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Topological Entropy Dimension and Directional Entropy Dimension for $\mathbb{Z}^2$-Subshifts

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Abstract: The notion of topological entropy dimension for a $\mathbb{Z}$-action has been introduced to measure the subexponential complexity of zero entropy systems. Given a $\mathbb{Z}^2$-action, along with a $\mathbb{Z}^2$-entropy dimension, we also consider a finer notion of directional entropy dimension arising from its subactions. The entropy dimension of a $\mathbb{Z}^2$-action and the directional entropy dimensions of its subactions satisfy certain inequalities. We present several constructions of strictly ergodic $\mathbb{Z}^2$-subshifts of positive entropy dimension with diverse properties of their subgroup actions. In particular, we show that there is a $\mathbb{Z}^2$-subshift of full dimension in which every direction has entropy 0.

Keywords: $\mathbb{Z}^2$-action; entropy dimension; directional entropy dimension; entropy generating shape; minimal

1. Introduction

Shannon introduced the notion of entropy to measure the information capacity of the process [1]. Since Kolmogorov brought the notion to dynamical systems, entropy provided the field with new perspectives and has played one of the central roles for understanding the chaoticity of measurable and topological dynamical systems [2,3]. Systems of positive entropy have been studied for several decades and many of the properties are well understood at least in the case of $\mathbb{Z}$-actions. Entropy has been studied for amenable group actions and more recently for nonamenable group actions [4–6].

In the case of measurable dynamics, zero entropy systems make a dense G_δ subset of the set of all ergodic systems. Given a full shift, the set of zero entropy subshifts is also a dense G_δ subset [7]. Moreover, zero entropy systems arise rather naturally in the study of general group actions. To understand the complexities of zero entropy $\mathbb{Z}^2$-actions, it is natural to ask the entropies of their non-cocompact subgroup actions. It is well-known that their subgroup actions exhibit diverse behaviors in their entropies. For example, the well-known three dot subshift ($x_{i,j} + x_{i,j+1} + x_{i+1,j} \equiv 0 \pmod 2$ for all $(i,j) \in \mathbb{Z}^2$) has entropy zero while all of its non-cocompact subgroup actions have positive entropy. In addition, there is a zero entropy $\mathbb{Z}^2$-subshift, all of whose directions have infinite entropy. In his study of cellular automaton maps, Milnor extended the entropy of noncocompact subgroup actions to irrational directions, and called it directional entropy [8]. It is easy to see that the three dot model also has positive directional entropy in all irrational directions. If a $\mathbb{Z}^2$-action has positive entropy, then each direction has infinite entropy. If a $\mathbb{Z}^2$-action has entropy zero, the entropy of its directions could be zero, positive, or infinite. We note that there exists a $\mathbb{Z}^2$-subshift of entropy zero that has directions of entropy zero, of positive entropy and of infinite entropy. Properties of directional entropies and the dynamics of subgroups have been investigated in [9–13].

Topological entropy dimension has been introduced and studied in [14,15] to classify the growth rate of the orbits of zero entropy systems. For example, any positive entropy $\mathbb{Z}^2$-subshift has the orbit growth rate in the order of 2^{n^2}, while the three dot model has the orbit growth rate in the order of 2^n. The model has intermediate growth rate with nontrivial directional dynamics. Zero entropy $\mathbb{Z}^2$-subshifts may contain subgroup actions whose directional entropy is 0. To understand the complexity of $\mathbb{Z}^2$-actions, we introduce topological entropy dimension analogous to the one for $\mathbb{Z}$-actions. As in the case of $\mathbb{Z}$-action, entropy dimension for $\mathbb{Z}^2$-action measures the intermediate growth rate, which is bigger than polynomial and less than exponential. If a system has a polynomial growth rate, then it has entropy dimension 0. Meyerovitch [15] has constructed a family of $\mathbb{Z}^2$-subshifts of entropy dimension α for all $\alpha \in [0,1]$. To measure the subexponential growth rate in all directions including the irrational directions, we define directional entropy dimension, which is the extension of the entropy dimension for the noncocompact subgroup actions.

Our main interest is to look into the complexity of given group actions of entropy zero together with their subgroup actions in terms of directional entropy dimension. In the case of $\mathbb{Z}^2$-actions, if a direction has positive entropy or has entropy dimension 1, then clearly the $\mathbb{Z}^2$-entropy dimension is greater than $1/2$. In general, we show that if X is a $\mathbb{Z}^2$-subshift with entropy dimension $D(X)$ and $D(\mathbf{v})$ is the directional entropy dimension of a direction vector $\mathbf{v} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, then the following inequalities hold: $D(\mathbf{v}) \leq 2D(X) \leq D(\mathbf{v}) + 1$ (see Theorem 2). We construct $\mathbb{Z}^2$-subshifts of different positive entropy dimensions for which the equality holds in the second inequality. In fact, for each $1/2 \leq \alpha \leq 1$, we present a $\mathbb{Z}^2$-subshift of entropy dimension α whose directional entropy dimension is $2\alpha - 1$ for every direction (see Example 5).

We present a $\mathbb{Z}^2$-subshift of entropy dimension 1, where the directional entropy is 0 for every direction (see Example 7). This example indicates that $\mathbb{Z}^2$-complexity may be spread out in all directions. It is interesting to compare the example with the three dot model whose entropy dimension is $1/2$. It also shows that there is a difference between zero entropy subshifts of entropy dimension 1 and positive entropy subshifts, as every directional entropy is infinite for the latter ones.

The paper is organized as follows. Section 2 presents necessary terminology for $\mathbb{Z}^2$-subshifts and the definitions of the entropy dimension and directional entropy dimension. In Section 3, we discuss equivalent definitions for entropy dimension. An inequality for entropy dimension and directional entropy dimension is presented in Section 4. In Section 5, we first present a general method to construct strictly ergodic $\mathbb{Z}^2$-subshifts with positive entropy dimension, and then construct $\mathbb{Z}^2$-subshifts exhibiting interesting behaviors in their directional entropy dimensions.

2. Topological Entropy Dimension for $\mathbb{Z}^2$-Actions

As we assume some familiarity with topological and symbolic dynamics, we introduce a few terminology and known results. For details on symbolic dynamics, see [16], and, for topological entropy dimension of $\mathbb{Z}$-actions, see [14].

A two-dimensional full shift is a set $\mathcal{A}^{\mathbb{Z}^2}$ for a finite set $\mathcal{A}$, together with the $\mathbb{Z}^2$-shift actions $\sigma^{\mathbf{i}} : \mathcal{A}^{\mathbb{Z}^2} \to \mathcal{A}^{\mathbb{Z}^2}$ given by translations $\sigma^{\mathbf{i}}(x)_{\mathbf{j}} = x_{\mathbf{i}+\mathbf{j}}$ for $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^2$. A $\mathbb{Z}^2$-subshift (or $\mathbb{Z}^2$-*shift space*) X is a closed σ-invariant subset of a full shift. A finite set $F \subset \mathbb{Z}^2$ is called a *shape*. A member of $\mathcal{A}^F$ is called a *pattern* on the shape F. For a shape $F \subset \mathbb{Z}^2$, denote by $\mathcal{B}_F(X)$ the set $\{x|_F : x \in X\}$ of all patterns on the shape F occurring in X. For $F \subset \mathbb{R}^2$, we denote by $\mathcal{B}_F(X)$ the set $\mathcal{B}_{F\cap\mathbb{Z}^2}(X)$ for notational simplicity. In particular, for $m, n \in \mathbb{N}$, let

$$R_{m,n} = \{\mathbf{v} = (v, w) \in \mathbb{Z}^2 : 0 \leq v < m \text{ and } 0 \leq w < n\}$$

be a rectangular shape in $\mathbb{Z}^2$ and

$$\mathcal{B}_{m,n}(X) = \{x|_{R_{m,n}} : x \in X\}$$

be the set of the patterns on the shape $R_{m,n}$ occurring in X. We simply put $\mathcal{B}_n(X) = \mathcal{B}_{n,n}(X)$.

The *(two-dimensional) topological entropy* of X is defined by

$$h(X) = \lim_{n\to\infty} \frac{\log |\mathcal{B}_n(X)|}{n^2}.$$

It is well known that the limit exists and equals the maximum of the measure-theoretic entropies of the shift-invariant probability measures. As in the case of $\mathbb{Z}$-actions, the entropy dimension of a $\mathbb{Z}^2$-subshift X is defined.

Definition 1. *The (two-dimensional) upper entropy dimension of X is defined by*

$$\overline{D}(X) = \limsup_{n\to\infty} \frac{\log\log |\mathcal{B}_n(X)|}{\log n^2}.$$

The lower entropy dimension $\underline{D}(X)$ is defined analogously by using lim inf *instead of* lim sup*. If $\overline{D}(X) = \underline{D}(X)$, we denote it by $D(X)$ and call it the (topological) entropy dimension of X.*

Note that the (upper and lower) entropy dimension of X lies in the interval $[0,1]$. They are invariant under topological conjugacy between two $\mathbb{Z}^2$-subshifts. One can check that $\overline{D}(X)$ is the unique critical value for α of the function

$$\overline{D}(X,\alpha) = \limsup_{n\to\infty} \frac{\log |\mathcal{B}_n(X)|}{(n^2)^{\alpha}},$$

that is,

$$\overline{D}(X) = \inf\{\alpha : \overline{D}(X,\alpha) = 0\} = \sup\{\alpha : \overline{D}(X,\alpha) = \infty\}.$$

The similar equivalences hold for $\underline{D}(X)$ and $D(X)$ using lim inf and lim, respectively. We note that if X has positive entropy, then it has entropy dimension 1.

We recall the definition of directional entropy introduced by Milnor [8,9]. For a $\mathbb{Z}^2$-subshift, the definition is stated much simpler. For $\mathbf{v} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, let $\mathbf{v}^{\perp}$ be a unit vector orthogonal to $\mathbf{v}$. Given $t > 0$ and $n > 0$, we let

$$E(\mathbf{v},n,t) = \{a\mathbf{v} + b\mathbf{v}^{\perp} \in \mathbb{R}^2 : 0 \le a < n \text{ and } 0 \le b < t\}.$$

Then, *directional entropy* $h(\mathbf{v})$ of a $\mathbb{Z}^2$-subshift X in the direction $\mathbf{v}$ is defined by

$$h(\mathbf{v}) = \sup_{t>0} \limsup_{n\to\infty} \frac{\log |\mathcal{B}_{E(\mathbf{v},n,t)}(X)|}{n} = \lim_{t\to\infty} \limsup_{n\to\infty} \frac{\log |\mathcal{B}_{E(\mathbf{v},n,t)}(X)|}{n}.$$

Note that there are two vectors orthogonal to $\mathbf{v}$, and $E(\mathbf{v},n,t)$ depends on the choice of $\mathbf{v}^{\perp}$. However, the set of patterns $\mathcal{B}_{E(\mathbf{v},n,t)}(X)$ in both cases are the same.

By definition, it is clear that $h(t\mathbf{v}) = th(\mathbf{v})$ for all $t > 0$. Note that, for $\mathbf{v} \in \mathbb{Z}^2$, $h(\mathbf{v})$ coincides with the entropy of the $\mathbb{Z}$-topological dynamical system $(X, \sigma^{\mathbf{v}})$. Analogously, we define directional entropy dimension as follows.

Definition 2. *Let X be a $\mathbb{Z}^2$-subshift and $\mathbf{v} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. The* directional upper entropy dimension of X in the direction $\mathbf{v}$ *is defined by*

$$\overline{D}(\mathbf{v}) = \sup_{t>0} \limsup_{n\to\infty} \frac{\log\log |\mathcal{B}_{E(\mathbf{v},n,t)}(X)|}{\log n}.$$

The directional lower entropy dimension $\underline{D}(\mathbf{v})$ is defined analogously using lim inf*. If $\overline{D}(\mathbf{v}) = \underline{D}(\mathbf{v})$, and we denote it by $D(\mathbf{v})$ and call it the directional entropy dimension of X in the direction $\mathbf{v}$.*

Using a similar argument as for entropy dimension, one can check that $\overline{D}(\mathbf{v})$ is equal to $\lim_{t\to\infty}\overline{D}(\mathbf{v},t)$ where $\overline{D}(\mathbf{v},t)$ is a unique critical value for α of the function

$$\overline{D}(\mathbf{v},t,\alpha)=\limsup_{n\to\infty}\frac{\log|\mathcal{B}_{E(\mathbf{v},n,t)}(X)|}{n^{\alpha}}.$$

As for the case of directional entropy, for $\mathbf{v}\in\mathbb{Z}^2$, $\overline{D}(\mathbf{v})$ coincides with the topological upper entropy dimension [14] of the $\mathbb{Z}$-topological dynamical system $(X,\sigma^{\mathbf{v}})$. One can see that $\overline{D}(\mathbf{v})=\overline{D}(t\mathbf{v})$ for all $t>0$. Hence, we may assume that $\mathbf{v}$ lies on the unit circle S^1 as far as the directional entropy dimension is concerned. The properties similar to the mentioned hold for $\underline{D}(\mathbf{v})$ and $D(\mathbf{v})$.

3. Equivalent Definitions for Entropy Dimension

In this section, we present equivalent formulations for two-dimensional entropy dimension using the entropy generating shape, which generalizes the notion of entropy generating sequence for one-dimensional case in [14]. The argument directly extends to the case of $\mathbb{Z}^d$-actions for any integer $d>2$. Throughout the paper, $\mathbb{N}$ denotes the set of nonnegative integers.

Let $S\subset\mathbb{N}^2$ be an infinite subset. For $\tau\geq 0$, we define a function

$$\overline{D}(S,\tau)=\limsup_{n\to\infty}\frac{|[0,n)^2\cap S|}{(n^2)^{\tau}}$$

and denote by $\overline{D}(S)$ the critical value for τ of the function $\overline{D}(S,\tau)$, that is,

$$\overline{D}(S)=\inf\{\tau:\overline{D}(S,\tau)=0\}=\sup\{\tau:\overline{D}(S,\tau)=\infty\}.$$

This definition is equivalent to

$$\overline{D}(S)=\limsup_{n\to\infty}\frac{\log|[0,n)^2\cap S|}{\log n^2}.$$

We call $\overline{D}(S)$ the *upper dimension* of S. The *lower dimension* $\underline{D}(S)$ and the *dimension* of S, $D(S)$, are defined similarly. Following [14], we say that S is an *entropy generating shape* of the $\mathbb{Z}^2$-subshift X if

$$\liminf_{n\to\infty}\frac{\log|\mathcal{B}_{[0,n)^2\cap S}(X)|}{|[0,n)^2\cap S|}>0.$$

As for the $\mathbb{Z}$-case, the intuitive idea of an entropy generating shape is to specify positions where the independence occurs. An infinite subset $S\subset\mathbb{N}^2$ is called a *weak entropy generating shape* of X if

$$\liminf_{n\to\infty}\frac{\log|\mathcal{B}_{[0,n)^2\cap S}(X)|}{|[0,n)^2\cap S|^{\beta}}>0 \text{ for all } 0<\beta<1.$$

It is easy to see that if S is a weak generating shape of X, then

$$\liminf_{n\to\infty}\frac{\log\log|\mathcal{B}_{[0,n)^2\cap S}(X)|}{\log|[0,n)^2\cap S|}=1.$$

Theorem 1. *Let X be a $\mathbb{Z}^2$-subshift. Then, the following three values are equal.*

1. $\overline{D}(X)$,
2. *the supremum of $\overline{D}(S)$ over all entropy generating shapes S of X,*
3. *the supremum of $\overline{D}(S)$ over all weak entropy generating shapes S of X.*

Proof. Let $\overline{D}_e$ (resp. $\overline{D}_e^*$) be the supremum of $\overline{D}(S)$ over the entropy generating shapes S (resp. weak entropy generating shapes S). Clearly, $\overline{D}_e \leq \overline{D}_e^*$. In ([14], Theorems 3.8 and 3.10), it was shown that if X is a $\mathbb{Z}$-subshift, then $\overline{D}(X)$ equals the supremum of $\overline{D}(S)$ over all entropy generating sequences S for X. One may check that the proof is valid for $\mathbb{Z}^2$-subshifts with a little modification. For each j-th step in ([14], Theorem 3.8), we can take $W_j \subset [0, n_{j+1})^2 \setminus [0, n_j]^2$. Then, $F = \bigcup W_j$ is an entropy generating shape.

Thus, it remains to show that $\overline{D}_e^* \leq \overline{D}(X)$. Suppose not. Then, there is a weak entropy generating shape S with $\overline{D}(S) > \overline{D}(X)$. Hence,

$$\begin{aligned}\overline{D}(X) &= \limsup_{n\to\infty} \frac{\log\log|\mathcal{B}_n(X)|}{\log n^2} \geq \limsup_{n\to\infty} \frac{\log\log|\mathcal{B}_{[0,n)^2\cap S}(X)|}{\log n^2} \\ &\geq \liminf_{n\to\infty} \frac{\log\log|\mathcal{B}_{[0,n)^2\cap S}(X)|}{\log|[0,n)^2\cap S|} \cdot \limsup_{n\to\infty} \frac{\log|[0,n)^2\cap S|}{\log n^2} = \overline{D}(S) > \overline{D}(X),\end{aligned}$$

which is a contradiction. Therefore, $\overline{D}_e^* \leq \overline{D}(X)$. □

4. Inequalities for Entropy Dimension and Directional Entropy Dimension

In this section, we present simple inequalities between the entropy dimension of a $\mathbb{Z}^2$-action and its directional entropy dimensions.

Theorem 2. *Let X be a $\mathbb{Z}^2$-subshift and let $\mathbf{v} \in S^1$. Then, we have*

$$\overline{D}(\mathbf{v}) \leq 2\overline{D}(X) \leq \overline{D}(\mathbf{v}) + 1$$

and

$$\underline{D}(\mathbf{v}) \leq 2\underline{D}(X) \leq \underline{D}(\mathbf{v}) + 1.$$

In particular, if X has entropy dimension, then we have

$$\overline{D}(\mathbf{v}) \leq 2D(X) \leq \underline{D}(\mathbf{v}) + 1.$$

Proof. First suppose that $\mathbf{v} = \mathbf{e}_1 = (1, 0)$. Then, it is clear that $|\mathcal{B}_{nt,t}(X)| \leq |\mathcal{B}_{nt,nt}(X)|$ for each $n, t \in \mathbb{N}$. Hence, we have

$$\limsup_{n\to\infty} \frac{\log\log|\mathcal{B}_{nt,t}(X)|}{\log nt} \leq \limsup_{n\to\infty} \frac{\log\log|\mathcal{B}_{nt,nt}(X)|}{\log nt} = 2\overline{D}(X)$$

for each fixed $t > 0$. Hence, by letting $t \to \infty$, we have the first inequality $\overline{D}(\mathbf{v}) \leq 2\overline{D}(X)$. On the other hand, each pattern on the shape $R_{nt,nt}$ is obtained by stacking n patterns on the shape $R_{nt,t}$. Hence, we have $|\mathcal{B}_{nt,nt}(X)| \leq |\mathcal{B}_{nt,t}(X)|^n$. Then,

$$2\overline{D}(X) = \limsup_{n\to\infty} \frac{\log\log|\mathcal{B}_{nt,nt}(X)|}{\log nt} \leq 1 + \limsup_{n\to\infty} \frac{\log\log|\mathcal{B}_{nt,t}(X)|}{\log nt}.$$

Hence, by taking supremum on $t > 0$, we have $2\overline{D}(X) \leq \overline{D}(\mathbf{v}) + 1$.

Let $\mathbf{v} \in S^1$. Then, one can find constants $\alpha, \beta > 0$ such that, for all $s > 0$,

$$\text{a translate of } [0, s\alpha]^2 \subset E(\mathbf{v}, s, s) \subset \text{a translate of } [0, s\beta]^2.$$

Then, for each $t > 0$ and $n \in \mathbb{N}$, we have

$$|\mathcal{B}_{E(\mathbf{v},nt,t)}(X)| \leq |\mathcal{B}_{[0,nt\beta]^2}(X)|,$$

from which we obtain $\overline{D}(\mathbf{v}) \le 2\overline{D}(X)$. On the other hand, since $E(\mathbf{v}, nt, nt) = \bigcup_{k=0}^{n-1} \left(E(\mathbf{v}, nt, t) + kt\mathbf{v}^{\perp}\right)$ and for each k

$$\left(E(\mathbf{v}, nt, t) + kt\mathbf{v}^{\perp}\right) \cap \mathbb{Z}^2 \subset \text{a translate of } \left(E(\mathbf{v}, nt+2, t+2) \cap \mathbb{Z}^2\right),$$

we have

$$|\mathcal{B}_{[0,nt\alpha]^2}(X)| \le |\mathcal{B}_{E(\mathbf{v},nt,nt)}(X)| \le |\mathcal{B}_{E(\mathbf{v},nt+2,t+2)}(X)|^n,$$

from which we obtain $2\overline{D}(X) \le \overline{D}(\mathbf{v}) + 1$.

The inequalities for lower entropy dimension are similarly proved. □

Remark 1. *Let X be a $\mathbb{Z}^k$-subshift and $G \subset \mathbb{R}^k$ a hyperplane of codimension ℓ. Then, one can define k-dimensional entropy dimension $D^{(k)}(X)$ of X and $(k-\ell)$-dimensional entropy dimension $D^{(k-\ell)}(G)$ of G as in Section 2. By the same argument as in the proof of the theorem, we see that*

$$(k-1)D^{(k-1)}(G) \le kD^{(k)}(X) \le (k-1)D^{(k-1)}(G) + 1,$$

for any subspace G of codimension 1, and, hence, for any subspace G of codimension ℓ, inductively we have

$$(k-\ell)D^{(k-\ell)}(G) \le kD^{(k)}(X) \le (k-\ell)D^{(k-\ell)}(G) + \ell.$$

We mentioned that the equality $D(\mathbf{v}) = 2D(X)$ is obtained if a direction $\mathbf{v}$ has the same complexity as X has, and the equality $2D(X) = D(\mathbf{v}) + 1$ is obtained if there is a certain independence along the direction $\mathbf{v}^{\perp}$.

We list simple examples of $\mathbb{Z}^2$-subshifts whose entropy dimension and directional entropy dimension can be easily calculated. In the examples below, there is a direction $\mathbf{v}$ for which the inequality $2D(X) \le D(\mathbf{v}) + 1$ is strict.

Example 1. Let $X \subset \{0,1\}^{\mathbb{Z}^2}$ be the three dot model (from §1). It is known that $h(X) = 0$ and $h(\mathbf{v}) > 0$ for each $\mathbf{v} \ne 0$. It follows that $D(\mathbf{v}) = 1$ for all $\mathbf{v} \in S^1$. For each R_n, the pattern on the half of the boundary (left and bottom of R_n) determines the whole pattern on R_n. It follows that $D(X) = 1/2$.

Example 2. Let (Z, T) be a $\mathbb{Z}$-subshift of positive entropy, and let X be the $\mathbb{Z}^2$-subshift generated by $\sigma^{\mathbf{e}_1} = T$ and $\sigma^{\mathbf{e}_2} =$ identity on Z. We know that the directional entropy is continuous [11]. Since $h(\mathbf{v}) > 0$, we have $D(\mathbf{v}) = 1$ for all $\mathbf{v}$ not parallel to $\mathbf{e}_2$. It is clear that $h(\mathbf{e}_2) = D(\mathbf{e}_2) = 0$. Hence, directional entropy dimension need not be upper-semicontinuous even when directional entropy is continuous on S^1.

Example 3. Let (Z, T) be a $\mathbb{Z}$-subshift of positive entropy, and let X be the orbit closure of the set

$$\{x \in \mathcal{A}^{\mathbb{Z}^2} : (x_{(i,k)})_{i\in\mathbb{N}} \in Z \text{ if } k \text{ is a square number and } (x_{(i,k)})_{i\in\mathbb{N}} = 0^{\infty} \text{ otherwise}\}.$$

Let $\mathcal{B}_n(Z)$ denote the set of blocks of length n occurring in Z. Since $|\mathcal{B}_n(Z)|^{\lfloor\sqrt{n}\rfloor} \le |\mathcal{B}_{n,n}(X)| \le n|\mathcal{B}_n(Z)|^{\sqrt{n}}$, one finds that $D(X) = 3/4$. It can be checked that $h(\mathbf{e}_1) = \infty$, $D(\mathbf{e}_1) = 1$ and $h(\mathbf{v}) = D(\mathbf{v}) = 0$ for all $\mathbf{v}$ not parallel to $\mathbf{e}_1$.

5. Constructions of Subshifts with Positive Entropy Dimension and Directional Entropy Dimension

In this section, we construct subshifts with positive topological entropy dimension with diverse properties in their subgroup actions. We first provide a framework with notations for a general construction of a family of subshifts. Then, we will modify the constructions depending on required properties. All the examples in this sections are minimal. We remark that, without the minimality requirement, the construction with similar properties can be carried out more easily.

The basic idea of our construction is a successive concatenation of previous patterns with well-chosen permuting positions as in [17,18]. In what follows, to simplify the notation, we omit the floor function notation on the square roots and write $\sqrt{N}$ instead of $\lfloor\sqrt{N}\rfloor$.

Fix a large number $l_1 \in \mathbb{N}$. Let $\mathcal{C}_1 \subset \{0,1\}^{R_{l_1}}$ denote a set of binary patterns on $l_1 \times l_1$ square R_{l_1}, and let N_1 denote the cardinality of $\mathcal{C}_1$. For the induction step, suppose that a set $\mathcal{C}_j$ of patterns on the $l_j \times l_j$ square R_{l_j} has been constructed and $N_j = |\mathcal{C}_j|$. Give an ordering on $\mathcal{C}_j$ and write $\mathcal{C}_j = \{u_i^{(j)} : 1 \le i \le \sqrt{N_j}^2\}$. We should note that this new $\mathcal{C}_j$ contains less elements than the old $\mathcal{C}_j$ unless N_j is a square number. We may abuse the notation since the cardinalities of both sets have the same asymptotic behavior, which only matters in what follows. Let $l_{j+1} = l_j \cdot \sqrt{N_j}$ and consider a new pattern $u_1^{(j+1)}$ on $R_{l_{j+1}}$ formed by concatenating all the patterns in $\mathcal{C}_j$ in the following way:

$$u_1^{(j+1)}|_{R_{l_j}+l_j\cdot(i_1,i_2)} = u^{(j)}_{i_2\sqrt{N_j}+i_1+1} \text{ for each } 0 \le i_1, i_2 < \sqrt{N_j}.$$

We choose a subset $\mathcal{P}_j \subset [0, \sqrt{N_j})^2 \cap \mathbb{N}^2$, which we call the set of *permuted positions at the j-th step* and let $\mathcal{P}_j = \bigcup_{i=1}^{q_j} \mathcal{P}_{j,i}$ be a partition of $\mathcal{P}_j$. The collection $\mathcal{C}_{j+1}$ consists of all patterns on the square $R_{l_{j+1}}$ obtained by permuting R_{l_j}-subpatterns of $u_1^{(j+1)}$ whose lower left corner is at the location $l_j \cdot (i_1, i_2)$ with $(i_1, i_2) \in \mathcal{P}_{j,i}$ for each $1 \le i \le q_j$. Then, we have iterative formulae for N_j and l_j

$$l_{j+1} = l_j \cdot \sqrt{N_j} \quad \text{and} \quad N_{j+1} = \prod_{i=1}^{q_j} |\mathcal{P}_{j,i}|!.$$

By the construction, $u_1^{(j)}$ is a subpattern of $u_1^{(j+1)}$ at the lower left corner for each j. If the cardinality of $\mathcal{P}_j$ grows fast enough to satisfy $\lim_{j\to\infty} l_j = \infty$, then, by compactness, there is a unique point $w \in \{0,1\}^{\mathbb{N}^2}$ such that $w|_{R_{l_j}} = u_1^{(j)}$ for all $j \in \mathbb{N}$. Let X^+ be the $\mathbb{N}^2$-subshift defined as the orbit closure of w and X the natural extension of X^+. Equivalently, we may let X be the set of all configurations $x \in \{0,1\}^{\mathbb{Z}^2}$ such that each subpattern of x occurs in some member of $\mathcal{C}_j$ for some $j \in \mathbb{N}$. Since each pattern $u_i^{(j)}$, for $i \le \sqrt{N_j}^2$, in $\mathcal{C}_j$ occurs in all patterns in $\mathcal{C}_{j+1}$, it follows that X is minimal.

We are free to choose $\mathcal{P}_j$ and its partition elements $\mathcal{P}_{j,i}$. By choosing them carefully, we may construct subshifts with prescribed entropy dimension and directional entropy dimensions. The following notations are useful for calculations. For $n, m \in \mathbb{N}$, let

$$\mathcal{B}^0_{n,m}(X) = \{u|_{R_{n,m}} : u \in \mathcal{C}_j \text{ for some } j \in \mathbb{N} \text{ with } l_j \ge n, m\}$$

and, for $n, m \in \mathbb{N}$ and $k \in \mathbb{N}$, let

$$\begin{aligned} \mathcal{B}^k_{n,m}(X) &= \{u|_{(p,q)\cdot l_k + R_{n,m}} : u \in \mathcal{C}_j \text{ for some } j > k \\ &\qquad \text{and } p, q \in \mathbb{N} \text{ with } \big((p,q)\cdot l_k + R_{n,m}\big) \subset R_{l_j}\} \\ &= \{w|_{(p,q)\cdot l_k + R_{n,m}} : p, q \in \mathbb{N}\}. \end{aligned}$$

That is, $\mathcal{B}^0_{n,m}(X)$ is the collection of $n \times m$ patterns of X which can be obtained by restricting the patterns in $\mathcal{C}_j$ to its lower left corner and $\mathcal{B}^k_{n,m}(X)$ is that of $n \times m$ subpatterns of $\mathcal{C}_j$ for some $j > k$ whose lower left corner is on the lattice $l_k\mathbb{Z}^2$. We list several inequalities between the cardinality of the sets aforementioned:

(a) Let $n = l_j$. Then $|\mathcal{B}^0_{n,n}(X)| = |\mathcal{C}_j| = N_j$ and $|\mathcal{B}_n(X)| \le n^2 \cdot |\mathcal{B}^0_{n,n}(X)|^4$.

(b) Let $n = k \cdot l_j$ for $0 \le k < \sqrt{N_j}$. Then $|\mathcal{B}_n(X)| \le (l_j)^2 \cdot |\mathcal{B}^j_{(k+1)l_j,(k+1)l_j}(X)|$.

(c) For $i, j \in \mathbb{N}$, we have $|\mathcal{B}_{l_i,l_j}(X)| \le l_i l_j |\mathcal{B}^{\min(i,j)}_{l_i,l_j}(X)|^4$.

We mention that in each of the examples in this section, $\mathcal{P} = \bigcup_{j\in\mathbb{N}} \mathcal{P}_j$ is a weak entropy generating shape.

Example 4. Let $\mathbf{v}_0 \in S^1$ be a rational direction. Then, there is a $\mathbb{Z}^2$-subshift X with $D(X) = 1/2$, $D(\mathbf{v}_0) = 1$ and $D(\mathbf{v}) = 0$ for all $\mathbf{v}$ not parallel to $\mathbf{v}_0$.

We only give a construction for the case $\mathbf{v}_0 = \mathbf{e}_1$ since the construction is similar when v_0 is an arbitrary rational direction. Let $\mathcal{P}_j = \{(i,0) : 0 \le i < \sqrt{N_j}\}$ with $q_j = 1$ and $\mathcal{P}_{j,1} = \mathcal{P}_j$. At the j-th step for $j \in \mathbb{N}$, a typical $(j+1)$-st pattern is obtained by permuting the $l_j \times l_j$ subpatterns (elements of $\mathcal{C}_j$) at the bottom of $u_1^{(j+1)}$. The iterative formula for N_j is given by $N_{j+1} = (\sqrt{N_j})!$. Hence, we have

$$\begin{aligned}\lim_{j\to\infty} \frac{\log\log|\mathcal{B}_{l_j}(X)|}{\log(l_j)^2} &= \lim_{j\to\infty} \frac{\log\log|\mathcal{B}^0_{l_j,l_j}(X)|}{\log(l_j)^2} = \lim_{j\to\infty} \frac{\log\log N_j}{\log(l_j)^2} \\ &= \lim_{j\to\infty} \frac{\log\left(\sqrt{N_{j-1}}\log\sqrt{N_{j-1}}\right)}{\log(l_{j-1}\sqrt{N_{j-1}})^2} \\ &= \lim_{j\to\infty} \frac{\log\sqrt{N_{j-1}}}{\log\sqrt{N_{j-1}}^2} = \frac{1}{2},\end{aligned} \tag{1}$$

where the first two equalities follow from property (a) and the third equality follows from Stirling's formula.

To show that $D(X) = 1/2$, fix $l \in \mathbb{N}$. Then, there is $j \in \mathbb{N}$ such that $l_j \le l < l_{j+1} = l_j\sqrt{N_j}$, and we may assume that $l = k \cdot l_j$ for $1 \le k < \sqrt{N_j}$. The number of $\mathcal{C}_j$-patterns at the permuted positions which are contained in each $l \times l$ pattern $u \in \mathcal{B}^0_{l,l}(X)$ is k, and that of $\mathcal{C}_j$-patterns at the permuted positions which are contained in each $u \in \mathcal{B}^j_{l,l}(X)$ is at most k. Hence, we have

$$\begin{aligned}P(\sqrt{N_j},k) = |\mathcal{B}^0_{l,l}(X)| \le |\mathcal{B}_l(X)| &\le (l_j)^2|\mathcal{B}^j_{(k+1)l_j,(k+1)l_j}(X)| \\ &\le (l_j)^2(\sqrt{N_j})^2 P(\sqrt{N_j},k+1),\end{aligned}$$

where $P(n,k)$ denotes the number of k-permutations of n. For all sufficiently large n and any k with $1 \le k \le n$, we have $k\log n - k < \log P(n,k) \le \log n^k = k\log n$. Hence, for large j and any $k < \sqrt{N_j}$, we have

$$\begin{aligned}\frac{\log k + \log\log(\sqrt{N_j}-1)}{2\log k + \log(l_j)^2} &\le \frac{\log\log|\mathcal{B}_l(X)|}{\log l^2} \\ &\le \frac{\log\left(\log(l_j)^2 + \log(\sqrt{N_j})^2 + (k+1)\log\sqrt{N_j}\right)}{2\log k + \log(l_j)^2} \\ &\le \frac{\log(k+4) + \log\log\sqrt{N_j}}{2\log k + \log(l_j)^2},\end{aligned}$$

from which this equation and (1), it follows that

$$\lim_{l\to\infty} \frac{\log\log|\mathcal{B}_l(X)|}{\log l^2} = \lim_{j\to\infty} \frac{\log\log N_j}{\log(l_j)^2} = \frac{1}{2}.$$

A similar calculation for $\mathbb{Z}$-subshifts can be found in ([18], Section 2).

Now, we calculate the directional entropy dimension. From the construction of $\mathcal{C}_{i+1}$ from $\mathcal{C}_i$, a pattern u in $\mathcal{B}^0_{l_{i+1},l_i}(X)$ can be uniquely extended to a pattern in $\mathcal{C}_{i+1}$ whose bottom equals u. By induction, for all $i > j$, each pattern $\mathcal{B}^0_{l_i,l_j}(X)$ can be uniquely extended to a pattern in $\mathcal{C}_i$. Hence, we have $|\mathcal{B}_{l_i,l_j}(X)| \geq |\mathcal{B}^0_{l_i,l_j}(X)| = N_i$. Hence, for each j

$$\lim_{i\to\infty} \frac{\log\log|\mathcal{B}_{l_i,l_j}(X)|}{\log l_i} \geq \lim_{i\to\infty} \frac{\log\log N_i}{\log l_i} = 1.$$

We can show that in general $\lim_l \frac{\log\log|\mathcal{B}_{l,l_j}(X)|}{\log l} \geq 1$ for any j by assuming $l = k \cdot l_i$ with $0 \leq k < \sqrt{N_i}$ and arguing as in the above. Hence, we have $D(\mathbf{e}_1) = 1$.

Now, we show $D(\mathbf{e}_2) = 0$. As there are $(\sqrt{N_{j-1}})^2$ different $l_{j-1} \times l_j$ subpatterns of members of $\mathcal{C}_j$ whose lower left corner is at $(k \cdot l_{j-1}, 0)$ for $0 \leq k < \sqrt{N_{j-1}}$, it follows that $|\mathcal{B}^{j-1}_{l_{j-1},l_j}(X)| \leq (\sqrt{N_{j-1}})^4$. By this and property (c), we have $|\mathcal{B}_{l_i,l_j}(X)| \leq |\mathcal{B}_{l_{j-1},l_j}(X)| \leq l_{j-1}l_j|\mathcal{B}^{j-1}_{l_{j-1},l_j}(X)|^4 \leq l_{j-1}l_j(\sqrt{N_{j-1}})^{16}$. This yields

$$\lim_{j\to\infty} \frac{\log\log|\mathcal{B}_{l_i,l_j}(X)|}{\log l_j} = 0$$

for each i; hence, $D(\mathbf{e}_2) = 0$.

Finally, let $\mathbf{v} \in S^1$ be not parallel to $\mathbf{e}_1$ and let θ be the angle between $\mathbf{v} \in S^1$ and the x-axis. It is enough to show the case when $\mathbf{v}$ is in the first quadrant. For each i and j with $j > i$, denote by $Q_{i,j}$ the parallelogram generated by the line segment from $\mathbf{0}$ to $(l_i, 0)$ and that from $\mathbf{0}$ to $\frac{l_j}{\sin\theta}\mathbf{v}$. Then, $Q_{i,j}$ has base l_i and height l_j. Let $\mathcal{Q}_{i,j} = \mathcal{B}_{Q_{i,j}}(X)$.

Note that $Q_{i,j}$ can intersect only finitely many $l_i \times l_i$ squares, say q (depending only on i), whose lower left corner is at $l_i\mathbb{Z} \times \{0\}$. *Put*

$$\mathcal{Q}^*_{i,j} = \{u|_{(Q_{i,j}\cap\mathbb{Z}^2)+l_i(a,0)} : u \in \mathcal{C}_k \textit{ for some } k > j \textit{ and } a \in \mathbb{N}\}.$$

The number of different upper subpatterns with height $l_j - l_i$ of members in $\mathcal{Q}^*_{i,j}$ is $\sqrt{N_{j-1}}$, since all the upper subpatterns with height $l_j - l_i$ of members in $\mathcal{C}_j$ are the same. On the other hand, the number of different lower subpatterns with height l_i of members in $\mathcal{Q}^*_{i,j}$ is at most $|\mathcal{C}_i|^q$.

As any pattern on $Q_{i,j}$ occurs as a subpattern on $Q_{2i,2j}$, we have

$$|\mathcal{Q}_{i,j}| \leq (l_j)^2|\mathcal{Q}^*_{i,j}|^4 \leq (l_j)^2(\sqrt{N_{j-1}} \cdot |\mathcal{C}_i|^q)^4.$$

By this, we obtain

$$\lim_{j\to\infty} \frac{\log\log|\mathcal{Q}_{i,j}|}{\log l_j} = 0$$

for each i—thus, $D(\mathbf{v}) = 0$, by taking the supremum over all i.

Remark 2. *At the j-th step of Example 4, instead of permuting the j-th patterns at the bottom row of $u_1^{(j+1)}$, we permute all the columns of $u_1^{(j+1)}$ and denote the collection by $\mathcal{C}_{j+1}$. By a column, we mean a tower of $\sqrt{N_j}$-many j-th patterns in $u_1^{(j+1)}$ whose lower left corner is at $(k \cdot l_j, 0)$ for $0 \leq k < \sqrt{N_j}$.*

The iterative formula for N_j is given by $N_{j+1} = (\sqrt{N_j})!$. Note that the cardinalities of the sets $\mathcal{C}_j$, $\mathcal{B}^0_{l_j,l_j}$ and $\mathcal{B}^0_{l_i,l_j}$ for each i, j are the same as those obtained in Example 4. The constructed system has entropy dimension $1/2$. *We expect that $D(\mathbf{e}_2) = 0$ and $D(\mathbf{v}) = 1$ for all $\mathbf{v}$ not parallel to $\mathbf{e}_1$.*

The following example shows that $\mathbb{Z}^2$-complexity may be spread out in all directions, in the sense that the inequality $2D(X) \leq D(\mathbf{v}) + 1$ in Theorem 2 can be an equality for all directions.

Example 5. Let $\frac{1}{2} < \alpha \leq 1$. Then, there is a $\mathbb{Z}^2$-subshift X with

$$D(X) = \alpha \text{ and } D(\mathbf{v}) = 2\alpha - 1 \text{ for all } \mathbf{v} \in S^1.$$

Let $r = \frac{1}{2\alpha-1} \geq 1$. Given j and $1 \leq i < (N_j)^{\frac{1}{2r}}$, we let

$$\mathcal{P}_{j,i} = \{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 \leq i^2 < (a+1)^2 + (b+1)^2 \\ \text{with } 0 \leq a, b < \sqrt{N_j}\},$$

and $\mathcal{P}_j = \bigcup_i \mathcal{P}_{j,i}$. Note that each $\mathcal{P}_{j,i}$ is the set of coordinates near the circle of radius i. We will only give an argument for $r = 2$ (i.e., $\alpha = 3/4$) for notational simplicity.

Each N_j satisfies

$$\prod_{k^2 \leq \sqrt{N_j}} \left(k^2\right)! \leq N_{j+1} \leq \prod_{k^2 \leq \sqrt{N_j}} \left(2 \cdot k^2\right)!,$$

and so

$$\log N_{j+1} \sim \sum_{k^2 \leq \sqrt{N_j}} k^2 \log(k^2) \sim (N_j)^{3/4} \log \sqrt{N_j}$$

for all large j, where we write $a(n) \sim b(n)$ if the ratio $a(n)/b(n)$ goes to some positive constant as $n \to \infty$. Hence we have

$$\lim_{j\to\infty} \frac{\log\log N_{j+1}}{\log(l_{j+1})^2} = \lim_{j\to\infty} \frac{\log N_j^{3/4} + \log\log N_j}{\log(l_j)^2 + \log \sqrt{N_j}^2} = \frac{3}{4}.$$

Hence, we have $D(X) = 3/4$. For general r, similar calculation gives $\log N_{j+1} \sim N_j^{\frac{r+1}{2r}} \log N_j$; hence, $D(X) = \frac{r+1}{2r} = \alpha$.

Now, we calculate directional entropy dimension. By the symmetry of permuted positions, it suffices to consider $\mathbf{v} = \mathbf{e}_1$. First, by Theorem 2, we have $D(\mathbf{v}) \geq 1/2$.

For each j, the number of $\mathcal{C}_j$ patterns at the permuted positions that are contained in each $l_{j+1} \times l_j$ subpattern of members of $\mathcal{B}_{l_{j+1},l_{j+1}}(X)$ whose lower left corner is at $\{0\} \times l_j\mathbb{Z}$ is at most $\sqrt{N_j}^{1/2}$. Hence, we have, for a fixed j and all $i > j+1$,

$$|\mathcal{B}^0_{l_{i+1},l_j}(X)| \leq |\mathcal{B}^0_{l_{i+1},l_i}(X)| = N_i^{\sqrt{N_i}^{1/2}}.$$

The number of $l_{i+1} \times l_i$ subpatterns of w whose lower left corner is at $\{0\} \times l_j\mathbb{Z}$ is at most $\sqrt{N_i} \cdot N_i^{\sqrt{N_i}^{1/2}}$. As in (c),

$$|\mathcal{B}_{l_{i+1},l_j}(X)| \leq l_i l_{i+1} \left(\sqrt{N_i} \cdot N_i^{\sqrt{N_i}^{1/2}}\right)^4.$$

Hence,

$$\limsup_{i\to\infty} \frac{\log\log \mathcal{B}_{l_{i+1},l_j}(X)}{\log l_{i+1}} \leq \lim_{i\to\infty} \frac{\log\left(\sqrt{N_i}^{1/2} \log N_i\right)}{\log l_{i+1}} = \frac{1}{2},$$

for each j, from which we have $D(\mathbf{v}) \leq 1/2$, as desired.

It is possible to construct a $\mathbb{Z}^2$-subshift with arbitrary entropy dimension. However, we are not able to compute its directional entropy dimension.

Example 6. There exists a $\mathbb{Z}^2$-subshift X with $D(X) = \alpha$ for any $0 < \alpha \leq 1$.

Let $r, s \geq 1$. Given j and $1 \leq i < (N_j)^{\frac{1}{2r}}$, we let

$$\mathcal{P}_{j,i} = \{(\lfloor i^r \rfloor, \lfloor m^s \rfloor) : m^s \leq i^r\}$$

and $\mathcal{P}_j = \bigcup_i \mathcal{P}_{j,i}$. As $N_{j+1} \sim \prod_{i \leq N_j^{1/2r}} (i^{\lfloor \frac{r}{s} \rfloor})!$, by a similar argument to the one in Example 5, one can check that

$$D(X) = \frac{r+s}{2rs}.$$

The result follows from the fact that any $\alpha \in (0, 1]$ can be written as $\alpha = \frac{r+s}{2rs}$ for some $r, s \geq 1$.

If X is a zero entropy $\mathbb{Z}^2$-subshift with $D(X) = 1$, then $D(\mathbf{v}) = 1$ for all $\mathbf{v}$ by Theorem 2. In the following, we construct such a $\mathbb{Z}^2$-subshift such that the directional entropy $h(\mathbf{v}) = 0$ for every $\mathbf{v}$.

Example 7. There is a $\mathbb{Z}^2$-subshift X with

$$D(X) = 1 \text{ and } h(\mathbf{v}) = 0 \text{ for all } \mathbf{v}.$$

For each $n \in \mathbb{N}$, let $p(n)$ be the n-th prime number, and $\pi(n)$ the number of prime numbers less than n. Given j and $1 \leq i < \pi(\sqrt{N_j})$, we let

$$\mathcal{P}_{j,i} = \{(a, b) \in \mathbb{Z}^2 : a^2 + b^2 \leq p(i)^2 < (a+1)^2 + (b+1)^2 \\ \text{with } 0 \leq a, b < \sqrt{N_j}\},$$

and $\mathcal{P}_j = \bigcup_i \mathcal{P}_{j,i}$. Then, the iterative formula is

$$N_{j+1} \sim \prod_{\substack{p \leq N_j^{1/2} \\ p:\text{prime}}} p!.$$

Hence, we have

$$\log N_{j+1} \sim \sum_{\substack{p \leq N_j^{1/2} \\ p:\text{prime}}} p \log p \sim N_j$$

for all large j. This yields $D(X) = 1$.

For the calculation of directional entropy, by symmetry, it suffices to consider when $\mathbf{v} = \mathbf{e}_1$. For each j, the number of $\mathcal{C}_j$ patterns at the permuted positions that are contained in each pattern in $\mathcal{B}^0_{j+1,j}(X)$ is $\pi(\sqrt{N_j})$. Hence, by a simple induction, we have, for all i and k,

$$|\mathcal{B}^0_{l_{i+k}, l_i}(X)| \leq |\mathcal{B}^0_{l_i, l_i}(X)|^{\prod_{j=i}^{i+k-1} \pi(\sqrt{N_j})}.$$

It is well known that there exists a constant c such that $\pi(x) \leq cx / \log x$ for all x:

$$\begin{aligned} \log |\mathcal{B}^0_{l_{i+k}, l_i}(X)| &\leq \log |\mathcal{B}^0_{l_i, l_i}(X)| \prod_{j=i}^{i+k-1} \pi(\sqrt{N_j}) \\ &\leq \log |\mathcal{B}^0_{l_i, l_i}(X)| c^k \prod_{j=i}^{i+k-1} \frac{\sqrt{N_j}}{\log \sqrt{N_j}}. \end{aligned}$$

As $l_{i+k} = l_i \cdot \prod_{j=i}^{i+k-1} \sqrt{N_j}$, we have

$$\frac{\log |\mathcal{B}^0_{l_{i+k},l_i}(X)|}{l_{i+k}} \le \frac{1}{l_i} \log |\mathcal{B}^0_{l_i,l_i}(X)| c^k \prod_{j=i}^{i+k-1} \left(\frac{1}{\log \sqrt{N_j}} \right) \to 0$$

for $k \to \infty$. As in Example 5, we also have

$$\frac{\log |\mathcal{B}_{l_{i+k},l_i}(X)|}{l_{i+k}} \to 0 \quad \text{for } k \to \infty.$$

Since this holds for all i, it follows that $h(\mathbf{e}_1) = 0$.

Example 5 gives a family of subshifts with $2D(X) = 2\alpha = D(\mathbf{v}) + 1$ for all directions for each $\alpha > 1/2$. In the following, we show that there is an example with the same property for $0 < \alpha \le 1/2$. Recall that three dot example satisfies $D(X) = 1/2$ and $D(\mathbf{v}) = 1$ for all directions. Our example shows that $\mathbb{Z}^2$-complexity may be spread out in all directions.

Example 8. Let $0 < \alpha \le 1/2$. Then, there is a $\mathbb{Z}^2$-subshift X with

$$D(X) = \alpha \text{ and } D(\mathbf{v}) = 0 \text{ for all } \mathbf{v} \in S^1.$$

Let $r = 1/2\alpha \ge 1$ and let $\mathcal{P}_j = \{(i^r, \lfloor i^r/j \rfloor) : 0 \le i \le (N_j)^{1/2r}\}$ with $q_j = 1$ and $\mathcal{P}_j = \mathcal{P}_{j,1}$. At the j-th step, we permute the $\mathcal{C}_j$ patterns on the line $y = \frac{1}{j}x$.

Then, the iterative formula for N_j is given by $N_{j+1} = ((N_j)^{1/2r})!$, from which it follows that $D(X) = 1/2r = \alpha$. As the number of $\mathcal{C}_j$ patterns at the permuted positions that are contained in each pattern in $\mathcal{B}^0_{l_{j+1},l_j}(X)$ is $j^{1/r} \le j$, we have

$$|\mathcal{B}^0_{l_{j+1},l_j}(X)| \le |\mathcal{C}_j|^j = (N_j)^j,$$

from which we have $|\mathcal{B}_{l_{j+1},l_j}(X)| \le l_j l_{j+1} (\sqrt{N_j} |\mathcal{B}^0_{l_{j+1},l_j}(X)|^j)^4 \le l_j l_{j+1} ((N_j)^{j+1/2})^4$. Hence, $D(\mathbf{e}_1) = 0$. When $\mathbf{v}$ is not parallel to $\mathbf{e}_1$, then its directional entropy dimension can be calculated similarly to Example 4.

The following table 1 summarizes the examples in this paper.

Table 1. Entropy dimension and directional entropy dimension.

Examples	$D(X)$	$D(\mathbf{v})$	$h(\mathbf{v})$ when $D(\mathbf{v}) = 1$
1	$1/2$	$D(\mathbf{v}) = 1$ for all $\mathbf{v}$	positive
2	$1/2$	$D(\mathbf{v}) = 1$ for all $\mathbf{v} \nparallel \mathbf{e}_2$	positive
		$D(\mathbf{e}_2) = 0$	
3	$3/4$	$D(\mathbf{e}_1) = 1$	$h(\mathbf{e}_1) = \infty$
		$D(\mathbf{v}) = 0$ for all $\mathbf{v} \nparallel \mathbf{e}_1$	
4	$1/2$	$D(\mathbf{v}_0) = 1$	$h(\mathbf{v}_0) = 0$
		$D(\mathbf{v}) = 0$ for all $\mathbf{v} \nparallel \mathbf{v}_0$	
5	$\frac{1}{2} < \alpha \le 1$	$D(\mathbf{v}) = 2\alpha - 1$ for all $\mathbf{v}$	$(*)$
7	1	$D(\mathbf{v}) = 1$ for all $\mathbf{v}$	$h(\mathbf{v}) = 0$
8	$0 < \alpha \le \frac{1}{2}$	$D(\mathbf{v}) = 0$ for all $\mathbf{v}$	

(*) For $\alpha = 1$, it seems that directional entropy depends on the arrangement of $u_i^{(j)}$ blocks in $u_1^{(j+1)}$.

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