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A Novel Numerical Approach for a Nonlinear Fractional Dynamical Model of Interpersonal and Romantic Relationships

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Abstract: In this paper, we propose a new numerical algorithm, namely q-homotopy analysis Sumudu transform method (q-HASTM), to obtain the approximate solution for the nonlinear fractional dynamical model of interpersonal and romantic relationships. The suggested algorithm examines the dynamics of love affairs between couples. The q-HASTM is a creative combination of Sumudu transform technique, q-homotopy analysis method and homotopy polynomials that makes the calculation very easy. To compare the results obtained by using q-HASTM, we solve the same nonlinear problem by Adomian's decomposition method (ADM). The convergence of the q-HASTM series solution for the model is adapted and controlled by auxiliary parameter $\hbar$ and asymptotic parameter n. The numerical results are demonstrated graphically and in tabular form. The result obtained by employing the proposed scheme reveals that the approach is very accurate, effective, flexible, simple to apply and computationally very nice.

Keywords: Nonlinear fractional dynamical model of interpersonal and romantic relationship; Caputo fractional derivative; q-homotopy analysis method; Sumudu transform; Adomian's decomposition method

1. Introduction

The theory of entropy was originally related with thermodynamics, but in recent years it has also been used in other areas of investigation such as information theory, psychodynamics, thermoeconomics, human relationships and many more. The second law of thermodynamics states that entropy increases with time. It indicates the instability of a system over a period of time if there is nothing to stabilize it. Similarly, in human relationships, we have daily interactions and these relationships also become disordered. We require input to stabilize any relationship, to iron out the wrinkles or differences, so that we don't possess and store things forever [1–4]. The study of interpersonal and romantic relations has been gaining popularity during the past ten years. Interpersonal relations occur in many ways, for example marriage, blood relations, close associations, work and clubs. Since 1957, marriage has been investigated scientifically and we can observe some general interpretations which lead to models of marital interactions. The authors were inspired to study why some married couples get a divorce, when some others couples do not divorce. Moreover, among married couples, some are happy, while some are not happy with each other. In today's scenario around the globe the number of divorce cases in interpersonal and romantic relations for marriage is increasing day to day. A survey in the United States revealed that for a 40 year duration the chances a first marriage will end in divorce are approximately 50 to 67 percent. For the case of a second marriage for the same duration the chance it will end in divorce is 10 percent higher than for a first marriage. Worldwide, the United States has the highest number of divorce cases. In these areas, experiments are inconvenient and may be restricted due to ethical considerations. To study the dynamics of interpersonal and romantic relations in marriage mathematical models can play a key role and in recent years, many scientists and researchers have studied dynamical modelling of interpersonal relations [5–8].

Fractional order models are very important to study natural problems. It is well known that the nature of the trajectory of the fractional order derivatives is non-local, which describes that the fractional order derivative has memory effect features [9–13], and any dynamical or physical system related to fractional order differential operators has a memory effect, meaning that the future states depend on the present as well as the past states [14–19]. In view of the great importance of fractional approaches in real world problems we were motivated to study the nonlinear dynamical model of interpersonal and romantic relationships. There are various analytical and numerical techniques to study such problems. In 1992, Liao proposed an analytic technique known as the homotopy analysis method (HAM) [20,21] for managing nonlinear problems. More recently, the authors of [22,23] have suggested an extension of HAM known as q-homotopy analysis method (q-HAM) to discuss nonlinear mathematical models. Standard analytical schemes requisite more computer memory and computational time, thus to overcome these limitations the analytical methods need to be merged with standard integral transform operators to study the nonlinear equations appearing in science and engineering [24–26].The key purpose of the present work is to propose a novel numerical approach namely q-homotopy analysis Sumudu transform method (q-HASTM) to solve the nonlinear fractional dynamical model of interpersonal and romantic relations in marriages. The q-HASTM is an elegant amalgamation of the q-HAM, the standard Sumudu transform and the homotopy polynomials. The supremacy of this algorithm is its potential of merging two robust computational approaches for investigating nonlinear fractional differential equations. Moreover, the proposed technique contains an asymptotic parameter n by which we can insure to convergence of a series of problem solutions. The paper is organized as follows: In Section 2, fractional calculus and the Sumudu transform are discussed. In Section 3, a fractional order nonlinear dynamical model of interpersonal and romantic relations in marriages is discussed. In Section 4, the fundamental plan of q-HASTM is proposed. In Section 5, a solution of the fractional model is obtained by employing q-HASTM. In Section 6, the fundamental plan of ADM is discussed. In Section 7, a solution of the mathematical model is obtained by employing ADM. Results and discussion are presented in Section 8. Lastly, in Section 9 the concluding observations are highlighted.

2. Basic Definitions

In this section, we discuss the fundamental definitions of integrals and derivatives of non-integer order and the Sumudu transform.

Definition 1. *The left-sided Riemann-Liouville fractional integral operator of order* $\mu > 0$, *of a function* $g(\xi) \in C_\alpha, \alpha \geq -1$ *is given in the following manner [9]:*

$$J^\mu g(\xi) = \frac{1}{\Gamma(\mu)} \int_0^\xi (\xi - \eta)^{\mu - 1} g(\eta) d\eta, \quad (\mu > 0), \tag{1}$$

Definition 2. *If $g(\xi)$ is a function of a variable ξ, then the Caputo fractional derivative of the function is expressed as follows [10]:*

$$D_{\xi}^{\mu}g(\xi) = J^{m-\mu}D^{m}g(\xi) = \frac{1}{\Gamma(m-\mu)}\int_{0}^{\xi}(\xi-\eta)^{m-\mu-1}g^{(m)}(\eta)d\eta, \tag{2}$$

for $m-1<\mu\leq m, m\in N, \xi>0$.

Now, we present the following relation for the fractional integral in the Riemann-Liouville sense and the fractional derivative operator in the Caputo sense:

$$J_{\xi}^{\mu}D_{\xi}^{\mu}g(\xi) = g(\xi) - \sum_{l=0}^{m-1} g^{(l)}(0+)\frac{\xi^{l}}{l!}. \tag{3}$$

Definition 3. *The Sumudu transform is a newly introduced integral operator, which was suggested and developed by Watugala [27].This newly integral transform is expressed and discussed over the set of functions:*

$$\Theta = \{g(\xi) : \exists \mathrm{N}, \delta_1, \delta_2 > 0, |g(\xi)| < \mathrm{N}\, \mathrm{e}^{|\xi|/\delta_{\mathrm{j}}}, \text{if } \xi \in (-1)^{\mathrm{j}} \times [0,\infty)\}$$

as follows:

$$G(\mathrm{u}) = S\,[g(\xi)] = \int_{0}^{\infty} g\,(\mathrm{u}\xi)\,\mathrm{e}^{-\xi}\,\mathrm{d}\xi, \mathrm{u} \in (-\delta_1, \delta_2). \tag{4}$$

The pioneer work in connection with the formation of important and basic results of the newly integral transform was conducted by Asiru [28], Belgacem et al. [29,30] and Srivastava et al. [31].

Definition 4. *If $g(\xi)$ is a function of ξ, then the Sumudu transform of the Caputo fractional operator $D_{\xi}^{\mu}g(\xi)$ is written as follows [32]:*

$$S[D_{\xi}^{\mu}g(\xi)] = u^{-\mu}S[g(\xi)] - \sum_{l=0}^{m-1} u^{-\mu+l}g^{(l)}(0+), (m-1<\mu\leq m). \tag{5}$$

3. Model Description

We consider a nonlinear dynamical model of interpersonal relations with arbitrary order of the form [8]:

$$D_{\xi}^{\mu}x(\xi) = -a_1x + b_1y(1-\varepsilon y^2) + c_1, 0<\mu\leq 1, D_{\xi}^{\mu}y(\xi) = -a_2y + b_2x(1-\varepsilon x^2) + c_2, 0<\mu\leq 1, \tag{6}$$

with initial conditions:

$$x(0) = 0, y(0) = 0.$$

Here variable x measures the love of individual 1 for his/her partner and y measures the love of individual 2 for his/her partner. The positive and negative measurement indicates the feelings of the partners towards each other. In the model, $a_i > 0, a_i, b_i, c_i (i = 1, 2)$ and ε are real constants. These parameters are described as the oblivion, reaction, and attraction constants. In this model, we suppose that in the absence of partners feelings of each other decay exponentially fast. The romantic style of individuals 1 and 2 specified by the parameters a_i, b_i and $c_i (i = 1, 2)$. For example, a_i represents the expanse to which the individual i is inspired by one's own feelings. The parameter b_i describes the expanse to which the individual i is inspired by the partner and it is expected that individual's partner are supportive in nature. It measures the inclination to seek or avoid the closeness in the love affair between a couple. Hence, $-a_i x_i$ indicates that love measures of i when the partner is not present, decay exponentially. The term $1/a_i$ indicates the time required for love to decay.

4. Basic Planof q-HASTM

To discuss the key plan of q-HASTM, we take a nonlinear fractional differential equation of the form:

$$D_{\xi}^{\mu} x(\xi) + Rx(\xi) + Nx(\xi) = \psi(\xi), \quad m-1 < \mu \leq m, \tag{7}$$

Here x is an unknown function of variable ξ and $D_{\xi}^{\mu} = \frac{d^{\mu}}{d\xi^{\mu}}$ denotes the fractional operator of order μ proposed by Caputo, $m \in N$, R indicates the bounded linear operator in ξ and N represents the general nonlinear differential operator in ξ Employing the Sumudu transform on Equation (7), we obtain:

$$S[D_{\xi}^{\mu}\, x] + S[Rx + Nx] = S[\psi(\xi)]. \tag{8}$$

Applying the differentiation aspect of the Sumudu transform, we have the following result:

$$u^{-\mu} S[x] - \sum_{l=0}^{m-1} u^{-\mu+l} x^{(l)}(0) + S[Rx + Nx] = S[\psi(\xi)]. \tag{9}$$

On simplification, we have:

$$S\,[x] - u^{\mu} \sum_{l=0}^{m-1} u^{-\mu+l} x^{(l)}(0) + u^{\mu}[S[Rx + Nx] - S[\psi(\xi)]] = 0. \tag{10}$$

We define the nonlinear operator by keeping view of Equation (10):

$$\wp[\vartheta(\xi;q)] = S[\vartheta(\xi;q)] - u^{\mu} \sum_{l=0}^{m-1} u^{-\mu+l} \vartheta^{(l)}(\xi;q)(0) + u^{\mu}[S[R\vartheta(\xi;q) + N\vartheta(\xi;q)] - S[\psi(\xi)]], \tag{11}$$

In Equation (11), $q \in \left[1, \frac{1}{n}\right]$ is the embedding parameter and $\vartheta(\xi;q)$ is known as a real function of ξ and q. Now we develop the homotopy which is given by Equation (12):

$$(1 - nq)\; S\;[\vartheta(\xi;q) - x_0] = \hbar\; N\;[x(\xi)], \tag{12}$$

where S indicates the Sumudu transform operator, $n \geq 1, \hbar \neq 0$ specify an auxiliary parameter, $x_0(\xi)$ show an initial approximation of $x(\xi)$. If we set the embedding parameter $q = 0$ and $q = \frac{1}{n}$, then we have:

$$\vartheta(\xi;0) = x_0(\xi), \quad \vartheta(\xi;\frac{1}{n}) = x(\xi), \tag{13}$$

Thus, when q varies from 0 to $\frac{1}{n}$, then the solution $\vartheta(\xi;q)$ takes place from the initial approximation $x_0(\xi)$ to the solution $x(\xi)$. Expanding the function $\vartheta(\xi;q)$ in the series form by exerting the Taylor series about q, we obtain:

$$\vartheta(\xi;q) = x_0(\xi) + \sum_{k=1}^{\infty} x_k(\xi) q^k, \tag{14}$$

where:

$$x_k(\xi) = \frac{1}{k!} \frac{\partial^k}{\partial q^k} \{\vartheta(\xi;q)\}\big|_{q=0}. \tag{15}$$

If the initial approximation $x_0(\xi)$, asymptotic parameter n and the auxiliary parameter $\hbar$ are chosen properly, then Equation (14) converges at $q = \frac{1}{n}$, then we get the following result:

$$x(\xi) = x_0(\xi) + \sum_{k=1}^{\infty} x_k(\xi) \left(\frac{1}{n}\right)^k, \tag{16}$$

which must be one of the solutions of the nonlinear fractional differential Equation (7). By taking into consideration Equation (16), the governing equation can be found from the Equation (12).

We define the vectors as follows:

$$\vec{x}_k = \{x_0(\xi),\ x_1(\xi), x_2(\xi), x_3(\xi), \ldots,\ x_k(\xi)\}. \tag{17}$$

Now differentiating the equation (12) k-times w.r.t. q, then divide by k! and finally setting $q = 0$, it yields the following result:

$$S\,[x_k(\xi) - \chi_k x_{k-1}(\xi)] = \hbar\, \Im_k(\vec{x}_{k-1}). \tag{18}$$

Now employing the inverse of the Sumudu transform operator on Equation (18), we obtain:

$$x_k(\xi) = \chi_k x_{k-1}(\xi) + \hbar\, S^{-1}[\Im_k(\vec{x}_{k-1})], \tag{19}$$

where χ_k is defined and given as follows:

$$\chi_k = \begin{cases} 0, k \leq 1 \\ n, k > 1, \end{cases} \tag{20}$$

We define the value of $\Im_k(\vec{x}_{k-1})$ in a new way as:

$$\Im_k(\vec{x}_{k-1}) = S\,[x_{k-1}(\xi)] - \left(1 - \frac{\chi_k}{n}\right) u^{\mu} \left(\sum_{l=0}^{m-1} u^{-\mu+l} x^{(l)}(0) + S[\phi(\xi)] \right) + u^{\mu} S[R x_{k-1} + P_{k-1}], \tag{21}$$

In the Equation (21), P_k represents the homotopy polynomial [33] and given in the following form:

$$P_k = \frac{1}{\Gamma(k)} \left[\frac{\partial^k}{\partial q^k} N\vartheta(\xi, q) \right]_{q=0}, \tag{22}$$

and:

$$\vartheta(\xi, q) = \vartheta_0 + q\vartheta_1 + q^2\vartheta_2 + \cdots. \tag{23}$$

Using Equation (21) in Equation (19), we get:

$$\begin{aligned} x_k(\xi) = (\chi_k + \hbar) x_{k-1}(\xi) - \hbar(1 - \tfrac{\chi_k}{n}) S^{-1} \left(u^{\mu} \sum_{l=0}^{m-1} u^{-\mu+l} x^{(l)}(\xi) + u^{\mu} S[\psi(\xi)] \right) \\ + \hbar S^{-1}[u^{\mu} S[R x_{k-1} + P_{k-1}]]. \end{aligned} \tag{24}$$

The novelty in our proposed approach is that a new correction function (24) is introduced by employing homotopy polynomials. Therefore, from Equation (24), we can calculate the distinct iterates $x_k(\xi)$ for $k > 1$ and the q-HASTM series solution is given as follows:

$$x(\xi) = \sum_{k=0}^{\infty} x_k(\xi) \left(\frac{1}{n}\right)^k \tag{25}$$

5. *q*-HASTM Solution for Nonlinear Fractional Dynamical Model of Interpersonal and Romantic Relationships

In this section, we find the solution of nonlinear fractional dynamical model of interpersonal and romantic relationship written in the following form:

$$\begin{cases} D_{\xi}^{\mu}x = -a_1x + b_1y(1-\varepsilon y^2) + c_1,\ 0<\mu\le 1 \\ D_{\xi}^{\mu}y = -a_2y + b_2x(1-\varepsilon x^2) + c_2,\ 0<\mu\le 1, \end{cases} \tag{26}$$

withinitial conditions:

$$\begin{cases} x(0) = 0 \\ y(0) = 0. \end{cases} \tag{27}$$

Employing the Sumudu transform operator on Equation (26) and using the initial conditions (27), we get the following results:

$$S[x(\xi)] - u^{\mu}c_1 - u^{\mu}S[-a_1x + b_1y(1-\varepsilon y^2)\ ] = 0, S[y(\xi)] - u^{\mu}c_2 - u^{\mu}S[-a_2y + b_2x(1-\varepsilon x^2)\ ] = 0. \tag{28}$$

Now, we write the nonlinear operator as follows:

$$\wp_1[\vartheta_1(\xi;q)] = S[\vartheta_1(\xi;q)] - u^{\mu}c_1 - u^{\mu}S[-a_1\vartheta_1(\xi;q) + b_1\vartheta_2(\xi;q)(1-\varepsilon\vartheta_2^2(\xi;q))\ ], \tag{29}$$

and:

$$\wp_2[\vartheta_2(\xi;q)] = S[\vartheta_2(\xi;q))] - u^{\mu}c_2 - u^{\mu}S[-a_2\vartheta_2(\xi;q) + b_2\vartheta_1(\xi;q)(1-\varepsilon\vartheta_1^2(\xi;q))\ ], \tag{30}$$

thus, we have:

$$\begin{gathered} \Re_{1k}(\overrightarrow{x}_{k-1}) = S[x_{k-1}(\xi)] - \left(1-\tfrac{\chi_k}{n}\right)u^{\mu}c_1 \\ -u^{\mu}S\left[-a_1x_{k-1}(\xi) + b_1y_{k-1}(\xi) - b_1\varepsilon\sum_{r=0}^{k-1}y_{k-1-r}\left(\sum_{j=0}^{r}y_jy_{r-j}\right)\right], \\ \Re_{2k}(\overrightarrow{y}_{k-1}) = S[y_{k-1}(\xi)] - \left(1-\tfrac{\chi_k}{n}\right)u^{\mu}c_2 \\ -u^{\mu}S\left[-a_2y_{k-1}(\xi) + b_2x_{k-1}(\xi) - b_2\varepsilon\sum_{r=0}^{k-1}x_{k-1-r}\left(\sum_{j=0}^{r}x_jx_{r-j}\right)\right]. \end{gathered} \tag{31}$$

The *m*-th–order deformation equations are given as follows:

$$S\ [x_k(\xi) - \chi_kx_{k-1}(\xi)] = \hbar\ \Re_{1k}(\overrightarrow{x}_{k-1}), \text{and } S\ [y_k(\xi) - \chi_ky_{k-1}(\xi)] = \hbar\ \Re_{2k}(\overrightarrow{y}_{k-1}). \tag{32}$$

Next, on employing the inverse Sumudu operator, this gives:

$$x_k(\xi) = \chi_kx_{k-1}(\xi) + \hbar\ S^{-1}[\Re_{1k}(\overrightarrow{x}_{k-1})], \text{and } y_k(\xi) = \chi_ky_{k-1}(\xi) + \hbar\ S^{-1}[\Re_{2k}(\overrightarrow{y}_{k-1})]. \tag{33}$$

Now using the initial approximation $x_0 = 0, y_0 = 0$ and recursive relation (33), we obtain the following iterations of the *q*-HASTM series solution:

$$x_1(\xi) = -c_1\hbar\frac{\xi^{\mu}}{\Gamma(1+\mu)}, y_1(\xi) = -c_2\hbar\frac{\xi^{\mu}}{\Gamma(1+\mu)}, \tag{34}$$

Following the same way the remaining components $x_k, y_k, k \ge 2$ of the *q*-HASTM solution can be easilyobtained, hence we obtain the complete solution. Finally, we arrive at the following series solution:

$$x(\xi) = \lim_{M\to\infty}\sum_{k=0}^{M}x_k(\xi)\left(\frac{1}{n}\right)^k \text{and } y(\xi) = \lim_{M\to\infty}\sum_{k=0}^{M}y_k(\xi)\left(\frac{1}{n}\right)^k. \tag{35}$$

6. Basic Idea of ADM

To demonstrate the ADM solution procedure [34,35], we use a fractional nonlinear differential equation with the initial condition of the form:

$$D_{\xi}^{\mu}x(\xi)+Rx(\xi)+Nx(\xi)=\psi(\xi),\quad m-1<\mu\leq m, \tag{36}$$

here x is an unknown function of variable ξ and $D_{\xi}^{\mu}=\frac{d^{\mu}}{d\xi^{\mu}}$ denotes the fractional operator of order μ proposed by Caputo, $m\in N, R$ indicates the bounded linear operator in ξ and N represents the general nonlinear differential operator in ξ.

Employing the operator J_{ξ}^{μ} on both sides of Equation (36) and making use of result (3), we get:

$$x(\xi)=\sum_{l=0}^{m-1}\left(\frac{d^{l}x}{d\xi^{l}}\right)_{\xi=0}\frac{\xi^{l}}{l!}+J_{\xi}^{\mu}\psi(\xi)-J_{\xi}^{\mu}[Rx(\xi)+Nx(\xi)]. \tag{37}$$

Further, we decompose the function $x(\xi)$ into sum of an infinite number of components expressed by the decomposition series as follows:

$$x=\sum_{n=0}^{\infty}x_{n}, \tag{38}$$

and the nonlinear term can be decomposed as:

$$Nx=\sum_{n=0}^{\infty}A_{n}, \tag{39}$$

where A_n denotes the Adomian polynomials, given as follows:

$$A_{n}=\frac{1}{n!}\left[\frac{d^{n}}{d\lambda^{n}}N\left(\sum_{i=0}^{n}\lambda^{i}x_{i}\right)\right]_{\lambda=0},n=0,1,2,\ldots \tag{40}$$

The components $x_0, x_1, x_2, \ldots$ are determined recursively by substituting (38) and (39) into (37) leads to:

$$\sum_{n=0}^{\infty}x_{n}=\sum_{l=0}^{m-1}\left(\frac{d^{l}x}{d\xi^{k}}\right)_{\xi=0}\frac{\xi^{l}}{l!}+J_{\xi}^{\mu}\psi(\xi)-J_{\xi}^{\mu}\left[R\left(\sum_{n=0}^{\infty}x_{n}\right)+\sum_{n=0}^{\infty}A_{n}\right]. \tag{41}$$

This can be written as:

$$x_{0}+x_{1}+x_{2}+\cdots=\sum_{l=0}^{m-1}\left(\frac{d^{l}x}{d\xi^{l}}\right)_{\xi=0}\frac{\xi^{l}}{l!}+J_{\xi}^{\mu}\psi(\xi)-J_{\xi}^{\mu}[R(x_{0}+x_{1}+x_{2}+\cdots)+(A_{0}+A_{1}+A_{2}+\cdots)]. \tag{42}$$

ADM uses the formal recursive relations as:

$$x_{0}=\sum_{l=0}^{m-1}\left(\frac{d^{l}x}{d\xi^{l}}\right)_{\xi=0}\frac{\xi^{l}}{l!}+J_{\xi}^{\mu}\psi(\xi),x_{n+1}=-J_{\xi}^{\mu}[R(x_{n})+A_{n}],\quad n\geq 0. \tag{43}$$

7. ADM Solution for Nonlinear Fractional Dynamical Model of Interpersonal and Romantic Relationship

To solve the nonlinear fractional dynamical model of interpersonal and romantic relations (26), we apply the operator J_ξ^μ on both sides of Equation (26) and use result (3) to obtain:

$$\begin{aligned} x &= \sum_{l=0}^{1-1} \frac{\xi^l}{l!}\left[D_\xi^l x\right]_{\xi=0} + J_\xi^\mu c_1 + J_\xi^\mu\left[-a_1 x + b_1 y(1-\varepsilon y^2)\right], \\ y &= \sum_{l=0}^{1-1} \frac{\xi^l}{l!}\left[D_\xi^l y\right]_{\xi=0} + +J_\xi^\mu c_2 + J_\xi^\mu\left[-a_2 y + b_1 x(1-\varepsilon x^2)\right]. \end{aligned} \tag{44}$$

This yields the following recursive relations using Equation (43):

$$x_0 = \sum_{l=0}^{0} \frac{\xi^l}{l!}\left[D_\xi^l x\right]_{\xi=0} + J_\xi^\mu c_1, y_0 = \sum_{l=0}^{0} \frac{\xi^l}{l!}\left[D_\xi^l y\right]_{\xi=0} + J_\xi^\mu c_2. \tag{45}$$

$$x_{n+1} = J_\xi^\mu[-a_1 x_n + b_1 y_n - \varepsilon b_1 A_n], n = 0,1,2,\ldots y_{n+1} = J_\xi^\mu[-a_2 y_n + b_2 x_n - \varepsilon b_2 A'_n], \tag{46}$$

where:

$$\sum_{n=0}^{\infty} A_n = y^3, \sum_{n=0}^{\infty} A'_n = x^3. \tag{47}$$

The first few components of the Adomian polynomials are given as follows:

$$A_0 = y_0^3, A_1 = 3y_0^2 y_1, A_2 = 3\left[y_0^2 y_2 + y_0 y_1^2\right], A'_0 = x_0^3, A'_1 = 3x_0^2 x_1, A'_2 = 3\left[x_0^2 x_2 + x_0 x_1^2\right], \tag{48}$$

The components of the ADM solution can be simply obtained by making use of the above recursive relation:

$$\begin{aligned} x_0 &= c_1 \tfrac{1}{\Gamma(1+\mu)}\xi^\mu, \\ y_0 &= c_2 \tfrac{1}{\Gamma(1+\mu)}\xi^\mu, \\ x_1 &= \tfrac{-a_1 c_1}{\Gamma(1+2\mu)}\xi^{2\mu} + \tfrac{b_1 c_2}{\Gamma(1+2\mu)}\xi^{2\mu} - \tfrac{\varepsilon b_1 c_2^3}{(\Gamma(1+\mu))^3}\tfrac{\Gamma(1+3\mu)}{\Gamma(1+4\mu)}\xi^{4\mu}, \\ y_1 &= \tfrac{-a_2 c_2}{\Gamma(1+2\mu)}\xi^{2\mu} + \tfrac{b_2 c_1}{\Gamma(1+2\mu)}\xi^{2\mu} - \tfrac{\varepsilon b_2 c_1^3}{(\Gamma(1+\mu))^3}\tfrac{\Gamma(1+3\mu)}{\Gamma(1+4\mu)}\xi^{4\mu}, \end{aligned} \tag{49}$$

and so on. In this way the remaining components of the ADM solution can be determined. Hence, the ADM solution of Equation (26) is given as follows:

$$\begin{aligned} x(\xi) &= c_1 \tfrac{1}{\Gamma(1+\mu)}\xi^\mu + \tfrac{-a_1 c_1}{\Gamma(1+2\mu)}\xi^{2\mu} + \tfrac{b_1 c_2}{\Gamma(1+2\mu)}\xi^{2\mu} \\ &\quad - \tfrac{\varepsilon b_1 c_2^3}{(\Gamma(1+\mu))^3}\tfrac{\Gamma(1+3\mu)}{\Gamma(1+4\mu)}\xi^{4\mu} + \ldots \\ y(\xi) &= c_2 \tfrac{1}{\Gamma(1+\mu)}\xi^\mu + \tfrac{-a_2 c_2}{\Gamma(1+2\mu)}\xi^{2\mu} + \tfrac{b_2 c_1}{\Gamma(1+2\mu)}\xi^{2\mu} \\ &\quad - \tfrac{\varepsilon b_2 c_1^3}{(\Gamma(1+\mu))^3}\tfrac{\Gamma(1+3\mu)}{\Gamma(1+4\mu)}\xi^{4\mu} + \ldots \end{aligned} \tag{50}$$

which can be recovered from the q-HASTM solution by setting $\hbar = -1$ and $n = 1$. However, most of the results obtained by using other analytical schemes such as HPM, VIM, DTM, and ADM converge to the corresponding numerical results in a very small region. Instead of these methods, in the case of q-HASTM, we can easily settle and restrict the region of convergence of series solution obtained by q-HASTM by setting appropriate values of $\hbar$ and n.

8. Numerical Simulations

In this section, we perform the numerical simulations for $x(\xi)$ and at the distinct fractional Brownian motions $\mu = 0.95$ and $\mu = 0.90$ and also for the standard motion $\mu = 1$. The special solution of fractional dynamical model of the problem is obtained by employing q-HASTM and ADM with different parameters as $a_1 = 0.05, b_1 = 0.04, c_1 = 0.2, a_2 = 0.07, b_2 = 0.06, c_2 = 0.3$ and $\varepsilon = 0.01$. The results are shown through Tables 1 and 2 and Figures 1–11. From Tables 1 and 2, it can be noticed that the values of the approximate solution at distinct grid points obtained by the q-HASTM and ADM are in a very good agreement. Figures 1 and 2 depict the behavior of $x(\xi)$ and $y(\xi)$ for the different values of μ. From Figure 1 we can observe that when we decrease the values of μ in the fractional dynamic model then the love of individual 1 for his/her partner decreases. From Figure 2 we can observe that when we decrease the value of μ in the fractional dynamic model then the love of individual 2 for his/her partner decreases. Figures 3–5 reveals the behavior of $x(\xi)$ and $y(\xi)$ for the different value of μ at $\hbar = -1$ and $n = 1$. From Figures 3–5, we see that when we decrease the values of μ in the fractional dynamic model then romantic relation between the couple decreases. Figures 6–8 demonstrate the $\hbar$-curves for $x(\xi)$ at distinct values of μ and n. Figures 9–11 show the $\hbar$-curves of $y(\xi)$ at distinct values of μ and n. The horizontal line segment in $\hbar$-curves shows the range of convergence of q-HASTM solution. From the Figures 6–11, it can be noticed that the range of convergence is directly proportional to the value of asymptotic parameter n. Hence, the results obtained by using the proposed technique converge very fast as compared to other existing analytical sachems.

Table 1. Comparative study between q-HASTM and ADM for x when $\mu = 1$, $a_1 = 0.05$, $b_1 = 0.04$, $c_1 = 0.2$, $a_2 = 0.07$, $b_2 = 0.06$, $c_2 = 0.3$ and $\varepsilon = 0.01$.

ξ	ADM	q-HASTM(for n=1 and $\hbar$=−1)	q-HASTM(for n=2 and $\hbar$=−1.99)
0	0	0	0
2	0.4033806333	0.4033788000	0.4033790454
4	0.8110188698	0.8109674666	0.8109706878
6	1.219148908	1.218802800	1.218817897
8	1.624049986	1.622732800	1.622778592
10	2.022138493	2.018416666	2.018525638
12	2.410163488	2.401324800	2.401546847
14	2.785587314	2.766738800	2.767144981
16	3.147219587	3.109751466	3.110437747
18	3.496167079	3.425266800	3.426357807
20	3.837175265	3.708000000	3.709652760

Table 2. Comparative study between q-HASTM and ADM for y when $\mu = 1$,$a_1 = 0.05$, $b_1 = 0.04$, $c_1 = 0.2$, $a_2 = 0.07$, $b_2 = 0.06$, $c_2 = 0.3$ and $\varepsilon = 0.01$.

ξ	ADM	q-HASTM(for n=1 and $\hbar$=−1)	q-HASTM(for n=2 and $\hbar$=−1.99)
0	0	0	0
2	0.5829289274	0.5829274000	0.5829287007
4	1.134891434	1.134838400	1.134860301
6	1.659550026	1.659119400	1.659232190
8	2.159330141	2.157414400	2.157773939
10	2.635727526	2.629625000	2.630507313
12	3.089599578	3.073910400	3.075746263
14	3.521399539	3.486687400	3.490096920
16	3.931326626	3.862630400	3.868457617
18	4.319398257	4.194671400	4.204018861
20	4.685515415	4.474000000	4.488263354

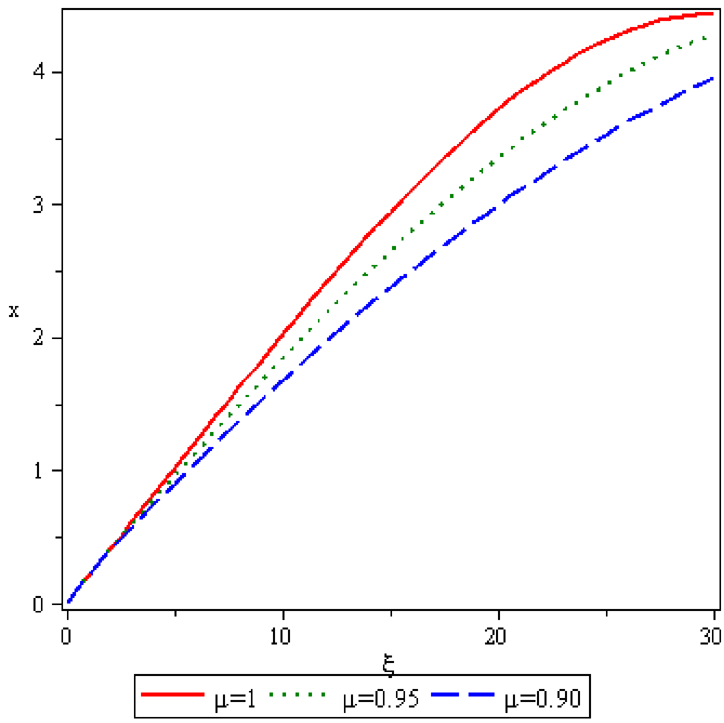

Figure 1. Behavior of $x(\xi)$ vs.time ξ for distinct values of μ when $n = 1$ and $\hbar = -1$.

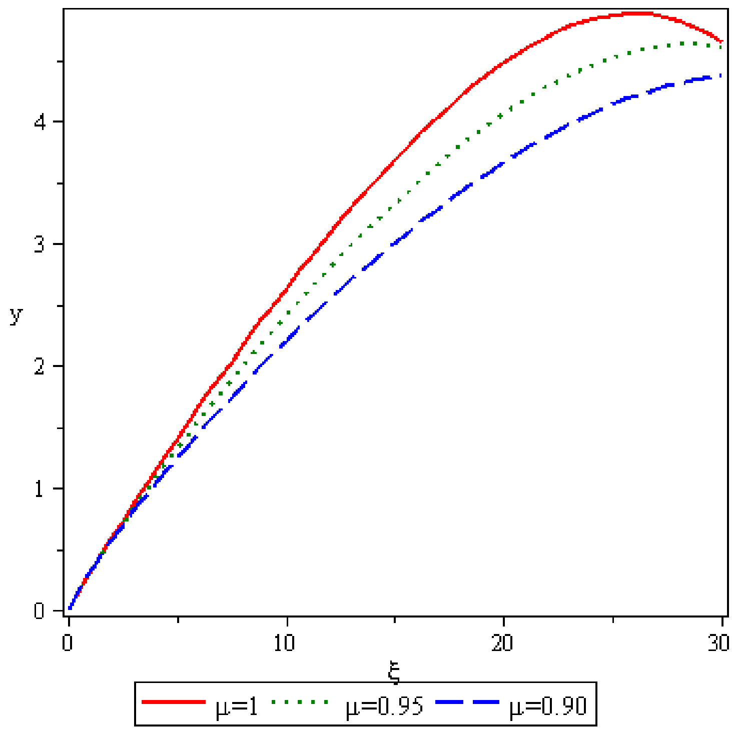

Figure 2. Nature of $y(\xi)$ vs.time ξ for distinct values of μ when $n = 1$ and $\hbar = -1$.

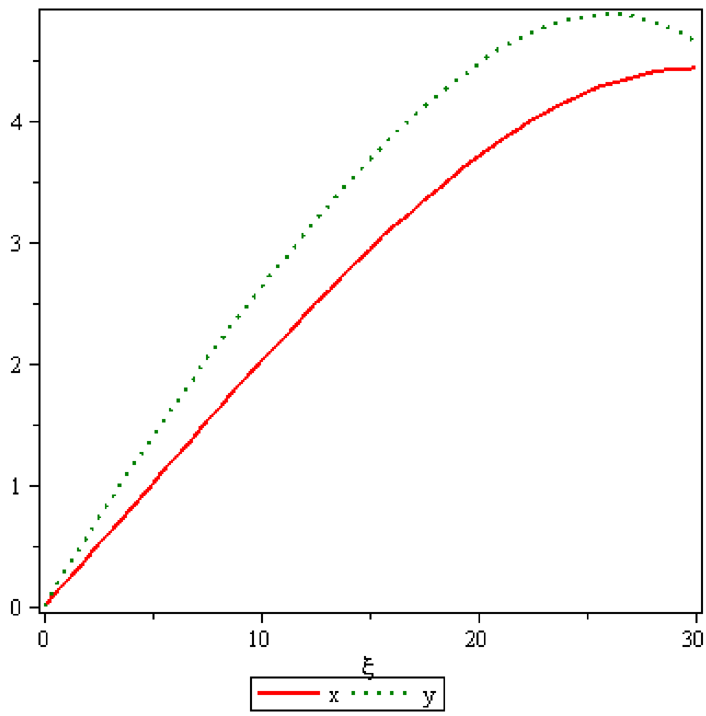

Figure 3. Behavior of $x(\xi)$ and $y(\xi)$ when $\mu = 1, n = 1$ and $\hbar = -1$.

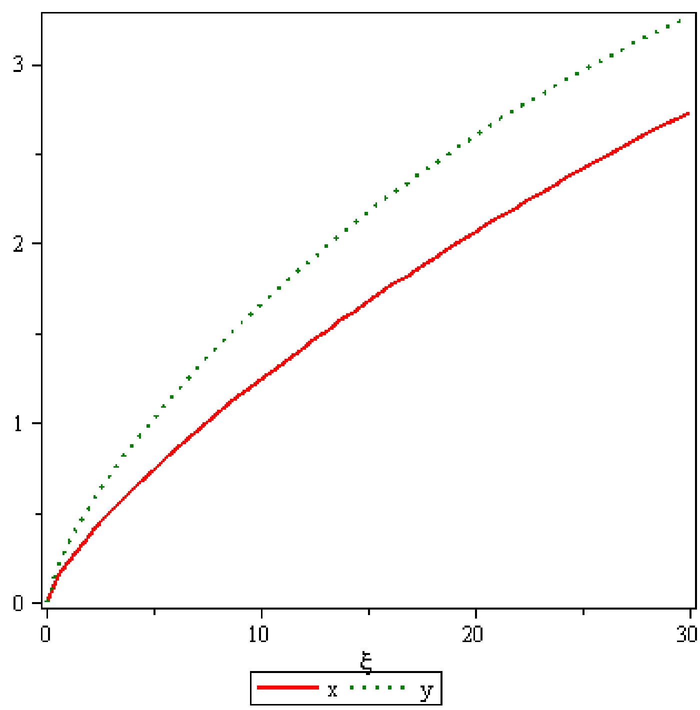

Figure 4. Characteristic of $x(\xi)$ and $y(\xi)$ when $\mu = 0.75, n = 1$ and $\hbar = -1$.

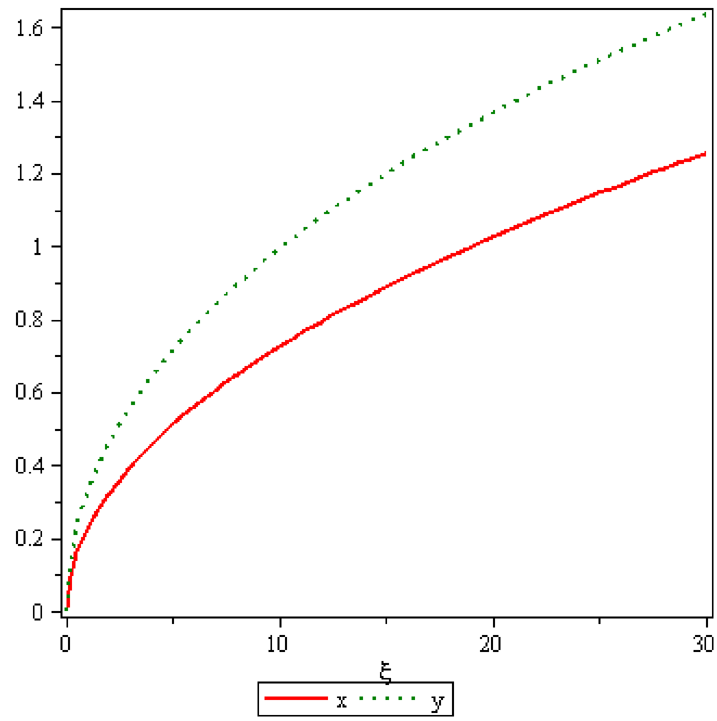

Figure 5. Response of $x(\xi)$ and $y(\xi)$ when $\mu = 0.50, n = 1$ and $\hbar = -1$.

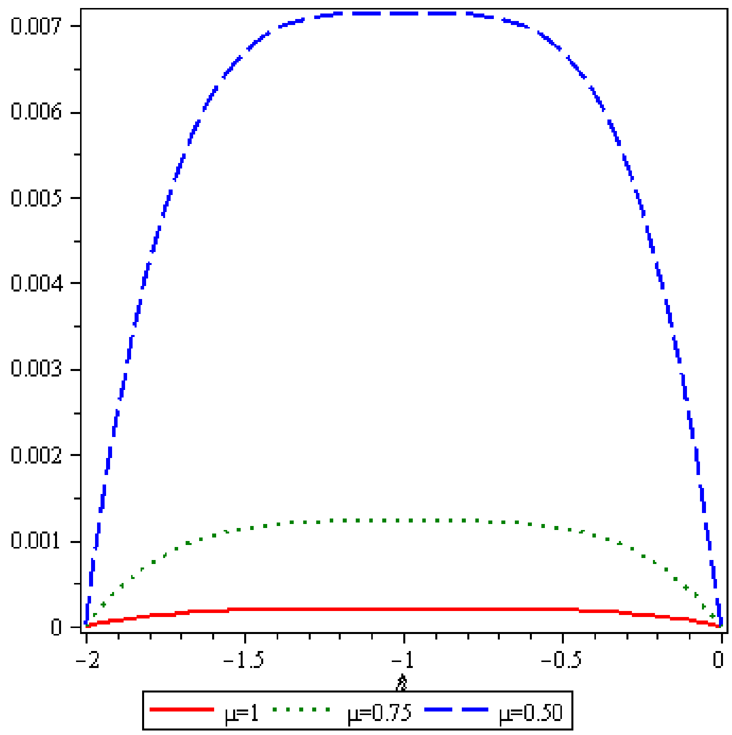

Figure 6. $\hbar$–curve of $x(\xi)$ for different values of μ at $n = 1$.

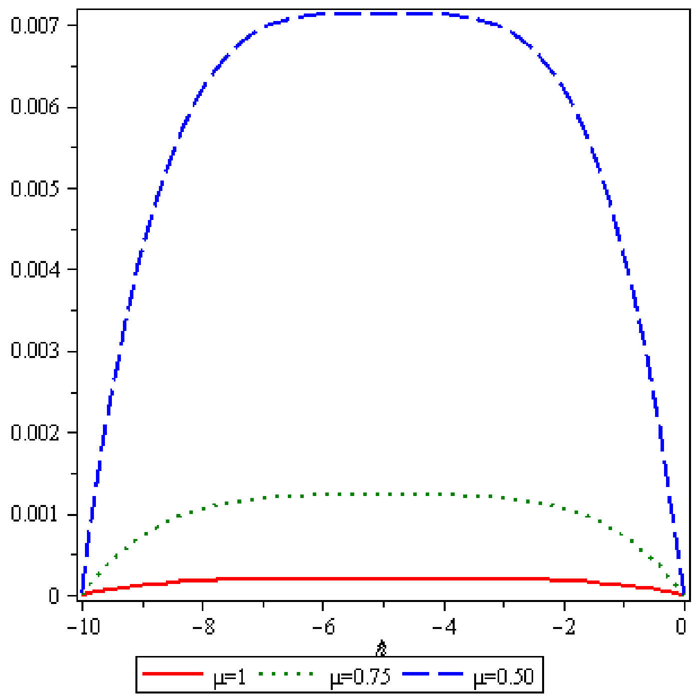

Figure 7. $\hbar$–curve of $x(\xi)$ for different values of μ at $n = 5$.

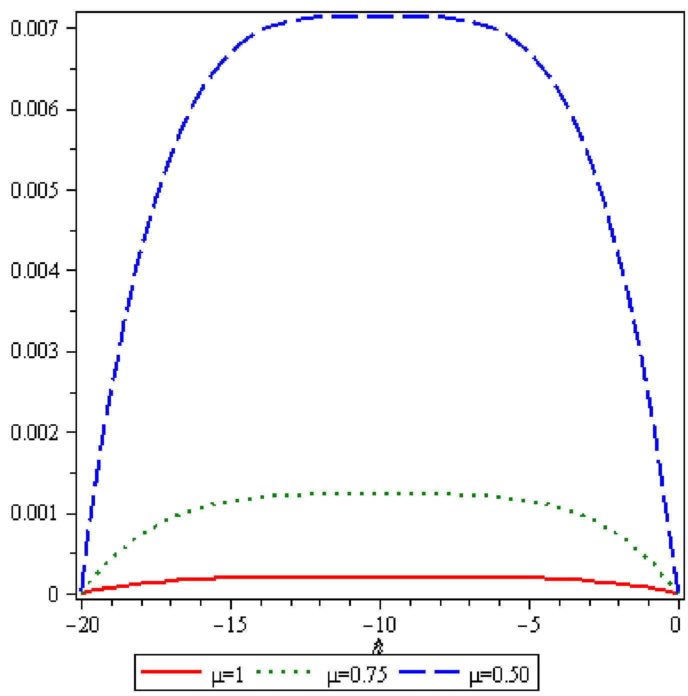

Figure 8. $\hbar$–curve of $x(\xi)$ for different values of μ at $n = 10$.

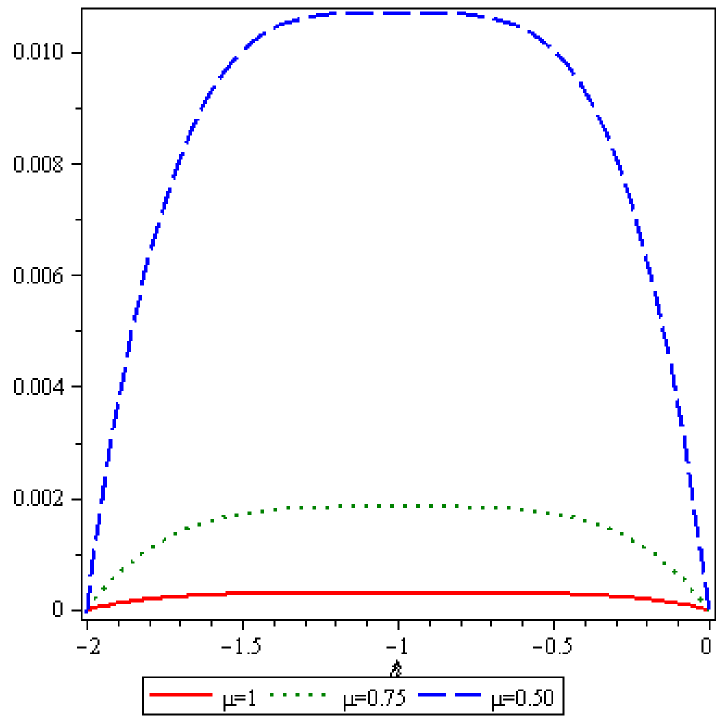

Figure 9. $\hbar$–curve of $y(\xi)$ for different values of μ at $n = 1$.

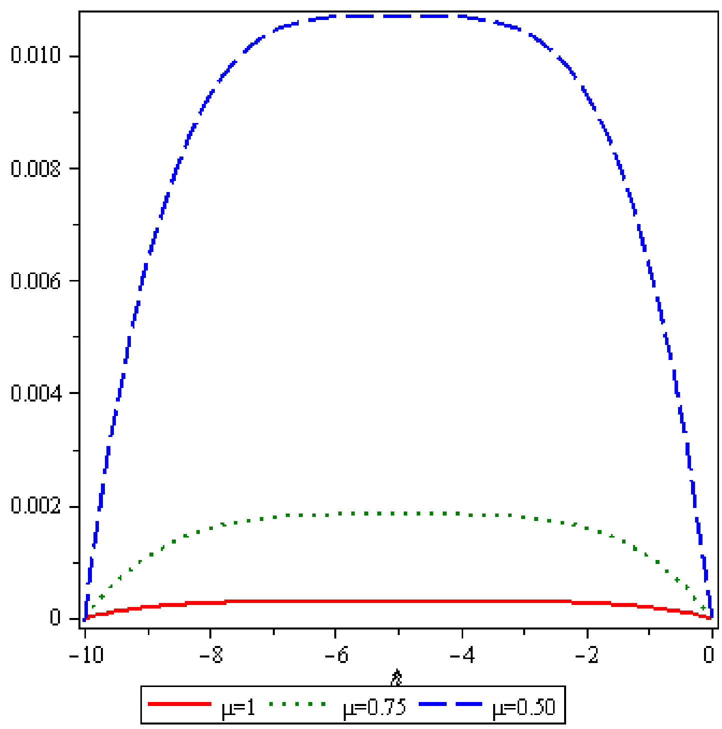

Figure 10. $\hbar$–curve of $y(\xi)$ for different values of μ at $n = 5$.

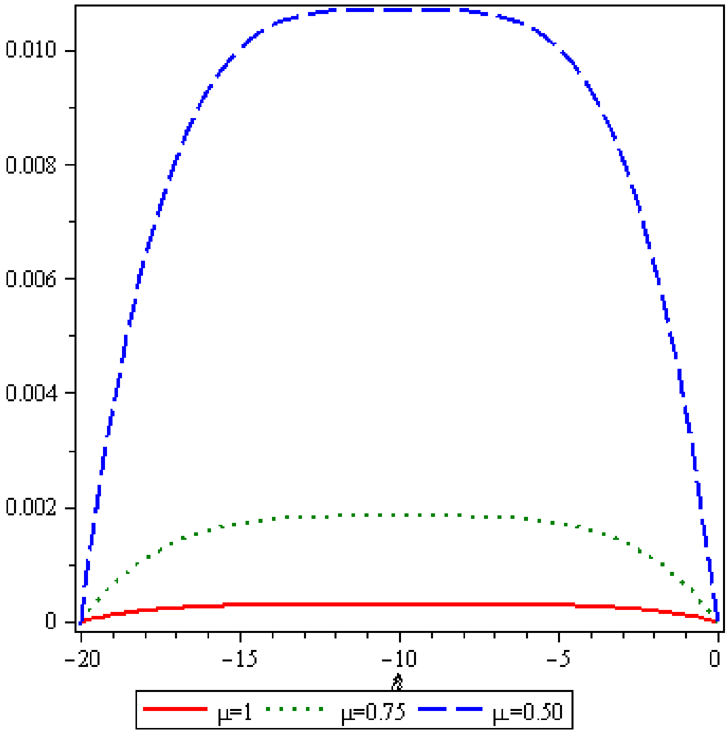

Figure 11. $\hbar$–curve of $y(\xi)$ for different values of μ at $n = 10$.

9. Conclusions

In this article, we have proposed a novel numerical method named q-HASTM to solve nonlinear dynamical models of interpersonal and romantic relationships for marriages with a fractional approach and compared the results with ADM. The numerical and graphical results reveal the successfully application of q-HASTM for solving the fractional dynamical model. The results derived with the aid of both the techniques are in an excellent agreement. The displacement shows a new nature for the time fractional derivative compared to the integer order derivative. The convergence of the q-HASTM solution can be adjusted and controlled with the aid of the auxiliary parameter $\hbar$ and asymptotic parameter n. The results presented in the form of graphs and $\hbar$-curves indicate that the suggested scheme is very efficient and accurate. Hence, it can be concluded that the proposed approach is highly logical and can be employed to investigate a wide range of nonlinear mathematical models of fractional order appearing in real world problems. Moreover, q-HASTM opens new doors in the fields of mathematical modeling and fractional calculus.

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