entropy

MDPI

Letter

Combinatorial Intricacies of Labeled Fano Planes

Metod Saniga

Astronomical Institute, Slovak Academy of Sciences, SK-05960 Tatranská Lomnica, Slovak Republic; msaniga@astro.sk; Tel.: +421-52-78791-28

Academic Editors: Antonio M. Scarfone and Kevin H. Knuth
Received: 20 July 2016; Accepted: 19 August 2016; Published: 23 August 2016

Abstract: Given a seven-element set $X = \{1,2,3,4,5,6,7\}$, there are 30 ways to define a Fano plane on it. Let us call a line of such a Fano plane—that is to say an unordered triple from X—ordinary or defective, according to whether the sum of two smaller integers from the triple is or is not equal to the remaining one, respectively. A point of the labeled Fano plane is said to be of the order s, $0 \leq s \leq 3$, if there are s *defective* lines passing through it. With such structural refinement in mind, the 30 Fano planes are shown to fall into eight distinct types. Out of the total of 35 lines, nine ordinary lines are of five different kinds, whereas the remaining 26 defective lines yield as many as ten distinct types. It is shown that no labeled Fano plane can have all points of zero-th order, or feature just one point of order two. A connection with prominent configurations in Steiner triple systems is also pointed out.

Keywords: labeled Fano planes; ordinary/defective lines; Steiner triple systems

1. Introduction

The Fano plane is the smallest projective plane. It consists of seven lines and seven points, with three points on a line and, dually, three lines per point, where every pair of points is connected by a line, every line intersects every other line, and there are four points such that no line contains more than two of them. It is well known [1–3] that there are thirty different Fano planes on a given seven-element set. Slightly rephrased, there are thirty different ways to label the points of the Fano plane by integers from 1 to 7, two labeled Fano planes having zero, one or three lines in common, and each line occurring in six Fano planes. The set of thirty labeled Fano planes can be uniquely partitioned into two sets (say A and B) of fifteen elements each, such that any two labeled Fanos in the same set have just one line in common. Following the labeling adopted by Polster [4], the two sets read

$$A_1: \{124, 136, 157, \overline{235}, 267, \overline{347}, 456\}, \qquad B_1: \{127, 136, \overline{145}, \overline{235}, \overline{246}, \overline{347}, 567\},$$

$$A_2: \{127, 136, \overline{145}, 234, 256, 357, 467\}, \qquad B_2: \{125, 136, 147, 234, 267, 357, 456\},$$

$$A_3: \{125, 136, 147, 237, \overline{246}, 345, 567\}, \qquad B_3: \{124, 136, 157, 237, 256, 345, 467\},$$

$$A_4: \{125, \overline{134}, \overline{167}, 236, 247, 357, 456\}, \qquad B_4: \{127, \overline{134}, \overline{156}, 236, 245, 357, 467\},$$

$$A_5: \{127, 135, 146, 236, 245, \overline{347}, 567\}, \qquad B_5: \{124, 135, \overline{167}, 236, \overline{257}, \overline{347}, 456\},$$

$$A_6: \{124, 137, \overline{156}, 236, \overline{257}, 345, 467\}, \qquad B_6: \{125, 137, 146, 236, 247, 345, 567\},$$

$$A_7: \{\overline{123}, 147, \overline{156}, 245, 267, 346, 357\}, \qquad B_7: \{\overline{123}, \overline{145}, \overline{167}, 247, 256, 346, 357\},$$

$$A_8: \{124, 135, \overline{167}, 237, 256, 346, 457\}, \qquad B_8: \{126, 135, 147, 237, 245, 346, 567\},$$

$$A_9: \{126, 137, \overline{145}, \overline{235}, 247, 346, 567\}, \qquad B_9: \{124, 137, \overline{156}, \overline{235}, 267, 346, 457\},$$

$$A_{10} : \{\overline{123}, \overline{145}, \overline{167}, \overline{246}, \overline{257}, \overline{347}, 356\}, \qquad B_{10} : \{\overline{123}, 146, 157, 245, 267, \overline{347}, 356\},$$
$$A_{11} : \{126, \overline{134}, 157, 237, 245, 356, 467\}, \qquad B_{11} : \{125, \overline{134}, \overline{167}, 237, \overline{246}, 356, 457\},$$
$$A_{12} : \{125, 137, 146, 234, 267, 356, 457\}, \qquad B_{12} : \{126, 137, \overline{145}, 234, \overline{257}, 356, 467\},$$
$$A_{13} : \{\overline{123}, 146, 157, 247, 256, 345, 367\}, \qquad B_{13} : \{\overline{123}, 147, \overline{156}, \overline{246}, \overline{257}, 345, 367\},$$
$$A_{14} : \{127, \overline{134}, \overline{156}, \overline{235}, \overline{246}, 367, 457\}, \qquad B_{14} : \{126, \overline{134}, 157, \overline{235}, 247, 367, 456\},$$
$$A_{15} : \{126, 135, 147, 234, \overline{257}, 367, 456\}, \qquad B_{15} : \{127, 135, 146, 234, 256, 367, 457\},$$

where for triples of integers we use a shorthand notation $\{a, b, c\} = abc$.

2. Refined Structure of Numbered Fano Planes

Given a line abc, where, without loss of generality, we can take $1 \leq a < b < c \leq 7$, we shall distinguish between the cases when $a + b = c$ and $a + b \neq c$ and call the former/latter ordinary/defective [5]; for the sake of convenience, in the above-given sets A and B, all ordinary lines are denoted by overbars. Furthermore, a point of the labeled Fano plane is said to be of order s, $0 \leq s \leq 3$ if there are s *defective* lines passing through it; hence, in addition to two different kinds of lines, a labeled Fano plane can potentially feature up to four distinct types of points. A detailed inspection of each of the 30 labeled Fano planes given above shows that they fall, in terms of such structural refinement, into eight different types, as portrayed in Figure 1 and summarized in Table 1. It is also interesting to compare the distributions of types within each 15-element set, given in Tables 2 and 3, as well as cardinalities of individual types, listed in Table 4; note a pronounced asymmetry between sets A and B.

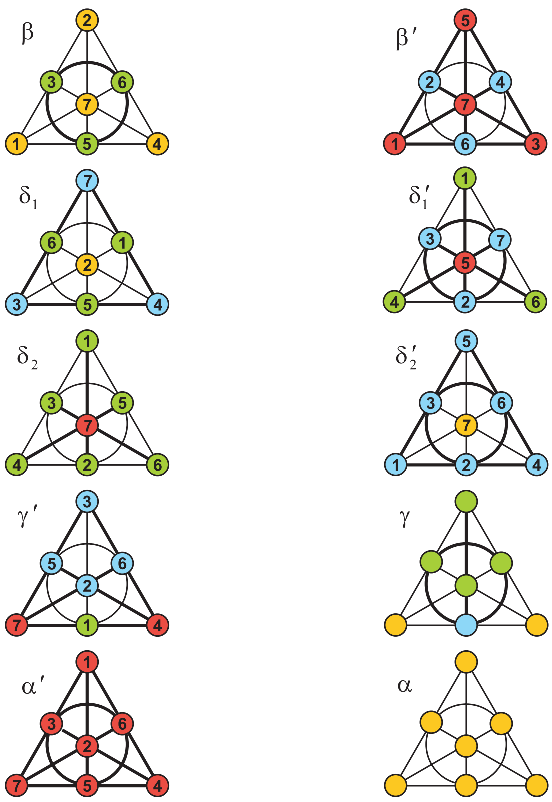

Figure 1. A diagrammatic illustration of representatives of eight distinct types of numbered Fano planes. A point of order three, two, one, and/or zero is represented by red, blue, green, and/or yellow color, respectively; heavy lines are defective. The types α and γ do not exist; i.e., it is impossible to label the Fano plane in such a way that all or just five of its lines would be ordinary.

Table 1. The eight distinct types of labeled Fano planes, each being uniquely characterized by the number of points of every particular order. To better understand the symmetry pattern, we arrange complementary types into pairs and also list the two non-existent types α and γ.

Type	**Points of Order**			
	0	**1**	**2**	**3**
(α)	(7)	(0)	(0)	(0)
α'	0	0	0	7
β	4	3	0	0
β'	0	0	3	4
(γ)	(2)	(4)	(1)	(0)
γ'	0	1	4	2
δ_1	1	3	3	0
δ_1'	0	3	3	1
δ_2	0	6	0	1
δ_2'	1	0	6	0

Table 2. Types of Fano planes in set A.

Plane	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Type	γ'	β'	β'	γ'	β'	γ'	γ'	β'	γ'	β	β'	α'	β'	δ_2	β'

Table 3. Types of Fano planes in set B.

Plane	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Type	δ_1	α'	α'	γ'	δ_2'	α'	δ_2'	α'	γ'	γ'	δ_1'	γ'	δ_1	γ'	α'

Table 4. Cardinalities of individual types of Fano planes.

Type	**(α)**	β	**(γ)**	δ_1	δ_2	α'	β'	γ'	δ_1'	δ_2'
Set *A*	-	1	-	0	1	1	7	5	0	0
Set *B*	-	0	-	2	0	5	0	5	1	2
Total	-	1	-	2	1	6	7	10	1	2

As easily discernible from Figure 1, the different types of Fano planes—and the dual pairs formed by them—are more readily ascertained by the number of defective lines they contain; the only exception is the case when there are three defective lines, where one has to distinguish whether these are concurrent or not. From this perspective, the lack of a plane of type γ becomes even more interesting. We can see through exhaustion that there is no such plane, but it would be really interesting to see if there are some geometric explanations for why it is impossible.

In addition to classifying planes, we can also classify lines in terms of the types of six planes passing through each of them, the corresponding findings being given in Table 5. We find that nine ordinary lines are of five different kinds, whereas the remaining 26 defective lines yield as many as ten distinct types.

To conclude the paper, let us have a look at Figure 1 and focus on that sub-geometry in each labeled Fano plane representative which is formed by defective lines and points lying on them. We shall find the Fano plane itself and, using the language of [6], all its derivatives depicted in Figure 2 that play a crucial role in classifying Steiner triple systems (e.g., [7,8]). As readily discerned from Figure 1, the semihead lies in a type-β' plane, the mia in a type-γ' plane, a sail is hosted by the type-δ_1' plane, the Pasch configuration is located in a type-δ_2' plane, the triangle in a type-δ_1 plane, and the Fano plane itself coincides with a type-α' plane. It should also be pointed out that if one instead considers sub-geometries formed by ordinary lines, then both the Fano plane and its derivate mia are absent.

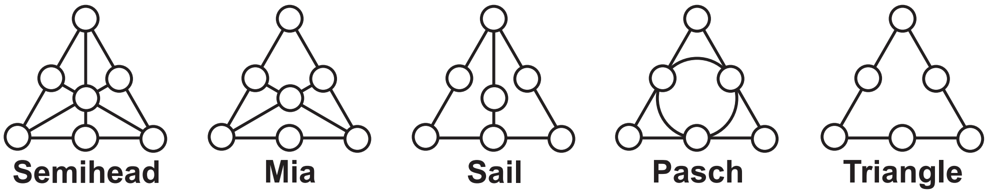

Figure 2. Five distinguished Fano derivatives, with their traditional names, found within labeled Fano planes.

Table 5. The types of lines; nine ordinary lines go first. Lines belong to a given type if they possess the same string of parameters.

Line	(α)	β	(γ)	δ_1	δ_2	α'	β'	γ'	δ_1'	δ_2'
123	-	1	-	1	0	0	1	2	0	1
145	-	1	-	1	0	0	1	2	0	1
257	-	1	-	1	0	0	1	2	0	1
347	-	1	-	1	0	0	1	2	0	1
156	-	0	-	1	1	0	0	4	0	0
235	-	0	-	1	1	0	0	4	0	0
246	-	1	-	2	1	0	1	0	1	0
167	-	1	-	0	0	0	1	1	1	2
134	-	0	-	0	1	0	1	3	1	0
124	-	0	-	0	0	1	1	3	0	1
236	-	0	-	0	0	1	1	3	0	1
247	-	0	-	0	0	1	1	3	0	1
346	-	0	-	0	0	1	1	3	0	1
357	-	0	-	0	0	1	1	3	0	1
456	-	0	-	0	0	1	1	3	0	1
136	-	0	-	1	0	2	2	1	0	0
147	-	0	-	1	0	2	2	1	0	0
345	-	0	-	1	0	2	2	1	0	0
567	-	0	-	1	0	2	2	1	0	0
126	-	0	-	0	0	1	2	3	0	0
157	-	0	-	0	0	1	2	3	0	0
245	-	0	-	0	0	1	2	3	0	0
467	-	0	-	0	0	1	2	3	0	0
135	-	0	-	0	0	2	3	0	0	1
237	-	0	-	0	0	2	3	0	0	1
256	-	0	-	0	0	2	3	0	0	1
127	-	0	-	1	1	1	2	1	0	0
367	-	0	-	1	1	1	2	1	0	0
146	-	0	-	0	0	3	2	1	0	0
234	-	0	-	0	0	3	2	1	0	0
137	-	0	-	0	0	2	0	4	0	0
267	-	0	-	0	0	2	0	4	0	0
356	-	1	-	0	0	1	1	2	1	0
457	-	0	-	0	1	2	1	1	1	0
125	-	0	-	0	0	3	1	1	1	0

3. Conclusions

The Fano plane occurs in algebraic geometry and geometric algebra in a number of disguises [9,10], providing a link between such important mathematical concepts as design theory, error-correcting codes, Latin squares, skew-Hadamard matrices, Klein's quartic curve, and Leech's eight-dimensional minimal sphere-packing lattice, being perhaps most recognized as a "gadget" completely describing the algebra structure of the octonions [11]. The above-described properties of labeled Fano planes clearly demonstrate that there is still much that this prominent object of discrete mathematics is likely to teach us. Although some readers may find the adopted labeling of Fano planes rather arbitrary or unnatural, there is already a piece of solid motivation for it stemming from its successful use in discovering an interesting sequence of nested finite geometries associated with Cayley–Dickson algebras [5]. An even more intriguing question is whether a similar strategy could also be applied to get deeper insight into so-called q-analogs of the Fano plane, where there are still many open problems, even for $q = 2$, to be properly addressed (for a recent review, see [12]).

Obviously, any finite geometry (point–line incidence structure) with lines of size three can be looked at this way. We have already performed the corresponding examination of the Möbius–Kantor (8_3)-configuration, the Pappus (9_3)-configuration, as well as the Desargues (10_3)-configuration, and plan to publish its outcome in a separate paper.

Acknowledgments: This work was supported by the Austrian Science Fund (Fonds zur Förderung der Wissenschaftlichen Forschung (FWF)), project number M1564–N27 and, in part, by the VEGA Grant Agency, project number 2/0003/16. I am grateful to my friend Petr Pracna for electronic versions of the figures and to the reviewers for a number of constructive comments/suggestions.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Hall, J.I. On identifying PG(3,2) and the complete 3-design on seven points. *Ann. Discret. Math.* **1980**, *7*, 131–141.
2. Lindner, C.C.; Rosa, A. Steiner triple systems having a prescribed number of triples in common. *Can. J. Math.* **1975**, *27*, 1166–1175.
3. Polster, B. Yea why try her raw wet hat: A tour of the smallest projective space. *Math. Intell.* **1999**, *21*, 38–43.
4. Polster, B. *A Geometrical Picture Book*; Springer: New York, NY, USA, 1998; pp. 77–79.
5. Saniga, M.; Holweck, F.; Pracna, P. From Cayley–Dickson algebras and finite geometry. *Mathematics* **2015**, *3*, 1192–1221.
6. Grannell, M.J.; Griggs, T.S.; Rosa, A. Three-line chromatic indices of Steiner triple systems. *Australas. J. Comb.* **2000**, *21*, 67–84.
7. Horak, P.; Phillips, N.; Wallis, W.D.; Yucas, J. Counting frequencies of configurations in Steiner triple systems. *Ars Comb.* **1997**, *46*, 65–75.
8. Forbes, A.D.; Grannell, M.J.; Griggs, T.S. Configurations and trades in Steiner triple systems. *Australas. J. Comb.* **2004**, *29*, 75–84.
9. Brown, E. The many names of $(7, 3, 1)$. *Math. Mag.* **2002**, *75*, 83–94.
10. Brown, E. Many more names of $(7, 3, 1)$. *Math. Mag.* **2015**, *88*, 103–120.
11. Baez, J. The Octonions. *Bull. Am. Math. Soc.* **2002**, *39*, 145–205.
12. Honold, T.; Kiermaier, M. On putative q-analogues of the Fano plane and related combinatorial structures. **2015**, arXiv:1504.06688.