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# Feynman's Ratchet and Pawl with Ecological Criterion: Optimal Performance versus Estimation with Prior Information

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**Abstract:** We study the optimal performance of Feynman's ratchet and pawl, a paradigmatic model in nonequilibrium physics, using ecological criterion as the objective function. The analysis is performed by two different methods: (i) a two-parameter optimization over internal energy scales; and (ii) a one-parameter optimization of the estimate for the objective function, after averaging over the prior probability distribution (Jeffreys' prior) for one of the uncertain internal energy scales. We study the model for both engine and refrigerator modes. We derive expressions for the efficiency/coefficient of performance (COP) at maximum ecological function. These expressions from the two methods are found to agree closely with equilibrium situations. Furthermore, the expressions obtained by the second method (with estimation) agree with the expressions obtained in finite-time thermodynamic models.

**Keywords:** finite-time thermodynamics; mesoscopic thermodynamics; Feynman's ratchet; nonequilibrium physics; optimization; ecological criterion; prior information; Jeffreys prior

## 1. Introduction

The subject of optimization in finite-time thermodynamics has received a lot of attention recently [1–23]. Here, the interest is to optimize the performance of heat devices operating with non-zero rates of, say, heat transfer, or energy conversion. The most commonly used optimization criteria are power output [1],  $\Omega$  criterion [24] and ecological criterion [25]. Some authors also use minimum entropy production criterion to optimize the performance of heat devices [26,27].

A standard method of optimization assumes a complete knowledge of the model, in the sense that the variables over which optimization is performed, such as intrinsic energy scales [28,29], or the intermediate temperatures of the working medium [30], or the times spent on the thermal contacts with the reservoirs [9], take on definite values; one just has to tune them to specific value(s) in order to optimize the objective function. Recently, one of the authors and coworkers [31–35] introduced a novel method of optimization, by which some variables can be assigned values, only in a probabilistic sense. This approach is based on interpreting the limited prior information about the system in the sense of subjective probability [36,37], and a prior distribution quantifies the uncertainty in these parameters. Then, an averaging procedure is performed on the target function using this distribution so as to eliminate these parameters. Following this approach, Curzon-Ahlborn efficiency for engine as well as coefficient of performance (COP) for the refrigerator [38,39] have been reproduced. In particular, for the problem of maximum work extraction from finite source and sink, the behavior of efficiency at maximum estimate of work shows universal features near equilibrium [34], e.g.,  $\eta = \eta_c/2 + \eta_c^2/8 + O[\eta_c^3]$ , where  $\eta_c$  is Carnot efficiency. Similarly, other expressions for efficiency

at maximum power, such as in irreversible models of stochastic engines [40–42], which obey a different universality near equilibrium, can also be reproduced from the inference based approach [32,43]. Thus, the latter approach provides an effective way of analyzing the performance of energy conversion systems. Here, the limited information can be interpreted in the sense of a limited control of the observer over the system [43]. Thus, this approach also points towards an entirely different origin of the figures of merit at optimal performance, which are usually obtained by an exact tuning of the variable parameters.

In this paper, we apply and compare the two approaches mentioned above for optimization, the standard two-parameter optimization, and the alternate, one-parameter optimization that also involves estimation based on prior information. We use Feynman's ratchet model [44–46] as the paradigmatic example for our investigation. The previous study [38] analyzed the power output in engine mode and corresponding criterion for refrigerator mode for this model. In this paper, we analyze the ecological criterion proposed by Angulo-Brown [25,47]  $E = P - T_2 \dot{S}_{\text{tot}}$  for an endoreversible Carnot heat engine, where  $P$  is the power output,  $T_2$  is the temperature of the cold reservoir and  $\dot{S}_{\text{tot}}$  is total rate of entropy production. The optimization of the ecological function represents the best compromise between the power output  $P$  and the loss of power  $T_2 \dot{S}_{\text{tot}}$ , due to the entropy production. Furthermore, Yan and Chen [48] optimized an endoreversible Carnot refrigerator under the ecological criterion  $\mathcal{E} = \dot{Q}_2 - \zeta_c T_0 \dot{S}_{\text{tot}}$ , where  $\dot{Q}_2$  is rate of refrigeration,  $\zeta_c$  is Carnot COP and  $T_0$  is environment temperature.

The paper is organized as follows. In Section 2, we describe the model of Feynman's ratchet as heat engine and discuss its optimal performance with ecological criterion [49–52]. In Section 2.1, two-parameter optimization of the ratchet is carried out. In Section 2.2, the approach based on prior information is applied to the case when the efficiency of the engine is fixed, but one of the internal energy scales is uncertain. The analysis is extended to the refrigerator mode, in Section 3 where we discuss performance based on two-parameter optimization of the ecological criterion as well as the estimation based on prior information. The final Section 4 is devoted to results and conclusions.

## 2. Optimal Performance of the Heat Engine

The model of Feynman's ratchet [44] consists of a vane, immersed in a hot reservoir at temperature  $T_1$ , and connected through an axle with a ratchet in contact with a cold reservoir at  $T_2$ . The ratchet is restricted to rotate in one direction due to a pawl, which, in turn, is connected to a spring. Let  $\epsilon_2$  be the amount of energy to overcome the elastic energy of the spring. Let the wheel rotate an angle  $\delta$  in each step and the torque induced by the weight be  $Z$ . Then, the system requires a minimum of  $\epsilon_1 = \epsilon_2 + Z\delta$  energy to lift the weight hanging from the axle. Hence, the rate of forward jumps of the ratchet is given as  $R_F = r_0 e^{-\epsilon_1/T_1}$ , where  $r_0$  is a rate constant and we have set Boltzmann's constant  $k_B = 1$ . In other words, temperature has the dimensions of energy. Similarly, the rate of the backward jumps is  $R_B = r_0 e^{-\epsilon_2/T_2}$ . One may regard  $Z\delta$  and  $-Z\delta$  as the work done by and on the system, respectively. Then, the rates of heat absorbed from the hot and the cold reservoirs are given as

$$\dot{Q}_1 = r_0 \epsilon_1 \left( e^{-\epsilon_1/T_1} - e^{-\epsilon_2/T_2} \right), \quad (1)$$

$$\dot{Q}_2 = r_0 \epsilon_2 \left( e^{-\epsilon_1/T_1} - e^{-\epsilon_2/T_2} \right). \quad (2)$$

The power output is defined as:  $P = \dot{Q}_1 - \dot{Q}_2$ . Then, the efficiency of the engine is given by

$$\eta = \frac{P}{\dot{Q}_1} = 1 - \frac{\epsilon_2}{\epsilon_1}. \quad (3)$$

The rate of total entropy production in this energy conversion is:

$$\dot{S}_{\text{tot}} = -\frac{\dot{Q}_1}{T_1} + \frac{\dot{Q}_2}{T_2}. \quad (4)$$

Then, the ecological criterion has been defined as [25]:

$$E = P - T_2 \dot{S}_{\text{tot}}. \quad (5)$$

### 2.1. Two-Parameter Ecological Optimization of Heat Engine

Here, we scale the parameters  $\epsilon_2$  and  $Z$  for simplification. Let  $\epsilon = \epsilon_2/T_2$  and  $z = Z\delta/T_1$ . In terms of  $\epsilon$  and  $z$ , we can write:

$$\eta = \frac{z}{z + \epsilon(1 - \eta_c)}, \quad (6)$$

$$P = r_o T_1 z \left[ e^{-\epsilon(1-\eta_c)-z} - e^{-\epsilon} \right], \quad (7)$$

$$\dot{Q}_1 = r_o T_1 [z + \epsilon(1 - \eta_c)] \left[ e^{-\epsilon(1-\eta_c)-z} - e^{-\epsilon} \right], \quad (8)$$

$$\dot{Q}_2 = \dot{Q}_1 - P. \quad (9)$$

Then, the ecological function  $E$  can be written as:

$$E = r_o T_1 [(2 - \eta_c)z - \eta_c(1 - \eta_c)\epsilon] \left[ e^{-\epsilon(1-\eta_c)-z} - e^{-\epsilon} \right]. \quad (10)$$

On optimizing  $E$  with respect to  $\epsilon$  and  $z$ , i.e., setting  $\partial E/\partial \epsilon = 0$  and  $\partial E/\partial z = 0$ , we get the following two equations respectively:

$$\left[ e^{\epsilon\eta_c - z} (\eta_c - 1) + 1 \right] [(2 - \eta_c)z - \eta_c(1 - \eta_c)\epsilon] = \eta_c(1 - \eta_c) \left[ e^{\epsilon\eta_c - z} - 1 \right], \quad (11)$$

$$e^{\epsilon\eta_c - z} [(2 - \eta_c)z - \eta_c(1 - \eta_c)\epsilon] = (2 - \eta_c) \left[ e^{\epsilon\eta_c - z} - 1 \right]. \quad (12)$$

Upon dividing Equation (11) by Equation (12), we can obtain

$$\epsilon\eta_c - z = \ln \left[ \frac{(2 - \eta_c)}{2(1 - \eta_c)} \right] \equiv k. \quad (13)$$

Eliminating  $e^{\epsilon\eta_c - z}$  from Equation (12) by using Equation (13), we get

$$(2 - \eta_c)z - \eta_c(1 - \eta_c)\epsilon = \eta_c. \quad (14)$$

The solution of Equations (13) and (14) is

$$z^* = \eta_c + (1 - \eta_c)k; \quad \epsilon^* = 1 + \frac{(2 - \eta_c)k}{\eta_c}. \quad (15)$$

Substituting from Equation (15) into Equation (6), we obtain the efficiency at the maximum ecological function

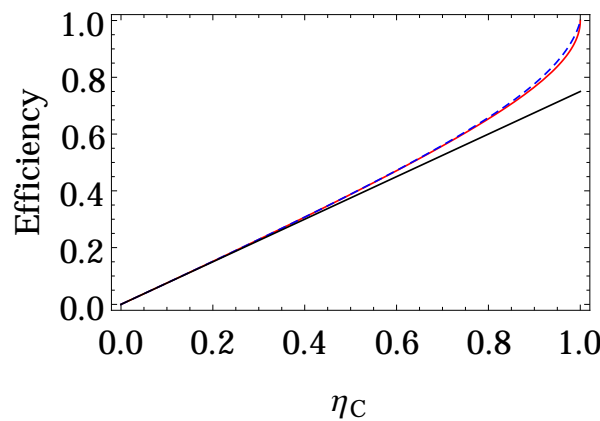
$$\eta^* = \frac{\eta_c + (1 - \eta_c)k}{\eta_c + 2(1 - \eta_c)k} \eta_c. \quad (16)$$

Close to equilibrium (small values of  $\eta_c$ ),  $\eta^*$  behaves as follows:

$$\eta^* = \frac{3\eta_c}{4} + \frac{\eta_c^2}{32} + \frac{19\eta_c^3}{768} + \mathcal{O}(\eta_c^4). \quad (17)$$

Note that the above value of efficiency depends only on  $\eta_c$ , or the ratio of the reservoir temperatures. The behavior of  $\eta^*$  close to equilibrium will be discussed below. Equation (16) is plotted in Figure 1. Similarly, from Equation (10), we obtain the maximum value of the ecological function

$$E^* = \frac{r_0 T_1 \eta_c^2 [2(1 - \eta_c)]^{2(1-\eta_c)/\eta_c}}{e(2 - \eta_c)^{(2-\eta_c)/\eta_c}}. \tag{18}$$



**Figure 1.** The efficiency at maximum ecological function, obtained from two different methods, is plotted versus  $\eta_c$ . The dashed curve represents the efficiency obtained from two-parameter optimization (Equation (16)). The solid curve is the corresponding efficiency when prior information approach is used (Equation (26)). The bottom straight line is  $3\eta_c/4$ . See also Equation (27).

2.2. Prior Information and Estimation for Heat Engine

Now, we consider a situation where the efficiency of the engine has some pre-specified value  $\eta$ , but the energy scales ( $\epsilon_1, \epsilon_2$ ) are not given to us in a priori information. Since  $\eta$  is known, the problem is reduced to a single uncertain parameter, due to Equation (3). One can cast the problem either in terms of  $\epsilon_1$  or  $\epsilon_2$ . In terms of the latter, we can write ecological function as

$$E(\eta, \epsilon_2) = \frac{r_0 \epsilon_2}{(1 - \eta)} (2\eta - \eta_c) \left( e^{-\epsilon_2/(1-\eta)T_1} - e^{-\epsilon_2/T_2} \right). \tag{19}$$

Based on the notion of prior information in Bayesian statistics, we assign a prior probability distribution for  $\epsilon_2$  in some arbitrary, but finite range of positive values:  $[\epsilon_{\min}, \epsilon_{\max}]$ . Later, we consider an asymptotic range in which the analysis becomes simplified and we observe universal features.

Now, consider two observers *A* and *B* who respectively assign a prior for  $\epsilon_1$  and  $\epsilon_2$ . Taking the simplifying assumption that each observer is in an equivalent state of knowledge, we can write [34,37,38]

$$\Pi(\epsilon_1) = \Pi(\epsilon_2) \left| \frac{d\epsilon_2}{d\epsilon_1} \right|, \tag{20}$$

where  $\Pi$  is the prior distribution function, taken to be of the same form for each observer. At a fixed known value of efficiency, it implies that  $\Pi(\epsilon_2) \propto 1/\epsilon_2$ . This is also well known as Jeffreys prior for a one-dimensional scale parameter [36,37,53].

Now, we define the expected value of function *E*, over this prior, as

$$\begin{aligned} \bar{E}(\eta) &= \int_{\epsilon_{\min}}^{\epsilon_{\max}} E(\eta, \epsilon_2) \Pi(\epsilon_2) d\epsilon_2 \\ &= \frac{C}{(1 - \eta)} (2\eta - \eta_c) \int_{\epsilon_{\min}}^{\epsilon_{\max}} \left( e^{-\epsilon_2/(1-\eta)T_1} - e^{-\epsilon_2/T_2} \right) d\epsilon_2, \end{aligned} \tag{21}$$

where

$$C = r_0 \left[ \ln \left( \frac{\epsilon_{\max}}{\epsilon_{\min}} \right) \right]^{-1}. \tag{22}$$

Upon performing the integration, we get

$$\bar{E}(\eta) = CT_1(2\eta - \eta_c) \left( e^{-\epsilon_{\min}/(1-\eta)T_1} - e^{-\epsilon_{\max}/(1-\eta)T_1} \right) + \frac{CT_2(2\eta - \eta_c)}{(1-\eta)} \left( e^{-\epsilon_{\max}/T_2} - e^{-\epsilon_{\min}/T_2} \right). \tag{23}$$

We are interested in the value of the efficiency at maximum of  $\bar{E}$ . Hence, on optimizing  $\bar{E}(\eta)$  with respect to  $\eta$ , we get

$$\begin{aligned} \frac{\partial \bar{E}}{\partial \eta} \equiv & 2T_1 \left( e^{-\epsilon_{\min}/(1-\eta)T_1} - e^{-\epsilon_{\max}/(1-\eta)T_1} \right) + \frac{T_2(2-\eta_c)}{(1-\eta)^2} \left( e^{-\epsilon_{\max}/T_2} - e^{-\epsilon_{\min}/T_2} \right) \\ & - \frac{2\eta-\eta_c}{(1-\eta)^2} \left( \epsilon_{\min} e^{-\epsilon_{\min}/(1-\eta)T_1} - \epsilon_{\max} e^{-\epsilon_{\max}/(1-\eta)T_1} \right) = 0. \end{aligned} \tag{24}$$

Now, we consider the asymptotic limit [31,38], in which the maximal allowed range for  $\epsilon_2$  is considered. In particular, we require  $\epsilon_{\max} \gg T_1$  and  $\epsilon_{\min} \ll T_2$ . In this limit, Equation (24) reduces to:

$$2T_1 - \frac{2T_2}{(1-\eta)^2} + \frac{T_2\eta_c}{(1-\eta)^2} = 0. \tag{25}$$

Putting  $T_2/T_1 = 1 - \eta_c$  and solving Equation (25) for  $\eta$ , we get

$$\tilde{\eta} = 1 - \sqrt{\frac{(1-\eta_c)(2-\eta_c)}{2}}. \tag{26}$$

The above expression is identical to the one obtained by Angulo-Brown [25].

Although the above expression for  $\tilde{\eta}$  is different from the one obtained via the two-parameter optimization [Equation (16)], we note that, near equilibrium, i.e.,  $\eta_c$  close to zero,

$$\tilde{\eta} = \frac{3\eta_c}{4} + \frac{\eta_c^2}{32} + \frac{3\eta_c^3}{128} + \mathcal{O}(\eta_c^4), \tag{27}$$

which shows the same universality up to the second order in  $\eta_c$  [54], as Equation (17). The interpretation of this result viz-à-viz the result from exact optimization is the following. If the experimentalist is unable to tune an internal parameter, or has a limited control over it, then it makes sense to consider an expected value of  $E$ , suitably averaged over the uncertain parameter. Then, it has been observed in the above that this average value,  $\bar{E}$ , takes its maximum value at a certain efficiency  $\tilde{\eta}$ , which is also the one obtained in purely thermodynamic models. We observe that the behavior of  $\tilde{\eta}$  is very similar to  $\eta^*$ , as a function of  $\eta_c$  close to equilibrium.

### 3. Optimal Performance as a Refrigerator

In this section, we consider the function of Feynman’s ratchet as a refrigerator [38,55–58]. We will optimize the corresponding ecological criterion [48]:  $\mathcal{E} = \dot{Q}_2 - \zeta_c T_1 \dot{S}_{\text{tot}}$ . The rate of refrigeration and rate of heat added to the hot reservoir are given respectively as:

$$\dot{Q}_2 = r_0 \epsilon_2 \left( e^{-\epsilon_2/T_2} - e^{-\epsilon_1/T_1} \right), \tag{28}$$

$$\dot{Q}_1 = r_0 \epsilon_1 \left( e^{-\epsilon_2/T_2} - e^{-\epsilon_1/T_1} \right). \tag{29}$$

Furthermore, for refrigerator, we have:

$$\dot{S}_{\text{tot}} = \frac{\dot{Q}_1}{T_1} - \frac{\dot{Q}_2}{T_2}. \tag{30}$$

The COP for given values of  $\epsilon_1$  and  $\epsilon_2$  is:  $\zeta = \dot{Q}_2 / (\dot{Q}_1 - \dot{Q}_2) = \epsilon_2 / (\epsilon_1 - \epsilon_2)$ , and Carnot COP is  $\zeta_c = T_2 / (T_1 - T_2)$ .

### 3.1. Two-Parameter Ecological Optimization for the Refrigerator

In terms of  $\epsilon$  and  $z$ , the scaled parameters introduced in Section 2.1, the ecological function for refrigerator is given by

$$\mathcal{E} = r_o \left[ e^{-\epsilon} - e^{-\epsilon \zeta_c / (1 + \zeta_c) - z} \right] [2T_2 \epsilon - \zeta_c T_1 z]. \tag{31}$$

On optimizing  $\mathcal{E}$  with respect to  $\epsilon$  and  $z$ , that is, setting  $\partial \mathcal{E} / \partial \epsilon = 0$  and  $\partial \mathcal{E} / \partial z = 0$ , we get the two following equations, respectively,

$$\left[ 1 - \frac{\zeta_c}{1 + \zeta_c} e^{\epsilon / (1 + \zeta_c) - z} \right] [2T_2 \epsilon - \zeta_c T_1 z] = 2T_2 \left[ 1 - e^{\epsilon / (1 + \zeta_c) - z} \right], \tag{32}$$

$$e^{\epsilon / (1 + \zeta_c) - z} [2T_2 \epsilon - \zeta_c T_1 z] = \zeta_c T_1 \left[ 1 - e^{\epsilon / (1 + \zeta_c) - z} \right]. \tag{33}$$

Comparing Equations (32) and (33), we can obtain

$$\frac{\epsilon}{1 + \zeta_c} - z = \ln \left( \frac{1 + \zeta_c}{2 + \zeta_c} \right) \equiv k'. \tag{34}$$

Using Equations (33) and (34), we get  $2\epsilon - (1 + \zeta_c)z = 1$ . Finally, we obtain the following optimum solution:

$$z^* = \frac{1 - 2(1 + \zeta_c)k'}{(1 + \zeta_c)}; \quad \epsilon^* = 1 - (1 + \zeta_c)k'. \tag{35}$$

Now, COP of the refrigerator can be written as

$$\zeta = \frac{\epsilon \zeta_c}{z(1 + \zeta_c)}. \tag{36}$$

Using Equation (35) in Equation (36), we derive COP at the maximum ecological function

$$\zeta^* = \frac{1 - (1 + \zeta_c)k'}{1 - 2(1 + \zeta_c)k'} \zeta_c. \tag{37}$$

We can write series expansion of  $\zeta^*$  with respect to  $\zeta_c$  as follows

$$\frac{\zeta^*}{\zeta_c} = \frac{2}{3} + \frac{1}{18\zeta_c} - \frac{2}{27\zeta_c^2} + O\left(\frac{1}{\zeta_c}\right)^3. \tag{38}$$

Substituting Equation (35) in Equation (31), we get the maximum ecological function

$$\mathcal{E}^* = \frac{r_o T_2 (1 + \zeta_c)^{1 + \zeta_c}}{e(2 + \zeta_c)^{2 + \zeta_c}}. \tag{39}$$

### 3.2. Prior Information and Estimation for Refrigerator

Similar to the case of the heat engine, we now obtain, using the prior based approach, the COP at optimal performance of Feynman’s ratchet as refrigerator. Again, we suppose that the figure of merit  $\zeta$  is fixed at some value and  $\epsilon_2$  is uncertain, within the range  $[\epsilon_{\min}, \epsilon_{\max}]$ . Then, Jeffreys prior for  $\epsilon_2$  can be argued, similar to Equation (20). In terms of  $\zeta$  and one of the scales say,  $\epsilon_2$ , the ecological-criterion  $\mathcal{E} = \dot{Q}_2(2 + \zeta_c) - \zeta_c \dot{Q}_1$  is given by

$$\mathcal{E} = r_0 \left( 2 - \frac{\zeta_c}{\zeta} \right) \epsilon_2 \left( e^{-\epsilon_2/T_2} - e^{-\epsilon_2(1+\zeta)/\zeta T_1} \right). \tag{40}$$

Then, we define the expected value of  $\mathcal{E}$  as

$$\bar{\mathcal{E}}(\zeta) = \int_{\epsilon_{\min}}^{\epsilon_{\max}} \mathcal{E}(\zeta, \epsilon_2) \Pi(\epsilon_2) d\epsilon_2 \tag{41}$$

$$= C \left( 2 - \frac{\zeta_c}{\zeta} \right) \int_{\epsilon_{\min}}^{\epsilon_{\max}} \left( e^{-\epsilon_2/T_2} - e^{-\epsilon_2(1+\zeta)/\zeta T_1} \right) d\epsilon_2, \tag{42}$$

where  $C$  is given by Equation (22). Upon integrating the above equation, we get

$$\begin{aligned} \bar{\mathcal{E}}(\zeta) &= CT_2 \left( 2 - \frac{\zeta_c}{\zeta} \right) \left( e^{-\epsilon_{\min}/T_2} - e^{-\epsilon_{\max}/T_2} \right) \\ &+ CT_1 \left( 2 - \frac{\zeta_c}{\zeta} \right) \frac{\zeta}{(1+\zeta)} \left( e^{-\epsilon_{\max}(1+\zeta)/\zeta T_1} - e^{-\epsilon_{\min}(1+\zeta)/\zeta T_1} \right). \end{aligned} \tag{43}$$

Then, the maximum of  $\bar{\mathcal{E}}$  with respect to  $\zeta$ , is evaluated as

$$\begin{aligned} \frac{\partial \bar{\mathcal{E}}}{\partial \zeta} &\equiv \frac{\zeta_c T_2}{\zeta^2} \left( e^{-\epsilon_{\min}/T_2} - e^{-\epsilon_{\max}/T_2} \right) + \frac{(2+\zeta_c)T_1}{(1+\zeta)^2} \left( e^{-\epsilon_{\max}(1+\zeta)/\zeta T_1} - e^{-\epsilon_{\min}(1+\zeta)/\zeta T_1} \right) \\ &+ \frac{2\zeta - \zeta_c}{\zeta^2(1+\zeta)} \left( \epsilon_{\max} e^{-\epsilon_{\max}(1+\zeta)/\zeta T_1} - \epsilon_{\min} e^{-\epsilon_{\min}(1+\zeta)/\zeta T_1} \right) = 0. \end{aligned} \tag{44}$$

In the asymptotic limit mentioned earlier, the above equation reduces to

$$\frac{\zeta_c T_2}{\zeta^2} = \frac{(2 + \zeta_c) T_1}{(1 + \zeta)^2}. \tag{45}$$

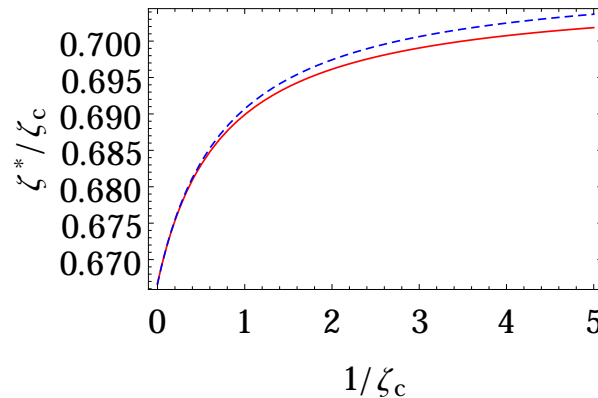
Putting  $T_1/T_2 = (1 + \zeta_c)/\zeta_c$  and solving for  $\zeta$ , we get

$$\tilde{\zeta} = \left( \frac{1 + \zeta_c}{\zeta_c} \sqrt{\frac{2 + \zeta_c}{1 + \zeta_c}} - 1 \right)^{-1}. \tag{46}$$

This is the same equation as obtained by Yan [48] when we take the environment temperature equal to the temperature of the hot reservoir. In a near-equilibrium regime, the Carnot COP  $\zeta_c$  in addition to  $\zeta^*$  become large in magnitude. One can then write the series expansion for  $\zeta^*$  relative to  $\zeta_c$  as follows:

$$\frac{\zeta^*}{\zeta_c} = \frac{2}{3} + \frac{1}{18\zeta_c} - \frac{17}{216\zeta_c^2} + \mathcal{O} \left( \frac{1}{\zeta_c} \right)^3, \tag{47}$$

which is similar to the Equation (38) up to the first two terms. In Figure 2, we compare the expressions from Equations (37) and (46). Before closing, we point out that, upon performing the same analysis in terms of  $\epsilon_1$  as the uncertain scale, we obtain a similar behavior in the asymptotic range of values, and the same figures of merit as  $\tilde{\eta}$  and  $\tilde{\zeta}$  are obtained with the choice of Jeffreys' prior.



**Figure 2.** The COP (relative to  $\zeta_c$ ) at maximum ecological function, obtained from two different methods, is plotted versus  $1/\zeta_c$ . The dashed curve represents the COP obtained from two-parameter optimization (Equation (37)). The solid curve is the corresponding COP when prior information approach is used (Equation (46)).

#### 4. Conclusions

We studied optimization of performance in Feynman's ratchet model using the ecological criterion. Earlier studies focused on optimization of power output (for engine mode) and the so-called  $\chi$ -criterion (for refrigerator mode). Our choice of the objective function is motivated by the fact that it represents the best compromise between, say, the power output and power loss in the engine mode. We performed our analysis by recourse to two methods: (i) the standard approach of two-parameter optimization over the two internal energy scales. We derived explicit expressions for the efficiency and coefficient of performance at the maximum of ecological criterion; (ii) we also performed an estimation over one uncertain energy scale followed by a single-parameter optimization. The latter approach is based on the quantification of (limited) prior information as in the Bayesian probability theory. Here, we are also able to find exact expressions for efficiency in some well-defined asymptotic limit. Remarkably, we obtain the well-known expressions of finite-time thermodynamic models where ecological function is optimized. Furthermore, we observe that the behavior of efficiency as predicted in the two methods is quite similar. In fact, close to equilibrium, the corresponding expressions match up to the second term. These observations are analogous to the ones made on the same model using optimization of a different objective function, such as power output. Since the presence of limited information may also be interpreted in the sense of limited control over the system, the agreement of figures of merit by the two methods suggests that an exact knowledge of the variable parameters is not essential to infer the optimal behavior. Thus, the present results provide further evidence that the estimation based approach can provide a robust and effective method to indicate the figures of merit at optimal performance of heat devices.

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**Author Contributions:** Ramandeep S. Johal conceived the problem; Varinder Singh performed the calculations; Ramandeep S. Johal and Varinder Singh jointly wrote the paper.

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