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Variational Principle for Relative Tail Pressure

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Abstract: We introduce the relative tail pressure to establish a variational principle for continuous bundle random dynamical systems. We also show that the relative tail pressure is conserved by the principal extension.

Keywords: relative tail pressure; relative entropy; variational principle; principal extension

1. Introduction

The notion of topological pressure for the potential was introduced by Ruelle [1] for expansive dynamical systems. Walters [2] generalized it to the general case and established the classical variational principle, which states that the topological pressure is the supremum of the measure-theoretic entropy together with the integral of the potential over all invariant measures. In the special case that the potential is zero, it reduces to the variational principle for topological entropy.

The entropy concepts can be localized by defining topological tail entropy to quantify the local complexity of a system at arbitrary small scales [3]. A variational principle for topological tail entropy was established in the case of homeomorphism from subtle results in the theory of entropy structure by Downarowicz [4]. An elementary proof of this variational principle for continuous transformations was obtained by Burguet [5] in term of essential partitions. Ledrappier [6] presented a variational principle between the topological tail entropy and the defect of upper semi-continuity of the measure-theoretic entropy on the cartesian square of the dynamical system involved, and proved that the tail entropy is an invariant under any principal extension. Kifer and Weiss [7] introduced the relative tail entropy for continuous bundle random dynamical systems (RDSs) by using the open covers and spanning subsets and deduced the equivalence between the two notions.

A relative version of the variational principle for topological pressure was given by Ledrappier and Walters [8] in the framework of the relativized ergodic theory, and it was extended by Bogenschütz [9] to random transformations acting on one place. Later, Kifer [10] gave the variational principle for random bundle transformations.

In this paper, we propose a relative variational principle for the relative tail pressure, which is introduced for random bundle transformations by using open random sets. The notion defined here enables us to treat the different open covers for different fibers. We deal with the product RDS generated by a given RDS and any other RDS with the same base. We obtain a variational inequality, which shows that the defect of the upper semi-continuity of the relative measure-theoretic entropy of any invariant measure together with the integral of the random continuous potential in the product RDS cannot exceed the relative tail pressure of the original RDS. In particular, when the two continuous-bundle RDSs coincide, we construct a maximal invariant measure in the product RDS to ensure that the relative tail pressure could be reached, and establish the variational principle. For the trivial probability space and the zero potential, the relative tail pressure is the topological tail entropy defined in [3] and the variational principle reduces to the version deduced by Ledrappier [6] in deterministic dynamical systems. As an application of the variational principle we show that the relative tail pressure is conserved by any principal extension.

The paper is organized as follows. In Section 2, we recall some background in the ergodic theory. In Section 3, we introduce the notion of the relative tail pressure with respect to open random covers and give the power rule. Section 4 is devoted to the proof of the variational principle and shows that the relative tail pressure is an invariant under principal extensions.

2. Relative Entropy

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete countably generated probability space together with a $\mathbb{P}$-preserving transformation ϑ and $(X, \mathcal{B})$ be a compact metric space with the Borel σ-algebra $\mathcal{B}$. Let $\mathcal{E}$ be a measurable subset of $\Omega \times X$ with respect to the product σ-algebra $\mathcal{F} \times \mathcal{B}$ and the fibers $\mathcal{E}_\omega = \{x \in X : (\omega, x) \in \mathcal{E}\}$ be compact. A continuous bundle random dynamical system (RDS) T over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is generated by the mappings $T_\omega : \mathcal{E}_\omega \to \mathcal{E}_{\vartheta\omega}$ so that the map $(\omega, x) \to T_\omega x$ is measurable and the map $x \to T_\omega x$ is continuous for $\mathbb{P}$-almost all (a.a.) ω. The family $\{T_\omega : \omega \in \Omega\}$ is called a random transformation and each T_ω maps the fiber $\mathcal{E}_\omega$ to $\mathcal{E}_{\vartheta\omega}$. The map $\Theta : \mathcal{E} \to \mathcal{E}$ defined by $\Theta(\omega, x) = (\vartheta\omega, T_\omega x)$ is called the skew product transformation. Observe that $\Theta^n(\omega, x) = (\vartheta^n\omega, T^n_\omega x)$, where $T^n_\omega = T_{\vartheta^{n-1}\omega} \circ \cdots T_{\vartheta\omega} \circ T_\omega$ for $n \geq 0$ and $T^0_\omega = id$.

Let $\mathcal{P}_\mathbb{P}(\Omega \times X)$ be the space of probability measures on $\Omega \times X$ having the marginal $\mathbb{P}$ on Ω and set $\mathcal{P}_\mathbb{P}(\mathcal{E}) = \{\mu \in \mathcal{P}_\mathbb{P}(\Omega \times X) : \mu(\mathcal{E}) = 1\}$. Denote by $\mathcal{I}_\mathbb{P}(\mathcal{E})$ the space of all Θ–invariant measures in $\mathcal{P}_\mathbb{P}(\mathcal{E})$.

Let $\mathcal{S}$ be a sub-σ-algebra of $\mathcal{F} \times \mathcal{B}$ restricted on $\mathcal{E}$, and $\mathcal{R} = \{R_i\}$ be a finite or countable partition of $\mathcal{E}$ into measurable sets. For $\mu \in \mathcal{P}_\mathbb{P}(\Omega \times X)$ the conditional entropy of $\mathcal{R}$ given σ-algebra $\mathcal{S}$ is defined as:

$$H_\mu(\mathcal{R} \mid \mathcal{S}) = -\int \sum_i E(1_{R_i} \mid \mathcal{S}) \log E(1_{R_i} \mid \mathcal{S}) d\mu,$$

where $E(1_{R_i} \mid \mathcal{S})$ is the conditional expectation of 1_{R_i} with respect to $\mathcal{S}$.

Let $\mu \in \mathcal{I}_\mathbb{P}(\mathcal{E})$ and let $\mathcal{S}$ be a sub-σ-algebra of $\mathcal{F} \times \mathcal{B}$ restricted on $\mathcal{E}$ satisfying $\Theta^{-1}\mathcal{S} \subset \mathcal{S}$. For a given measurable partition $\mathcal{R}$ of $\mathcal{E}$, the conditional entropy $H_\mu(\mathcal{R}^{(n)} \mid \mathcal{S})$ is a non-negative sub-additive sequence, where $\mathcal{R}^{(n)} = \bigvee_{i=0}^{n-1} (\Theta^i)^{-1}\mathcal{R}$. The *relative entropy* $h_\mu(\mathcal{R} \mid \mathcal{S})$ *of* Θ *with respect to a partition* $\mathcal{R}$ is defined as:

$$h_\mu(\mathcal{R} \mid \mathcal{S}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{R}^{(n)} \mid \mathcal{S}) = \inf_n \frac{1}{n} H_\mu(\mathcal{R}^{(n)} \mid \mathcal{S}).$$

The *relative entropy of* Θ is defined by the formula:

$$h_\mu(\Theta \mid \mathcal{S}) = \sup_{\mathcal{R}} h_\mu(\mathcal{R} \mid \mathcal{S}),$$

where the supremum is taken over all finite or countable measurable partitions $\mathcal{R}$ of $\mathcal{E}$ with finite conditional entropy $H_\mu(\mathcal{R} \mid \mathcal{S}) < \infty$. The *defect of upper semi-continuity of the relative entropy* $h_\mu(\Theta \mid \mathcal{S})$ is defined on $\mathcal{I}_\mathbb{P}(\mathcal{E})$ as:

$$h^*_m(\Theta \mid \mathcal{S}) = \begin{cases} \limsup\limits_{\mu\to m} h_\mu(\Theta \mid \mathcal{S}) - h_m(\Theta \mid \mathcal{S}), & \text{if } h_m(\Theta \mid \mathcal{S}) < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

Any $\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E})$ on $\mathcal{E}$ disintegrates $d\mu(\omega, x) = d\mu_\omega(x) d\mathbb{P}(\omega)$ (see [11] (Section 10.2)), where $\omega \mapsto \mu_\omega$ is the disintegration of μ with respect to the σ-algebra $\mathcal{F}_\mathcal{E}$ formed by all sets $(F \times X) \cap \mathcal{E}$ with $F \in \mathcal{F}$. This means that μ_ω is a probability measure on $\mathcal{E}_\omega$ for $\mathbb{P}$-almost all (a.a.) ω and for any measurable set $R \in \mathcal{E}$, $\mathbb{P}$-a.s. $\mu_\omega(R(\omega)) = E(R \mid \mathcal{F}_\mathcal{E})(\omega)$, where $R(\omega) = \{x : (\omega, x) \in R\}$ and so $\mu(R) = \int \mu_\omega(R(\omega))d\mathbb{P}(\omega)$. The conditional entropy of $\mathcal{R}$ given the σ-algebra $\mathcal{F}_\mathcal{E}$ can be written as:

$$H_\mu(\mathcal{R} \mid \mathcal{F}_\mathcal{E}) = -\int \sum_i E(R_i \mid \mathcal{F}_\mathcal{E}) \log E(R_i \mid \mathcal{F}_\mathcal{E}) d\mathbb{P} = \int H_{\mu_\omega}(\mathcal{R}(\omega))d\mathbb{P},$$

where $\mathcal{R}(\omega) = \{R_i(\omega)\}$, $R_i(\omega) = \{x \in \mathcal{E}_\omega : (\omega, x) \in R_i\}$ is a partition of $\mathcal{E}_\omega$.

Let $(Y, \mathcal{C})$ be a compact metric space with the Borel σ-algebra $\mathcal{C}$ and $\mathcal{G}$ be a measurable, with respect to the product σ-algebra $\mathcal{F} \times \mathcal{C}$, subset of $\Omega \times Y$ with the fibers $\mathcal{G}_\omega$ being compact. The continuous bundle RDS S over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is generated by the mappings $S_\omega : \mathcal{G}_\omega \to \mathcal{G}_{\vartheta\omega}$ so that the map $(\omega, y) \to S_\omega y$ is measurable and the map $y \to S_\omega y$ is continuous for $\mathbb{P}$-almost all (a.a.) ω. The skew product transformation $\Lambda : \mathcal{G} \to \mathcal{G}$ is defined as $\Lambda(\omega, y) = (\vartheta\omega, S_\omega y)$.

Definition 1. *Let T, S be two continuous bundle RDSs over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ on $\mathcal{E}$ and $\mathcal{G}$, respectively. T is said to be a factor of S, or S is an extension of T, if there exists a family of continuous surjective maps $\pi_\omega : \mathcal{G}_\omega \to \mathcal{E}_\omega$ such that the map $(\omega, y) \to \pi_\omega y$ is measurable and $\pi_{\vartheta\omega} S_\omega = T_\omega \pi_\omega$. The map $\pi : \mathcal{G} \to \mathcal{E}$ defined by $\pi(\omega, y) = (\omega, \pi_\omega y)$ is called the factor or extension transformation from $\mathcal{G}$ to $\mathcal{E}$. The skew product system $(\mathcal{E}, \Theta)$ is called a factor of $(\mathcal{G}, \Lambda)$ or $(\mathcal{G}, \Lambda)$ is an extension of $(\mathcal{E}, \Theta)$.*

Denote by $\mathcal{A}$ the restriction of $\mathcal{F} \times \mathcal{B}$ on $\mathcal{E}$ and set $\mathcal{A}_\mathcal{G} = \{\pi^{-1}A : A \in \mathcal{A}\}$.

Definition 2. *A continuous bundle RDS T on $\mathcal{E}$ is called a principal factor of S on $\mathcal{G}$, or that S is a principal extension of T, if for any $\Lambda-$invariant probability measure m in $\mathcal{I}_\mathbb{P}(\mathcal{G})$, the relative entropy of Λ with respect to $\mathcal{A}_\mathcal{G}$ vanishes, i.e., $h_m(\Lambda \mid \mathcal{A}_\mathcal{G}) = 0$.*

Let T and S be two continuous bundle RDSs over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ on $\mathcal{E}$ and $\mathcal{G}$, respectively. Let $\mathcal{H} = \{(\omega, y, x) : y \in \mathcal{G}_\omega, x \in \mathcal{E}_\omega\}$ and $\mathcal{H}_\omega = \{(y, x) : (\omega, y, x) \in \mathcal{H}\}$. It is not hard to see that $\mathcal{H}$ is a measurable subset of $\Omega \times Y \times X$ with respect to the product σ-algebra $\mathcal{F} \times \mathcal{C} \times \mathcal{B}$ (as a graph of a measurable multifunction; see [12] (Proposition III.13)). The continuous bundle RDS $S \times T$ over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is generated by the family of mappings $(S \times T)_\omega : \mathcal{H}_\omega \to \mathcal{H}_{\vartheta\omega}$ with $(y, x) \to (S_\omega y, T_\omega x)$. The map $(\omega, y, x) \to (S_\omega y, T_\omega x)$ is measurable and the map $(y, x) \to (S_\omega y, T_\omega x)$ is continuous in (y, x) for $\mathbb{P}$-a.a. ω. The skew product transformation Γ generated by Θ and Λ from $\mathcal{H}$ to itself is defined as $\Gamma(\omega, y, x) = (\vartheta\omega, S_\omega y, T_\omega x)$.

Let $\pi_\mathcal{E} : \mathcal{H} \to \mathcal{E}$ be the natural projection with $\pi_\mathcal{E}(\omega, y, x) = (\omega, x)$, and $\pi_\mathcal{G} : \mathcal{H} \to \mathcal{G}$ with $\pi_\mathcal{G}(\omega, y, x) = (\omega, y)$. Then, $\pi_\mathcal{E}$ and $\pi_\mathcal{G}$ are two factor transformations from $\mathcal{H}$ to $\mathcal{E}$ and $\mathcal{G}$, respectively. Denote by $\mathcal{D}$ the restriction of $\mathcal{F} \times \mathcal{C}$ on $\mathcal{G}$ and set $\mathcal{D}_\mathcal{H} = \pi_\mathcal{G}^{-1}(\mathcal{D}) = \{(D \times X) \cap \mathcal{H} : D \in \mathcal{D}\}$, $\mathcal{A}_\mathcal{H} = \pi_\mathcal{E}^{-1}(\mathcal{A}) = \{(A \times Y) \cap \mathcal{H} : A \in \mathcal{A}\}$ and $\mathcal{F}_\mathcal{H} = \{(F \times Y \times X) \cap \mathcal{H} : F \in \mathcal{F}\}$.

The *relative entropy of* Γ *given the σ-algebra* $\mathcal{D}_H$ is defined by:

$$h_\mu(\Gamma \mid \mathcal{D}_H) = \sup_\mathcal{R} h_\mu(\mathcal{R} \mid \mathcal{D}_H),$$

where,

$$h_\mu(\mathcal{R} \mid \mathcal{D}_H) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} (\Gamma^i)^{-1}\mathcal{R} \mid \mathcal{D}_H)$$

is the *relative entropy of* Γ *with respect to a measurable partition* $\mathcal{R}$, and the supremum is taken over all finite or countable measurable partitions $\mathcal{R}$ of $\mathcal{H}$ with finite conditional entropy $H_\mu(\mathcal{R} \mid \mathcal{D}_H) < \infty$.

Let $\mathcal{E}^{(2)} = \{(\omega, x, y) : x, y \in \mathcal{E}_\omega\}$, which is also a measurable subset of $\Omega \times X^2$ with respect to the product σ-algebra $\mathcal{F} \times \mathcal{B}^2$. Let $\Theta^{(2)} : \mathcal{E}^{(2)} \to \mathcal{E}^{(2)}$ be a skew-product transformation with $\Theta^{(2)}(\omega, x, y) = (\vartheta\omega, T_\omega x, T_\omega y)$. The map $(\omega, x, y) \to (T_\omega x, T_\omega y)$ is measurable and the map $(x, y) \to (T_\omega x, T_\omega y)$ is continuous in (x, y) for $\mathbb{P}$-a.a. ω. Let $\mathcal{E}_1, \mathcal{E}_2$ be two copies of $\mathcal{E}$, i.e., $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, and $\pi_{\mathcal{E}_i}$ be the natural projection from $\mathcal{E}^{(2)}$ to $\mathcal{E}_i$ with $\pi_{\mathcal{E}_i}(\omega, x_1, x_2) = (\omega, x_i)$, i = 1, 2. Denote by $\mathcal{A}_{\mathcal{E}^{(2)}} = \{(A \times X) \cap \mathcal{E}^{(2)} : A \in \mathcal{F} \times \mathcal{B}\}$. The *relative entropy of* $\Theta^{(2)}$ *given the σ-algebra* $\mathcal{A}_{\mathcal{E}^{(2)}}$ is defined by:

$$h_\mu(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) = \sup_{\mathcal{R}} h_\mu(\mathcal{R} \mid \mathcal{A}_{\mathcal{E}^{(2)}}),$$

where,

$$h_\mu(\mathcal{R} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} ((\Theta^{(2)})^i)^{-1}\mathcal{R} \mid \mathcal{A}_{\mathcal{E}^{(2)}})$$

is the *relative entropy of* $\Theta^{(2)}$ *with respect to a measurable partition* $\mathcal{R}$, and the supremum is taken over all finite or countable measurable partitions $\mathcal{R}$ of $\mathcal{E}^{(2)}$ with finite conditional entropy $H_\mu(\mathcal{R} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) < \infty$.

3. Relative Tail Pressure

A (closed) random set Q is a measurable set valued map $\mathcal{Q} : \Omega \to 2^X$, or the graph of Q denoted by the same letter, taking values in the (closed) subsets of compact metric space X. An open random set U is a set valued map $U : \Omega \to 2^X$ whose complement U^c is a closed random set. A measurable set Q is an open (closed) random set if the fiber Q_ω is an open (closed) subset of $\mathcal{E}_\omega$ in its induced topology from X for $\mathbb{P}$-almost all ω (see [13] (Lemma 2.7)). A random cover $\mathcal{Q}$ of $\mathcal{E}$ is a finite or countable family of random sets $\{Q\}$, such that $\mathcal{E}_\omega = \bigcup_{Q\in\mathcal{Q}} Q(\omega)$ for all $\omega \in \Omega$, and it will be called an open random cover if all $Q \in \mathcal{Q}$ are open random sets. Set $\mathcal{Q}(\omega) = \{Q(\omega)\}$, $\mathcal{Q}^{(n)} = \bigvee_{i=0}^{n-1}(\Theta^i)^{-1}\mathcal{Q}$ and $\mathcal{Q}^{(n)}(\omega) = \bigvee_{i=0}^{n-1}(T_\omega^i)^{-1}\mathcal{Q}(\vartheta^i\omega)$. Denote by $\mathfrak{P}(\mathcal{E})$ the set of random covers and $\mathfrak{U}(\mathcal{E})$ the set of open random covers. For $\mathcal{R}, \mathcal{Q} \in \mathfrak{P}(\mathcal{E})$, $\mathcal{R}$ is said to be finer than $\mathcal{Q}$, which we will write $\mathcal{R} \succ \mathcal{Q}$ if each element of $\mathcal{R}$ is contained in some element of $\mathcal{Q}$.

For each measurable in (ω, x) and continuous in $x \in \mathcal{E}_\omega$ function f on $\mathcal{E}$, let:

$$\|f\| = \int \|f(\omega)\|_\infty \, d\mathbf{P}, \quad \text{where} \quad \|f(\omega)\|_\infty = \sup_{x\in\mathcal{E}_\omega} \mid f(\omega, x) \mid,$$

and $\mathbf{L}^1_{\mathcal{E}}(\Omega, \mathcal{C}(X))$ be the space of such functions f with $\|f\| < \infty$ and identify f and g provided $\|f - g\| = 0$; then $\mathbf{L}^1_{\mathcal{E}}(\Omega, \mathcal{C}(X))$ is a Banach space with the norm $\| \cdot \|$. Any such f will be called a random continuous function from $\mathcal{E}$ to $\mathbb{R}$.

Let $f \in L^1_{\mathcal{E}}(\Omega, \mathcal{C}(X))$ and $n \in \mathbb{N}$. Denote by:

$$S_n f(\omega, x) = \sum_{i=0}^{n-1} f(\vartheta^i\omega, T_\omega^i x) = \sum_{i=0}^{n-1} f \circ \Theta^i(\omega, x).$$

For any non-empty set $U \subset \mathcal{E}$ and a random cover $\mathcal{R} \in \mathfrak{P}(\mathcal{E})$, set:

$$P_\Theta^f(\omega, n, \mathcal{R}, U) = \inf\{\sum_{S\in\eta} \sup_{x\in S(\omega)\cap U(\omega)} e^{S_n f(\omega,x)} : \eta \text{ is a random subcover of } \mathcal{R}^{(n)}\}.$$

For $\mathcal{R}, \mathcal{Q} \in \mathfrak{P}(\mathcal{E})$, let:

$$P_\Theta^f(\omega, n, \mathcal{R}, \mathcal{Q}) = \max_{Q\in\mathcal{Q}^{(n)}} P_\Theta^f(\omega, n, \mathcal{R}, Q)\}.$$

For an open random cover $\mathcal{R}$, $P_\Theta^f(\omega, n, \mathcal{R}, \mathcal{Q})$ is measurable in ω. The following proof is similar to [10] (Proposition 1.6).

Lemma 1. *Let* $\mathcal{R} \in \mathfrak{U}(\mathcal{E})$ *and* $\mathcal{Q} \in \mathfrak{P}(\mathcal{E})$. *The function* $\omega \to P_\Theta^f(\omega, n, \mathcal{R}, \mathcal{Q})$ *is measurable.*

Proof. Fix $n \in \mathbb{N}$. Let $Q \in \mathcal{Q}^{(n)}$ and $\mathcal{R} = \{R_1, \ldots, R_l\}$. Notice that $\mathcal{R}^{(n)}(\omega)$ is the open cover of $\mathcal{E}_\omega$ consisting of sets,

$$R_{(j_0,\ldots,j_{n-1})}(\omega) = \bigcap_{i=0}^{n-1} (T_\omega^i)^{-1} R_{j_i}(\vartheta^i \omega)$$

Since each R_j is a random set, then the sets,

$$R_{(j_0,\ldots,j_{n-1})} = \{(\omega, x) : x \in R_{(j_0,\ldots,j_{n-1})}(\omega)\}$$

are measurable sets of $\mathcal{E}$. It follows from Lemma III.39 in [12] that the function:

$$\psi_{(j_0,\ldots,j_{n-1})}(\omega) = \sup\{e^{S_n f(\omega,x)} : x \in R_{(j_0,\ldots,j_{n-1})}(\omega) \cap Q(\omega)\}$$

is measurable in ω, where $\psi_{(j_0,\ldots,j_{n-1})}(\omega) = 0$ if $R_{(j_0,\ldots,j_{n-1})}(\omega) \cap Q(\omega) = \emptyset$. Since $Q \in \mathcal{F} \times \mathcal{B}$, it follows that (see [12] (Theorem III.30)) for any collection of $n-$strings $\mathbf{j}^i = (j_0^i, \ldots, j_{n-1}^i)$, $i = 1, \ldots, k$, the set:

$$\Omega_{\mathbf{j}^1,\ldots,\mathbf{j}^k} = \{\omega : Q(\omega) \subset \bigcup_{i=1}^{k} R_{\mathbf{j}^i}(\omega)\} = \Omega \setminus \{\omega : (X \setminus \bigcup_{i=1}^{k} R_{\mathbf{j}^i}(\omega)) \cap Q(\omega) \neq \emptyset\}$$

belongs to $\mathcal{F}$. Since l^n is finite, One obtains a finite partition of Ω into measurable sets Ω^J, where J is a finite family of n—strings such that $\Omega^J = \bigcap_{(\mathbf{j}^1,\ldots,\mathbf{j}^k)\in J} \Omega_{\mathbf{j}^1,\ldots,\mathbf{j}^k}$. Thus for each $\omega \in \Omega^J$,

$$P_\Theta^f(\omega, n, \mathcal{R}, Q) = \min_{(\mathbf{j}^1,\ldots,\mathbf{j}^k)\in J, k\in\mathbb{N}} \sum_{i=1}^{k} \psi_{\mathbf{j}^i},$$

and so this function is measurable in ω.

Since for each $t \in \mathbb{R}$,

$$\{\omega : P_\Theta^f(\omega, n, \mathcal{R}, \mathcal{Q}) > t\} = \bigcup_{Q \in \mathcal{Q}^{(n)}} \{\omega : P_\Theta^f(\omega, n, \mathcal{R}, Q) > t\},$$

Then the function $P_\Theta^f(\omega, n, \mathcal{R}, \mathcal{Q})$ is measurable in ω. $\square$

For each ω, the sequence $\log P_\Theta^f(\omega, n, \mathcal{R}, \mathcal{Q})$ is subadditive. Indeed, if β is a random cover of $\bigvee_{i=0}^{n-1}(T_\omega^i)^{-1}\mathcal{R}(\vartheta^i\omega)$ on $\mathcal{E}_\omega$ and γ is a random cover of $\bigvee_{i=0}^{k-1}(T_{\vartheta^n\omega}^i)^{-1}\,\mathcal{R}(\vartheta^{i+n}\omega)$ on $\mathcal{E}_{\vartheta^n\omega}$, then $\beta \vee (T_\omega^n)^{-1}\gamma$ is a finite subcover of $\bigvee_{i=0}^{n+k-1}(T_\omega^i)^{-1}\mathcal{R}(\vartheta^i\omega)$ on $\mathcal{E}_\omega$, and for each $Q \in \mathcal{Q}^{(n+k)}$,

$$\sum_{D \in \beta \vee (T_\omega^n)^{-1}\gamma} \sup_{x \in D \cap Q(\omega)} e^{S_{n+k} f(\omega,x)} \leq \sum_{B \in \beta} \sup_{x \in B \cap Q(\omega)} e^{S_n f(\omega,x)} \sum_{C \in \gamma} \sup_{x \in C \cap Q(\vartheta^n\omega)} e^{S_k f(\vartheta^n\omega,x)},$$

which implies:

$$\log P_\Theta^f(\omega, n+k, \mathcal{R}, Q) \leq \log P_\Theta^f(\omega, n, \mathcal{R}, Q) + \log P_\Theta^f(\vartheta^n\omega, k, \mathcal{R}, Q),$$

and so $P_\Theta^f(\omega, n+k, \mathcal{R}, \mathcal{Q})$ is also subadditive.

By the subadditive ergodic theorem (see [14,15]) the following limit:

$$P_\Theta^f(\omega, \mathcal{R}, \mathcal{Q}) = \lim_{n\to\infty} \frac{1}{n} \log P_\Theta^f(\omega, n, \mathcal{R}, \mathcal{Q})$$

$\mathbb{P}$-a.s. exists and,

$$\pi_\Theta^f(\mathcal{R}, \mathcal{Q}) = \lim_{n\to\infty} \frac{1}{n} \int \log P_\Theta^f(\omega, n, \mathcal{R}, \mathcal{Q}) d\mathbb{P} = \int P_\Theta^f(\omega, \mathcal{R}, \mathcal{Q}) d\mathbb{P},$$

which will be called *relative topological conditional pressure of* Θ *of an open random cover* $\mathcal{R}$ *given a random cover* $\mathcal{Q}$. If $\mathcal{Q}$ is a trivial random cover, then $\pi_{\Theta}^{f}(\mathcal{R},\mathcal{Q})$ is called the *relative topological pressure* $\pi_{\Theta}^{f}(\mathcal{R})$ *of an open random cover* $\mathcal{R}$ *(under the action of* Θ*)*. Observe that $\pi_{\Theta}^{f}(\mathcal{R},\mathcal{Q}) \leq \pi_{\Theta}^{f}(\mathcal{R})$ for all $\mathcal{Q} \in \mathfrak{P}(\mathcal{E})$.

Notice that $\pi_{\Theta}^{f}(\mathcal{R},\mathcal{Q})$ is increasing in $\mathcal{R}$ in the sense of the refinement. There exists a limit (finite or infinite) over the directed set $\mathfrak{U}(\mathcal{E})$,

$$\pi_{\Theta}^{f}(\mathcal{Q}) = \lim_{\mathcal{R}\in\mathfrak{U}(\mathcal{E})} \pi_{\Theta}^{f}(\mathcal{R},\mathcal{Q}) = \sup_{\mathcal{R}\in\mathfrak{U}(\mathcal{E})} \pi_{\Theta}^{f}(\mathcal{R},\mathcal{Q}),$$

which will be called the *relative topological conditional pressure of* Θ *given a random cover* $\mathcal{Q}$. If $\mathcal{Q}$ is trivial, $\pi_{\Theta}^{f}(\mathcal{Q})$ will be abbreviated as $\pi_{\Theta}(f)$ and be called the *relative topological pressure of* Θ. Since $\pi_{\Theta}^{f}(\mathcal{Q})$ is decreasing in $\mathcal{Q}$, one can take the limit again:

$$\pi_{\Theta}^{*}(f) = \lim_{\mathcal{Q}\in\mathfrak{P}(\mathcal{E})} \pi_{\Theta}^{f}(\mathcal{Q}) = \inf_{\mathcal{Q}\in\mathfrak{P}(\mathcal{E})} \pi_{\Theta}^{f}(\mathcal{Q}),$$

which is called the *relative tail pressure of* Θ. It is clear that $\pi_{\Theta}^{*}(f) \leq \pi_{\Theta}(f)$.

Remark 1. *For each open cover* $\xi = \{A_1, \ldots, A_k\}$ *of the compact space* X, $\{(\Omega \times A_i) \cap \mathcal{E}\}_{i=1}^{k}$ *naturally form an open random cover of* $\mathcal{E}$. *In this case, the above definition of relative topological pressure reduces to that given in [10].*

Proposition 1. *Let* T *be a continuous bundle RDS on* $\mathcal{E}$, $\mathcal{Q}$ *be a random cover of* $\mathcal{E}$ *and* $f \in L^1_{\mathcal{E}}(\Omega, \mathcal{C}(X))$. *Then for each* $m \in \mathbb{N}$,

$$\pi_{\Theta^m}^{S_m f}(\mathcal{Q}^{(m)}) = m\pi_{\Theta}^{f}(\mathcal{Q}),$$

where $\mathcal{Q}^{(m)} = \bigvee_{i=0}^{m-1}(\Theta^i)^{-1}\mathcal{Q}$.

Proof. Let $\mathcal{R}$ be an open random cover of $\mathcal{E}$. Since,

$$\bigvee_{j=0}^{n-1} (\Theta^{mj})^{-1} \Big(\bigvee_{i=0}^{m-1} (\Theta^i)^{-1}\mathcal{R} \Big) = \bigvee_{i=0}^{nm-1} (\Theta^i)^{-1}\mathcal{R}$$

and $\sum_{j=0}^{n-1}(S_m f)(\Theta^m)^j(\omega,x) = S_{nm}f(\omega,x)$, then,

$$P_{\Theta^m}^{S_m f}(\omega, n, \mathcal{R}^{(m)}, \mathcal{Q}^{(m)}) = P_{\Theta}^{f}(\omega, nm, \mathcal{R}, \mathcal{Q}).$$

By the definition of the relative topological conditional pressureof open random cover $\mathcal{R}^{(m)}$ given $\mathcal{Q}^{(m)}$, under the action of Θ^m, we have:

$$\begin{aligned}
\pi_{\Theta^m}^{S_m f}(\mathcal{R}^{(m)}, \mathcal{Q}^{(m)}) &= \lim_{n\to\infty} \frac{1}{n} \int \log P_{\Theta^m}^{S_m f}(\omega, n, \mathcal{R}^{(m)}, \mathcal{Q}^{(m)}) d\mathbb{P} \\
&= \lim_{n\to\infty} \frac{1}{n} \int \log P_{\Theta}^{f}(\omega, nm, \mathcal{R}, \mathcal{Q}) d\mathbb{P} \\
&= \lim_{n\to\infty} m \frac{1}{nm} \int \log P_{\Theta}^{f}(\omega, nm, \mathcal{R}, \mathcal{Q}) d\mathbb{P} \\
&= m\pi_{\Theta}^{f}(\mathcal{R}, \mathcal{Q}).
\end{aligned}$$

Then,

$$m\pi_{\Theta}^{f}(\mathcal{Q}) = \sup_{\mathcal{R}} \pi_{\Theta^m}^{S_m f}(\mathcal{R}^{(m)}, \mathcal{Q}^{(m)}) \leq \pi_{\Theta^m}^{S_m f}(\mathcal{Q}^{(m)}),$$

where the supremum is taken over all open random covers $\mathcal{R}$ of $\mathcal{E}$.

Since $\mathcal{R} \prec \mathcal{R}^{(m)}$, then:

$$P_{\Theta^m}^{S_m f}(\omega, n, \mathcal{R}, \mathcal{Q}^{(m)}) \leq p_{\Theta^m}^{S_m f}(\omega, n, \mathcal{R}^{(m)}, \mathcal{Q}^{(m)}),$$

and so,

$$\pi_{\Theta^m}^{S_m f}(\mathcal{R}, \mathcal{Q}^{(m)}) \leq \pi_{\Theta^m}^{S_m f}(\mathcal{R}^{(m)}, \mathcal{Q}^{(m)}) = m\pi_{\Theta}^{f}(\mathcal{R}, \mathcal{Q}).$$

Thus, $\pi_{\Theta^m}^{S_m f}(\mathcal{Q}^{(m)}) \leq m\pi_{\Theta}^{f}(\mathcal{Q})$ and the result follows. $\square$

The relative tail pressure has the following power rule.

Proposition 2. *Let T be a continuous bundle RDS on $\mathcal{E}$ and $f \in L^1_{\mathcal{E}}(\Omega, \mathcal{C}(X))$. Then for each $m \in \mathbb{N}$, $\pi^*_{\Theta^m}(S_m f) = m\pi^*_{\Theta}(f)$.*

Proof. By Proposition 1,

$$\inf_{\mathcal{Q}} \pi_{\Theta^m}^{S_m f}(\mathcal{Q}^{(m)}) = \inf_{\mathcal{Q}} m\pi_{\Theta}^{f}(\mathcal{Q}) = m\pi^*_{\Theta}(f),$$

where the infimum is taken over all random covers of $\mathcal{E}$. Then, $\pi^*_{\Theta^m}(S_m f) \leq m\pi^*_{\Theta}(f)$.

Since $\mathcal{Q} \prec \mathcal{Q}^{(m)}$, then,

$$\pi_{\Theta^m}^{S_m f}(\mathcal{Q}) \geq \pi_{\Theta^m}^{S_m f}(\mathcal{Q}^{(m)}) = m\pi_{\Theta}^{f}(\mathcal{Q}).$$

By taking infimum on the inequality over all random covers of $\mathcal{E}$, one gets $\pi^*_{\Theta^m}(S_m f) \geq m\pi^*_{\Theta}(f)$ and the equality holds. $\square$

We need the following lemma which shows the basic connection between the relative entropy and relative tail pressure.

Lemma 2. *Let T be a continuous bundle RDS on $\mathcal{E}$ and $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$. Suppose that $\mathcal{R}$, $\mathcal{Q}$ are two finite measurable partitions of $\mathcal{E}$ and $f \in L^1_{\mathcal{E}}(\Omega, \mathcal{C}(X))$, then,*

$$H_\mu(\mathcal{R} \mid \mathfrak{Q} \vee \mathcal{F}_{\mathcal{E}}) + \int f d\mu \leq \int \max_{Q \in \mathcal{Q}} \log \sum_{R \in \mathcal{R}} e^{\alpha(R(\omega))} d\mathbb{P},$$

where $\alpha(R(\omega)) = \sup_{x \in R(\omega) \cap Q(\omega)} f(\omega, x)$ and $\mathfrak{Q}$ is the sub-σ-algebra generated by the partition $\mathcal{Q}$.

Proof. A simple calculation (see for instance [16] (Section 14.2)) shows that,

$$E(1_R \mid \mathfrak{Q} \vee \mathcal{F}_{\mathcal{E}}) = \sum_{Q \in \mathcal{Q}} 1_Q \frac{E(1_{R \cap Q} \mid \mathcal{F}_{\mathcal{E}})}{E(1_Q \mid \mathcal{F}_{\mathcal{E}})}.$$

Then,

$$H_\mu(\mathcal{R} \mid \mathfrak{Q} \vee \mathcal{F}_{\mathcal{E}}) = \int \sum_{R \in \mathcal{R}} \sum_{Q \in \mathcal{Q}} -E(1_{R \cap Q} \mid \mathcal{F}_{\mathcal{E}}) \log \frac{E(1_{R \cap Q} | \mathcal{F}_{\mathcal{E}})}{E(1_Q | \mathcal{F}_{\mathcal{E}})} d\mu.$$

Let $\alpha(R(\omega)) = \sup_{x \in R(\omega) \cap Q(\omega)} f(\omega, x)$. Notice that μ can disintegrate $d\mu(\omega, x) = d\mu_\omega(x) d\mathbb{P}(\omega)$, $E(1_{R \cap Q} \mid \mathcal{F}_{\mathcal{E}}) = \mu_\omega((R \cap Q)(\omega))$ and $E(1_Q \mid \mathcal{F}_{\mathcal{E}}) = \mu_\omega(Q(\omega))$ $\mathbb{P}-$a.s. Then,

$$\begin{aligned}
&H_\mu(\mathcal{R} \mid \mathfrak{Q} \vee \mathcal{F}_{\mathcal{E}}) + \int f d\mu \\
&\leq \int \sum_{R \in \mathcal{R}} \sum_{Q \in \mathcal{Q}} -\mu_\omega((R \cap Q)(\omega)) \log \frac{\mu_\omega((R \cap Q)(\omega))}{\mu_\omega(Q(\omega))} d\mathbb{P} + \int \sum_{Q \in \mathcal{Q}} \sum_{R \in \mathcal{R}} \alpha(R(\omega)) d\mathbb{P} \\
&= \int \sum_{Q \in \mathcal{Q}} \mu_\omega(Q(\omega)) \sum_{R \in \mathcal{R}} \frac{\mu_\omega((R \cap Q)(\omega))}{\mu_\omega(Q(\omega))} \Big(-\log \frac{\mu_\omega((R \cap Q)(\omega))}{\mu_\omega(Q(\omega))} + \alpha(R(\omega)) \Big) d\mathbb{P}
\end{aligned}$$

$$\leq \int \sum_{Q\in\mathcal{Q}} \mu_\omega(Q(\omega)) \log \sum_{R\in\mathcal{R}} e^{\alpha(R(\omega))} d\mathbb{P} \leq \int \max_{Q\in\mathcal{Q}} \log \sum_{R\in\mathcal{R}} e^{\alpha(R(\omega))} d\mathbb{P}. \qquad \square$$

4. Variational Principle for Relative Tail Pressure

We now take up the consideration of the relationship between the relative entropy and relative tail pressure on the measurable subset $\mathcal{H}$ of $\Omega \times Y \times X$ with respect to the product σ-algebra $\mathcal{F} \times \mathcal{C} \times \mathcal{B}$.

Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$. A partition $\mathcal{P}$ is called γ—contains a partition $\mathcal{Q}$ if there exists a partition $\mathcal{R} \preceq \mathcal{P}$ such that $\inf \sum_i \mu(R_i^* \triangle Q_i^*) < \gamma$, where the infimum is taken over all ordered partitions $\mathcal{R}^*$, $\mathcal{Q}^*$ obtained from $\mathcal{R}$ and $\mathcal{Q}$.

The following lemma comes essentially from the argument of Theorem 4.18 in [17] and Lemma 4.15 in [15]. We omit the proof.

Lemma 3. *Given $\epsilon > 0$ and $k \in \mathbb{N}$. There exists $\gamma = \gamma(\epsilon, k) > 0$, such that if the measurable partition $\mathcal{P}$ γ—contains $\mathcal{Q}$, where $\mathcal{Q}$ is a finite measurable partition with k elements, then $H_\mu(\mathcal{Q} \mid \mathcal{P}) < \epsilon$.*

We need the following result, which has appeared already at several places (see for instance [8,10]).

Lemma 4. *Let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a finite measurable partition of $\mathcal{H}$. Given $m \in \mathcal{P}_{\mathbb{P}}(\mathcal{H})$ satisfying $m(\partial R_i) = 0$ for each $1 \leq i \leq k$, where ∂ denotes the boundary and $m(\partial R) = \int m_\omega(\partial R(\omega)) d\mathbb{P}(\omega)$, then m is a upper semi-continuity point of the function $\mu \to H_\mu(\mathcal{R} \mid \mathcal{D}_\mathcal{H})$ defined on $\mathcal{P}_{\mathbb{P}}(\mathcal{H})$, i.e.,*

$$\limsup_{\mu\to m} H_\mu(\mathcal{R} \mid \mathcal{D}_\mathcal{H}) \leq H_m(\mathcal{R} \mid \mathcal{D}_\mathcal{H}).$$

Lemma 5. *Let $S \times T$ be the continuous bundle RDSs on $\mathcal{H}$ and $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{H})$. Suppose that $\mathcal{R} = \{R\}$, $\mathcal{Q} = \{Q\}$ are two finite measurable partitions of $\mathcal{E}$ and $f \in L^1_{\mathcal{E}}(\Omega, \mathcal{C}(X))$, then,*

$$H_\mu(\pi_\mathcal{E}^{-1}\mathcal{R} \mid \mathcal{D}_\mathcal{H}) + \int f \circ \pi_\mathcal{E} d\mu \leq H_\mu(\pi_\mathcal{E}^{-1}\mathcal{Q} \mid \mathcal{D}_\mathcal{H}) + \int \max_{Q\in\mathcal{Q}} \log \sum_{R\in\mathcal{R}} e^{\alpha(R(\omega))} d\mathbb{P}.$$

where $\alpha(R(\omega)) = \sup_{x\in R(\omega)\cap Q(\omega)} f(\omega, x)$.

Proof. Let $\mathfrak{Q}$ be the sub-σ-algebra generated by the partition $\mathcal{Q}$. Since $\mathcal{F}_\mathcal{H}$ is a sub-σ-algebra of $\mathcal{D}_\mathcal{H}$ and $\mathcal{F}_\mathcal{H} = \pi_\mathcal{E}^{-1}\mathcal{F}_\mathcal{E}$, then,

$$\begin{aligned}
& H_\mu(\pi_\mathcal{E}^{-1}\mathcal{R} \mid \mathcal{D}_\mathcal{H}) + \int f \circ \pi_\mathcal{E} d\mu \\
\leq & H_\mu(\pi_\mathcal{E}^{-1}\mathcal{Q} \mid \mathcal{D}_\mathcal{H}) + H_\mu(\pi_\mathcal{E}^{-1}\mathcal{R} \mid \pi_\mathcal{E}^{-1}\mathfrak{Q} \vee \mathcal{D}_\mathcal{H}) + \int f \circ \pi_\mathcal{E} d\mu \\
\leq & H_\mu(\pi_\mathcal{E}^{-1}\mathcal{Q} \mid \mathcal{D}_\mathcal{H}) + H_\mu(\pi_\mathcal{E}^{-1}\mathcal{R} \mid \pi_\mathcal{E}^{-1}\mathfrak{Q} \vee \mathcal{F}_\mathcal{H}) + \int f \circ \pi_\mathcal{E} d\mu \\
= & H_\mu(\pi_\mathcal{E}^{-1}\mathcal{Q} \mid \mathcal{D}_\mathcal{H}) + H_\mu(\pi_\mathcal{E}^{-1}\mathcal{R} \mid \pi_\mathcal{E}^{-1}\mathfrak{Q} \vee \pi_\mathcal{E}^{-1}\mathcal{F}_\mathcal{E}) + \int f \circ \pi_\mathcal{E} d\mu \\
\leq & H_\mu(\pi_\mathcal{E}^{-1}\mathcal{Q} \mid \mathcal{D}_\mathcal{H}) + H_{\pi_\mathcal{E}\mu}(\mathcal{R} \mid \mathfrak{Q} \vee \mathcal{F}_\mathcal{E}) + \int f d\pi_\mathcal{E}\mu
\end{aligned}$$

Let $\nu = \pi_\mathcal{E}\mu$, then $\nu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$. By Lemma 2, one has,

$$H_\mu(\pi_\mathcal{E}^{-1}\mathcal{R} \mid \mathcal{D}_\mathcal{H}) + \int f \circ \pi_\mathcal{E} d\mu \leq H_\mu(\pi_\mathcal{E}^{-1}\mathcal{Q} \mid \mathcal{D}_\mathcal{H}) + \int \max_{Q\in\mathcal{Q}} \log \sum_{R\in\mathcal{R}} e^{\alpha(R(\omega))} d\mathbb{P}.$$

$\square$

Proposition 3. *Let $S \times T$ be the continuous bundle RDS on $\mathcal{H}$, $\mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{H})$ and $f \in L^1_{\mathcal{E}}(\Omega, \mathcal{C}(X))$. Then for each finite measurable partition $\mathcal{Q}$ of $\mathcal{E}$,*

$$h_\mu(\Gamma \mid \mathcal{D}_{\mathcal{H}}) + \int f \circ \pi_{\mathcal{E}} d\mu \leq h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{Q} \mid \mathcal{D}_{\mathcal{H}}) + \pi^f_\Theta(\mathcal{Q}).$$

Proof. Let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a measurable partition of $\mathcal{E}$ and $\nu = \pi_{\mathcal{E}}\mu$.

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ can be viewed as a Borel subset of the unit interval $[0,1]$. Then $\nu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ is also a probability measure on the compact space $[0,1] \times X$ with the marginal $\mathbb{P}$ on $[0,1]$. Let $\epsilon > 0$ and $\gamma > 0$ as desired in Lemma 3. Since ν is regular, there exists a compact subset $P_i \subset R_i$ with $\nu(R_i \setminus P_i) < \frac{\gamma}{2k}$ for each $1 \leq i \leq k$. Denote by $P_0 = \mathcal{E} \setminus \bigcup_{i=1}^k P_i$. Then $\mathcal{P} = \{P_0, P_1, \ldots, P_k\}$ is a measurable partition of $\mathcal{E}$ and $\sum_{i=1}^k \nu(R_i \setminus P_i) + \nu(P_0) < \frac{\gamma}{2k} \cdot k + \frac{\gamma}{2} = \gamma$. By Lemma 3, $H_\nu(\mathcal{R} \mid \mathcal{P}) < \epsilon$.

Let $\tau(\omega) = \min_{1 \leq i \neq j \leq k} d(P_i(\omega), P_j(\omega))$. Choose $\delta(\omega) > 0$ with $\delta(\omega) < \frac{\tau(\omega)}{2}$ such that $d(x,y) < \delta(\omega)$ implies $\mid f(\omega,x) - f(\omega,y) \mid < \epsilon$. Fix $n \in \mathbb{N}$. Since $\mathcal{E}_\omega$ is compact, for each $Q(\omega) \in \mathcal{Q}^{(n)}(\omega)$, there exists a finite $(n, \delta(\omega))-$separated subset $E_Q(\omega)$ in $Q(\omega)$, which fails to be $(n, \delta(\omega))-$separated when any point is added. Recall that $\mathcal{Q}^{(n)}(\omega) = \bigvee_{i=0}^{n-1} (T^i_\omega)^{-1}\mathcal{Q}(\vartheta^i\omega)$.

For each $P(\omega) \in \mathcal{P}^{(n)}(\omega)$, let $\alpha(P(\omega)) = \sup_{x \in \overline{P(\omega) \cap Q(\omega)}} S_n f(\omega, x)$. Choose some point $x \in \overline{P(\omega) \cap Q(\omega)}$ with $S_n f(\omega, x) = \alpha(P(\omega))$, and an element $y(P(\omega)) \in E_Q(\omega)$ with $d^n_\omega(x, y(P(\omega))) \leq \delta(\omega)$, where d^n_ω is the Bowen metric defined as $d^n_\omega(x,y) = \max_{0 \leq i < n} d(T^i_\omega x, T^i_\omega y)$ for $x, y \in X$. Then, $\alpha(P(\omega)) \leq S_n f(\omega, y(P(\omega))) + n\epsilon$. Since each ball of radius $\delta(\omega)$ meets at most the closure of two members of $\mathcal{P}(\omega)$, then for each $y \in E_Q(\omega)$, the cardinality of the set $\{P(\omega) \in \mathcal{P}^{(n)}(\omega) \mid y(P(\omega)) = y\}$ cannot exceed 2^n. Therefore,

$$\sum_{P \in \mathcal{P}^{(n)}} e^{\alpha(P(\omega)) - n\epsilon} \leq \sum_{P \in \mathcal{P}^{(n)}} e^{S_n f(\omega, y(P(\omega)))} \leq 2^n \sum_{y \in E_Q(\omega)} e^{S_n f(\omega, y)}$$

and so,

$$\log \sum_{P \in \mathcal{P}^{(n)}} e^{\alpha(P(\omega))} \leq \log \sum_{y \in E_Q(\omega)} e^{S_n f(\omega,y)} + n \log 2 + n\epsilon.$$

Hence by Lemma 5, one has,

$$\begin{aligned}
&\frac{1}{n} H_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{P}^{(n)} \mid \mathcal{D}_{\mathcal{H}}) + \int f \circ \pi_{\mathcal{E}} d\mu \\
=&\frac{1}{n} H_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{P}^{(n)} \mid \mathcal{D}_{\mathcal{H}}) + \frac{1}{n} \int S_n f \circ \pi_{\mathcal{E}} d\mu \\
\leq&\frac{1}{n} H_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{Q}^{(n)} \mid \mathcal{D}_{\mathcal{H}}) + \frac{1}{n} \int \max_{Q \in \mathcal{Q}^{(n)}} \log \sum_{y \in E_Q(\omega)} e^{S_n f(\omega,y)} d\mathbb{P} + \log 2 + \epsilon.
\end{aligned} \tag{1}$$

Let $\mathcal{U} = \{U\}$ be an open random cover with $\operatorname{diam}(U(\omega)) < \delta(\omega)$, then each $U(\omega) \in \mathcal{U}^{(n)}(\omega)$ contains at most one element of $E_Q(\omega)$. Thus,

$$\max_{Q \in \mathcal{Q}^{(n)}} \sum_{y \in E_Q(\omega)} e^{S_n f(\omega,y)} \leq P^f_\Theta(\omega, n, \mathcal{U}, \mathcal{Q}),$$

and by the inequality (1), one has,

$$\begin{aligned}
&\frac{1}{n} H_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{P}^{(n)} \mid \mathcal{D}_{\mathcal{H}}) + \int f \circ \pi_{\mathcal{E}} d\mu \\
\leq&\frac{1}{n} H_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{Q}^{(n)} \mid \mathcal{D}_{\mathcal{H}}) + \frac{1}{n} \int \log P^f_\Theta(\omega, n, \mathcal{U}, \mathcal{Q}) d\mathbb{P} + \log 2 + \epsilon.
\end{aligned}$$

Since $\Theta\pi_{\mathcal{E}} = \pi_{\mathcal{E}}\Gamma$, then,

$$\begin{aligned}
&h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{P} \mid \mathcal{D}_{\mathcal{H}}) + \int f \circ \pi_{\mathcal{E}} d\mu \\
\leq &h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{Q} \mid \mathcal{D}_{\mathcal{H}}) + \pi_{\Theta}^{f}(\mathcal{U}, \mathcal{Q}) + \log 2 + \epsilon \\
\leq &h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{Q} \mid \mathcal{D}_{\mathcal{H}}) + \pi_{\Theta}^{f}(\mathcal{Q}) + \log 2 + \epsilon.
\end{aligned} \tag{2}$$

Since,

$$\begin{aligned}
h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{R} \mid \mathcal{D}_{\mathcal{H}}) &\leq h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{P} \mid \mathcal{D}_{\mathcal{H}}) + h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{R} \mid \pi_{\mathcal{E}}^{-1}\mathcal{P}) \\
&\leq h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{P} \mid \mathcal{D}_{\mathcal{H}}) + h_\nu(\mathcal{R} \mid \mathcal{P}),
\end{aligned}$$

then by the inequality (2), one has,

$$h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{R} \mid \mathcal{D}_{\mathcal{H}}) + \int f \circ \pi_{\mathcal{E}} d\mu \leq h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{Q} \mid \mathcal{D}_{\mathcal{H}}) + \pi_{\Theta}^{f}(\mathcal{Q}) + \log 2 + 2\epsilon.$$

Let $\mathcal{R}_1 \prec \cdots \prec \mathcal{R}_n \prec \cdots$ be an increasing sequence of finite measurable partitions with $\bigvee_{i=1}^{\infty} \mathcal{R}_n = \mathcal{A}$, by Lemma 1.6 in [14], one has,

$$h_\mu(\Gamma \mid \mathcal{D}_{\mathcal{H}}) + \int f \circ \pi_{\mathcal{E}} d\mu \leq h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{Q} \mid \mathcal{D}_{\mathcal{H}}) + \pi_{\Theta}^{f}(\mathcal{Q}) + \log 2 + 2\epsilon. \tag{3}$$

Since,

$$\bigvee_{j=0}^{n-1} (\Gamma^{mj})^{-1} (\bigvee_{i=0}^{m-1} (\Gamma^i)^{-1} \pi_{\mathcal{E}}^{-1}\mathcal{Q}) = \bigvee_{i=0}^{nm-1} (\Gamma^i)^{-1} \pi_{\mathcal{E}}^{-1}\mathcal{Q},$$

it is not hard to see that,

$$h_{\mu,\Gamma^m}(\bigvee_{i=0}^{m-1} (\Gamma^i)^{-1} \pi_{\mathcal{E}}^{-1}\mathcal{Q} \mid \mathcal{D}_{\mathcal{H}}) = m h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{Q} \mid \mathcal{D}_{\mathcal{H}}), \tag{4}$$

where $h_{\mu,\Gamma^m}(\xi \mid \mathcal{D}_{\mathcal{H}})$ denotes the relative entropy of Γ^m with respect to the partition ξ.

By Lemma 1.4 in [14], for each $m \in \mathbb{N}$,

$$h_\mu(\Gamma^m \mid \mathcal{D}_{\mathcal{H}}) = m h_\mu(\Gamma \mid \mathcal{D}_{\mathcal{H}}), \tag{5}$$

where $h_\mu(\Gamma^m \mid \mathcal{D}_{\mathcal{H}})$ is the relative entropy of Γ^m.

By the equality (4), (5) and Proposition 1, and applying Γ^m, Θ^m, $\mathcal{Q}^{(m)}$ and $S_m f$ to the inequality (3), dividing by m and letting m go to infinity, one has:

$$h_\mu(\Gamma \mid \mathcal{D}_{\mathcal{H}}) + \int f \circ \pi_{\mathcal{E}} d\mu \leq h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{Q} \mid \mathcal{D}_{\mathcal{H}}) + \pi_{\Theta}^{f}(\mathcal{Q}),$$

and we complete the proof. □

Now, we can give the variational inequality between defect of upper semi-continuity of the relative entropy function on invariant measures and the relative tail pressure.

Theorem 1. *Let $S \times T$ be the continuous bundle RDS on $\mathcal{H}$, $m \in \mathcal{I}_{\mathbb{P}}(\mathcal{H})$ and $f \in L^1_{\mathcal{E}}(\Omega, C(X))$. Then $h_m^*(\Gamma \mid \mathcal{D}_{\mathcal{H}}) + \int f \circ \pi_{\mathcal{E}} dm \leq \pi_{\Theta}^{*}(f)$.*

Proof. Let $\mathcal{Q}$ be a finite random cover of $\mathcal{E}$ and $\nu = \pi_{\mathcal{E}} m$. Choose a finite measurable partition $\mathcal{R}$ of $\mathcal{E}$ with $\mathcal{Q} \prec \mathcal{R}$ and $\nu(\partial R) = 0$ for each $R \in \mathcal{R}$. By Proposition 3 and $\pi_{\mathcal{E}}\Gamma = \Theta\pi_{\mathcal{E}}$, for each $\mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{H})$ and $n \in \mathbb{N}$,

$$\begin{aligned}
h_\mu(\Gamma \mid \mathcal{D}_{\mathcal{H}}) + \int f \circ \pi_{\mathcal{E}} d\mu &\leq h_\mu(\pi_{\mathcal{E}}^{-1}\mathcal{R} \mid \mathcal{D}_{\mathcal{H}}) + \pi_{\Theta}^{f}(\mathcal{R}) \\
&\leq \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} (\Gamma^i)^{-1} \pi_{\mathcal{E}}^{-1}\mathcal{R} \mid \mathcal{D}_{\mathcal{H}}) + \pi_{\Theta}^{f}(\mathcal{Q})
\end{aligned}$$

Then by Lemma 4,

$$\begin{aligned}
&\limsup_{\mu\to m} h_\mu(\Gamma \mid \mathcal{D}_{\mathcal{H}}) + \int f\circ \pi_{\mathcal{E}} dm \\
&\le \limsup_{\mu\to m} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} (\Gamma^i)^{-1}\pi_{\mathcal{E}}^{-1}\mathcal{R} \mid \mathcal{D}_{\mathcal{H}}) + \pi_\Theta^f(\mathcal{Q}) \\
&\le \frac{1}{n} H_m(\bigvee_{i=0}^{n-1} (\Gamma^i)^{-1}\pi_{\mathcal{E}}^{-1}\mathcal{R} \mid \mathcal{D}_{\mathcal{H}}) + \pi_\Theta^f(\mathcal{Q}).
\end{aligned}$$

Thus,

$$\limsup_{\mu\to m} h_\mu(\Gamma \mid \mathcal{D}_{\mathcal{H}}) + \int f\circ \pi_{\mathcal{E}} dm \le h_m(\Gamma \mid \mathcal{D}_{\mathcal{H}}) + \pi_\Theta^f(\mathcal{Q}).$$

Since the partition $\mathcal{Q}$ is arbitrary, then $h_m^*(\Gamma \mid \mathcal{D}_{\mathcal{H}}) + \int f\circ\pi_{\mathcal{E}} dm \le \pi_\Theta^*(f)$. □

Next, we are concerned with the variational principle relating the relative entropy of $\mathcal{E}^{(2)}$ and the relative tail pressure of Θ. Recall that $\mathcal{E}^{(2)} = \{(\omega,x,y) : x,y\in\mathcal{E}_\omega\}$ is a measurable subset of $\Omega\times X^2$ with respect to the product $\sigma-$algebra $\mathcal{F}\times\mathcal{B}^2$ and $\mathcal{A}_{\mathcal{E}^{(2)}} = \{(A\times X)\cap\mathcal{E}^{(2)} : A\in\mathcal{F}\times\mathcal{B}\}$. The skew product transformation $\Theta^{(2)} : \mathcal{E}^{(2)} \to \mathcal{E}^{(2)}$ is given by $\Theta^{(2)}(\omega,x,y) = (\vartheta\omega, T_\omega x, T_\omega y)$. Let $\mathcal{E}_1,\mathcal{E}_2$ be two copies of $\mathcal{E}$, i.e., $\mathcal{E}_1=\mathcal{E}_2=\mathcal{E}$, and $\pi_{\mathcal{E}_i}$ be the natural projection from $\mathcal{E}^{(2)}$ to $\mathcal{E}_i$ with $\pi_{\mathcal{E}_i}(\omega,x_1,x_2) = (\omega,x_i)$, i = 1, 2.

The following important proposition relating the relative tail pressure and the relative entropy is necessary for the proof of the variational principle.

Proposition 4. *Let T be a continuous bundle RDS on $\mathcal{E}$, $\mathcal{Q}=\{Q_1,\dots,Q_k\}$ be an open random cover of $\mathcal{E}$ and $f\in L^1_{\mathcal{E}}(\Omega,\mathcal{C}(X))$. There exists a probability measure $\mu_{\mathcal{Q}}\in\mathcal{I}_{\mathbb{P}}(\mathcal{E}^{(2)})$ such that,*

$h_{\mu_{\mathcal{Q}}}(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f\circ\pi_{\mathcal{E}} d\mu_{\mathcal{Q}} \ge \pi_\Theta^f(\mathcal{Q}) - \frac{1}{k}$,
$\mu_{\mathcal{Q}}$ is supported on the set $\bigcup_{j=1}^{k} \{(\omega,x,y)\in\mathcal{E}^{(2)} : x,y\in\overline{Q_j}(\omega)\}$.

Proof. Choose an open random cover $\mathcal{P}=\{P_1,\dots,P_l\}$ of $\mathcal{E}$ with $\mathcal{P}(\omega)=\{P_1(\omega),\dots,P_l(\omega)\}$ such that, $\pi_\Theta^f(\mathcal{P},\mathcal{Q}) \ge \pi_\Theta^f(\mathcal{Q}) - \frac{1}{k}$. Recall that $\mathfrak{U}(\mathcal{E})$ is the collection of all open random covers on $\mathcal{E}$, $\mathcal{Q}^{(n)} = \bigvee_{i=0}^{n-1}(\Theta^i)^{-1}\mathcal{Q}$ and $\mathcal{Q}^{(n)}(\omega) = \bigvee_{i=0}^{n-1}(T_\omega^i)^{-1}\mathcal{Q}(\vartheta^i\omega)$

Let $n\in\mathbb{N}$ and $\omega\in\Omega$. Choose one element $Q(\omega)\in\mathcal{Q}^{(n)}(\omega)$ with $P_\Theta^f(\omega,n,\mathcal{P},Q) = P_\Theta^f(\omega,n,\mathcal{P},\mathcal{Q})$, and a point $x\in Q(\omega)$. Since $\mathcal{P}$ is an open random cover of $\mathcal{E}$, by the compactness of $\mathcal{E}_\omega$, there exists a Lebesgue number $\delta(\omega)$ for the open cover $\{P_1(\omega),\dots,P_l(\omega)\}$ and a maximal $(n,\delta)-$separated subset $E_n(\omega)$ in $Q(\omega)$ such that,

$$Q(\omega) \subset \bigcup_{y\in E_n(\omega)} B_y(\omega,n,\delta),$$

where $B_y(\omega,n,\delta)$ denotes the open ball in $\mathcal{E}_\omega$ center at y of radius 1 with respect to the Bowen metric $d^\omega_{\delta,n}(x,y) = \max_{0\le k<n}\{d(T^k_\omega x, T^k_\omega y)(\delta(\vartheta^k\omega))^{-1}\}$ for each $x,y\in\mathcal{E}_\omega$, *i.e.*, $B_y(\omega,n,\delta(\omega)) = \bigcap_{i=0}^{n-1}(T^i_\omega)^{-1}B(T^i_\omega y, \delta(\vartheta^i\omega))$. Let,

$$\tau_\delta(\omega) = \sup\{\mid f(\omega,x) - f(\omega,y)\mid : d(x,y) < \delta(\omega)\}.$$

Notice that for each $0\le i\le n-1$, the open ball $B(T^i_\omega y, \delta(\vartheta^i\omega))$ is contained in some element of $\mathcal{P}(\vartheta^i\omega)$, then $B_y(\omega,n,\delta)$ must be contained in some element of $\mathcal{P}^{(n)}(\omega)$. This means that,

$$\sum_{P\in\mathcal{P}} \sup_{x\in P(\omega)\cap Q(\omega)} \exp(S_n f(\omega,x) - \sum_{i=0}^{n-1}\tau_\delta(\vartheta^i\omega)) \le \sum_{y\in E_n(\omega)} \exp S_n f(\omega,y),$$

and so,

$$P_{\Theta}^{f}(\omega,n,\mathcal{P},\mathcal{Q})\cdot\exp(-\sum_{i=0}^{n-1}\tau_\delta(\vartheta^i\omega))\le\sum_{y\in E_n(\omega)}\exp S_nf(\omega,y).$$

Consider the probability measures $\sigma^{(n)}$ of $\mathcal{E}^{(2)}$ via their disintegrations:

$$\sigma_\omega^{(n)}=\frac{\sum_{z\in E_n(\omega)}\exp(S_nf\circ\pi_{\mathcal{E}_2}(\omega,x,z))\delta_{(\omega,x,z)}}{\sum_{y\in E_n(\omega)}\exp(S_nf\circ\pi_{\mathcal{E}_2}(\omega,x,y))}$$

so that $d\sigma^{(n)}(\omega,x,y)=d\sigma_\omega^{(n)}(x,y)d\mathbb{P}(\omega)$, and let,

$$\mu^{(n)}=\frac{1}{n}\sum_{i=0}^{n-1}(\Theta^{(2)})^i\sigma^{(n)},$$

By the Krylov–Bogolyubov procedure for continuous RDS (see [18] (Theorem 1.5.8) or [10] (Lemma 2.1 (i))), one can choose a subsequence $\{n_j\}$ such that $\mu^{(n_j)}$ convergence to some probability measure is $\mu_{\mathcal{Q}}\in\mathcal{I}_{\mathbb{P}}(\mathcal{E}^{(2)})$. Next we will verify that the measure $\mu_{\mathcal{Q}}$ satisfies (i) and (ii).

Let $\nu=\pi_{\mathcal{E}_2}\mu_{\mathcal{Q}}$. Choose a finite measurable partition $\mathcal{R}=\{R_1,\dots,R_q\}$ of $\mathcal{E}$ with $\operatorname{diam}R_i(\omega)<\delta(\omega)$ for each ω and $\nu(\partial R_i)=0$, $0\le i\le q$, in the sense of $\nu(\partial R)=\int\nu_\omega(\partial R(\omega))d\mathbb{P}(\omega)$, where ∂ denotes the boundary. Set $\xi^{(n)}=\bigvee_{i=0}^{n-1}(\Theta^{(2)})^{-i}\pi_{\mathcal{E}_2}^{-1}\mathcal{R}$. Since $\pi_{\mathcal{E}_2}\Theta^{(2)}=\Theta\pi_{\mathcal{E}_2}$, then $\xi^{(n)}=\pi_{\mathcal{E}_2}^{-1}\bigvee_{i=0}^{n-1}\Theta^{-i}\mathcal{R}=\pi_{\mathcal{E}_2}^{-1}\mathcal{R}^{(n)}$. Denote by $\xi^{(n)}=\{D\}$. For each ω, let $\pi_{X_1}^{-1}\mathcal{B}(\omega)=\{(B\times X_2)\cap\mathcal{E}_\omega^{(2)}:B\in\mathcal{B}\}$, where X_1,X_2 are two copies of the space X and π_{X_1} is the natural projection from the product space $X_1\times X_2$ to the space X_1. We abbreviate it as $\pi_{X_1}^{-1}\mathcal{B}$ for convenience.

Since each element of $\mathcal{R}^{(n)}(\omega)$ contains at most one element of $E_n(\omega)$, one has,

$$E(1_{D(\omega)}\mid\pi_{X_1}^{-1}\mathcal{B})(x,y)=\sigma_\omega^{(n)}(D(\omega)).\tag{6}$$

Then,

$$\begin{aligned}
&H_{\sigma_\omega^{(n)}}(\xi^{(n)}(\omega))+\int S_nf\circ\pi_{\mathcal{E}_2}d\sigma_\omega^{(n)}\\
&=\int\sum_{D(\omega)\in\xi^{(n)}(\omega)}-E(1_{D(\omega)}\mid\pi_{X_1}\mathcal{B})\log E(1_{D(\omega)}\mid\pi_{X_1}\mathcal{B})d\sigma_\omega^{(n)}+\int S_nf\circ\pi_{\mathcal{E}_2}d\sigma_\omega^{(n)}\\
&=\sum_{D(\omega)\in\xi^{(n)}(\omega)}-\sigma_\omega^{(n)}(D(\omega))\log\sigma_\omega^{(n)}(D(\omega))+\sum_{D(\omega)\in\xi^{(n)}(\omega)}\int_{D(\omega)}S_nf\circ\pi_{\mathcal{E}_2}d\sigma_\omega^{(n)}\\
&=\log\sum_{y\in E_n(\omega)}\exp S_nf\circ\pi_{\mathcal{E}_2}(\omega,x,y)\\
&=\log\sum_{y\in E_n(\omega)}\exp S_nf(\omega,y).
\end{aligned}$$

Since for each $G\in\mathcal{A}_{\mathcal{E}^{(2)}}$,

$$\begin{aligned}
\int_G E(1_D\mid\mathcal{A}_{\mathcal{E}^{(2)}})d\sigma^{(n)}&=\int 1_G\times E(1_D\mid\mathcal{A}_{\mathcal{E}^{(2)}})d\sigma^{(n)}\\
&=\int E(1_G\times 1_D\mid\mathcal{A}_{\mathcal{E}^{(2)}})d\sigma^{(n)}\\
&=\int 1_{G\cap D}(\omega,x,y)d\sigma^{(n)}\\
&=\iint 1_{(G\cap D)(\omega)}(x,y)d\sigma_\omega^{(n)}d\mathbb{P}\\
&=\iint E(1_{(G\cap D)(\omega)}\mid\pi_{X_1}\mathcal{B})(x,y)d\sigma_\omega^{(n)}d\mathbb{P}
\end{aligned}$$

$$
\begin{aligned}
&= \iint 1_{G(\omega)}(x,y) \times E(1_{D(\omega)} \mid \pi_{X_1}\mathcal{B})(x,y)) d\sigma_\omega^{(n)} d\mathbb{P} \\
&= \iint 1_G(\omega,x,y) E(1_{D(\omega)} \mid \pi_{X_1}\mathcal{B})(x,y)) d\sigma_\omega^{(n)} d\mathbb{P} \\
&= \iint_G E(1_{D(\omega)} \mid \pi_{X_1}\mathcal{B})(x,y)) d\sigma_\omega^{(n)} d\mathbb{P} \\
&= \int_G E(1_{D(\omega)} \mid \pi_{X_1}\mathcal{B}) d\sigma^{(n)}.
\end{aligned}
$$

Then,

$$
E(1_D \mid \mathcal{A}_{\mathcal{E}^{(2)}})(\omega,x,y) = E(1_{D(\omega)} \mid \pi_{X_1}\mathcal{B})(x,y) \ \mathbb{P}-a.s.
$$

Therefore,

$$
\begin{aligned}
& H_{\sigma^{(n)}}(\xi^{(n)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int S_n f \circ \pi_{\mathcal{E}_2} d\sigma^{(n)} \\
&= \int \sum_{D \in \xi^{(n)}} -E(1_D \mid \mathcal{A}_{\mathcal{E}^{(2)}}) \log E(1_D \mid \mathcal{A}_{\mathcal{E}^{(2)}}) d\sigma^{(n)} + \int S_n f \circ \pi_{\mathcal{E}_2} d\sigma^{(n)} \\
&= \iint \sum_{D(\omega) \in \xi^{(n)}(\omega)} -E(1_{D(\omega)} \mid \pi_{X_1}\mathcal{B}) \log E(1_{D(\omega)} \mid \pi_{X_1}\mathcal{B}) d\sigma_\omega^{(n)} d\mathbb{P} \\
&\qquad + \iint S_n f \circ \pi_{\mathcal{E}_2} d\sigma_\omega^{(n)} d\mathbb{P} \\
&= \int \big(H_{\sigma_\omega^{(n)}}(\xi^{(n)}(\omega)) + \int S_n f \circ \pi_{\mathcal{E}_2} d\sigma_\omega^{(n)}\big) d\mathbb{P} \\
&= \int \log \sum_{y \in E_n(\omega)} \exp S_n f(\omega,y) d\mathbb{P} \geq \int \log P_\Theta^f(\omega,n,\mathcal{P},\mathcal{Q}) - \sum_{i=0}^{n-1} \tau_\delta(\vartheta^i\omega) d\mathbb{P}.
\end{aligned} \tag{7}
$$

For $0 \leq j < m < n$, one can cut the segment $(0, n-1)$ into disjoint union of $[\frac{n}{m}] - 2$ segments $(j, j+m-1), \ldots, (j+km, j+(k+1)m-1), \ldots$ and less than $3m$ other natural numbers. Then,

$$
\begin{aligned}
& H_{\sigma^{(n)}}(\xi^{(n)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int S_n f \circ \pi_{\mathcal{E}_2} d\sigma^{(n)} \\
&\leq \sum_{k=0}^{[\frac{n}{m}]-2} H_{\sigma^{(n)}}\Big(\bigvee_{i=j+km}^{j+(k+1)m-1} (\Theta^{(2)})^{-i} \pi_{\mathcal{E}_2}^{-1}\mathcal{R} \mid \mathcal{A}_{\mathcal{E}^{(2)}}\Big) + \int S_n f \circ \pi_{\mathcal{E}_2} d\sigma^{(n)} + 3m \log q \\
&\leq \sum_{k=0}^{[\frac{n}{m}]-2} H_{(\Theta^{(2)})^{j+km}\sigma^{(n)}}(\xi^{(m)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int S_n f \circ \pi_{\mathcal{E}_2} d\sigma^{(n)} + 3m \log q.
\end{aligned}
$$

By summing over all j, $0 \leq j < m$ and considering the concavity of the entropy function $H_{(\cdot)}$, one has,

$$
\begin{aligned}
& m H_{\sigma^{(n)}}(\xi^{(n)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + m \int S_n f \circ \pi_{\mathcal{E}_2} d\sigma^{(n)} \\
&\leq \sum_{k=0}^{n-1} H_{(\Theta^{(2)})^k\sigma^{(n)}}(\xi^{(m)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + m \int S_n f \circ \pi_{\mathcal{E}_2} d\sigma^{(n)} + 3m^2 \log q \\
&\leq n H_{\mu^{(n)}}(\xi^{(m)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + m \int S_n f \circ \pi_{\mathcal{E}_2} d\sigma^{(n)} + 3m^2 \log q.
\end{aligned}
$$

Then, by inequality (7),

$$
\begin{aligned}
& \frac{1}{m} H_{\mu^{(n)}}(\xi^{(m)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}_2} d\mu^{(n)} \\
&\geq \frac{1}{n} \int \log P_\Theta^f(\omega,n,\mathcal{P},\mathcal{Q}) - \sum_{i=0}^{n-1} \tau_\delta(\vartheta^i\omega) d\mathbb{P} - \frac{3m}{n} \log q.
\end{aligned}
$$

Replacing the sequence $\{n\}$ by the above selected subsequence $\{n_j\}$, letting $j \to \infty$ and $\delta \to 0$, by Lemma 4, one has,

$$\frac{1}{m} H_{\mu_{\mathcal{Q}}}(\xi^{(m)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}_2} d\mu_{\mathcal{Q}} \geq \pi_{\Theta}^{f}(\mathcal{P}, \mathcal{Q}) \geq \pi_{\Theta}^{f}(\mathcal{Q}) - \frac{1}{k}.$$

By letting $m \to \infty$, one gets,

$$h_{\mu_{\mathcal{Q}}}(\pi_{\mathcal{E}_2}^{-1}\mathcal{R} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}_2} d\mu_{\mathcal{Q}} \geq \pi_{\Theta}^{f}(\mathcal{Q}) - \frac{1}{k}.$$

Let $\mathcal{R}_1 \prec \cdots \prec \mathcal{R}_n \prec \cdots$ be an increasing sequence of finite measurable partitions with $\bigvee_{i=1}^{\infty} = \mathcal{A}$, by Lemma l.6 in [14] one has,

$$h_{\mu_{\mathcal{Q}}}(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}_2} d\mu_{\mathcal{Q}} \geq \pi_{\Theta}^{f}(\mathcal{Q}) - \frac{1}{k},$$

which shows that the measure $\mu_{\mathcal{Q}}$ satisfies property (i).

For the other part of this proposition, let $n \in \mathbb{N}$. Recall that $Q \in \mathcal{Q}^{(n)}$ and notice that $\mathcal{Q}^{(n)} \succ (\Theta^{j})^{-1}\mathcal{Q}$ for all $0 \leq j < n$. Let $Q^{(2)} = \{(\omega, x, y) \in \mathcal{E}^{(2)} : x, y \in Q(\omega)\}$ and $Q_i^{(2)} = \{(\omega, x, y) \in \mathcal{E}^{(2)} : x, y \in Q_i(\omega)\}$, $1 \leq i \leq k$. All of them are the measurable subsets of $\mathcal{E}^{(2)}$ with the product $\sigma-$algebra $\mathcal{F} \times \mathcal{B}^2$, and $Q^{(2)}$ is contained in $(\Theta^{(2)})^{-j}Q_i^{(2)}$ for some $1 \leq i \leq k$ and $0 \leq j < n$. It follows from the construction of $\mu^{(n)}$ that,

$$\begin{aligned}
\mu^{(n)}(\bigcup_{i=1}^{k} Q_i^{(2)}) &= \frac{1}{n}\sum_{j=0}^{n-1} \sigma^{(n)}((\Theta^{(2)})^{-j}(\bigcup_{i=1}^{k} Q_i^{(2)})) \\
&\geq \frac{1}{n}\sum_{j=0}^{n-1} \sigma^{(n)}(Q^{(2)}) = \sigma^{(n)}(Q^{(2)}) = 1.
\end{aligned}$$

Then,

$$\mu^{(n)}(\bigcup_{i=1}^{k} \{(\omega, x, y) : x, y \in \overline{Q_i}(\omega)\} = 1).$$

Therefore, the probability measure $\mu_{\mathcal{Q}}$ satisfies the property (ii) and we complete the proof. □

Proposition 5. *Let T be a continuous bundle RDS on $\mathcal{E}$ and $f \in Ł_{\mathcal{E}}^1(\Omega, \mathcal{C}(X))$. There exists a probability measure $m \in \mathcal{I}_{\mathbb{P}}(\mathcal{E}^{(2)})$, which is supported on $\{(\omega, x, x) \in \mathcal{E}^{(2)} : x \in \mathcal{E}_\omega\}$, and satisfies $h_m^*(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}} dm = \pi_{\Theta}^*(f)$.*

Proof. Let $\mathcal{Q}_1 \prec \cdots \prec \mathcal{Q}_n \prec \cdots$ be an increasing sequence of open random covers of $\mathcal{E}$. Denote by $\mathcal{Q}_n = \{Q_j^{(n)}\}_{j=1}^{k_n}$. By Property 4, for each $n \in \mathbb{N}$, there exists a probability measure $\mu_n \in \mathcal{I}_{\mathbb{P}}(\mathcal{E}^{(2)})$ such that

$$h_{\mu_n}(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}} d\mu_n \geq \pi_{\Theta}^{f}(\mathcal{Q}_n) - \frac{1}{k_n}$$

and μ_n is supported on $\bigcup_{j=1}^{k_n}\{(\omega, x, y) : x, y \in \overline{Q_j^{(n)}}(\omega)\}$. Let m be some limit point of the sequence of μ_n, then $m \in \mathcal{I}_{\mathbb{P}}(\mathcal{E}^{(2)})$ (see [10] (Lemma 2.1 (i))) and

$$\begin{aligned}
&\limsup_{\mu \to m}(h_{\mu}(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}} d\mu) \\
&\geq \liminf_{n \to \infty}(h_{\mu_n}(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}} d\mu_n) \\
&\geq \inf_{n} \pi_{\Theta}^{f}(\mathcal{Q}_n) = \pi_{\Theta}^*(f).
\end{aligned}$$

On the other hand, notice that the support of m,

$$\mathrm{supp} m = \bigcap_{l=1}^{\infty} \bigcup_{j=1}^{k_{n_l}} \{(\omega, x, y) : x, y \in \overline{Q_j^{(n_l)}}(\omega)\},$$

where $\{n_l\}$ is the subsequence of $\{n\}$ such that μ_{n_l} convergence to m in the sense of the narrow topology. Since $\{\mathcal{Q}_{n_l}\}$ is a refining sequence of measurable partition on $\mathcal{E}$, then,

$$\mathrm{supp} m = \{(\omega, x, x) \in \mathcal{E}^{(2)} : x \in \mathcal{E}_\omega\}.$$

Thus for every finite measurable partition $\xi = \{\xi_1, \cdots, \xi_k\}$ on $\mathcal{E}$,

$$m(\pi_{\mathcal{E}_1}^{-1}\xi_i) = m(\pi_{\mathcal{E}_1}^{-1}\xi_i \cap \mathrm{supp} m) = m(\pi_{\mathcal{E}_2}^{-1}\xi_i),\ 1 \le i \le k,$$

This means that $\pi_{\mathcal{E}_1}^{-1}\xi$ and $\pi_{\mathcal{E}_2}^{-1}\xi$ coincide up to sets of $m-$measure zero. Observe that $E(1_{\pi_{\mathcal{E}_1}^{-1}\xi_i} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) = 1_{\pi_{\mathcal{E}_1}^{-1}\xi_i}$ $\mathbb{P}-$a.s. for all $1 \le i \le k$. Then,

$$H_m(\pi_{\mathcal{E}_2}^{-1}\xi \mid \mathcal{A}_{\mathcal{E}^{(2)}}) = H_m(\pi_{\mathcal{E}_1}^{-1}\xi \mid \mathcal{A}_{\mathcal{E}^{(2)}}) = 0,$$

and $h_m(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) = 0$ by the definition of the relative entropy. Hence,

$$\begin{aligned} & h_m^*(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}} dm \\ = & \limsup_{\mu \to m} h_\mu(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) - h_m(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}} dm \ge \pi_\Theta^*(f). \end{aligned}$$

By Theorem 1, $h_m^*(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}} dm \le \pi_\Theta^*(f)$ and we complete the proof. □

The following variational principle comes directly from Theorem 1 and Proposition 5.

Theorem 2. *Let T be a continuous bundle RDS on $\mathcal{E}$ and $f \in L^1_{\mathcal{E}}(\Omega, \mathcal{C}(X))$. Then,*

$$\max\{h_\mu^*(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}} d\mu : \mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{E}^{(2)})\} = \pi_\Theta^*(f).$$

We are now in a position to prove that the relative tail pressure of a continuous bundle RDS is equal to that of its factor under the principal extension.

Theorem 3. *Let T, S be two continuous bundle RDSs over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ on $\mathcal{E}$ and $\mathcal{G}$, respectively. Suppose that S is a principal extension of T via the factor transformation π, then for each $f \in L^1_{\mathcal{E}}(\Omega, \mathcal{C}(X))$, $\pi_\Lambda^*(f \circ \pi) = \pi_\Theta^*(f)$.*

Proof. Denote by $\mathcal{G}^{(2)} = \{(\omega, y, z) : y, z \in \mathcal{G}_\omega\}$, which is a measurable subset of $\Omega \times Y \times Y$ with respect to the product $\sigma-$algebra $\mathcal{F} \times \mathcal{C}^2$. Let $\phi : \mathcal{G}^{(2)} \to \mathcal{E}^{(2)}$ be the map induced by the factor transformation π as $\phi(\omega, y, z) = (\omega, \pi_\omega y, \pi_\omega z)$. Then ϕ is a factor transformation from $\mathcal{G}^{(2)}$ to $\mathcal{E}^{(2)}$.

Let $m \in \mathcal{I}_{\mathbb{P}}(\mathcal{G}^{(2)})$ and $\alpha : \mathcal{G}^{(2)} \to \mathcal{G}$ be the natural projection defined as $\alpha(\omega, y, z) = (\omega, y)$. By the equality 4.18 in [19], for each $m \in \mathcal{I}_{\mathbb{P}}(\mathcal{G}^{(2)})$, $h_m(\Lambda^{(2)} \mid \mathcal{D}_{\mathcal{G}^{(2)}}) = h_{\alpha m}(\Lambda, \mathcal{G})$, where $h_{\alpha m}(\Lambda, \mathcal{G})$ is the usual measure-theoretical entropy. Let $\beta : \mathcal{E}^{(2)} \to \mathcal{E}$ be the natural projection defined as $\beta(\omega, x, u) = (\omega, x)$. Then $\phi m \in \mathcal{I}_{\mathbb{P}}(\mathcal{E}^{(2)})$ and $h_{\phi m}(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) = h_{\beta(\phi m)}(\Theta, \mathcal{E})$.

Notice that $\pi\alpha = \beta\phi$. One obtains $h_{\beta(\phi m)}(\Theta, \mathcal{E}) = h_{\pi\alpha m}(\Theta, \mathcal{E})$. Since the continuous bundle RDS S is a principal extension of the RDS T via the factor transformation π, by the Abramov-Rokhlin formula (see [20,21]) one has $h_{\pi\alpha m}(\Theta, \mathcal{E}) = h_{\alpha m}(\Lambda, \mathcal{G})$. It follows that $h_m(\Lambda^{(2)} \mid \mathcal{D}_{\mathcal{G}^{(2)}}) = h_{\phi m}(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}})$, and then $h^*_m(\Lambda^{(2)} \mid \mathcal{D}_{\mathcal{G}^{(2)}}) = h^*_{\phi m}(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}})$. Observe that $\pi\pi_{\mathcal{G}} = \pi_{\mathcal{E}}\phi$ and $\int f \circ \pi \circ \pi_{\mathcal{G}} dm = \int f \circ \pi_{\mathcal{E}} d\phi m$, then

$$h^*_m(\Lambda^{(2)} \mid \mathcal{D}_{\mathcal{G}^{(2)}}) + \int f \circ \pi \circ \pi_{\mathcal{G}} dm = h^*_{\phi m}(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}} d\phi m.$$

Thus by Theorem 2,

$$\begin{aligned} \pi^*_{\Lambda}(f \circ \pi) &= \max_{m \in \mathcal{I}_{\mathbb{P}}(\mathcal{G}^{(2)})} \{h^*_m(\Lambda^{(2)} \mid \mathcal{D}_{\mathcal{G}^{(2)}}) + \int f \circ \pi \circ \pi_{\mathcal{G}} dm\} \\ &\leq \max_{\mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{E}^{(2)})} \{h^*_{\mu}(\Theta^{(2)} \mid \mathcal{A}_{\mathcal{E}^{(2)}}) + \int f \circ \pi_{\mathcal{E}} d\mu\} = \pi^*_{\Theta}(f). \end{aligned}$$

For each $\mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{E}^{(2)})$, there exists some $m \in \mathcal{I}_{\mathbb{P}}(\mathcal{G}^{(2)})$ such that $\phi m = \mu$. Therefore, the other part of the above inequality holds and we complete the proof. $\square$

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References

1. Ruelle, D. Statistical mechanics on a compact set with z^v action satisfying expansiveness and specification. *Trans. Am. Math. Soc.* **1973**, *187*, 237–251.
2. Walters, P. A variational principle for the pressure of continuous transformations. *Am. J. Math.* **1975**, *97*, 937–971.
3. Misiurewicz, M. Topological conditional entropy. *Stud. Math.* **1976**, *55*, 175–200.
4. Downarowicz, T. Entropy structure. *J. Anal. Math.* **2005**, *96*, 57–116.
5. Burguet, D. A direct proof of the tail variational principle and its extension to maps. *Ergod. Theory Dynam. Syst.* **2009**, *29*, 357–369.
6. Ledrappier, F. *A Variational Principle for the Topological Conditional Entropy*; Vol. 729 of Lecture Notes in Math; Springer: New York, NY, USA, 1979; pp. 78–88.
7. Kifer, Y.; Weiss, B. Generating partitions for random transformations. *Ergod. Theory Dynam. Syst.* **2002**, *22*, 1813–1830.
8. Ledrappier, F.; Walters, P. A Relativised Variational Principle for Continuous Transformations. *J. Lond. Math. Soc. 1977*, *2*, 568–576.
9. Bogenschütz, T. Entropy, pressure, and a variational principle for random dynamical systems. *Random Comput. Dyn.* **1992**, *1*, 99–116.
10. Kifer, Y. On the topological pressure for random bundle transformations. In *Rokhlin's Memorial Volume*; American Mathematical Society: Providence, RI, USA, 2001.
11. Dudley, R.M. *Real Analysis and Probability*; Wadsworth and Brooks/Cole: Monterey, CA, USA, 1989.
12. Castaing, C.; Valadier, M. *Convex Analysis and Measurable Multifunctions*; Vol. 580 of Lecture Notes in Mathematics; Springer: New York, NY, USA, 1977.
13. Crauel, H. *Random Probability Measures on Polish Spaces*; Taylor & Francis: London, UK, 2002.
14. Kifer, Y. *Ergodic Thoery of Random Transformations*; Birkhäuser: Berlin, Germany, 1986.
15. Walters, P. *An Introduction to Ergodic Thoery*; Springer: New York, NY, USA, 1982.
16. Glasner, E. *Ergodic Theory via Joinings*; American Mathematical Society: Providence, RI, USA, 2003.
17. Smorodinsky, M. *Ergodic Theory, Entropy*; Vol. 214 of Lecture Notes in Math; Springer: Berlin, Germany, 1971.
18. Arnold, L. *Random Dynamical Systems*; Springer: New York, NY, USA, 1998.
19. Downarowicz, T. *Entropy in Dynamical Systems*; New Mathematical Monographs 18; Cambridge University Press: Cambridge, UK, 2011.

20. Bogenschütz, T.; Crauel, H. *The Abromov-Rokhlin Formula*; Vol. 1514 of Lecture Notes in Mathematics; Springer: New York, NY, USA, 1992; pp. 32–35.
21. Liu, P. A note on the entropy of factors of random dynamical systems. *Ergod. Theory Dynam. Syst.* **2005**, *25*, 593–603.