phase) to lower values (phantom one). In general, we have some interesting properties about entropy and temperature concerning the phantom dark energy scenario. On one hand, as the internal phantom energy is negative, whereas the phantom temperature is definitely negative, and then hotter than any other sources in the Universe, its entropy is always positive, even though holding of the second law is not guaranteed by quantum-mechanical reasons [54]. On the other hand, concerning the observable matter in the Universe, the cosmic phantom field can be regarded as a cosmological source of information and negative entropy [54].

Trapping horizon (R_T), a hypersurface foliated by marginal spheres, is characterized by [55]:

$$\partial_+ R_T = 0, \ \text{ i.e., } \dot{R}_T = \sqrt{1 + f(r)}, \tag{39}$$

$$\partial_\pm = \sqrt{2}\left(\partial_t \mp \frac{\sqrt{1+f(r)}}{R'}\partial_r\right), \tag{40}$$

are the null vectors to the two-sphere. Thus, both the horizons coincide for the LTB model as in the FRW spacetime.

The Misner–Sharp gravitational mass (in units of $G = 1$) is defined as: [55,56]

$$m(r,t) = \frac{R}{2}(1 - h^{ab}\partial_a R \partial_b R). \tag{41}$$

One might note that this mass "m" is an invariant quantity on the 2D hypersurface normal to the two-sphere and $m = m_H = \frac{R_A}{2}$ on the horizon. Furthermore, one can introduce another invariant scalar associated with the horizon in the normal hypersurface known as dynamic surface gravity which is defined as [28]:

$$\kappa_D = \frac{1}{2}R|_H = \frac{1}{2\sqrt{-h}}\partial_a(\sqrt{-h}h^{ab}\partial_b R)|_H, \tag{42}$$

which, for the given model, has the expression

$$\kappa_D = \left[\frac{1}{2R'}\left\{-\partial_t(\dot{R}R') + \frac{1}{2}f'(2)\right\}\right]_H. \tag{43}$$

For the LTB model with a decomposed metric (33), the Kodama vector κ has components

$$\kappa^a(r,t) = \frac{1}{\sqrt{-h}}\epsilon^{ab}\partial_b R\,, \;\; \kappa^\theta = 0 = \kappa^\phi, \tag{44}$$

i.e.,

$$\kappa^\mu = \left(\sqrt{1+f(r)}, -\frac{\dot{R}}{R'}\sqrt{1+f(r)}, 0, 0\right). \tag{45}$$

Thus, $||\kappa^\mu||^2 = \dot{R}^2 - 1 - f(r)$, i.e., the Kodama vector is time like, null or space like for inside, on the surface or outside the apparent (i.e., trapping) horizon, respectively. Note that the Kodama vector is very similar to the time like Killing vector for stationary BH spacetime and, consequently, an invariant energy associated with a particle is defined by the scalar on the normal space as [57]:

$$\omega = -\kappa^a \partial_a I. \tag{46}$$

It should be noted that energy in general relativity is not an invariant. This is a consequence of the equivalence principle [58], which states that one can always choose a coordinate system, i.e., local Lorentz coordinate system, where the gravitational field is null. Thus, the gravitational energy cannot be localized [58]. One measures a particle's energy, with respect to some natural timelike vector field. However, in Hawking's black hole radiation, one should not always use the natural timelike Killing vector for a stationary black hole.

Here, the classical action I of the massless particle satisfies the HJ equation:

$$\begin{aligned} h^{ab}\partial_a I \partial_b I &= 0, \\ \text{i.e.,} \quad \left(\frac{\partial I}{\partial t}\right)^2 - \left\{\frac{1+f(r)}{R'^2}\right\}\left(\frac{\partial I}{\partial r}\right)^2 &= 0. \end{aligned} \tag{47}$$

Hence, solving (47), we have

$$\frac{\partial I}{\partial r} = \frac{\omega R'}{\sqrt{1+f(r)}\left\{\dot{R} - \sqrt{1+f(r)}\right\}} \quad \textit{and} \quad \frac{\partial I}{\partial t} = \frac{\omega}{\left\{\dot{R} - \sqrt{1+f(r)}\right\}}, \tag{48}$$

where there is a pole at the horizon. The full classical action of an outgoing massless particle is

$$I = \int_\gamma \partial_a I dx^a, \tag{49}$$

where γ is an oriented curve with positive orientation along the increasing values of x^a=(t,r). As for massless particles, the radial motion is along a null direction, so from the metric, we have

$$\begin{aligned} 0 = ds^2 &= -dt^2 + \frac{R'^2}{1+f(r)}dr^2, \\ \text{i.e.,} \quad dt &= \pm\frac{R'}{\sqrt{1+f(r)}}dr, \end{aligned} \tag{50}$$

for outgoing and ingoing particles, respectively. Thus, using Equation (48) in Equation (49), we have for outgoing particles

$$I = 2\int dr(\partial_r I) = -2\int_\gamma \frac{\omega R'}{1+f(r)} \frac{dr}{\left\{1 - \frac{\dot{R}}{\sqrt{1+f(r)}}\right\}}. \tag{51}$$

Now, expanding $G(r,t) = 1 - \frac{\dot{R}}{\sqrt{1+f(r)}}$ in the neighbourhood of the horizon along a null direction, we obtain

$$\begin{aligned} G(r,t) &\simeq \left[-\frac{\ddot{R}}{\sqrt{1+f}}\Delta t - \left(\frac{\dot{R}'}{\sqrt{1+f}} - \frac{\dot{R}f'(r)}{2(1+f)^{\frac{3}{2}}}\right)\Delta r\right]_H + \cdots, \\ &= \left[-\frac{\ddot{R}R'}{(1+f)} - \frac{\dot{R}'}{\sqrt{1+f}} + \frac{\dot{R}f'(r)}{2(1+f)^{\frac{3}{2}}}\right]\Delta r|_H + \cdots, \\ &= \frac{2R'}{(1+f)}\kappa_D(r-r_H) + \cdots. \end{aligned} \tag{52}$$

Then, from Equation (51),

$$I = -\int_\gamma dr \frac{\omega}{\kappa_D(r - r_H - i0)}, \tag{53}$$

which has a simple pole at $r = r_H$. Using Feynmann's $i\epsilon$ prescription, the imaginary part of the action (the real part has no physical consequence) can be written as [52]

$$Im\ I = -\frac{\pi\omega_H}{\kappa_D}. \tag{54}$$

Hence, one might interpret $T = -\frac{\kappa_D}{2\pi} > 0$ as the dynamical temperature associated with LTB spacetimes.

Moreover, we consider LTB tunnelling computation in the coordinate system $(\tilde{r},t,\theta,\phi)$, where $\tilde{r} = R$. The metric ansatz (in $\gamma-$gauge) becomes

$$\begin{aligned} ds^2 &= -Adt^2 - 2Bd\tilde{r}dt^2 + C(d\tilde{r})^2 + \tilde{r}^2 d\Omega_2{}^2 \\ &= q_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_2{}^2, \end{aligned} \tag{55}$$

where

$$\begin{aligned} A &= 1 - \tfrac{\dot{R}^2}{1+f(r)} = A(\tilde{r},t), \\ B &= \tfrac{\dot{R}}{1+f(r)} = B(\tilde{r},t), \\ C &= \tfrac{1}{1+f(r)} = C(\tilde{r},t). \end{aligned}$$

The horizon is located at $\chi = 0$, where

$$\chi = q^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = q^{\tilde{r}\tilde{r}} = \frac{A}{AC+B^2}, \tag{56}$$

i.e., $A_H = 0$ gives the horizon. The Kodama vector is given by

$$\kappa^\mu = \left(\frac{1}{\sqrt{B^2+AC}}, \hat{0}\right), \tag{57}$$

and the invariant energy has the expression

$$\omega = -\frac{\partial_t I}{\sqrt{B^2+AC}}. \tag{58}$$

The dynamical surface gravity on the horizon is

$$\kappa_D = \left[\frac{1}{2B^3}\left(A'B + \frac{1}{2}\dot{A}C\right)\right]|_H, \tag{59}$$

where "dash" and "dot" represent partial derivatives with respect to $\tilde{r}$ and t, respectively. The HJ equation for a massless particle along a radial trajectory has the explicit form

$$-C(\partial_t I)^2 - 2B(\partial_t I)(\partial_{\tilde{r}} I) + A(\partial_{\tilde{r}} I)^2 = 0. \tag{60}$$

Now, integration over the temporal coordinate gives real contribution to the particle action (which has no physical significance). An imaginary contribution comes only from integration along the radial direction. Now, eliminating $\partial_t I$ between Equations (58) and (60), we obtain

$$\partial_{\tilde{r}} I = -\frac{\omega(B + \sqrt{B^2 + AC})}{A}\sqrt{B^2 + AC}. \tag{61}$$

For the metric (55), by making a null expansion on the horizon, we obtain

$$0 = ds^2 = -2dtd\tilde{r} + \frac{d\tilde{R}^2}{1 + f(r)}, \tag{62}$$

and

$$\begin{aligned}\left(1 - \frac{\dot{R}^2}{1 + f(r))}\right) &\simeq -\frac{2\dot{R}\ddot{R}}{1 + f(r)}dt - \left\{\frac{2\dot{R}\dot{R}'}{1 + f(r)} - \frac{f'(r)\dot{R}^2}{(1 + f(r))^2}d\tilde{r}\right\} \\ &= -\left\{\frac{\dot{R}\ddot{R}}{(1 + f(r))^{\frac{3}{2}}} + \frac{2\dot{R}\dot{R}'}{1 + f(r)} - \frac{f'(r)\dot{R}^2}{(1 + f(r))^2}\right\}d\tilde{r} \\ &= 2\kappa_{DH}(\tilde{r} - \tilde{r}_H).\end{aligned} \tag{63}$$

Hence,

$$Im\ I = -\int_\gamma d\tilde{r}\frac{\omega(B + \sqrt{B^2 + AC})}{2\kappa_{DH}(\tilde{r} - \tilde{r}_H - i0)} = -\frac{\pi\omega_H}{\kappa_{DH}}. \tag{64}$$

Thus, we have the same result as in Equation (54).

5. Determination of Entropy: Area Law

In this section, we shall examine the semiclassical Bekenstein–Hawking area law [1,2,18] for the non-static FRW spacetime at the apparent horizon. In addition, we shall calculate the corrections to the semiclassical entropy due to quantum effects of the Hawking-like temperature. Thus, we start with the thermodynamical law that expresses the energy conservation as

$$dM = T_h dS_A. \tag{65}$$

Here, T_h is the temperature of the horizon (with quantum corrections), S_A is the entropy of the horizon, and M is chosen to be the Misner–Sharp gravitational mass, defined as [59]

$$dM = \frac{dR_A}{2} \tag{66}$$

on the horizon. Now, if we identify M in the quantum corrections (29) for the temperature as the above Misner–Sharp mass, then integrating (65), we have

$$
\begin{aligned}
S_A &= \int \frac{dM}{T_h} = \int \frac{4\pi M}{\hbar}\left[1+\Sigma\alpha_k\left(\frac{\hbar}{M^2}\right)^k\right]dM \\
&= \frac{2\pi M^2}{\hbar} + 4\pi\alpha_1 ln\ M - 4\pi\frac{\alpha_2\hbar}{M^2} + O(\hbar^2) \\
&= \frac{1}{2}\frac{A}{4\hbar} + 4\pi\alpha_1 ln\ M - 16\pi\frac{\alpha_2\hbar}{R_A^2} + O(\hbar^2).
\end{aligned} \tag{67}
$$

In the above equation, the first term on the right hand side is the semiclassical relation between area and entropy, which is the usual Bekenstein–Hawking area law with a discrepancy of factor $\frac{1}{2}$. The second term is the leading order quantum correction term and is the standard logarithmic correction term in black hole thermodynamics [45,60–62]. The higher order correction terms are in the inverse powers of the area of the horizon.

6. Tunnelling Approach

The basic idea in tunnelling method is that following the standard approach, i.e., in the semiclassical approximation (WKB approximation), the emission rate of tunnelling of a massless particle across the horizon can be related to the imaginary part of the action of the system. In the s-wave approximation, the particles are considered as massless shells, moving along a radial null geodesic.

Thus, for the metric (55), the radial null geodesic is characterized by

$$
\dot{\tilde{r}} = \frac{B \pm \sqrt{B^2+AC}}{C}, \tag{68}
$$

where as before $\pm$ sign indicates an outgoing or incoming null geodesic. Due to the tunnelling of the particles from the outside to the inside of the horizon, here we shall consider only an incoming geodesic. As we have seen that in the present context we need the imaginary part of the action produced by the tunnelling particles (the remaining part is always real), i.e., particles tunnelling through a barrier (the classically forbidden region), thus we obtain [27]:

$$
Im\ S = Im\int_{\tilde{r}_i}^{\tilde{r}_f} p_{\tilde{r}}d\tilde{r} = Im\int_{\tilde{r}_c}^{\tilde{r}_f}\int_0^{\tilde{p}_r} dp'_{\tilde{r}}d\tilde{r}. \tag{69}
$$

Here, the particle with radial momentum $p_{\tilde{r}}$ tunnels from the initial position $\tilde{r}_i$ just outside the horizon to the final point at $\tilde{r}_f$ which is a classical turning point, i.e., in the semiclassical analysis, the trajectory can represent a classically allowed motion

$$
\dot{\tilde{r}} = \frac{d\tilde{H}}{dp_{\tilde{r}}}|_{\tilde{r}}, \tag{70}
$$

where the Hamiltonian $\tilde{H}$ is the generator of the cosmic time "t". Then, using Equations (68) and (70) in Equation (69), we write [27]

$$
Im\ S = Im\int_{\tilde{r}_i}^{\tilde{r}_f} d\tilde{r}\int\frac{d\tilde{H}}{\dot{\tilde{r}}} = Im\int_{\tilde{r}_i}^{\tilde{r}_f}\frac{d\tilde{r}}{\dot{\tilde{r}}}\omega\sqrt{B^2+AC} = -\omega Im\int_{\tilde{r}_i}^{\tilde{r}_f}\frac{C\sqrt{B^2+AC}}{\{\sqrt{B^2+AC}-B\}} = \pi\tilde{r}_H\omega. \tag{71}
$$

One may note that in order to perform the integration over the Hamiltonian H, we get the energy as $\omega\sqrt{B^2+AC}$, as measured by an observer with the Kodama vector. Then, using the emission rate, i.e., $\Gamma \sim exp\{-2Im\ S\}$, we have

$$
T = \frac{\omega}{2Im\ S} = \frac{1}{2\pi\tilde{r}_H}, \tag{72}
$$

which is the HT as derived previously in Section 2.

7. Conclusions

In this work, we have studied Hawking-like radiation from the homogeneous FRW model and the inhomogeneous LTB model of the Universe using both the radial null geodesic method (tunnelling approach) and the HJ formalism approaches. In both of the methods, we have obtained identical Hawking-like temperature at the semiclassical level. "The factor of two" problem in the tunnelling approach has been overcome by considering the Kodama vector instead of time-like vectors and associated energy as the energy of the tunnelling particle. We have obtained quantum corrections to the Hawking-like temperature using the HJ method, and these corrections are similar to those for general static BHs. In addition, the quantum corrected entropy formula has been evaluated from the law of thermodynamics, and it is found that the usual entropy law with quantum corrections are the same as in BH thermodynamics with some discrepancies in the multiplicative factor. In the LTB model, the HT is measured by an observer with the Kodama vector (Equation (45)) inside the trapping (apparent) horizon. In both of the models, the trapping horizon of the Universe is related to the HT in contrast to the BH case, where HR is associated with an event horizon. The present work supports that of the FRW Universe [21] in the literature. In the LTB model, making a coordinate transformation (given by metric Equation (55)), we reach the same conclusion, indicating that HR is independent of coordinate choice. For future work, we shall attempt to find interpretation of the parameters involved in the quantum corrections. It is worthy to generalize the tunnelling approach for non-zero mass particles (i.e., time-like geodesics) as well as to incorporate the quantum correction terms (back reaction effects). Finally, we shall try to resolve the ambiguity in the multiplicative factors for the entropy-area formula.

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Appendix. Calculation of Kodama Vector and Surface Gravity

In a time-dependent spacetime, there is no (asymptotically timelike) Killing vector to define a preferred time coordinate. Kodama came forward with a divergence free vector field for any time-dependent spherically symmetric spacetime. The Kodama vector lies on $(1+1)$-dimensional radial-temporal plane and is defined as

$$K^a = \epsilon^{ab}\nabla_b R, \tag{A1}$$

where ϵ^{ab} is the $(1+1)$-dimensional Levi–Civita tensor in the radial-temporal plane. It is easy to see that $\nabla_a K^a = 0$, i.e., Kodama vector is free.

Furthermore, with the Kodama vector, there is an associated conserved current, namely,

$$J^a = G^{ab} K_b,$$

with $\nabla_a J^a = 0$.

For the FRW model of spacetime with metrics given by Equation (1), the Kodama vector is given by

$$K^b = \left[-a \left(\frac{\partial}{\partial t} \right)^b + HR \left(\frac{\partial}{\partial r} \right)^b \right].$$

Similarly, for the metric (2), the Kodama vector is given by Equation (3).

To find surface gravity, we write down the FRW metric (1) as

$$ds^2 = h_{ab}(x^a)dx^a dx^b + R^2 d\Omega_2{}^2,$$

where a, b can take values 0 and 1. The two-dimensional metric

$$dh^2 = h_{ab}(x^a)dx^a dx^b,$$

with h_{ab} =diag$\{-1, \frac{a^2}{1-\kappa r^2}\}$ is referred to as the normal metric with $x^0 = t$, $x^1 = r$. Let us consider the following scalar on the normal space

$$\chi(x) = h^{ab}\partial_a R \partial_b R = 1 - \left(H^2 + \frac{\kappa}{a^2} \right) R^2.$$

Then, the surface gravity on the apparent horizon is defined as

$$\kappa_A = -\frac{1}{2}\frac{\partial \chi}{\partial R}|_{R=R_A} = \frac{1}{R_A}.$$

Hence, the Hawking temperature on the apparent horizon is given by

$$T_A = \frac{1}{2\pi R_A},$$

as shown in Equation (8). Clearly, the surface gravity on the apparent horizon is constant.

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