entropy

MDPI

Article

The Information Loss Problem: An Analogue Gravity Perspective

Stefano Liberati [1,2,3,*], **Giovanni Tricella** [1,2,3,*] and **Andrea Trombettoni** [1,2,3,4,*]

[1] SISSA-International School for Advanced Studies,Via Bonomea 265, 34136 Trieste, Italy
[2] INFN Sezione di Trieste, Via Valerio 2, 34127 Trieste, Italy
[3] IFPU-Institute for Fundamental Physics of the Universe, Via Beirut 2, 34014 Trieste, Italy
[4] CNR-IOM DEMOCRITOS Simulation Center, Via Bonomea 265, I-34136 Trieste, Italy
* Correspondence: liberati@sissa.it (S.L.); gtricell@sissa.it (G.T.); andreatr@sissa.it (A.T.)

Received: 21 August 2019; Accepted: 22 September 2019; Published: 25 September 2019

Abstract: Analogue gravity can be used to reproduce the phenomenology of quantum field theory in curved spacetime and in particular phenomena such as cosmological particle creation and Hawking radiation. In black hole physics, taking into account the backreaction of such effects on the metric requires an extension to semiclassical gravity and leads to an apparent inconsistency in the theory: the black hole evaporation induces a breakdown of the unitary quantum evolution leading to the so-called information loss problem. Here, we show that analogue gravity can provide an interesting perspective on the resolution of this problem, albeit the backreaction in analogue systems is not described by semiclassical Einstein equations. In particular, by looking at the simpler problem of cosmological particle creation, we show, in the context of Bose–Einstein condensates analogue gravity, that the emerging analogue geometry and quasi-particles have correlations due to the quantum nature of the atomic degrees of freedom underlying the emergent spacetime. The quantum evolution is, of course, always unitary, but on the whole Hilbert space, which cannot be exactly factorized a posteriori in geometry and quasi-particle components. In analogy, in a black hole evaporation one should expect a continuous process creating correlations between the Hawking quanta and the microscopic quantum degrees of freedom of spacetime, implying that only a full quantum gravity treatment would be able to resolve the information loss problem by proving the unitary evolution on the full Hilbert space.

Keywords: analogue gravity; Bose-Einstein condensation; information loss; cosmological particle creation

1. Introduction

Albeit being discovered more than 40 years ago, Hawking radiation is still at the center of much work in theoretical physics due to its puzzling features and its prominent role in connecting general relativity, quantum field theory, and thermodynamics. Among the new themes stimulated by Hawking's discovery, two have emerged as most pressing: the so-called transplanckian problem and the information loss problem.

The transplanckian problem stems from the fact that infrared Hawking quanta observed at late times at infinity seems to require the extension of relativistic quantum field theories in curved spacetime well within the UV completion of the theory, i.e., the Hawking calculation seems to require a strong assumption about the structure of the theory at the Planck scale and beyond.

With this open issue in mind, in 1981, Unruh introduced the idea to simulate in condensed matter systems black holes spacetime and the dynamics of fields above them [1]. Such analogue models of gravity are provided by several condensed-matter/optical systems in which the excitations propagate in an effectively relativistic fashion on an emergent pseudo-Riemannian geometry induced by the medium. Indeed, analogue gravity has played a pivotal role in the past years by providing a test bench for many open issues in quantum field theory in curved spacetime and in demonstrating the robustness of Hawking radiation and cosmological primordial spectrum of perturbations stemming from inflation against possible UV completions of the theory (see, e.g., the work by the authors of [2] for a comprehensive review). In recent years, the same models have offered a valuable framework within which current ideas about the emergence of spacetime and its dynamics could be discussed via convenient toy models [3–6].

Among the various analogue systems, a preeminent role has been played by Bose–Einstein condensates (BEC), because these are macroscopic quantum systems whose phonons/quasi-particle excitations can be meaningfully treated quantum mechanically. Therefore, they can be used to fully simulate the above mentioned quantum phenomena [7–9] and also as an experimental test bench of these ideas [10–13].

In what follows, we shall argue that these systems cannot only reproduce Hawking radiation and address the transplanckian problem, but can also provide a precious insight into the information loss problem. For gravitational black holes, the latter seems to be a direct consequence of the backreaction of Hawking radiation, leading to the decrease of the black hole mass and of the region enclosed by the horizon. The natural endpoint of such process into a complete evaporation of the object leads to a thermal bath over a flat spacetime, which appears to be incompatible with a unitary evolution of the quantum fields from the initial state to the final one. (We are not considering here alternative solutions such as long-living remnants, as these are as well problematic in other ways [14–18], or they imply deviations from the black hole structure at macroscopic scale, see, e.g., the work by the authors of [19]).

Of course, the BEC system at the fundamental level cannot violate unitary evolution. However, it is obvious that one can conceive analogue black holes provided with singular regions for the emergent spacetime where the description of quasi-particles propagating on an analogue geometry fails. (For example, one can describe flows characterized by regions where the hydrodynamical approximation fails even without necessarily having loss of atoms from the systems). In such cases, despite the full dynamics being unitary, it seems that a trace over the quasi-particle falling in these "analogue singularities" would be necessary, so leading to an apparent loss of unitarity from the analogue system point of view. The scope of the present investigation is to describe how such unitarity evolution is preserved on the full Hilbert space.

However, in addressing the information loss problem in gravity, the spacetime geometry and the quantum fields are implicitly assumed to be separated sectors of the Hilbert space. In the BEC analogue, this assumption is reflected in the approximation that the quantum nature of the operator $\hat{a}_{k=0}$, creating particles in the background condensate, can be neglected. Therefore, in standard analogue gravity, a description of the quantum evolution on the full Hilbert space seems precluded.

Nonetheless, it is possible to retain the quantum nature of the condensate operator as well as to describe their possible correlations with quasi-particles within an improved Bogoliubov description, namely, the number-conserving approach [20]. Remarkably, we shall show that the analogue gravity framework can be extended also in this context. Using the simpler setting of a cosmological particle creation, we shall describe how this entails the continuous generation of correlations between the condensate atoms and the quasi-particles. Such correlations are responsible for (and in turn consequence of) the nonfactorizability of the Hilbert space and are assuring in any circumstances the unitary evolution of the full system. The lesson to be drawn is that in gravitational systems only a full quantum gravity description could account for the mixing between gravitational and matter quantum degrees of freedom and resolve, in this way, the apparent paradoxes posed by black hole evaporation in quantum field theory on curved spacetime.

The paper is organized as follows. In Section 2, we briefly recall the analogue gravity model for a nonrelativistic BEC in mean-field approximation. In Section 3, we review the time-dependent orbitals formalism and how it is employed in the general characterization of condensates and in the description of the dynamics. In Section 4, we introduce the number-conserving formalism. In Section 5, we discuss the conditions under which we can obtain an analogue gravity model in this general case. Finally, we analyze, in Section 6, how in analogue gravity the quasi-particle dynamics affects the condensate and, in Section 7, how the condensate and the excited part of the full state are entangled, showing that the unitarity of the evolution is a feature of the system considered in its entirety. Section 8 is devoted to a discussion of the obtained results and of the perspectives opened by our findings.

2. Analogue Gravity

In this Section, we briefly review how to realize a set-up for analogue gravity [2] with BEC in the Bogoliubov approximation, with a bosonic low-energy (nonrelativistic) atomic system. In particular, we consider the simplest case, where the interaction potential is given by a local 2-body interaction. The Hamiltonian operator, the equation of motion, and commutation relations in second quantization formalism are as usual:

$$H = \int \mathrm{d}x \left[\phi^\dagger(x)\left(-\frac{\nabla^2}{2m}\phi(x)\right) + \frac{\lambda}{2}\phi^\dagger(x)\phi^\dagger(x)\phi(x)\phi(x)\right], \tag{1}$$

$$i\partial_t\phi(x) = [\phi(x), H] = -\frac{\nabla^2}{2m}\phi(x) + \lambda\phi^\dagger(x)\phi(x)\phi(x), \tag{2}$$

$$\left[\phi(x), \phi^\dagger(y)\right] = \delta(x-y). \tag{3}$$

For notational convenience, we dropped the time dependence of the bosonic field operator, ϕ, while retaining the dependence from the spatial coordinates x and y, and for simplicity, we omit the hat notation for the operators. Moreover, m is the atomic mass and the interaction strength λ, proportional to the scattering length [20], and could be taken as time-dependent. We also set $\hbar = 1$.

In the Bogoliubov approximation [20,21], the field operator is split into two contributions: a classical mean-field and a quantum fluctuation field (with vanishing expectation value),

$$\phi(x) = \langle\phi(x)\rangle + \delta\phi(x), \tag{4}$$

$$\left[\delta\phi(x), \delta\phi^\dagger(y)\right] = \delta(x-y). \tag{5}$$

The exact equations for the dynamics of these objects could be obtained from the full Equation (2), but two approximations ought to be considered:

$$i\partial_t\langle\phi\rangle = -\frac{\nabla^2}{2m}\langle\phi\rangle + \lambda\left\langle\phi^\dagger\phi\phi\right\rangle \approx -\frac{\nabla^2}{2m}\langle\phi\rangle + \lambda\overline{\langle\phi\rangle}\langle\phi\rangle\langle\phi\rangle, \tag{6}$$

$$i\partial_t\delta\phi = -\frac{\nabla^2}{2m}\delta\phi + \lambda\left(\phi^\dagger\phi\phi - \left\langle\phi^\dagger\phi\phi\right\rangle\right) \approx -\frac{\nabla^2}{2m}\langle\phi\rangle + 2\lambda\overline{\langle\phi\rangle}\langle\phi\rangle\delta\phi + \lambda\langle\phi\rangle^2\delta\phi^\dagger \tag{7}$$

the bar denoting complex conjugation. The first equation is the Gross–Pitaevskii equation, and we refer to the second as the Bogoliubov–de Gennes (operator) equation. The mean-field term represents the condensate wave function, and the approximation in Equation (6) is to remove from the evolution the backreaction of the fluctuation on the condensate. The second approximation is to drop all the nonlinear terms (of order higher than $\delta\phi$) from Equation (7), which then should be diagonalized to solve the time evolution.

The standard set-up for analogue gravity is obtained by describing the mean-field of the condensate wave function, and the fluctuations on top of it, in terms of number density and phase, as defined in the so-called Madelung representation

$$\langle\phi\rangle = \rho_0^{1/2}e^{i\theta_0}, \tag{8}$$

$$\delta\phi = \rho_0^{1/2}e^{i\theta_0}\left(\frac{\rho_1}{2\rho_0} + i\theta_1\right), \tag{9}$$

$$[\theta_1(x), \rho_1(y)] = -i\delta(x-y). \tag{10}$$

From the Gross–Pitaevskii equation, we obtain two equations for the real classical fields θ_0 and ρ_0:

$$\partial_t \rho_0 = -\frac{1}{m} \nabla \left(\rho_0 \nabla \theta_0 \right) , \tag{11}$$

$$\partial_t \theta_0 = -\lambda \rho_0 + \frac{1}{2m} \rho_0^{-1/2} \left(\nabla^2 \rho_0^{1/2} \right) - \frac{1}{2m} \left(\nabla \theta_0 \right) \left(\nabla \theta_0 \right) . \tag{12}$$

These are the quantum Euler equations for the superfluid. Equation (11) can be easily interpreted as a continuity equation for the density of the condensate, whereas Equation (12) is the Bernoulli equation for the phase of the superfluid, which generates the potential flow: the superfluid has velocity $\left(\nabla \theta_0 \right) / m$. From the Bogoliubov–de Gennes Equation, we obtain two equations for the real quantum fields θ_1 and ρ_1

$$\partial_t \rho_1 = -\frac{1}{m} \nabla \left(\rho_1 \nabla \theta_0 + \rho_0 \nabla \theta_1 \right) , \tag{13}$$

$$\partial_t \theta_1 = -\left(\lambda \rho_0 + \frac{1}{4m} \nabla \left(\rho_0^{-1} \left(\nabla \rho_0 \right) \right) \right) \frac{\rho_1}{\rho_0} + \frac{1}{4m} \nabla \left(\rho_0^{-1} \left(\nabla \rho_1 \right) \right) - \frac{1}{m} \left(\nabla \theta_0 \right) \left(\nabla \theta_1 \right) . \tag{14}$$

If in Equation (14) the "quantum pressure" term $\nabla \left(\rho_0^{-1} \left(\nabla \rho_1 \right) \right) / 4m$ is negligible, as usually assumed, by substitution we obtain

$$\rho_1 = -\frac{\rho_0}{\lambda \rho_0 + \frac{1}{4m} \nabla \left(\rho_0^{-1} \left(\nabla \rho_0 \right) \right)} \left(\left(\partial_t \theta_1 \right) + \frac{1}{m} \left(\nabla \theta_0 \right) \left(\nabla \theta_1 \right) \right) , \tag{15}$$

$$\begin{aligned} 0 = \partial_t & \left(\frac{\rho_0}{\lambda \rho_0 + \frac{1}{4m} \nabla \left(\rho_0^{-1} \left(\nabla \rho_0 \right) \right)} \left(\left(\partial_t \theta_1 \right) + \frac{1}{m} \left(\nabla \theta_0 \right) \left(\nabla \theta_1 \right) \right) \right) \\ & + \nabla \left(\frac{\rho_0}{\lambda \rho_0 + \frac{1}{4m} \nabla \left(\rho_0^{-1} \left(\nabla \rho_0 \right) \right)} \frac{1}{m} \left(\nabla \theta_0 \right) \left(\left(\partial_t \theta_1 \right) + \frac{1}{m} \left(\nabla \theta_0 \right) \left(\nabla \theta_1 \right) \right) - \frac{\rho_0}{m} \left(\nabla \theta_1 \right) \right) \end{aligned} \tag{16}$$

with Equation (16) being a Klein–Gordon equation for the field θ_1. Equation (16) can be written in the form

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \theta_1 \right) = 0 , \tag{17}$$

where we have introduced the prefactor $1/\sqrt{-g}$ with g being as usual the determinant of the Lorentzian metric $g_{\mu\nu}$. This equation describes an analogue system for a scalar field in curved spacetime as the quantum field θ_1 propagates on a curved geometry with a metric given by

$$\widetilde{\lambda} = \lambda + \frac{1}{4m} \rho_0^{-1} \nabla \left(\rho_0^{-1} \left(\nabla \rho_0 \right) \right) , \tag{18}$$

$$v_i = \frac{1}{m} \left(\nabla \theta_0 \right)_i , \tag{19}$$

$$\sqrt{-g} = \sqrt{\frac{\rho_0^3}{m^3 \widetilde{\lambda}}} , \tag{20}$$

$$g_{tt} = -\sqrt{\frac{\rho_0}{m \widetilde{\lambda}}} \left(\frac{\widetilde{\lambda} \rho_0}{m} - v^2 \right) , \tag{21}$$

$$g_{ij} = \sqrt{\frac{\rho_0}{m \widetilde{\lambda}}} \delta_{ij} , \tag{22}$$

$$g_{ti} = -\sqrt{\frac{\rho_0}{m \widetilde{\lambda}}} v_i , \tag{23}$$

where the latin indices i and j characterize spatial components.

If the condensate is homogeneous the superfluid velocity vanishes, the coupling is homogeneous in space ($\widetilde{\lambda} = \lambda$), the number density ρ_0 is constant in time, and the only relevant behavior of the condensate wave function is in the time-dependent phase θ_0. Furthermore, in this case there is also no need to neglect the quantum pressure term in Equation (14), as it will be handled easily after Fourier-transforming and it will simply introduce a modified dispersion relation—directly derived from the Bogoliubov spectrum.

If the condensate has an initial uniform number density but is not homogeneous—meaning that the initial phase depends on the position—the evolution will introduce inhomogeneities in the density ρ_0, as described by the continuity equation Equation (11), and the initial configuration will be deformed in time. However, as long as $\nabla^2\theta_0$ is small, also the variations of ρ_0 are small as well: while there is not a nontrivial stationary analogue metric, the scale of the inhomogeneities will define a timescale for which one could safely assume stationarity. Furthermore, the presence of an external potential $V_{ext}(x)$ in the Hamiltonian, via a term of the form $\int \mathrm{d}x V_{ext}(x)\,\phi^{\dagger}(x)\,\phi(x)$, would play a role in the dynamical equation for θ_0, leaving invariant those for ρ_0, ρ_1, and θ_1.

3. Time-Dependent Natural Orbitals

The mean-field approximation presented in the previous section is a solid and consistent formulation for studying weakly interacting BEC [22]. It requires, however, that the quantum state has peculiar features which need to be taken into account. In analogue gravity, these assumptions are tacitly considered, but as they play a crucial role for our treatment, we present a discussion of them in some detail to lay down the ground and the formalism in view of next sections.

As is well known [22], the mean-field approximation, consisting in substituting the operator $\phi(x)$ with its expectation value $\langle\phi(x)\rangle$, is strictly valid when the state considered is coherent, meaning it is an eigenstate of the atomic field operator ϕ:

$$\phi(x)\,|coh\rangle = \langle\phi(x)\rangle\,|coh\rangle \ . \tag{24}$$

For states satisfying this equation, the Gross–Pitaevskii Equation (6) is exact (whereas Equation (7) is still a linearized approximation). Note that the coherent states $|coh\rangle$ are not eigenstates of the number operator, but they are rather quantum superpositions of states with different number of atoms. This is necessary because ϕ is an operator that—in the nonrelativistic limit—destroys a particle. We also observe that the notion of coherent state is valid instantaneously, but it may be in general not preserved along the evolution in presence of an interaction.

The redefinition of the field operator, as in Equation (4), provides a description where the physical degrees of freedom are concealed: the new degrees of freedom are not the excited atoms, but the quantized fluctuations over a coherent state. Formally, this is a simple and totally legit redefinition, but for our discussion, we stress that the quanta created by the operator $\delta\phi$ do not have a direct interpretation as atoms.

Given the above discussion, it is useful to remember that coherent states are not the only states to express the condensation, i.e., the fact that a macroscopic number of particles occupies the same state. As it is stated in the Penrose–Onsager criterion for off-diagonal long-range-order [23,24], the condensation phenomenon is best defined considering the properties of the 2-point correlation functions.

The 2-point correlation function is the expectation value on the quantum state of an operator composed of the creation of a particle in a position x after the destruction of a particle in a different position y: $\langle\phi^{\dagger}(x)\,\phi(y)\rangle$. As, by definition, the 2-point correlation function is Hermitian, $\overline{\langle\phi^{\dagger}(y)\,\phi(x)\rangle} = \langle\phi^{\dagger}(x)\,\phi(y)\rangle$, it can always be diagonalized as

$$\left\langle\phi^{\dagger}(x)\,\phi(y)\right\rangle = \sum_I \langle N_I\rangle\,\overline{f_I}(x)\,f_I(y)\,, \tag{25}$$

with

$$\int \mathrm{d}x\overline{f_I}(x)\, f_J(x) = \delta_{IJ}\,. \tag{26}$$

The orthonormal functions f_I, eigenfunctions of the 2-point correlation function, are known as the natural orbitals, and define a complete basis for the 1-particle Hilbert space. In the case of a time-dependent Hamiltonian (or during the dynamics), they are in turn time-dependent. As for the field operator, to simplify the notation, we will not always explicitly write the time dependence of f_I.

The eigenvalues $\langle N_I \rangle$ are the occupation numbers of these wave functions. The sum of these eigenvalues gives the total number of particles in the state (or the mean value, in the case of superposition of quantum states with different number of particles):

$$\langle N \rangle = \sum_I \langle N_I \rangle\ . \tag{27}$$

The time-dependent orbitals define creation and destruction operators, and consequently the relative number operators (having as expectation values the eigenvalues of the 2-point correlation function):

$$a_I = \int \mathrm{d}x\overline{f_I}(x)\,\phi(x)\ , \tag{28}$$

$$\left[a_I, a_J^\dagger\right] = \delta_{IJ}\,, \tag{29}$$

$$\left[a_I, a_J\right] = 0\,, \tag{30}$$

$$N_I = a_I^\dagger a_I\,. \tag{31}$$

The state is called "condensate" [23] when one of these occupation numbers is macroscopic (comparable with the total number of particles) and the others are small when compared to it.

In the weakly interacting limit, the condensed fraction $\langle N_0 \rangle / \langle N \rangle$ is approximately equal to 1, and the depletion factor $\sum_{I\neq 0} \langle N_I \rangle / \langle N \rangle$ is negligible. This requirement is satisfied by coherent states that define perfect condensates, as the 2-point correlation functions are a product of the mean-field and its conjugate:

$$\langle coh|\, \phi^\dagger(x)\,\phi(y)\,|coh\rangle = \overline{\langle \phi(x) \rangle}\,\langle \phi(y) \rangle\ , \tag{32}$$

with

$$f_0(x) = \langle N_0 \rangle^{-1/2} \langle \phi(x) \rangle\ , \tag{33}$$

$$\langle N_0 \rangle = \int \mathrm{d}y \overline{\langle \phi(y) \rangle}\,\langle \phi(y) \rangle\ , \tag{34}$$

$$\langle N_{I\neq 0} \rangle = 0\,. \tag{35}$$

Therefore, in this case, the set of time-dependent orbitals is given by the proper normalization of the mean-field function with a completion that is the basis for the subspace of the Hilbert space orthogonal to the mean-field. The latter set can be redefined arbitrarily, as the only nonvanishing eigenvalue of the 2-point correlation function is the one relative to mean-field function. The fact that there is a nonvanishing macroscopic eigenvalue implies that there is total condensation, i.e., $\langle N_0 \rangle / \langle N \rangle = 1$.

3.1. Time-Dependent Orbitals Formalism

It is important to understand how we can study the condensate state even if we are not considering coherent states and how the description is related to the mean-field approximation. In this framework, we shall see that the mean-field approximation is not a strictly necessary theoretical requirement for analogue gravity.

With respect to the basis of time-dependent orbitals and their creation and destruction operators, we can introduce a new expression for the atomic field operator, projecting it on the sectors of the Hilbert space as

$$\begin{aligned}\phi(x) &= \phi_0(x) + \phi_1(x) \\ &= f_0(x)\, a_0 + \sum_I f_I(x)\, a_I \\ &= f_0(x)\left(\int \mathrm{d}y \overline{f_0}(y)\,\phi(y)\right) + \sum_{I\neq 0} f_I(x)\left(\int \mathrm{d}y \overline{f_I}(y)\,\phi(y)\right).\end{aligned} \tag{36}$$

The two parts of the atomic field operator so defined are related to the previous mean-field $\langle\phi\rangle$ and quantum fluctuation $\delta\phi$ expressions given in Section 2. The standard canonical commutation relation of the background field is of order V^{-1}, where V denotes the volume of the system

$$\left[\phi_0(x), \phi_0^\dagger(y)\right] = f_0(x)\,\overline{f_0}(y) = \mathcal{O}\left(V^{-1}\right). \tag{37}$$

Note that although the commutator (37) does not vanish identically, it is negligible in the limit of large V.

In the formalism (36), the condensed part of the field is described by the operator ϕ_0 and the orbital producing it through projection, the 1-particle wave function f_0. The dynamics of the function f_0, the 1-particle wave function that best describes the collective behavior of the condensate, can be extracted by using the relations

$$\left\langle \phi^\dagger(x)\,\phi(y)\right\rangle = \sum_I \langle N_I\rangle\, \overline{f_I}(x)\, f_I(y)\,, \tag{38}$$

$$\left\langle \left[a_K^\dagger a_J, H\right]\right\rangle = i\partial_t \langle N_J\rangle\,\delta_{JK} + i\left(\langle N_K\rangle - \langle N_J\rangle\right)\left(\int \mathrm{d}x \overline{f_J}(x)\, \dot{f}_K(x)\right) \tag{39}$$

and the evolution of the condensate 1-particle wave function

$$\begin{aligned} i\partial_t f_0(x) &= \left(\int \mathrm{d}y \overline{f_0}(y)\left(i\partial_t f_0(y)\right)\right) f_0(x) + \sum_{I\neq 0}\left(\int \mathrm{d}y \overline{f_0}(y)\left(i\partial_t f_I(y)\right)\right) f_I(x) \\ &= \left(\int \mathrm{d}y \overline{f_0}(y)\left(i\partial_t f_0(y)\right)\right) f_0(x) + \sum_{I\neq 0}\frac{1}{\langle N_0\rangle - \langle N_I\rangle}\left\langle\left[a_0^\dagger a_I, H\right]\right\rangle f_I(x) \\ &= -\frac{i}{2}\frac{\partial_t\langle N_0\rangle}{\langle N_0\rangle} f_0(x) + \left(-\frac{\nabla^2}{2m} f_0(x)\right) + \frac{1}{\langle N_0\rangle}\left\langle a_0^\dagger\left[\phi(x), V\right]\right\rangle + \\ &\quad + \sum_{I\neq 0}\frac{\langle N_I\rangle\left\langle a_0^\dagger\left[a_I, V\right]\right\rangle + \langle N_0\rangle\left\langle\left[a_0^\dagger, V\right] a_I\right\rangle}{\langle N_0\rangle\left(\langle N_0\rangle - \langle N_I\rangle\right)} f_I(x) \end{aligned} \tag{40}$$

we assumed at any time $\langle N_0\rangle \neq \langle N_{I\neq 0}\rangle$. The above equation is valid for a condensate when the dynamics is driven by a Hamiltonian operator composed of a kinetic term and a generic potential V, but we are interested in the case of Equation (1). Furthermore, $f_0(x)$ can be redefined through an overall phase transformation, $f_0(x) \to e^{i\Theta} f_0(x)$, with any arbitrary time-dependent real function Θ. We have chosen the overall phase to satisfy the final expression Equation (40), as it is the easiest to compare with the Gross–Pitaevskii Equation (6).

3.2. Connection with the Gross–Pitaevskii Equation

In this section, we discuss the relation between the function f_0—the eigenfunction of the 2-point correlation function with macroscopic eigenvalue—and the solution of the Gross–Pitaevskii equation, approximating the mean-field function for quasi-coherent states. In particular, we aim at comparing the equations describing their dynamics, detailing under which approximations they show the same behavior. This discussion provides a preliminary technical basis for the study of the effect of the quantum correlations between the background condensate and the quasi-particles, which are present when the quantum nature of the condensate field operator is retained and it is not just approximated by a number, as done when performing the standard Bogoliubov approximation. We refer to the work by the authors of [21] for a review on the Bogoliubov approximation in weakly imperfect Bose gases and to the work by the authors of [25] for a presentation of rigorous results on the condensation properties of dilute bosons.

The Gross–Pitaveskii Equation (6) is an approximated equation for the mean-field dynamics. It holds when the backreaction of the fluctuations $\delta\phi$ on the condensate—described by a coherent state—is negligible. This equation includes a notion of number conservation, meaning that the approximation of the interaction term implies that the spatial integral of the squared norm of the solution of the equation is conserved:

$$\left\langle \phi^\dagger(x)\,\phi(x)\,\phi(x)\right\rangle \approx \overline{\langle\phi(x)\rangle}\,\langle\phi(x)\rangle\,\langle\phi(x)\rangle \tag{41}$$

$$\Downarrow$$

$$i\partial_t \int \mathrm{d}x\overline{\langle\phi(x)\rangle}\,\langle\phi(x)\rangle = 0\,. \tag{42}$$

This depends on the fact that only the leading term of the interaction is included in the equation. Therefore we can compare the Gross–Pitaevskii equation for the mean-field with the equation for $\langle N_0\rangle^{1/2} f_0(x)$ approximated to leading order, i.e., $\langle\phi(x)\rangle$ should be compared to the function $f_0(x)$ under the approximation that there is no depletion from the condensate. If we consider the approximations

$$\left\langle a_0^\dagger\,[\phi(x)\,,V]\right\rangle \approx \lambda\,\langle N_0\rangle^2\,\overline{f_0}(x)\,f_0(x)\,f_0(x)\;, \tag{43}$$

$$i\partial_t\,\langle N_0\rangle = \left\langle \left[a_0^\dagger a_0, V\right]\right\rangle \approx 0\,, \tag{44}$$

$$\sum_{I\neq 0} \frac{\langle N_I\rangle\,\langle a_0^\dagger\,[a_I,V]\rangle + \langle N_0\rangle\,\langle [a_0^\dagger,V]\,a_I\rangle}{\langle N_0\rangle - \langle N_I\rangle} f_I(x) \approx 0\,, \tag{45}$$

we obtain that $\langle N_0\rangle^{1/2} f_0(x)$ satisfies the Gross–Pitaevskii equation.

The approximation in Equation (43) is easily justified, as we are retaining only the leading order of the expectation value $\langle a_0^\dagger\,[\phi,V]\rangle$ and neglecting the others, which depend on the operators ϕ_1 and $\phi_1^\dagger$ and are of order smaller than $\langle N_0\rangle^2$. The second equation Equation (44) is derived from the previous one as a direct consequence, as the depletion of N_0 comes from the subleading terms $\langle a_0^\dagger\phi_1^\dagger\phi_1 a_0\rangle$ and $\langle a_0^\dagger a_0^\dagger\phi_1\phi_1\rangle$. The first of these two terms is of order $\langle N_0\rangle$, having its main contributions from separable expectation values—$\langle a_0^\dagger\phi_1^\dagger\phi_1 a_0\rangle \approx \langle N_0\rangle\,\langle\phi_1^\dagger\phi_1\rangle$—and the second is of the same order due to the dynamics. The other terms are even more suppressed, as can be argued considering that they contain an odd number of operators ϕ_1. Taking their time derivatives, we observe that they arise from the second order in the interaction, making these terms negligible in the regime of weak interaction.

The terms $\langle a_0^\dagger a_0^\dagger a_0\phi_1\rangle$ are also subleading with respect to those producing the depletion, since the separable contributions—$\langle a_0^\dagger a_0\rangle\,\langle a_0^\dagger\phi_1\rangle$—vanish by definition, and the remaining describe the correlation between small operators, acquiring relevance only while the many-body quantum state is mixed by the depletion of the condensate:

$$\left\langle a_0^\dagger a_0^\dagger a_0\phi_1\right\rangle = \left\langle a_0^\dagger\phi_1\,(N_0 - \langle N_0\rangle)\right\rangle\,. \tag{46}$$

Using the same arguments we can assume the approximation in Equation (45) to hold, as the denominator of order $\langle N_0 \rangle$ is sufficient to suppress the terms in the numerator, which are negligible with respect to the leading term in Equation (43).

The leading terms in Equation (45) do not affect the depletion of N_0, but they may be of the same order. They depend on the expectation value

$$\left\langle a_0^\dagger \left[a_I, V \right] \right\rangle \approx \lambda \left(\int \mathrm{d}x \overline{f_I} \left(x \right) \overline{f_0} \left(x \right) f_0 \left(x \right) f_0 \left(x \right) \right) \left\langle N_0 \right\rangle^2 . \tag{47}$$

Therefore, these terms with the mixed action of ladder operators relative to the excited part and the condensate are completely negligible when the integral $\int \overline{f_I f_0} f_0 f_0$ is sufficiently small. This happens when the condensed 1-particle state is approximately $f_0 \approx V^{-1/2} e^{i\theta_0}$, i.e., when the atom number density of the condensate is approximately homogeneous.

Moreover, in many cases of interest, it often holds that the terms in the LHS of Equation (45) vanish identically: if the quantum state is an eigenstate of a conserved charge, e.g., total momentum or total angular momentum, the orbitals must be labeled with a specific value of charge. The relative ladder operators act by adding or removing from the state such charge, and for any expectation value not to vanish the charges must cancel out. In the case of homogeneity of the condensate and translational invariance of the Hamiltonian, this statement regards the conservation of momentum. In particular, if the state is invariant under translations, we have

$$\left\langle a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \right\rangle = \delta_{k_1+k_2, k_3+k_4} \left\langle a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \right\rangle , \tag{48}$$

$$\left\langle a_0^\dagger a_0^\dagger a_0 a_k \right\rangle = 0 \tag{49}$$

(for $k \neq 0$).

In conclusion, we obtain that for a condensate with a quasi-homogeneous density a good approximation for the dynamics of the function $\langle N_0 \rangle^{1/2} f_0$, the rescaled 1-particle wave function macroscopically occupied by the condensate, is provided by

$$\begin{aligned} i\partial_t \left(\langle N_0 \rangle^{1/2} f_0 \left(x \right) \right) \quad &= -\tfrac{\nabla^2}{2m} \left(\langle N_0 \rangle^{1/2} f_0 \left(x \right) \right) + \lambda \left\langle N_0 \right\rangle^{3/2} \overline{f_0} \left(x \right) f_0 \left(x \right) f_0 \left(x \right) \\ &+ \lambda \left\langle N_0 \right\rangle^{-1} \left(\langle a_0^\dagger \phi_1 \left(x \right) \phi_1^\dagger \left(x \right) a_0 \rangle + \langle a_0^\dagger \phi_1^\dagger \left(x \right) \phi_1 \left(x \right) a_0 \rangle \right) \left(\langle N_0 \rangle^{1/2} f_0 \left(x \right) \right) \\ &+ \lambda \left\langle N_0 \right\rangle^{-1} \langle a_0^\dagger \phi_1 \left(x \right) a_0^\dagger \phi_1 \left(x \right) \rangle \left(\langle N_0 \rangle^{1/2} \overline{f_0} \left(x \right) \right) + \mathcal{O} \left(\lambda \left\langle N_0 \right\rangle^{1/2} V^{-3/2} \right) . \end{aligned} \tag{50}$$

This equation is equivalent to the Gross–Pitaveskii equation Equation (6) when we consider only the leading terms, i.e., the first line of Equation (50). If we also consider the remaining lines of the Equation (50), i.e., if we include the effect of the depletion, we obtain an equation that should be compared to the equation for the mean-field function up to the terms quadratic in the operators $\delta\phi$. The two equations are analogous when making the identification:

$$\langle N_0 \rangle^{1/2} f_0 \sim \langle \phi \rangle , \tag{51}$$

$$\left\langle \phi_0^\dagger \phi_1^\dagger \phi_1 \phi_0 \right\rangle \sim \overline{\langle \phi \rangle} \langle \phi \rangle \left\langle \delta\phi^\dagger \delta\phi \right\rangle , \tag{52}$$

$$\left\langle \phi_0^\dagger \phi_0^\dagger \phi_1 \phi_1 \right\rangle \sim \overline{\langle \phi \rangle \langle \phi \rangle} \left\langle \delta\phi \delta\phi \right\rangle . \tag{53}$$

The possible ambiguities in comparing the two equations come from the arbitrariness in fixing the overall time-dependent phases of the functions f_0 and $\langle \phi \rangle$, and from the fact that the commutation relations for the operators ϕ_1 and the operators $\delta\phi$ differ from each other by a term going as $\overline{f_0} f_0$, as seen in Equation (37). This causes an apparent difference when comparing the two terms:

$$\lambda \left\langle N_0 \right\rangle^{-1} \left(\langle a_0^\dagger \phi_1 \left(x \right) \phi_1^\dagger \left(x \right) a_0 \rangle + \langle a_0^\dagger \phi_1^\dagger \left(x \right) \phi_1 \left(x \right) a_0 \rangle \right) \left(\langle N_0 \rangle^{1/2} f_0 \left(x \right) \right) \sim 2\lambda \left\langle \delta\phi^\dagger \left(x \right) \delta\phi \left(x \right) \right\rangle \left\langle \phi \left(x \right) \right\rangle . \tag{54}$$

However, the difference can be reabsorbed—manipulating the RHS—in a term which only affects the overall phase of the mean-field, not the superfluid velocity.

The equation for the depletion can be easily derived for the number-conserving approach and compared to the result in the Bogoliubov approach. As seen, the dynamical equation for f_0 contains the information for the time derivative of its occupation number. Projecting the derivative along the function itself and taking the imaginary part, one gets

$$\begin{aligned}\frac{1}{2}i\partial_t \langle N_0 \rangle &= \frac{1}{2}\left\langle \left[a_0^\dagger a_0, V\right]\right\rangle \\ &= i\mathrm{Im}\left(\int \mathrm{d}x \,\langle N_0\rangle^{1/2}\,\overline{f_0}\,(x)\, i\partial_t \left(\langle N_0\rangle^{1/2} f_0\,(x)\right)\right) \\ &\approx \frac{\lambda}{2}\left(\int \mathrm{d}x \left\langle \phi_0^\dagger\,(x)\,\phi_0^\dagger\,(x)\,\phi_1\,(x)\,\phi_1\,(x)\right\rangle - \left\langle \phi_1^\dagger\,(x)\,\phi_1^\dagger\,(x)\,\phi_0\,(x)\,\phi_0\,(x)\right\rangle\right). \end{aligned} \tag{55}$$

We can now compare this equation to the one for the depletion in mean-field description

$$\begin{aligned}\frac{1}{2}i\partial_t N &= \frac{1}{2}i\partial_t\left(\int \mathrm{d}x \overline{\langle\phi\,(x)\rangle}\,\langle\phi\,(x)\rangle\right) \\ &= i\mathrm{Im}\left(\int \overline{\langle\phi\,(x)\rangle}\, i\partial_t\,\langle\phi\,(x)\rangle\right) \\ &\approx \frac{\lambda}{2}\left(\int \overline{\langle\phi\,(x)\rangle\langle\phi\,(x)\rangle}\,\langle\delta\phi\,(x)\,\delta\phi\,(x)\rangle - \langle\phi\,(x)\rangle\,\langle\phi\,(x)\rangle\left\langle\delta\phi^\dagger\,(x)\,\delta\phi^\dagger\,(x)\right\rangle\right). \end{aligned} \tag{56}$$

The two expressions are consistent with each other

$$\left\langle \phi_0^\dagger\,(x)\,\phi_0^\dagger\,(x)\,\phi_1\,(x)\,\phi_1\,(x)\right\rangle \sim \overline{\langle\phi\,(x)\rangle\langle\phi\,(x)\rangle}\,\langle\delta\phi\,(x)\,\delta\phi\,(x)\rangle\,. \tag{57}$$

For coherent states, one expects to find equivalence between $\delta\phi$ and ϕ_1. To do so, we need to review the number-conserving formalism that can provide the same description used for analogue gravity in the general case, e.g., when there is a condensed state with features different from those of coherent states.

4. Number-Conserving Formalism

Within the mean-field framework, the splitting of the field obtained by translating the field operator ϕ by the mean-field function produces the new field $\delta\phi$. This redefinition of the field also induces a corresponding one of the Fock space of the many-body states which, to a certain extent, hides the physical atom degrees of freedom, as the field $\delta\phi$ describes the quantum fluctuations over the mean-field wave function instead of atoms.

Analogue gravity is defined, considering this field and its Hermitian conjugate, and properly combined, to study the fluctuation of phase. The fact that $\delta\phi$ is obtained by translation provides this field with canonical commutation relation. The mean-field description for condensates holds for coherent states and is a good approximation for quasi-coherent states.

When we consider states with fixed number of atoms, and therefore not coherent states, it is better to consider different operators to study the fluctuations. One can do it following the intuition that the fluctuation represents a shift of a single atom from the condensate to the excited fraction and vice versa. Our main reason here to proceed this way is that we are forced to retain the quantum nature of the condensate. We therefore want to adopt the established formalism of number-conserving ladder operators (see, e.g., the work by the authors of [20]) to obtain a different expression for the Bogoliubov–de Gennes equation, studying the excitations of the condensate in these terms. We can adapt this discussion to the time-dependent orbitals.

An important remark is that the qualitative point of introducing the number-conserving approach is conceptually separated from the fact that higher-order terms are neglected by Bogoliubov approximations anyway [20,26]. Indeed, neglecting the commutation relations for a_0 would always imply the impossibility to describe the correlations between quasi-particles and condensate, even when going beyond the Bogoliubov approximation (e.g., by adding terms with three quasi-particle operators). Including such terms, in a growing level of accuracy (and complexity), the main difference would be that the true quasi-particles of the systems do no longer coincide with the Bogoliubov ones. From a practical point of view, this makes clearly a (possibly) huge quantitative difference for the energy spectrum, correlations between quasi-particles, transport properties and observables. Nevertheless, this does not touch at heart that the quantum nature of the condensate is not retained. A discussion of the terms one can include beyond Bogoliubov approximation, and the resulting hierarchy of approximations, is presented in the work by the authors of [26]. Here, our point is rather of principle, i.e., the investigation of the consequences of retaining the operator nature of a_0. Therefore, we used the standard Bogoliubov approximation, improved via the introduction of number-conserving operators.

If we consider the ladder operators a_I, satisfying by definition the relations in Equations (29)–(30), and keep as reference the state a_0 for the condensate, it is a straightforward procedure to define the number-conserving operators $\alpha_{I\neq 0}$, one for each excited wave function, according to the relations

$$\alpha_I = N_0^{-1/2} a_0^\dagger a_I \,, \tag{58}$$

$$\left[\alpha_I, \alpha_J^\dagger\right] = \delta_{IJ} \qquad \forall I, J \neq 0 \,, \tag{59}$$

$$\left[\alpha_I, \alpha_J\right] = 0 \qquad \forall I, J \neq 0 \,. \tag{60}$$

The degree relative to the condensate is absorbed into the definition, from the hypothesis of number conservation. These relations hold for $I, J \neq 0$, and obviously there is no number-conserving ladder operator relative to the condensed state. The operators α_I are not a complete set of operators to describe the whole Fock space, but they span any subspace of given number of total atoms. To move from one another it would be necessary to include the operator a_0.

This restriction to a subspace of the Fock space is analogous to what is implicitly done in the mean-field approximation, where one considers the subspace of states which are coherent with respect to the action of the destruction operator associated to the mean-field function.

In this set-up, we need to relate the excited part described by ϕ_1 to the usual translated field $\delta\phi$, and obtain an equation for its dynamics related to the Bogoliubov–de Gennes equation. To do so, we need to study the linearization of the dynamics of the operator ϕ_1, combined with the proper operator providing the number conservation

$$\begin{aligned} N_0^{-1/2} a_0^\dagger \phi_1 &= N_0^{-1/2} a_0^\dagger \left(\phi - \phi_0\right) \\ &= \sum_{I\neq 0} f_I \alpha_I \,. \end{aligned} \tag{61}$$

As long as the approximations needed to write a closed dynamical equation for ϕ_1 are compatible with those under which the equation for the dynamics of f_0 resembles the Gross–Pitaevskii equation, i.e., as long as the time derivative of the operators α_I can be written as a combination of the α_I themselves, we can expect to have a set-up for analogue gravity. In fact, in this case, the functional form of the dynamical equations of the system will allow following the standard steps of the derivation reviewed in Section 2.

Therefore, we consider the order of magnitude of the various contributions to the time derivative of $\left(N_0^{-1/2} a_0^\dagger \phi_1\right)$. We have already discussed the time evolution of the function f_0: from the latter it depends the evolution of the operator a_0, since it is the projection along f_0 of the full field operator ϕ.

At first we observed that the variation in time of N_0 must be of smaller order, both for the definition of condensation and because of the approximations considered in the previous section:

$$\begin{aligned} i\partial_t N_0 &= i\partial_t \left(\int \mathrm{d}x\mathrm{d}y f_0(x) \overline{f_0}(y) \phi^\dagger(x) \phi(y) \right) \\ &= \left[a_0^\dagger a_0, V \right] + \sum_{I\neq 0} \frac{1}{\langle N_0 \rangle - \langle N_I \rangle} \left(\left\langle \left[a_0^\dagger a_I, V \right] \right\rangle a_I^\dagger a_0 + \left\langle \left[a_I^\dagger a_0, V \right] \right\rangle a_0^\dagger a_I \right) . \end{aligned} \tag{62}$$

Of these terms, the summations are negligible due to the prefactors $\langle [a_I^\dagger a_0, V] \rangle$ and their conjugates. Moreover, as $\langle a_I^\dagger a_0 \rangle$ is vanishing, the dominant term is the first, which has contributions of at most the order of the depletion factor.

The same can be argued for both the operators a_0 and $N_0^{-1/2}$: their derivatives do not provide leading terms when we consider the derivative of the composite operator $\left(N_0^{-1/2} a_0^\dagger \phi_1 \right)$, only the derivative of the last operator ϕ_1 being relevant. At leading order, we have

$$i\partial_t \left(N_0^{-1/2} a_0^\dagger \phi_1 \right) \approx i \left(N_0^{-1/2} a_0^\dagger \right) (\partial_t \phi_1) . \tag{63}$$

We, therefore, have to analyze the properties of $\partial_t \phi$, considering the expectation values between the orthogonal components ϕ_0 and ϕ_1 and their time derivatives:

$$\left\langle \phi_0^\dagger(y) \left(i\partial_t \phi_0(x) \right) \right\rangle = \left(\langle N_0 \rangle^{1/2} \overline{f_0}(y) \right) i\partial_t \left(\langle N_0 \rangle^{1/2} f_0(x) \right) , \tag{64}$$

$$\begin{aligned} \left\langle \phi_0^\dagger(y) \left(i\partial_t \phi_1(x) \right) \right\rangle &= - \sum_{I\neq 0} \overline{f_0}(y) f_I(x) \frac{\langle N_I \rangle \langle a_0^\dagger [a_I, V] \rangle + \langle N_0 \rangle \langle [a_0^\dagger, V] a_I \rangle}{\langle N_0 \rangle - \langle N_I \rangle} \\ &= - \left\langle \left(i\partial_t \phi_0^\dagger(y) \right) \phi_1(x) \right\rangle , \end{aligned} \tag{65}$$

$$\begin{aligned} \left\langle \phi_1^\dagger(y) \left(i\partial_t \phi_1(x) \right) \right\rangle &= \left\langle \phi_1^\dagger(y) \left(-\frac{\nabla^2}{2m} \phi_1(x) \right) \right\rangle + \left\langle \phi_1^\dagger(y) [\phi_1(x), V] \right\rangle \\ &\quad - \sum_{I\neq 0} \overline{f_I}(y) f_0(x) \frac{\langle N_I \rangle \langle [a_I^\dagger a_0, V] \rangle}{\langle N_0 \rangle - \langle N_I \rangle} . \end{aligned} \tag{66}$$

The first equation shows that the function $\langle N_0 \rangle^{1/2} f_0(x)$ assumes the same role of the solution of Gross–Pitaevskii equation in the mean-field description. As long as the expectation value $\langle [a_0^\dagger a_I, V] \rangle$ is negligible, we have that the mixed term described by the second equation is also negligible—as it can be said for the last term in the third equation—so that the excited part ϕ_1 can be considered to evolve separately from ϕ_0 in first approximation. Leading contributions from $\langle \phi_1^\dagger [\phi_1, V] \rangle$ must be those quadratic in the operators ϕ_1 and $\phi_1^\dagger$, and therefore the third equation can be approximated as

$$\left\langle \phi_1^\dagger i\partial_t \phi_1 \right\rangle \approx \left\langle \phi_1^\dagger \left(-\frac{\nabla^2}{2m} \phi_1 + 2\lambda \phi_0^\dagger \phi_0 \phi_1 + \lambda \phi_1^\dagger \phi_0 \phi_0 \right) \right\rangle . \tag{67}$$

This equation can be compared to the Bogoliubov–de Gennes equation. If we rewrite it in terms of the number-conserving operators, and we consider the fact that the terms mixing the derivative of ϕ_1 with ϕ_0 are negligible, we can write an effective linearized equation for $N_0^{-1/2} a_0^\dagger \phi_1$:

$$\begin{aligned} i\partial_t \left(N_0^{-1/2} a_0^\dagger \phi_1(x) \right) &\approx -\frac{\nabla^2}{2m} \left(N_0^{-1/2} a_0^\dagger \phi_1(x) \right) + 2\lambda \rho_0(x) \left(N_0^{-1/2} a_0^\dagger \phi_1(x) \right) \\ &\quad + \lambda \rho_0(x) e^{2i\theta_0(x)} \left(\phi_1^\dagger(x) a_0 N_0^{-1/2} \right) . \end{aligned} \tag{68}$$

In this equation, we use the functions ρ_0 and θ_0, which are obtained from the condensed wave function, by writing it as $\langle N_0\rangle^{1/2} f_0 = \rho_0^{1/2} e^{i\theta_0}$. One can effectively assume the condensed function to be the solution of the Gross–Pitaevskii equation, as the first corrections will be of a lower power of $\langle N_0\rangle$ (and include a backreaction from this equation itself).

Assuming that ρ_0 is, at first approximation, homogeneous, implies that the term $\left\langle \left[a_0^\dagger a_I, V\right]\right\rangle$ is negligible. If ρ_0 and θ_0 are ultimately the same as those obtained from the Gross–Pitaevskii equation, the same equation that holds for the operator $\delta\phi$ can be assumed to hold for the operator $N_0^{-1/2} a_0^\dagger \phi_1$. The solution for the mean-field description of the condensate is therefore a general feature of the system in studying the quantum perturbation of the condensate, not strictly reserved to coherent states.

Although having strongly related dynamical equations, the substantial difference between the operators $\delta\phi$ of Equation (4) and $N_0^{-1/2} a_0^\dagger \phi_1$ is that the number-conserving operator does not satisfy the canonical commutation relations with its Hermitian conjugate, as we have extracted the degree of freedom relative to the condensed state

$$\left[\phi_1(x), \phi_1^\dagger(y)\right] = \delta(x,y) - f_0(x)\overline{f_0}(y)\,. \tag{69}$$

Although this does not imply a significant obstruction, one must remind that the field ϕ_1 should never be treated as a canonical quantum field. What has to be done, instead, is considering its components with respect to the basis of time-dependent orbitals. Each mode of the projection ϕ_1 behaves as if it is a mode of a canonical scalar quantum field in a curved spacetime. Keeping this in mind, we can safely retrieve analogue gravity.

5. Analogue Gravity with Atom Number Conservation

In the previous section, we discussed the equivalent of the Bogoliubov–de Gennes equation in a number-conserving framework. Our aim in the present Section is to extend this description to analogue gravity.

The field operators required for analogue gravity will differ from those relative to the mean-field description, and they should be defined considering that we have by construction removed the contribution from the condensed 1-particle state f_0. The dynamical equation for the excited part in the number-conserving formalism Equation (68) appears to be the same as for the case of coherent states (as in Equation (7)), but instead of the field $\delta\phi$ one has $N_0^{-1/2} a_0^\dagger \phi_1$, where we remind that $N_0 = a_0^\dagger a_0$.

Using the Madelung representation, we may redefine the real functions ρ_0 and θ_0 from the condensed wave function f_0 and the expectation value $\langle N_0\rangle$. Approximating at the leading order, we can obtain their dynamics as in the quantum Euler Equations (11)–(12):

$$\langle N_0\rangle^{1/2} f_0 = \rho_0^{1/2} e^{i\theta_0}\,. \tag{70}$$

These functions enter in the definition of the quantum operators θ_1 and ρ_1, which take a different expression from the usual Madelung representation when we employ the set of number-conserving ladder operators

$$\begin{aligned}\theta_1 &= -\frac{i}{2}\langle N_0\rangle^{-1/2}\sum_{I\neq 0}\left(\frac{f_I}{f_0}N_0^{-1/2}a_0^\dagger a_I - \frac{\overline{f_I}}{\overline{f_0}}a_I^\dagger a_0 N_0^{-1/2}\right)\\ &= -\frac{i}{2}\left(\frac{N_0^{-1/2}\phi_0^\dagger\phi_1 - \phi_1^\dagger\phi_0 N_0^{-1/2}}{\langle N_0\rangle^{1/2}\overline{f_0}f_0}\right),\end{aligned} \tag{71}$$

$$\begin{aligned}\rho_1 &= \langle N_0\rangle^{1/2}\sum_{I\neq 0}\left(\overline{f_0}f_I N_0^{-1/2}a_0^\dagger a_I + f_0\overline{f_I}a_I^\dagger a_0 N_0^{-1/2}\right) = \\ &= \langle N_0\rangle^{1/2}\left(N_0^{-1/2}\phi_0^\dagger\phi_1 + \phi_1^\dagger\phi_0 N_0^{-1/2}\right)\,.\end{aligned} \tag{72}$$

From Equations (71) and (72), we observe that the structure of the operators θ_1 and ρ_1 consists of a superposition of modes, each dependent on a different eigenfunction f_I of the 2-point correlation function, with a sum over the index $I \neq 0$.

The new fields θ_1 and ρ_1 do not satisfy the canonical commutation relations since the condensed wave function f_0 is treated separately by definition. However, these operators could be analyzed mode-by-mode, and therefore be compared in full extent to the modes of quantum fields in curved spacetime to which they are analogous. Their modes satisfy the relations

$$[\theta_I, \rho_J] = -i\overline{f_I} f_I \delta_{JI} \qquad \forall I, J \neq 0 . \tag{73}$$

Equation (73) is a basis-dependent expression, which can, in general, be found for the fields of interest. In the simplest case of homogeneous density of the condensate ρ_0, this commutation relations reduce to $-i\delta_{IJ}$, and the Fourier transform provides the tools to push the description to full extent where the indices labeling the functions are the momenta k.

The equations for analogue gravity are found under the usual assumptions regarding the quantum pressure, i.e., the space gradients of the atom densities are assumed to be small. When considering homogeneous condensates this requirement is of course satisfied. In nonhomogeneous condensates, we require

$$\nabla \left(\rho_0^{-1} \left(\nabla \rho_0 \right) \right) \ll 4 m \lambda \rho_0 \, , \tag{74}$$

$$\nabla \left(\rho_0^{-1} \left(\nabla \rho_1 \right) \right) \ll 4 m \lambda \rho_1 \, . \tag{75}$$

Making the first assumption (74), the effective coupling constant $\widetilde{\lambda}$ is a global feature of the system with no space dependence. This means that all the inhomogeneities of the system are encoded in the velocity of the superfluid, the gradient of the phase of the condensate. As stated before, the continuity equation can induce inhomogeneities in the density if there are initial inhomogeneities in the phase, but for sufficiently short intervals of time, the assumption is satisfied. Another effect of the first assumption (74) is that the term $\int \mathrm{d}x \overline{f_I f_0} f_0 f_0$ is negligible. The more ρ_0 is homogeneous, the closer this integral is to vanishing, making the description more consistent. The second assumption (75) is a general requirement in analogue gravity, needed to have local Lorentz symmetry, and therefore a proper Klein–Gordon equation for the field θ_1. When ρ_0 is homogeneous, this approximation means considering only small momenta, for which we have the usual dispersion relation.

Under these assumptions, the usual equations for analogue gravity are obtained:

$$\rho_1 = -\frac{1}{\lambda} \left((\partial_t \theta_1) + \frac{1}{m} (\nabla \theta_0) (\nabla \theta_1) \right) , \tag{76}$$

$$(\partial_t \rho_1) = -\frac{1}{m} \nabla \left(\rho_1 (\nabla \theta_0) + \rho_0 (\nabla \theta_1) \right) \tag{77}$$

$$\Downarrow$$

$$0 = \partial_t \left(-\frac{1}{\lambda} (\partial_t \theta_1) - \frac{\delta^{ij}}{m\lambda} (\nabla_j \theta_0) (\nabla_i \theta_1) \right) + \nabla_j \left(-\frac{\delta^{ij}}{m\lambda} (\nabla_i \theta_0)(\partial_t \theta_1) + \left(\frac{\delta^{ij} \rho_0}{m} - \frac{1}{\lambda} \frac{\delta^{il}}{m} \frac{\delta^{jm}}{m} (\nabla_l \theta_0)(\nabla_m \theta_0) \right) (\nabla_i \theta_1) \right) \tag{78}$$

so that θ_1 is the analogue of a scalar massless field in curved spacetime. However, the operator θ_1 is intrinsically unable to provide an exact full description of a massless field since it is missing the mode f_0. Therefore, the operator θ_1 is best handled when considering the propagation of its constituent modes, and relating them to those of the massless field.

The viability of this description as a good analogue gravity set-up is ensured, ultimately, by the fact that the modes of θ_1, i.e., the operators describing the excited part of the atomic field, have a closed dynamics. The most important feature in the effective dynamics of the number-conserving operators $N_0^{-1/2}a_0^\dagger a_I$, as described in equation Equation (68), is that its time derivative can be written as a composition of the same set of number operators, and this enables the analogue model.

In the following, we continue with the case of an homogenous condensate, which is arguably the most studied case in analogue gravity. The description is enormously simplified by the fact that the gradient of the condensed wave function vanishes, since $f_0 = V^{-1/2}$, meaning that the condensed state is fully described by the state of null momentum $k = 0$. In a homogeneous BEC, all the time-dependent orbitals are labeled by the momenta they carry, and at every moment in time, we can apply the same Fourier transform to transform the differential equations in the space of coordinates to algebraic equations in the space of momenta. We expect that the number-conserving treatment of the inhomogenous condensate follows along the same lines, albeit being technically more complicated.

6. Simulating Cosmology in Number-Conserving Analogue Gravity

With an homogeneous condensate we can simulate a cosmology with a scale factor changing in time—as long as we can control and modify in time the strength of the 2-body interaction λ—and we can verify the prediction of quantum field theory in curved spacetime that in an expanding universe one should observe a cosmological particle creation [9,27–30] . In this set-up, there is no ambiguity in approximating the mixed term of the interaction potential, as discussed in the Section 3.2.

To further proceed, we apply the usual transformation to pass from the Bogoliubov description of the atomic system to the set-up of analogue gravity, and we then proceed considering number-conserving operators. It is convenient to adopt a compact notation for the condensate wave function and its approximated dynamics, as discussed previously in Equations (11)–(12) and in Equation (70).

$$f_0\left(x\right)\left\langle N_0\right\rangle^{1/2} \equiv \phi_0 = \rho_0^{1/2} e^{i\theta_0}\,, \tag{79}$$

$$\partial_t \rho_0 = 0\,, \tag{80}$$

$$\partial_t \theta_0 = -\lambda\rho_0\,. \tag{81}$$

To study the excitations described by the operator $\theta_1\left(x\right)$ we need the basis of time-dependent orbitals, which in the case of a homogeneous condensate is given by the plane waves, the set of orthonormal functions which define the Fourier transform and are labeled by the momenta. By Fourier transforming the operator $\phi_1\left(x\right)$, orthogonal to the condensate wave function, we have

$$\begin{aligned}\delta\phi_k &\equiv \int \frac{\mathrm{d}x}{\sqrt{V}} e^{-ikx} N_0^{-1/2} a_0^\dagger \phi_1\left(x\right)\\ &= \int \frac{\mathrm{d}x}{\sqrt{V}} e^{-ikx} N_0^{-1/2} a_0^\dagger \sum_{q\neq 0}\frac{e^{iqx}}{\sqrt{V}} a_q\\ &= N_0^{-1/2} a_0^\dagger a_k\,,\end{aligned} \tag{82}$$

$$\left[\delta\phi_k, \delta\phi_{k'}^\dagger\right] = \delta_{k,k'} \qquad\qquad \forall k,k' \neq 0\,. \tag{83}$$

Notice that with the notation of Equation (58), $\delta\phi_k$ would be just α_k and one sees the dependence on the condensate operator a_0.

Following the same approach discussed in the Section 5, we define θ_k and ρ_k. These number-conserving operators are labeled with a nonvanishing momentum and act in the atomic Fock space, in a superposition of two operations, extracting momentum k from the state or introducing momentum $-k$ to it. All the following relations are defined for $k, k' \neq 0$.

$$\theta_k = -\frac{i}{2}\left(\frac{\delta\phi_k}{\phi_0} - \frac{\delta\phi^\dagger_{-k}}{\overline{\phi_0}}\right), \tag{84}$$

$$\rho_k = \rho_0\left(\frac{\delta\phi_k}{\phi_0} + \frac{\delta\phi^\dagger_{-k}}{\overline{\phi_0}}\right), \tag{85}$$

$$[\theta_k, \rho_{k'}] = -i\left[\delta\phi_k, \delta\phi^\dagger_{-k'}\right] = -i\delta_{k,-k'}, \tag{86}$$

$$\left\langle \delta\phi^\dagger_k(t)\,\delta\phi_{k'}(t)\right\rangle = \delta_{k,k'}\left\langle N_k\right\rangle . \tag{87}$$

Again, we remark that these definitions of θ_k and ρ_k do not provide, through an inverse Fourier transform of these operators, a couple of conjugate real fields, $\theta_1(x)$ and $\rho_1(x)$, with the usual commutation relations as in Equation (10), because they are not relative to a set of functions that form a complete basis of the 1-particle Hilbert space, as the mode $k = 0$ is not included. However, these operators, describing each mode with $k \neq 0$, can be studied separately and they show the same behavior of the components of a quantum field in curved spacetime: the commutation relations in Equation (86) are the same as those that are satisfied by the components of a quantum scalar field.

From the Bogoliubov–de Gennes Equation (68), we get the two coupled dynamical equations for θ_k and ρ_k:

$$\partial_t\theta_k = -\frac{1}{2}\left(\frac{k^2}{2m} + 2\lambda\rho_0\right)\frac{\rho_k}{\rho_0}, \tag{88}$$

$$\partial_t\frac{\rho_k}{\rho_0} = \frac{k^2}{m}\theta_k . \tag{89}$$

Combining these gives the analogue Klein–Gordon equation for each mode $k \neq 0$:

$$\partial_t\left(-\frac{1}{\lambda\rho_0 + \frac{k^2}{4m}}\left(\partial_t\theta_k\right)\right) = \frac{k^2}{m}\theta_k . \tag{90}$$

In this equation, the term due to quantum pressure is retained for convenience, as the homogeneity of the condensed state makes it easy to maintain it in the description. It modifies the dispersion relation and breaks Lorentz symmetry, but the usual expression is found in the limit $\frac{k^2}{2m} \ll 2\lambda\rho_0$.

When the quantum pressure is neglected, the analogue metric tensor is

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \sqrt{\frac{\rho_0}{m\lambda}}\left(-\frac{\lambda\rho_0}{m}\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) . \tag{91}$$

This metric tensor is clearly analogous to that of a cosmological spacetime, where the evolution is given by the time dependence of the coupling constant λ. This low-momenta limit is the regime in which we are mostly interested, because when these conditions are realized the quasi-particles, the excitations of the field θ_k, behave most similarly to particles in a curved spacetime with local Lorentz symmetry.

6.1. Cosmological Particle Production

We now consider a set-up for which the coupling constant varies from an initial value λ to a final value λ' through a transient phase. λ is assumed asymptotically constant for both $t \to \pm\infty$. This set-up has been studied in the Bogoliubov approximation in the works by the authors of [9,27–30] and can be experimentally realized with. e.g., via Feshbach resonance. For one-dimensional Bose gases where significant corrections to the Bogoliubov approximation are expected far from the weakly interacting limit, a study of the large time evolution of correlations was presented in the work by the authors of [31]. Here, our aim is to study the effect of the variation of the coupling constant in the number-conserving framework.

There will be particle creation and the field in general takes the expression

$$\theta_k\left(t\right) = \frac{1}{\mathcal{N}_k\left(t\right)}\left(e^{-i\Omega_k(t)}c_k + e^{i\Omega_{-k}(t)}c^{\dagger}_{-k}\right)\,, \tag{92}$$

where the operators c_k are the creation and destruction operators for the quasi-particles at $t \to -\infty$. For the time $t \to +\infty$, there will be a new set of operators c',

$$\theta_k\left(t \to -\infty\right) = \frac{1}{\mathcal{N}_k}\left(e^{-i\omega_k t}c_k + e^{i\omega_k t}c^{\dagger}_{-k}\right)\,, \tag{93}$$

$$\theta_k\left(t \to +\infty\right) = \frac{1}{\mathcal{N}'_k}\left(e^{-i\omega'_k t}c'_k + e^{i\omega'_k t}c'^{\dagger}_{-k}\right)\,. \tag{94}$$

From these equations, in accordance with Equation (88), we obtain

$$\rho_k = -\frac{2\rho_0}{\frac{k^2}{2m} + 2\lambda\rho_0}\partial_t\theta_k \tag{95}$$

and the two following asymptotic expressions for ρ_k,

$$\rho_k\left(t \to -\infty\right) = \frac{2i\omega_k\rho_0}{\frac{k^2}{2m} + 2\lambda\rho_0}\frac{1}{\mathcal{N}_k}\left(e^{-i\omega_k t}c_k - e^{i\omega_k t}c^{\dagger}_{-k}\right)\,, \tag{96}$$

$$\rho_k\left(t \to +\infty\right) = \frac{2i\omega'_k\rho_0}{\frac{k^2}{2m} + 2\lambda'\rho_0}\frac{1}{N'_k}\left(e^{-i\omega'_k t}c'_k - e^{i\omega'_k t}c'^{\dagger}_{-k}\right)\,. \tag{97}$$

With the previous expressions for θ_k and ρ_k and imposing the commutation relations in Equation (86), we retrieve the energy spectrum $\omega_k = \sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m} + 2\lambda\rho_0\right)}$ as expected and the (time-dependent) normalization prefactor $\mathcal{N}$:

$$\mathcal{N}_k = \sqrt{4\rho_0\sqrt{\frac{\frac{k^2}{2m}}{\frac{k^2}{2m} + 2\lambda\rho_0}}}\,. \tag{98}$$

The expected commutation relations for the operators c and c' are found (again not including the mode $k = 0$):

$$0 = \left[c_k, c_{k'}\right] = \left[c'_k, c'_{k'}\right]\,, \tag{99}$$

$$\delta_{k,k'} = \left[c_k, c^{\dagger}_{k'}\right] = \left[c'_k, c'^{\dagger}_{k'}\right]\,. \tag{100}$$

It is found

$$c'_k = \cosh\Theta_k c_k + \sinh\Theta_k e^{i\varphi_k}c^{\dagger}_{-k} \tag{101}$$

with

$$\cosh\Theta_k = \cosh\Theta_{-k}\,, \tag{102}$$

$$\sinh\Theta_k e^{i\varphi_k} = \sinh\Theta_{-k}e^{i\varphi_{-k}}\,. \tag{103}$$

The initial state in which we are interested is the vacuum of quasi-particles, so that each quasi-particle destruction operators c_k annihilates the initial state (To make contact with the standard

Bogoliubov approximation, if there one denotes by γ_k the quasi-particles one has that the γ_k are a combination of the atom operators a_k, a_{-k} of the form $\gamma_k = u_k a_k + v_k a^\dagger_{-k}$ [20]. Correspondingly, in the number-conserving formalism the quasi-particle operators c_k are a combination of the atom operators $\delta\phi_k \equiv \alpha_k$, $\delta\phi_{-k} \equiv \alpha_{-k}$):

$$c_k \left|in\right\rangle \equiv 0 \qquad \forall k \neq 0\,. \tag{104}$$

To realize this initial condition, we should impose constraints, in principle, on every correlation function. We focus on the 2-point correlation functions $\langle \delta\phi^\dagger \delta\phi \rangle$ and $\langle \delta\phi \delta\phi \rangle$. In particular, the first of the two determines the number of atoms with momentum k in the initial state:

$$\left\langle \delta\phi^\dagger_k \delta\phi_k \right\rangle = \left\langle a^\dagger_k a_0 N_0^{-1} a^\dagger_0 a_k \right\rangle = \left\langle a^\dagger_k a_k \right\rangle = \langle N_k \rangle\ . \tag{105}$$

In order for the state to be condensed with respect to the state with momentum 0, it must be that $\langle N_k \rangle \ll \langle N_0 \rangle = \rho_0 V$. When the vacuum condition Equation (104) holds, the 2-point correlation functions can be easily evaluated to be

$$\begin{aligned}\left\langle \delta\phi^\dagger_k \delta\phi_{k'} \right\rangle &= \left(\frac{1}{2} \frac{\frac{k^2}{2m} + \lambda\rho_0}{\sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m} + 2\lambda\rho_0\right)}} - \frac{1}{2} \right) \delta_{k,k'} \\ &\approx \frac{1}{4} \sqrt{\frac{2\lambda\rho_0}{\frac{k^2}{2m}}}\, \delta_{k,k'}\,,\end{aligned} \tag{106}$$

$$\begin{aligned}\langle \delta\phi_{-k} \delta\phi_{k'} \rangle &= -\frac{e^{2i\theta_0}}{4} \frac{2\lambda\rho_0}{\sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m} + 2\lambda\rho_0\right)}} \delta_{k,k'} \\ &\approx -e^{2i\theta_0} \left\langle \delta\phi^\dagger_k \delta\phi_{k'} \right\rangle\,,\end{aligned} \tag{107}$$

where in the last line we have used $\frac{k^2}{2m} \ll 2\lambda\rho_0$, the limit in which the quasi-particles propagate in accordance with the analogue metric Equation (91), and one has to keep into account that the phase of the condensate is time dependent and consequently the last correlator is oscillating.

We now see that the conditions of condensation $\langle N_k \rangle \ll \langle N_0 \rangle$ and of low-momenta translate into

$$\frac{2\lambda\rho_0}{16\,\langle N_0 \rangle^2} \ll \frac{k^2}{2m} \ll 2\lambda\rho_0\,. \tag{108}$$

The range of momenta that should be considered is, therefore, set by the number of condensate atoms, the physical dimension of the atomic system, and the strength of the 2-body interaction.

The operators θ_k satisfying Equation (90)—describing the excitations of quasi-particles over a BEC—are analogous to the components of a scalar quantum field in a cosmological spacetime. In particular, if we consider a cosmological metric given in the usual form of

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}\tau^2 + a^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j\,, \tag{109}$$

the analogy is realized for a specific relation between the coupling $\lambda(t)$ and the scale factor $a(\tau)$, which then induces the relation between the laboratory time t and the cosmological time τ. These relations are given by

$$a(\tau(t)) = \left(\frac{\rho_0}{m\lambda(t)}\right)^{1/4}\frac{1}{C}, \tag{110}$$

$$d\tau = \frac{\rho_0}{ma(\tau(t))}\frac{1}{C^2}dt, \tag{111}$$

for an arbitrary constant C.

In cosmology, the evolution of the scale factor leads to the production of particles by cosmological particle creation, as implied by the Bogoliubov transformation relating the operators which, at early and late times, create and destroy the quanta we recognize as particles. The same happens for the quasi-particles over the condensate, as discussed in Section 6, because the coupling λ is time-dependent and the definition itself of quasi-particles changes from initial to final time. The ladder operators associated to these quasi-particles are related to each other by the Bogoliubov transformation introduced in Equation (101), fully defined by the parameters Θ_k and φ_k (which must also satisfy Equations (102) and (103)).

6.2. Scattering Operator

The exact expressions of Θ_k and φ_k depend on the behavior of $\lambda(t)$, which is a function of the cosmological scale parameter, and is therefore different for each cosmological model. They can in general be evaluated with the well-established methods used in quantum field theory in curved spacetimes [32]. In general, it is found that $\cosh\Theta_k > 1$, as the value $\cosh\Theta_k = 1$ (i.e., $\sinh\Theta_k = 0$) is restricted to the case in which λ is a constant for the whole evolution, and the analogue spacetime is simply flat.

The unitary operator describing the evolution from initial to final time is $U(t_{out}, t_{in})$ when $t_{out} \to +\infty$ and $t_{in} \to -\infty$, and the operator, U, is the scattering operator, S. This is exactly the operator acting on the quasi-particles, defining the Bogoliubov transformation in which we are interested

$$c'_k = S^\dagger c_k S. \tag{112}$$

The behavior of c'_k, describing the quasi-particles at late times, can therefore be understood from the behavior of the initial quasi-particle operators c_k when the expression of the scattering operator is known. In particular, the phenomenon of cosmological particle creation is quantified considering the expectation value of the number operator of quasi-particles at late times in the vacuum state as defined by early times operators [32].

Consider as initial state the vacuum of quasi-particles at early times, satisfying the condition Equation (104). It is analogous to a Minkowski vacuum, and the evolution of the coupling $\lambda(t)$ induces a change in the definition of quasi-particles. We find that, of course, the state is not a vacuum with respect to the final quasi-particles c'. It is

$$S^\dagger c_k^\dagger c_k S = c_k'^\dagger c'_k = \left(\cosh\Theta_k c_k^\dagger + \sinh\Theta_k e^{-i\varphi_k} c_{-k}\right)\left(\cosh\Theta_k c_k + \sinh\Theta_k e^{i\varphi_k} c_{-k}^\dagger\right) \tag{113}$$

and

$$\left\langle S^\dagger c_k^\dagger c_k S\right\rangle = \sinh^2\Theta_k \left\langle c_{-k} c_{-k}^\dagger\right\rangle = \sinh^2\Theta_k > 0. \tag{114}$$

We are interested in the effect that the evolution of the quasi-particles have on the atoms. The system is fully characterized by the initial conditions and the Bogoliubov transformation: we have the initial occupation numbers, the range of momenta which we should consider, and the relation between initial and final quasi-particles.

What is most significant is that the quasi-particle dynamics affects the occupation number of the atoms. Considering that for sufficiently large t we are already in the final regime, the field takes the following values,

$$\delta\phi_k\left(t\rightarrow-\infty\right)=i\rho_0^{1/2}e^{i\theta_0(t)}\frac{1}{\mathcal{N}_k}\left(\left(\mathcal{F}_k+1\right)e^{-i\omega_k t}c_k-\left(\mathcal{F}_k-1\right)e^{i\omega_k t}c_{-k}^{\dagger}\right) \tag{115}$$

$$\delta\phi_k\left(t\rightarrow+\infty\right)=i\rho_0^{1/2}e^{i\theta_0(t)}\frac{1}{\mathcal{N}_k'}\left(\left(\mathcal{F}_k'+1\right)e^{-i\omega_k' t}c_k'-\left(\mathcal{F}_k'-1\right)e^{i\omega_k' t}c_{-k}'^{\dagger}\right) \tag{116}$$

where $\mathcal{F}_k\equiv\frac{\omega_k}{\frac{k^2}{2m}+2\lambda\rho_0}$ and $\mathcal{F}_k'\equiv\frac{\omega_k'}{\frac{k^2}{2m}+2\lambda'\rho_0}$, with $\omega_k'=\sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m}+2\lambda'\rho_0\right)}$. One finds

$$\left\langle\delta\phi_k^{\dagger}\left(t\right)\delta\phi_k\left(t\right)\right\rangle=\frac{\frac{k^2}{2m}+\lambda'\rho_0}{2\omega_k'}\cosh\left(2\Theta_k\right)-\frac{1}{2}+\frac{\lambda'\rho_0\sinh\left(2\Theta_k\right)}{2\omega_k'}\cos\left(2\omega_k't-\varphi_k\right)\,. \tag{117}$$

In Equation (117), the last term is oscillating symmetrically around 0—meaning that the atoms will leave and rejoin the condensate periodically in time—whereas the first two are stationary.

An increase in the value of the coupling λ therefore has deep consequences. It appears explicitly in the prefactor and more importantly it affects the hyperbolic functions $\cosh\Theta_k>1$, which implies that the mean value is larger than the initial one, differing from the equilibrium value corresponding to the vacuum of quasi-particles.

This result is significant because it explicitly shows that the quasi-particle dynamics influences the underlying structure of atomic particles. Even assuming that the backreaction of the quasi-particles on the condensate is negligible for the dynamics of the quasi-particles themselves, the mechanism of extraction of atoms from the condensate fraction is effective and increases the depletion (as also found in the standard Bogoliubov approach). This extraction mechanism can be evaluated in terms of operators describing the quasi-particles, that can be defined a posteriori, without notion of the operators describing the atoms.

The fact that analogue gravity can be reproduced in condensates independently from the use of coherent states enhances the validity of the discussion. It is not strictly necessary that we have a coherent state to simulate the effects of curvature with quasi-particles, but, in the more general case of condensation, the condensed wave function provides a support for the propagation of quasi-particles. From an analogue gravity point of view, its intrinsic role is that of seeding the emergence of the analogue scalar field [2].

7. Squeezing and Quantum State Structure

The Bogoliubov transformation in Equation (101) leading to the quasi-particle production describes the action of the scattering operator on the ladder operators, relating the operators at early and late times. The linearity of this transformation is obtained by the linearity of the dynamical equation for the quasi-particles, which is particularly simple in the case of homogeneous condensate.

The scattering operator S is unitary by definition, as it is easily checked by its action on the operators c_k. Its full expression can be found from the Bogoliubov transformation, finding the generators of the transformation when the arguments of the hyperbolic functions, the parameters Θ_k, are infinitesimal:

$$S^{\dagger}c_kS=c_k'=\cosh\Theta_k c_k+\sinh\Theta_k e^{i\varphi_k}c_{-k}^{\dagger}\,. \tag{118}$$

It follows

$$S=\exp\left(\frac{1}{2}\sum_{k\neq0}\left(-e^{-i\varphi_k}c_kc_{-k}+e^{i\varphi_k}c_k^{\dagger}c_{-k}^{\dagger}\right)\Theta_k\right)\,. \tag{119}$$

The scattering operator is particularly simple and takes the peculiar expression that is required for producing squeezed states. This is the general functional expression that is found in cosmological particle creation and in its analogue gravity counterparts, whether they are realized in the usual Bogoliubov framework or in its number-conserving reformulation. As discussed previously, the number-conserving formalism is more general, reproduces the usual case when the state is an eigenstate of the destruction operator a_0, and includes the notion that the excitations of the condensate move condensate atoms to the excited part.

The expression in Equation (119) has been found under the hypothesis that the mean value of the operator N_0 is macroscopically larger than the other occupation numbers. Instead of using the quasi-particle ladder operators, S can be rewritten easily in terms of the atom operators. In particular, we remind that the time-independent operators c_k depend on the condensate operator a_0 and can be defined as compositions of number-conserving atom operators $\delta\phi_k\,(t)$ and $\delta\phi^\dagger_{-k}\,(t)$ defined in Equation (82). At any time, there can be a transformation from one set of operators to the another. It is significant that the operators c_k commute with the operators $N_0^{-1/2} a_0^\dagger$ and $a_0 N_0^{-1/2}$, which are therefore conserved in time (as long as the linearized dynamics for $\delta\phi_k$ is a good approximation)

$$\left[\delta\phi_k, N_0^{-1/2}\,(t)\,a_0^\dagger\,(t)\right] = 0\,, \tag{120}$$

$$\Downarrow$$

$$\left[c_k, N_0^{-1/2}\,(t)\,a_0^\dagger\,(t)\right] = 0\,, \tag{121}$$

$$\Downarrow$$

$$\left[S, N_0^{-1/2}\,(t)\,a_0^\dagger\,(t)\right] = 0\,. \tag{122}$$

The operator S cannot have other terms apart for those in Equation (119), even if it is defined for its action on the operators c_k, and therefore on a set of functions, which is not a complete basis of the 1-particle Hilbert space. Nevertheless, the notion of number conservation implies its action on the condensate and on the operator a_0.

One could investigate whether it is possible to consider a more general expression with additional terms depending only on a_0 and $a_0^\dagger$, i.e., assuming the scattering operator to be

$$S = \exp\left(\frac{1}{2}\sum_{k\neq 0}\left(Z_k c_k c_{-k} + c_k^\dagger c_{-k}^\dagger Z_{-k}^\dagger\right) + G_0\right)\,, \tag{123}$$

where we could assume that the coefficients of the quasi-particle operators are themselves depending on only a_0 and $a_0^\dagger$, and so G_0. However, the requirement that S commutes with the total number of atoms N implies that so do its generators, and therefore Z and G_0 must be functionally dependent on N_0, and not on a_0 and $a_0^\dagger$ separately, as they do not conserve the total number. Therefore, it must hold that

$$0 = \left[\left(\frac{1}{2}\sum_{k\neq 0}\left(Z_k c_k c_{-k} + c_k^\dagger c_{-k}^\dagger Z_{-k}^\dagger\right) + G_0\right), N\right]\,. \tag{124}$$

The only expressions in agreement with the linearized dynamical equation for $\delta\phi$ imply that Z and G_0 are multiple of the identity; otherwise, they would modify the evolution of the operators $\delta\phi_k = N_0^{-1/2} a_0^\dagger a_I$, as they do not commute with N_0. This means that that corrections to the scattering operator are possible only involving higher-order corrections (in terms of $\delta\phi$).

The fact that the operator S as in Equation (119) is the only number-conserving operator satisfying the dynamics is remarkable because it emphasizes that the production of quasi-particles is a phenomenon that holds only in terms of excitations of atoms from the condensate to the excited part, with the number of transferred atoms evaluated in the previous subsection. The expression of the scattering operator shows that the analogue gravity system produces states in which the final state presents squeezed quasi-particle states; however, the occurrence of this feature in the emergent dynamics happens only introducing correlations in the condensate, with each quanta of the analogue field θ_1 entangling atoms in the condensate with atoms in the excited part.

The quasi-particle scattering operator obtained in the number-conserving framework is functionally equivalent to that in the usual Bogoliubov description, and the difference between the two appears when considering the atom operators, depending on whether a_0 is treated for its quantum nature or it is replaced with the number $\langle N_0^{1/2} \rangle$. This reflects that the dynamical equations are functionally the same when the expectation value $\langle N_0 \rangle$ is macroscopically large.

There are no requirements on the initial density matrix of the state, and it is not relevant whether the state is a coherent superposition of infinite states with different number of atoms or it is a pure state with a fixed number of atoms in the same 1-particle state. The quasi-particle description holds the same and it provides the same predictions. This is useful for implementing analogue gravity systems, but also a strong hint in interpreting the problem of information loss. When producing quasi-particles in analogue gravity one can, in first approximation, reconstruct the initial expectation values of the excited states and push the description to include the backreaction on the condensate. What we are intrinsically unable to do is reconstruct the entirety of the initial atom quantum state, i.e., how the condensate is composed.

We know that in analogue gravity the evolution is unitary, the final state is uniquely determined by the initial state. Knowing all the properties of the final state we could reconstruct the initial state, and yet the intrinsic inability to infer all the properties of the condensate atoms from the excited part shows that the one needs to access the full correlation properties of the condensate atoms with the quasi-particles to fully appreciate (and retrieve) the unitarity of the evolution.

7.1. Correlations

In the previous section, we made the standard choice of considering as initial state the quasi-particle vacuum. To characterize it with respect to the atomic degrees of freedom, the quasi-particle ladders operators have to be expressed as compositions of the number-conserving atomic operators, manipulating Equations (84) and (92).

By definition, at any time, both sets of operators satisfy the canonical commutation relations (87) and (100) $\forall k, k' \neq 0$. Therefore, it must exist a Bogoliubov transformation linking the quasi-particle and the number-conserving operators, which, in general, are written as

$$c_k = e^{-i\alpha_k} \cosh \Lambda_k \delta\phi_k + e^{i\beta_k} \sinh \Lambda_k \delta\phi^\dagger_{-k} \,. \tag{125}$$

The transformation is defined through a set of functions Λ_k, constant in the stationary case, and the phases α_k and β_k, inheriting their time dependence from the atomic operators. These functions can be obtained from Equations (84) and (92):

$$\cosh \Lambda_k = \left(\frac{\omega_k + \left(\frac{k^2}{2m} + 2\lambda\rho_0 \right)}{4\omega_k} \right) \frac{\mathcal{N}_k}{\phi_0} \,. \tag{126}$$

If the coupling changes in time, the quasi-particle operators during the transient are defined knowing the solutions of the Klein–Gordon equation. With the Bogoliubov transformation of Equation (125), it is possible to find the quasi-particle vacuum-state $|\varnothing\rangle_{qp}$ in terms of the atomic degrees of freedom

$$\begin{aligned} |\varnothing\rangle_{qp} &= \prod_k \frac{e^{-\frac{1}{2} e^{i(\alpha_k+\beta_k)} \tanh \Lambda_k \delta\phi^\dagger_k \delta\phi^\dagger_{-k}}}{\cosh \Lambda_k} |\varnothing\rangle_a \\ &= \exp \sum_k \left(-\frac{1}{2} e^{i(\alpha_k+\beta_k)} \tanh \Lambda_k \delta\phi^\dagger_k \delta\phi^\dagger_{-k} - \ln \cosh \Lambda_k \right) |\varnothing\rangle_a \,, \end{aligned} \tag{127}$$

where $|\varnothing\rangle_a$ should be interpreted as the vacuum of excited atoms.

From Equation (127), it is clear that in the basis of atom occupation number, the quasi-particle vacuum is a complicated superposition of states with different number of atoms in the condensed 1-particle state (and a corresponding number of coupled excited atoms, in pairs of opposite momenta). Every correlation function is therefore dependent on the entanglement of this many-body atomic state.

This feature is enhanced by the dynamics, as can be observed from the scattering operator in Equation (119) relating early and late times. The scattering operator acts on atom pairs and the creation of quasi-particles affects the approximated vacuum differently depending on the number of atoms occupying the condensed 1-particle state. The creation of more pairs modifies further the superposition of the entangled atomic states depending on the total number of atoms and the initial number of excited atoms.

We can observe this from the action of the condensed state ladder operator, which does not commute with the the creation of coupled quasi-particles $c_k^\dagger c_{-k}^\dagger$, which is described by the combination of the operators $\delta\phi_k^\dagger\delta\phi_k$, $\delta\phi_k^\dagger\delta\phi_{-k}^\dagger$, and $\delta\phi_k\delta\phi_{-k}$. The ladder operator $a_0^\dagger$ commutes with the first, but not with the others:

$$\left(\delta\phi_k^\dagger\delta\phi_k\right) a_0^\dagger = a_0^\dagger \left(\delta\phi_k^\dagger\delta\phi_k\right) , \tag{128}$$

$$\left(\delta\phi_k^\dagger\delta\phi_{-k}^\dagger\right)^n a_0^\dagger = a_0^\dagger \left(\delta\phi_k^\dagger\delta\phi_{-k}^\dagger\right)^n \left(\frac{N_0+1}{N_0+1-2n}\right)^{1/2} , \tag{129}$$

$$\left(\delta\phi_k\delta\phi_{-k}\right)^n a_0^\dagger = a_0^\dagger \left(\delta\phi_k\delta\phi_{-k}\right)^n \left(\frac{N_0+1}{N_0+1+2n}\right)^{1/2} . \tag{130}$$

The operators a_0 and $a_0^\dagger$ do not commute with the number-conserving atomic ladder operators, and therefore the creation of couples and the correlation functions, up to any order, will present corrections of order $1/N$ to the values that could be expected in the usual Bogoliubov description. Such corrections appear in correlation functions between quasi-particle operators and for correlations between quasi-particles and condensate atoms. This is equivalent to saying that a condensed state, which is generally not coherent, will present deviations from the expected correlation functions predicted by the Bogoliubov theory, due to both the interaction and the features of the initial state itself (through contributions coming from connected expectation values).

7.2. Entanglement Structure in Number-Conserving formalism

Within the Bogoliubov description discussed in Section 3, the mean-field approximation for the condensate is most adequate for states close to coherence, thus allowing a separate analysis for the mean-field. The field operator is split in the mean-field function $\langle\phi\rangle$ and the fluctuation operator $\delta\phi$, which is assumed not to affect the mean-field through backreaction. Therefore, the states in this picture can be written as

$$|\langle\phi\rangle\rangle_{mf} \otimes \left|\delta\phi, \delta\phi^\dagger\right\rangle_{a\,Bog} , \tag{131}$$

meaning that the state belongs to the product of two Hilbert spaces: the mean-field defined on one and the fluctuations on the other, with $\delta\phi$ and $\delta\phi^\dagger$ ladder operators acting only on the second. The Bogoliubov transformation from atom operators to quasi-particles allows to rewrite the state as shown in Equation (127). The transformation only affects its second part:

$$|\langle N\rangle\rangle_{mf} \otimes |\varnothing\rangle_{qp\,Bog} = |\langle N\rangle\rangle_{mf} \otimes \sum_{lr} a_{lr} |l, r\rangle_{a\,Bog} . \tag{132}$$

With such transformation, the condensed part of the state is kept separate from the superposition of coupled atoms (which here are denoted l and r for brevity) forming the excited part, a separation, that is maintained during the evolution in the Bogoliubov description. Also, the Bogoliubov transformation from early-times quasi-particles to late-times quasi-particles affects only the second part

$$|\langle N\rangle\rangle_{mf} \otimes \sum_{lr} a_{lr} \, |l,r\rangle_{a\,Bog} \Rightarrow |\langle N\rangle\rangle_{mf} \otimes \sum_{lr} a'_{lr} \, |l,r\rangle_{a\,Bog} \,. \tag{133}$$

In the number-conserving framework there is not such a splitting of the Fock space, and there is no separation between the two parts of the state. In this case, the best approximation for the quasi–particle vacuum is given by a superposition of coupled excitations of the atom operators, but the total number of atoms cannot be factored out:

$$|N;\emptyset\rangle_{qp} \approx \sum_{lr} a_{lr} \, |N-l-r,l,r\rangle_a \,. \tag{134}$$

The term in the RHS is a superposition of states with N total atoms, of which $N-l-r$ are in the condensed 1-particle state, and the others occupy excited atomic states and are coupled with each other analogously to the previous Equation (132) (the difference being the truncation of the sum, required for a sufficiently large number of excited atoms, implying a different normalization).

The evolution does not split the Hilbert space, and the final state will be a different superposition of atomic states:

$$\sum_{lr} a_{lr} \, |N,l,r\rangle_a \Rightarrow \sum_{lr} a'_{lr} \left(1+\mathcal{O}\left(N^{-1}\right)\right) |N-l-r,l,r\rangle_a \,. \tag{135}$$

We remark that in the RHS the final state must include corrections of order $1/N$ with respect to the Bogoliubov prediction, due to the fully quantum behavior of the condensate ladder operators. These are small corrections, but we expect that the difference from the Bogoliubov prediction will be relevant when considering many-point correlation functions.

Moreover, these corrections remark the fact that states with different number of atoms in the condensate are transformed differently. If we consider a superposition of states of the type in Equation (134) with different total atom numbers so to reproduce the state in Equation (132), therefore replicating the splitting of the state, we would find that the evolution produces a final state with a different structure, because every state in the superposition evolves differently. Therefore, also assuming that the initial state could be written as

$$\sum_N \frac{e^{-N/2}}{\sqrt{N!}} \, |N;\emptyset\rangle_{qp} \approx |\langle N\rangle\rangle_{mf} \otimes |\emptyset\rangle_{qp\,Boq} \,, \tag{136}$$

anyway, the final state would unavoidably have different features:

$$\sum_N \frac{e^{-N/2}}{\sqrt{N!}} \sum_{lr} a'_{lr} \left(1+\mathcal{O}\left(N^{-1}\right)\right) |N-l-r,l,r\rangle_a \neq |\langle N\rangle\rangle_{mf} \otimes \sum_{lr} a'_{lr} \, |l,r\rangle_{a\,Bog} \,. \tag{137}$$

We remark that our point is qualitative. Indeed, it is true that also in the weakly interacting limit the contribution coming from the interaction of Bogoliubov quasi-particles may be quantitatively larger than the $\mathcal{O}\left(N^{-1}\right)$ term in Equation (137). However, even if one treats the operator a_0 as a number disregarding its quantum nature, then one cannot have the above discussed entanglement. In that case, the Hilbert space does not have a sector associated to the condensed part and no correlation between the condensate and the quasi-particles is present. To have them one has to keep the quantum nature of a_0, and its contribution to the Hilbert space.

Alternatively, let us suppose to have an interacting theory of bosons for which no interactions between quasi-particles are present (as in principle one could devise and engineer similar models based on solvable interacting bosonic systems [33]). Even in that case one would have a qualitative difference (and the absence or presence of the entanglement structure here discussed) if one retains or not the quantum nature of a_0 and its contribution to the Hilbert space. Of course one could always argue that in principle the coupling between the quantum gravity and the matter degrees of freedom may be such to preserve the factorization of an initial state. This is certainly possible in principle, but it would require a surprisingly high degree of fine tuning at the level of the fundamental theory.

In conclusion, in the Bogoliubov description the state is split in two sectors, and the total density matrix is therefore a product of two contributions, of which the one relative to the mean-field can be traced away without affecting the other. The number-conserving picture shows instead that unavoidably the excited part of the system cannot be manipulated without affecting the condensate. Tracing away the quantum degrees of freedom of the condensate would imply a loss of information even without tracing away part of the couples created by analogous curved spacetime dynamics. In other words, when one considers the full Hilbert space and the full dynamics, the final state ρ_{fin} is obtained by an unitary evolution. But now, unlike the usual Bogoliubov treatment, one can trace out in ρ_{fin} the condensate degrees of freedom of the Hilbert space, an operation that we may denote by "$\mathrm{Tr}_0[\ldots]$". So

$$\rho_{fin}^{reduced} = \mathrm{Tr}_0[\rho_{fin}] \tag{138}$$

is not pure, as a consequence of the presence of the correlations. So one has $\mathrm{Tr}[\rho_{fin}^2] = 1$, at variance with $\mathrm{Tr}[(\rho_{fin}^{reduced})^2] \neq 1$. The entanglement between condensate and excited part is an unavoidable feature of the evolution of these states.

8. Discussion and Conclusions

The general aim of analogue gravity is to reproduce the phenomenology of quantum field theory on curved spacetime with laboratory-viable systems. In this framework, the geometry is given by a metric tensor assumed to be a classical tensor field without quantum degrees of freedom, implying that geometry and matter—the two elements of the system—are decoupled, i.e., the fields belong to distinct Hilbert spaces.

The usual formulation of analogue gravity in Bose–Einstein condensates reproduces this feature. In the analogy between the quasi-particle excitations on the condensate and those of scalar quantum fields in curved spacetime, the curvature is simulated by the effective acoustic metric derived from the classical condensate wave function. The condensate wave function itself does not belong to the Fock space of the excitations, instead it is a distinct classical function.

Moreover, analogue gravity with Bose–Einstein condensates is usually formulated assuming a coherent initial state, with a formally well-defined mean-field function identified with the condensate wave function. The excited part is described by operators obtained translating the atom field by the mean-field function and linearizing its dynamics; the quasi-particles studied in analogue gravity emerge from the resulting Bogoliubov–de Gennes equations. (Moreover, let us notice that the relation between several quantum gravity scenarios and analogue gravity in Bose–Einstein condensates appears to be even stronger than expected, as in many of these models a classical spacetime is recovered by considering an expectation value of the geometrical quantum degrees of freedom over a global coherent state the same way that the analogue metric is introduced by taking the expectation value of the field on a coherent ground state (see, e.g., the works by the authors of [34,35]). It is also interesting to note that within the AdS/CFT correspondence a deep connection between the analogue gravity system built from the hydrodynamics on the boundary and gravity in the bulk has emerged in recent work (see, e.g., the work by the authors of [36]). It would be interesting to extend the lessons of this work to these settings) The mean-field drives the evolution of the quasi-particles, which have a negligible backreaction on the condensate, and can be assumed to evolve independently, in accordance with the Gross–Pitaevskii equation. Neglecting the quantum nature of the operator creating particles in the condensate, one still has unitary dynamics occurring in the Hilbert space of noncondensed atoms (or, equivalently, of the quasi-particles).

However, one could still expect an information loss problem to arise whenever simulating an analogue black hole system entailing the complete loss of the ingoing Hawking partners, e.g., by having a flow with a region where the hydrodynamical description at the base of the standard analogue gravity formalism fails. The point is that the Bogoliubov theory is not exact as much as quantum field theory in curved spacetime is not a full description of quantum gravitational and matter degrees of freedom.

Within the number-conserving formalism, analogue gravity provides the possibility to develop a more complete description in which one is forced to retain the quantum nature of the operator creating particles in the condensate. While this is not per se a quantum gravity analogue (in the sense that it cannot reproduce the full dynamical equations of the quantum system), it does provide a proxy for monitoring the possible development of entanglement between gravitational and matter degrees of freedom.

It has already been conjectured in quantum gravity that degrees of freedom hidden from the classical spacetime description, but correlated to matter fields, are necessary to maintain unitarity in the global evolution and prevent the information loss [37]. To address the question of whether particle production induces entanglement between gravitation and matter degrees of freedom, we have carefully investigated the number-conserving formalism and studied the simpler process of cosmological particle production in analogue gravity, realized by varying the coupling constant from an initial value to a final one. We verified that one has a structure of quasi-particles, whose operators now depend on the operator a_0 destroying a particle in the natural orbital associated to the largest, macroscopic eigenvalue of the 1-point correlation functions.

We have shown that also in the number-conserving formalism one can define a unitary scattering operator, and thus the Bogoliubov transformation from early-times to late-times quasi-particles. The scattering operator provided in Equation (119) not only shows the nature of quasi-particle creation as a squeezing process of the initial quasi-particle vacuum, but also that the evolution process as a whole is unitary precisely, because it entangles the quasi-particles with the condensate atoms constituting the geometry over which the former propagate.

The correlation between the quasi-particles and the condensate atoms is a general feature, it is not realized just in a regime of high energies—analogous to the late stages of a black hole evaporation process or to sudden cosmological expansion—but it happens during all the evolution (Indeed, the transplanckian problem in black hole radiation may suggest that Hawking quanta might always probe the fundamental degrees of freedom of the underlying the geometry), albeit they are suppressed in the number of atoms, N, relevant for the system and are hence generally negligible. When describing the full Fock space, there is not unitarity breaking, and the purity of the state is preserved: it is not retrieved at late times nor is it spoiled in the transient of the evolution. Nonetheless, such a state after particle production will not factorise into the product of two states—a condensate (geometrical) and quasi-particle (matter) one—but, as we have seen, it will be necessarily an entangled state. This implies, as we have discussed at the end of the previous section, that an observer unable to access the condensate (geometrical) quantum degrees of freedom would define a reduced density matrix (obtained by tracing over the latter), which would no more be compatible with an unitary evolution.

In practice, in cases such as the cosmological particle creation, where the phenomenon happens on the whole spacetime, N is the (large) number of atoms in the whole condensate, and thus the correlations between the substratum and the quasi-particles are negligible. Therefore, the number-conserving formalism or the Bogoliubov one in this case may be practically equivalent. In the black hole case, a finite region of spacetime is associated to the particle creation, thus N is not only finite but decreases as a consequence of the evaporation making the correlators between geometry and Hawking quanta more and more non-negligible in the limit in which one simulates a black hole at late stages of its evaporation. This implies that tracing over the quantum geometry degrees of freedom could lead to non-negligible violation of unitary even for regular black hole geometries (i.e., for geometries without inner singularities, see, e.g., the works by the authors of [38–40]).

The Bogoliubov limit corresponds to taking the quantum degrees of freedom of the geometry as classical. This is not per se a unitarity violating operation, as it is equivalent to effectively recover the factorization of the above mentioned state. Indeed, the squeezing operator so recovered (which corresponds to the one describing particle creation on a classical spacetime) is unitarity preserving. However, the two descriptions are no longer practically equivalent when a region of quantum gravitational evolution is somehow simulated. In this case, having the possibility of tracking the quantum degrees of freedom underlying the background enables to describe the full evolution; whereas, in the analogue of quantum field theory in curved spacetime, a trace over the ingoing Hawking quanta is necessary with the usual problematic implications for the preservation of unitary evolution.

In the analogue gravity picture, the above alternatives would correspond to the fact that the number-conserving evolution can keep track of the establishment of correlations between the atoms and the quasi-particles that cannot be accounted for in the standard Bogoliubov framework. Hence, this analogy naturally leads to the conjecture that a full quantum gravitational description of a black hole creation and evaporation would leave not just a thermal bath on a Minkowski spacetime but rather a highly entangled state between gravitational and matter quantum degrees of freedom corresponding to the same classical geometry (With the possible exception of the enucleation of a disconnected baby universe which would lead to a sort of trivial information loss); a very complex state, but nonetheless a state that can be obtained from the initial one (for gravity and matter) via a unitary evolution.

In conclusion, the here presented investigation strongly suggests that the problems of unitarity breaking and information loss encountered in quantum field theory on curved spacetimes can only be addressed in a full quantum gravity description able to keep track of the correlations between quantum matter fields and geometrical quantum degrees of freedom developed via particle creation from the vacuum; these degrees of freedom are normally concealed by the assumption of a classical spacetime, but underlay it in any quantum gravity scenario.

Author Contributions: All the authors contributed equally to this work.

Funding: This research received no external funding.

Acknowledgments: Discussions with Renaud Parentani, Matt Visser, and Silke Weinfurtner are gratefully acknowledged.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Unruh, W. Experimental black hole evaporation. *Phys. Rev. Lett.* **1981**, *46*, 1351–1353. [CrossRef]
2. Barceló, C.; Liberati, S.; Visser, M. Analogue gravity. *Living Rev. Rel.* **2005**, *8*, 12. [CrossRef] [PubMed]
3. Visser, M.; Barceló, C.; Liberati, S. Analog models of and for gravity. *Gen. Rel. Grav.* **2002**, *34*, 1719–1734. [CrossRef]
4. Girelli, F.; Liberati, S.; Sindoni, L. Gravitational dynamics in Bose Einstein condensates. *Phys. Rev.* **2008**, *D78*, 084013. [CrossRef]
5. Finazzi, S.; Liberati, S.; Sindoni, L. Cosmological Constant: A Lesson from Bose-Einstein Condensates. *Phys. Rev. Lett.* **2012**, *108*, 071101. [CrossRef] [PubMed]
6. Belenchia, A.; Liberati, S.; Mohd, A. Emergent gravitational dynamics in a relativistic Bose-Einstein condensate. *Phys. Rev.* **2014**, *D90*, 104015. [CrossRef]
7. Garay, L.; Anglin, J.; Cirac, J.; Zoller, P. Black holes in Bose-Einstein condensates. *Phys. Rev. Lett.* **2000**, *85*, 4643–4647. [CrossRef] [PubMed]
8. Garay, L.; Anglin, J.; Cirac, J.; Zoller, P. Sonic black holes in dilute Bose-Einstein condensates. *Phys. Rev.* **2001**, *A63*, 023611. [CrossRef]
9. Barceló, C.; Liberati, S.; Visser, M. Probing semiclassical analog gravity in Bose-Einstein condensates with widely tunable interactions. *Phys. Rev.* **2003**, *A68*, 053613. [CrossRef]
10. Lahav, O.; Itah, A.; Blumkin, A.; Gordon, C.; Steinhauer, J. Realization of a sonic black hole analogue in a Bose-Einstein condensate. *Phys. Rev. Lett.* **2010**, *105*, 240401. [CrossRef]

11. Steinhauer, J. Measuring the entanglement of analogue Hawking radiation by the density-density correlation function. *Phys. Rev.* **2015**, *D92*, 024043. [CrossRef]
12. Steinhauer, J. Observation of quantum Hawking radiation and its entanglement in an analogue black hole. *Nat. Phys.* **2016**, *12*, 959. [CrossRef]
13. Muñoz de Nova, J.R.; Golubkov, K.; Kolobov, V.I.; Steinhauer, J. Observation of thermal Hawking radiation and its temperature in an analogue black hole. *Nature* **2019**, *569*, 688–691. [CrossRef]
14. Giddings, S.B. Comments on information loss and remnants. *Phys. Rev.* **1994**, *D49*, 4078–4088. [CrossRef] [PubMed]
15. Giddings, S.B. Constraints on black hole remnants. *Phys. Rev.* **1994**, *D49*, 947–957. [CrossRef] [PubMed]
16. Susskind, L. Trouble for remnants. *arXiv* **1995**, arXiv:hep-th/9501106.
17. Chen, P.; Ong, Y.C.; Yeom, D. Black Hole Remnants and the Information Loss Paradox. *Phys. Rept.* **2015**, *603*, 1–45. [CrossRef]
18. Hossenfelder, S.; Smolin, L. Conservative solutions to the black hole information problem. *Phys. Rev.* **2010**, *D81*, 064009. [CrossRef]
19. Almheiri, A.; Marolf, D.; Polchinski, J.; Sully, J. Black holes: Complementarity or firewalls? *J. High Energy Phys.* **2013**, *2013*, 62. [CrossRef]
20. Pethick, C.J.; Smith, H. *Bose-Einstein Condensation in Dilute Gases*, 2nd ed.; Cambridge Cambridge University Press Press: Cambridge, UK, 2008.
21. Zagrebnov, V.A. The Bogoliubov model of weakly imperfect Bose gas. *Phys. Rep.* **2001**, *350*, 291–434. [CrossRef]
22. Leggett, A. *Quantum Liquids: Bose Condensation and Cooper Pairing in Condensed-Matter Systems*; Oxford University Press: Oxford, UK, 2008. [CrossRef]
23. Penrose, O.; Onsager, L. Bose-Einstein Condensation and Liquid Helium. *Phys. Rev.* **1956**, *104*, 576–584. [CrossRef]
24. Pitaevskii, L.P.; Stringari, S. *Bose-Einstein Condensation*; Clarendon Press: Oxford, UK, 2003.
25. Lieb, E.H.; Seiringer, R.; Solovej, J.P.; Yngvason, J. *The Mathematics of the Bose Gas and its Condensation*; Oberwolfach Seminar Series; Birkhäuser: Basel, Switzerland, 2006.
26. Proukakis, N.P.; Jackson, B. Finite-temperature models of Bose–Einstein condensation. *J. Phys. B* **2008**, *41*, 203002. [CrossRef]
27. Jain, P.; Weinfurtner, S.; Visser, M.; Gardiner, C.W. Analog model of a Friedmann-Robertson-Walker universe in Bose-Einstein condensates: Application of the classical field method. *Phys. Rev. A* **2007**, *76*, 033616. [CrossRef]
28. Weinfurtner, S.; Jain, P.; Visser, M.; Gardiner, C.W. Cosmological particle production in emergent rainbow spacetimes. *Class. Quantum Gravity* **2009**, *26*, 065012. [CrossRef]
29. Carusotto, I.; Balbinot, R.; Fabbri, A.; Recati, A. Density correlations and analog dynamical Casimir emission of Bogoliubov phonons in modulated atomic Bose-Einstein condensates. *Eur. Phys. J. D* **2010**, *56*, 391–404. [CrossRef]
30. Jaskula, J.C.; Partridge, G.B.; Bonneau, M.; Lopes, R.; Ruaudel, J.; Boiron, D.; Westbrook, C.I. Acoustic Analog to the Dynamical Casimir Effect in a Bose-Einstein Condensate. *Phys. Rev. Lett.* **2012**, *109*, 220401. [CrossRef] [PubMed]
31. Sotiriadis, S.; Calabrese, P. Validity of the GGE for quantum quenches from interacting to noninteracting models. *J. Stat. Mech. Theory Exp.* **2014**, *2014*, P07024. [CrossRef]
32. Birrell, N.D.; Davies, P.C.W. *Quantum Fields in Curved Space*; Cambridge Cambridge University Press Press: Cambridge, UK, 1984. [CrossRef]
33. Richardson, R.W. Exactly Solvable Many? Boson Model. *J. Math. Phys.* **1968**, *9*, 1327–1343. [CrossRef]
34. Alesci, E.; Bahrami, S.; Pranzetti, D. Quantum Gravity Predictions for Black Hole Interior Geometry. *arXiv* **2019**, arXiv:gr-qc/1904.12412.
35. Gielen, S.; Oriti, D.; Sindoni, L. Homogeneous cosmologies as group field theory condensates. *JHEP* **2014**, *6*, 13. [CrossRef]
36. Ge, X.H.; Sun, J.R.; Tian, Y.; Wu, X.N.; Zhang, Y.L. Holographic interpretation of acoustic black holes. *Phys. Rev. D* **2015**, *92*, 084052. [CrossRef]
37. Perez, A. No firewalls in quantum gravity: The role of discreteness of quantum geometry in resolving the information loss paradox. *Class. Quant. Grav.* **2015**, *32*, 084001. [CrossRef]

38. Frolov, V.P.; Vilkovisky, G.A. Spherically Symmetric Collapse in Quantum Gravity. *Phys. Lett.* **1981**, *106*, 307–313. [CrossRef]
39. Hayward, S.A. Formation and evaporation of regular black holes. *Phys. Rev. Lett.* **2006**, *96*, 031103. [CrossRef] [PubMed]
40. Carballo-Rubio, R.; Di Filippo, F.; Liberati, S.; Pacilio, C.; Visser, M. On the viability of regular black holes. *JHEP* **2018**, *7*, 23. [CrossRef]