

Article

On Products of Random Matrices

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Abstract: We introduce a family of models, which we name matrix models associated with children’s drawings—the so-called dessin d’enfant. Dessins d’enfant are graphs of a special kind drawn on a closed connected orientable surface (in the sky). The vertices of such a graph are small disks that we call stars. We attach random matrices to the edges of the graph and get multimatrix models. Additionally, to the stars we attach source matrices. They play the role of free parameters or model coupling constants. The answers for our integrals are expressed through quantities that we call the “spectrum of stars”. The answers may also include some combinatorial numbers, such as Hurwitz numbers or characters from group representation theory.

Keywords: random complex and random unitary matrices; matrix models; products of random matrices; Schur polynomial; Hurwitz number; generalized hypergeometric functions; integrable systems

1. Introduction

Interest in matrix integrals arose in different contexts and at different times. These are problems of statistics in biology (Wishart), and problems of quantum chaos (Wigner, Dyson, Gorkov, Eliashberg, Efetov), and purely mathematical problems of representation theory (see the textbook [1]). At the end of the previous century, applications were added in the theory of elementary particles (t’Hooft [2]), statistical physics (Migdal, Itsikson and Zuber [3], and Kazakov and Kostov [4]) and string theory (Kazakov and Brezin [5], and Migdal and Gross [6]). Now new applications have been added: in the theory of information transfer and theory of quantum information ([7] or lectures [8] (Ch. 10 on quantum Shannon theory)). We allow ourselves not to provide an ocean of links for each of these areas.

Moving closer to the point, we only refer to works on the use of random matrices in [9], works about products of complex random matrices [10–12], a review [13–16] and works close to ours from a mathematical point of view [17–22].

In the theory of information, the product of the matrices describes the cascade transformation of signals, and averaging over the matrices means introducing interference and noise. The task is to calculate various correlation functions in such models.

We offer a family of models that in a sense can be called exactly solvable (cf. [20]). They are built according to the so-called children’s drawings, more precisely, clean children’s drawings, which in combinatorics are also called maps. This is a graph drawn on a closed orientable surface, which has the following property: if we cut it along the edges, the surface decomposes into regions homeomorphic to disks (that is, they can be turned into disks by continuous transformation). In addition to this picture, we turn the vertices of the graph into small disks; we call these disks stars.

The edges of the graph are assigned matrices over which the integration is performed. Depending on the graph we draw, we will get one model or another.

Surprisingly, it turns out that studying the products of random matrices with sources is an easier and more natural task. Writing answers for such integrals turns out to be a faster task if we have these matrices. The absence of these matrices is equivalent to some additional averaging that needs to be specifically monitored.

Writing out some answers requires knowledge of certain combinatorial numbers for which tables exist. In one case, these are the so-called Hurwitz numbers; in the other, these are the characters of representations of a symmetric group. However, sometimes the answers are simplified and written in a fairly simple form—in the form of determinants or in the form of products.

Adding source matrices leads to a variety of interesting relationships with differential operators. We briefly mention this in the Appendix E.

2. Technical Tools

2.1. Partitions; Power Sums and Schur Functions; Hurwitz Numbers

Here we follow [23]. Further technical details are in Appendices A and B

Partitions. The *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ is a set of nonnegative integers λ_i which are called parts of λ and which are ordered as $\lambda_i \geq \lambda_{i+1}$. The number of non-vanishing parts of λ is called the length of the partition λ , and will be denoted by $\ell(\lambda)$. The number $|\lambda| = \sum_i \lambda_i$ is called the weight of λ . The set of all partitions will be denoted by Y and the set of partitions of weight d by Y_d . Example: the partition $(4, 4, 1)$ belongs to Y_9 . The length of $(4, 4, 1)$ is 3.

We shall use Greek letters for partitions.

Sometimes, it is convenient to use a different notation

$$\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \dots),$$

where m_k is the number of times an integer k occurs as a part of λ . For instance, $(4, 4, 1)$ may be written as $(1^1 4^2)$. The number

$$z_\lambda = \prod_{k>0} k^{m_k} m_k! \tag{1}$$

plays an important role hereinafter.

Example: $z_{(4,4,1)} = z_{(1^1 4^2)} = 1^1 1! 4^2 2! = 32$.

Partitions can be perceived visually using Young diagrams (YDs): a partition λ with parts $\lambda_1, \dots, \lambda_\ell$ matches the rows of the corresponding YD of the length $\lambda_1, \dots, \lambda_\ell$, respectively; see [23] for details. The weight of λ is the area of the YD of λ .

Power sums. For a matrix $X \in GL_N(\mathbb{C})$, we put

$$p_k(X) := \text{tr} X^k = x_1^k + \dots + x_N^k, \quad k = 1, 2, \dots, \tag{2}$$

which are the Newton sums of its eigenvalues.

Then, for a partition $\Delta = (\Delta_1, \dots, \Delta_k)$, we introduce

$$\mathbf{p}_\Delta(X) = \text{tr}(X^{\Delta_1}) \dots \text{tr}(X^{\Delta_\ell}). \tag{3}$$

We put $\mathbf{p}_{(0)}(X) = 1$.

Remark 1. Let us note that $\mathbf{p}_\Delta(X)$ is a polynomial in the eigenvalues of X and also a polynomial in entries of X . We consider $\mathbf{p}_\Delta(X)$ to be a map $GL_N(\mathbb{C}) \rightarrow \mathbb{C}$.

The polynomial \mathbf{p}_Δ is a symmetric function of the eigenvalues x_1, \dots, x_N and called the *power sum* labeled with a multi-index Δ .

If one assigns a degree 1 to each x_i , then the degree of $\mathbf{p}_\Delta(X)$ is $|\Delta|$.

In many problems, the set p_1, p_2, \dots is considered as an independent set of complex parameters, which are in no way associated with any matrix. In this case, instead of $\mathbf{p}(X)$ we write simply $\mathbf{p} = (p_1, p_2, \dots)$. The degree of p_m is equal to m .

Schur functions. Next, let us recall the definition of the Schur polynomial, parametrised by a partition. Consider the equality

$$e^{\sum_{m>0} \frac{1}{m} z^m p_m} = 1 + zp_1 + z^2 \frac{p_1^2 + p_2}{2} + \dots = \sum_{m \geq 0} z^m s_{(m)}(\mathbf{p}).$$

The polynomials $s_{(m)}$ are called elementary Schur functions. As we can see, $s_{(0)}(\mathbf{p}) = 1$. Let $\lambda = [\lambda_1, \lambda_2, \dots]$ be a partition, and define the polynomial s_λ by

$$s_\lambda(\mathbf{p}) = \det \left(s_{(\lambda_i - i + j)}(\mathbf{p}) \right)_{i,j \geq 1}. \tag{4}$$

(On the right-hand side, it is assumed that $s_{(m)} = 0$ for negative m .) Here we define the Schur function as a function of the set of free parameters $\mathbf{p} = (p_1, p_2, \dots)$.

Let us remember that we chose the variables $\mathbf{p} = (p_1, p_2, \dots)$ to be related to a matrix $X \in \mathbb{GL}_N(\mathbb{C})$. Then s_λ is a map $\mathbb{GL}_N(\mathbb{C}) \rightarrow \mathbb{C}$; we will write

$$s_\lambda(X) := s_\lambda(\mathbf{p}(X)).$$

Remark 2. Let us note that $s_\lambda(X)$ is a polynomial in entries of X and a symmetric polynomial in the eigenvalues x_1, \dots, x_N of degree $d = |\lambda|$.

Remark 3 ([23]). The Schur functions $s_\lambda(x_1, \dots, x_N)$, where $\ell(\lambda) \geq N$, form a \mathbb{Z} -basis of the space of symmetric polynomials in x_1, \dots, x_N of degree d .

In terms of the eigenvalues x_1, \dots, x_N , the Schur function reads as

$$s_\lambda(X) = \frac{\det \left[x_j^{\lambda_i - i + N} \right]_{i,j \leq N}}{\det \left[x_j^{-i + N} \right]_{i,j \leq N}} \tag{5}$$

for $\ell(\lambda) \leq N$, and vanishes for $\ell(\lambda) > N$. One can see that $s_\lambda(x)$ is a symmetric homogeneous polynomial of degree $|\lambda|$ in the variables x_1, \dots, x_N .

Remark 4. The polynomial s_λ is the character of the irreducible representation of the $\mathbb{GL}_N(\mathbb{C})$ group labeled by λ .

Character map relation. Relation (4) relates polynomials s_λ and \mathbf{p}_Δ of the same degree $d = |\lambda| = |\Delta|$. Explicitly, one can write

$$\mathbf{p}_\Delta = \sum_{\lambda \in Y_d} \frac{\dim \lambda}{d!} z_\Delta \varphi_\lambda(\Delta) s_\lambda(\mathbf{p}) \tag{6}$$

and

$$s_\lambda(\mathbf{p}) = \frac{\dim \lambda}{d!} \sum_{\Delta \in Y_d} \varphi_\lambda(\Delta) \mathbf{p}_\Delta. \tag{7}$$

The last relation is called the character map relation. Here

$$\frac{\dim \lambda}{d!} := \frac{\prod_{i < j \leq N} (\lambda_i - \lambda_j - i + j)}{\prod_{i=1}^N (\lambda_i - i + N)!} = s_\lambda(\mathbf{p}_\infty) \tag{8}$$

(see Example 1 in Section 1 and Example 5 in Section 3 of Chapter I in [23]), where

$$\mathbf{p}_\infty = (1, 0, 0, \dots). \tag{9}$$

and where $N \geq \ell(\lambda)$. As one can check, the right-hand side does not depend on N . (We recall that $\lambda_i = 0$ in case $i > \ell(\lambda)$). The number $\dim \lambda$ is an integer.

The factors $\varphi_\lambda(\Delta)$ satisfy the following orthogonality relations.

$$\sum_{\lambda \in Y_d} \left(\frac{\dim \lambda}{d!} \right)^2 \varphi_\lambda(\mu) \varphi_\lambda(\Delta) = \frac{\delta_{\Delta, \mu}}{z_\Delta} \tag{10}$$

and

$$\left(\frac{\dim \lambda}{d!} \right)^2 \sum_{\Delta \in Y_d} z_\Delta \varphi_\lambda(\Delta) \varphi_\mu(\Delta) = \delta_{\lambda, \mu}. \tag{11}$$

Remark 5. Equality (7) expresses the $\mathbb{G}\mathbb{L}_N$ character s_λ in terms of characters χ_λ of the symmetric group S_d labeled by the same partition λ and evaluated on cycle classes $\Delta \in Y_d$. This explains the name of (7). The numbers $\varphi_\lambda(\Delta)$ are called the normalized characters:

$$\chi_\lambda(\Delta) = \frac{\dim \lambda}{d!} z_\Delta \varphi_\lambda(\Delta).$$

The integer $\dim \lambda$ is the dimension of the representation λ ; that is, $\dim \lambda = \chi_\lambda((1^d))$. Equations (10) and (11) are the orthogonality relation for the characters.

Hypergeometric tau functions and determinantal formulas.

Here we follow [24,25]. Let r be a function on the lattice \mathbb{Z} . Consider the following series

$$1 + r(n+1)x + r(n+1)r(n+2)x^2 + r(n+1)r(n+2)r(n+3)x^3 + \dots =: \tau_r(n, x) \tag{12}$$

Here n is an arbitrary integer. This is just a Taylor series for a function τ_r with a given n written in a form that is as close as possible to typical hypergeometric series. Here n is just a parameter that will be of use later. If as r we take a rational (or trigonometric) function, we get a generalized (or basic) hypergeometric series. Indeed, take

$$r(n) = \frac{\prod_{i=1}^p (a_i + n)}{n \prod_{i=1}^q (b_i + n)} \tag{13}$$

We obtain

$$\tau_r(n, x) = \sum_{m \geq 0} \frac{\prod_{i=1}^p (a_i + n)_m}{m! \prod_{i=1}^q (b_i + n)_m} x^m = {}_pF_q \left(\begin{matrix} a_1 + n, \dots, a_p + n \\ b_1 + n, \dots, b_q + n \end{matrix} \middle| x \right)$$

where

$$(a)_n = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{14}$$

is the Pochhammer symbol.

One can prove the following formula which express a certain series over partitions in terms of (12) (see, for instance, [24]):

$$\tau_r(n, X, Y) := \sum_{\lambda} r_{\lambda}(n) s_{\lambda}(X) s_{\lambda}(Y) = \frac{c_n}{c_{n-N}} \frac{\det [\tau_r(n - N + 1, x_i y_j)]_{i,j \leq N}}{\det [x_i^{N-k}]_{i,k \leq N} \det [y_i^{N-k}]_{i,k \leq N}} \tag{15}$$

where $c_k = \prod_{i=0}^{k-1} (r(i))^{i-k}$ and where

$$r_{\lambda}(n) := \prod_{(i,j) \in \lambda} r(n + j - i), \quad r_{(0)}(n) = 1 \tag{16}$$

where the product ranging over all nodes of the Young diagram λ is the so-called content product (which has the meaning of the generalized Pochhammer symbol related to λ). Let us test the formula for the simplest case $r \equiv 1$:

$$\det [1 - X \otimes Y] = \sum_{\lambda} s_{\lambda}(X) s_{\lambda}(Y) = \frac{\det [(1 - x_i y_j)^{-1}]_{i,j \leq N}}{\det [x_i^{N-k}]_{i,k \leq N} \det [y_i^{N-k}]_{i,k \leq N}} \tag{17}$$

Sometimes we also use infinite sets of power sums $\mathbf{p}^i = (p_1^{(i)}, p_1^{(i)}, \dots)$ and instead of matrices X and Y set:

$$\tau_r(n, \mathbf{p}^1, \mathbf{p}^2) := \sum_{\lambda} r_{\lambda}(n) s_{\lambda}(\mathbf{p}^1) s_{\lambda}(\mathbf{p}^2) \tag{18}$$

An example $r \equiv 1$:

$$\tau_1(n, \mathbf{p}^1, \mathbf{p}^2) := e^{\sum_{m>0} \frac{1}{m} p_m^{(1)} p_m^{(2)}} = \sum_{\lambda} s_{\lambda}(\mathbf{p}^1) s_{\lambda}(\mathbf{p}^2) \tag{19}$$

In case the function r has zeroes, there exists a determinantal representation. Suppose $r(0)$; then $r_{\lambda}(n) = 0$ if $\ell(\lambda) > n$. Then

$$\tau_r(n, \mathbf{p}^1, \mathbf{p}^2) = \sum_{\ell(\lambda) \leq n} r_{\lambda}(n) s_{\lambda}(\mathbf{p}^1) s_{\lambda}(\mathbf{p}^2) = c_n \det \left[\frac{\partial^a}{\partial p_1^{(1)}} \frac{\partial^b}{\partial p_1^{(2)}} \tau_r(1, \mathbf{p}^1, \mathbf{p}^2) \right]_{a,b=0, \dots, n-1}, \tag{20}$$

where $c_k = \prod_{i=1}^{k-1} (r(i))^{i-k}$, and where

$$\tau_r(1, \mathbf{p}^1, \mathbf{p}^2) = 1 + \sum_{m>0} r(1) \cdots r(m) s_{(m)}(\mathbf{p}^1) s_{(m)}(\mathbf{p}^2);$$

see [26].

In addition, there is the following formula:

$$\tau_r(n, X, \mathbf{p}) = \sum_{\lambda} r_{\lambda}(n) s_{\lambda}(X) s_{\lambda}(\mathbf{p}) = \frac{\det [x_i^{N-k} \tau_r(n - k + 1, x_i, \mathbf{p})]_{i,k \leq N}}{\det [x_i^{N-k}]_{i,k \leq N}} \tag{21}$$

where

$$\tau_r(n, x_i, \mathbf{p}) = 1 + \sum_{m>0} r(n) \cdots r(n + m) x_i^m s_{(m)}(\mathbf{p}) \tag{22}$$

Let us test it for $r \equiv 1, \mathbf{p} = \mathbf{p}_\infty := (1, 0, 0, \dots)$:

$$e^{\text{tr}X} = \sum_{\lambda} s_{\lambda}(X) s_{\lambda}(\mathbf{p}_\infty) = \frac{\det \left[x_i^{N-k} e^{x_i} \right]_{i,k \leq N}}{\det \left[x_i^{N-k} \right]} \tag{23}$$

There are similar series

$$\sum_{\lambda} r_{\lambda}(n) s_{\lambda}(X)$$

which can be written as a Pfaffian [27]; however, we will not use them in the present text.

Some properties of the Schur functions. Let us consider

Lemma 1. For $X \in \mathbb{GL}_N$, where $\det X \neq 0$, and for $\lambda = (\lambda_1, \dots, \lambda_N)$, $\ell(\lambda) \leq N$, we have

$$s_{\lambda}(X) \det(X^{\alpha}) = s_{\lambda+\alpha}(X) \tag{24}$$

where α is a nonnegative integer and where $\lambda + \alpha$ denotes the partition with parts $(\lambda_1 + \alpha, \lambda_2 + \alpha, \dots, \lambda_N + \alpha)$, in particular

$$s_{\lambda}(\mathbb{I}_N) = s_{\lambda+\alpha}(\mathbb{I}_N). \tag{25}$$

Additionally

$$\frac{s_{\lambda}(\mathbf{p}_\infty)}{s_{\lambda+\alpha}(\mathbf{p}_\infty)} = \prod_{i=1}^N \frac{\Gamma(h_i + 1 + \alpha)}{\Gamma(h_i + 1)} = \frac{(N + \alpha)_{\lambda}}{(N)_{\lambda}} \tag{26}$$

Hurwitz number. Let $E \leq 2$ be an integer and $\Delta^1 = (\Delta_1^1, \Delta_2^1, \dots), \dots, \Delta^k = (\Delta_1^k, \Delta_2^k, \dots) \in Y_d$. One can show that

$$H_E(\Delta^1, \dots, \Delta^k) = \sum_{\lambda \in Y_d} \left(\frac{\dim \lambda}{d!} \right)^E \varphi_{\lambda}(\Delta^1) \cdots \varphi_{\lambda}(\Delta^k) \tag{27}$$

is a rational number. This number is called the Hurwitz number, which is a popular combinatorial object in many fields of mathematics (see, for instance, [28,29]) and also in physics [30]. The explanations of the geometrical and combinatorial meanings of Hurwitz numbers may be found in Appendices C and D. We also need a weighted Hurwitz number

$$H_E(\Delta^1, \dots, \Delta^k | m) = \sum_{\lambda \in Y_d} \left(\frac{\dim \lambda}{d!} \right)^E \varphi_{\lambda}(\Delta^1) \cdots \varphi_{\lambda}(\Delta^k) \left(\frac{1}{(N)_{\lambda}} \right)^m \tag{28}$$

2.2. Mixed Ensembles of Random Matrices

A complex number $\mathbb{E}_{n_1, n_2} \{f\}$ is the notation of the expectation values of a function f which depends on the entries of matrices $Z_1, \dots, Z_{n_1} \in \mathbb{GL}_N(\mathbb{C})$ and of matrices $U_1, \dots, U_{n_2} \in \mathbb{U}_N$:

$$\mathbb{E}_{n_1, n_2} \{f\} = \int f(Z_1, \dots, Z_{n_1}, U_1, \dots, U_{n_2}) d\Omega_{n_1, n_2}, \tag{29}$$

$$d\Omega_{n_1, n_2} = \prod_{i=1}^{n_1} d\mu(Z_i) \prod_{i=1}^{n_2} d_* U_i, \tag{30}$$

where $d_* U_i$ ($i = 1, \dots, n_2$) is the Haar measure on \mathbb{U}_N and where

$$d\mu(Z_i) = c \prod_{a,b} e^{-\hbar^{-1} |(Z_i)_{a,b}|^2} d^2(Z_i)_{a,b}$$

is the Gaussian measure. Here \hbar is a parameter; usually it is chosen to be N^{-1} .

Each set of Z_i and $d\mu(Z_i)$ is called complex Ginibre ensemble and the whole set Z_1, \dots, Z_{n_1} and $d\mu(Z_1), \dots, d\mu(Z_{n_1})$ are called n_1 independent complex Ginibre ensembles. The set U_1, \dots, U_{n_2} and the measure $d_*U_1, \dots, d_*U_{n_2}$ are called n_2 independent circular ensembles. We assume each $\int d_*U_i = \int d\mu(Z_i) = 1$.

The ensemble of the matrices $Z_1, \dots, Z_{n_1}, U_1, \dots, U_{n_2}$ together with the probability measure $d\Omega_{n_1, n_2}$ (that is, with expectation values defined by (29)) we call a (n_1, n_2) mixed ensemble. We will consider mixed ensembles with $n = n_1 + n_2$ random matrices.

2.3. Integrals of Schur Functions and Integrals of Power Sums

In what follows we study expectation values of Schur functions and of power sums. We need four key lemmas.

These lemmas should be known in parts corresponding to Schur functions, but at the moment I have not found all the lemmas along with the proofs, so I will try to fill this gap.

In the Lemmas 2–5 below A and B are $N \times N$ complex matrices; this point is the key.

Everywhere in this section, $d = 1, \dots, N$.

Lemma 2. (I) For any $\lambda \in Y_d$ we have

$$\mathbb{E}_{1,0} \left\{ s_\lambda(ZAZ^\dagger B) \right\} = \hbar^d \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(\mathbf{p}_\infty)}, \tag{31}$$

where $s_\lambda(\mathbf{p}_\infty)$ is given in (8).

(II) Equivalently, for any $\Delta \in Y_d$ we have

$$\mathbb{E}_{1,0} \left\{ \mathbf{p}_\Delta(ZAZ^\dagger B) \right\} = z_\Delta \hbar^d \sum_{\Delta^a, \Delta^b \in Y_d} H_2(\Delta, \Delta^a, \Delta^b) \mathbf{p}_{\Delta^a}(A) \mathbf{p}_{\Delta^b}(B), \tag{32}$$

where

$$H_2(\Delta, \Delta^a, \Delta^b) = \sum_{\lambda \in Y_d} \left(\frac{\dim \lambda}{d!} \right)^2 \varphi_\lambda(\Delta) \varphi_\lambda(\Delta^a) \varphi_\lambda(\Delta^b) \tag{33}$$

is the three-point Hurwitz number (27).

Note that Hurwitz numbers do not depend on the order of its arguments, in particular, for (33), $H_2(\Delta, \Delta^a, \Delta^b) = H_2(\Delta, \Delta^b, \Delta^a) = \dots = H_2(\Delta^b, \Delta^a, \Delta)$.

Proof. (i) Relation (31) is known for Hermitian A, B ; see [23], Chap. VII, Section 5, Example 5. Taking into account Remark 2, we see that both sides of (31) can be analytically continued as functions of matrix entries of A and B . Therefore, (31) is correct for $A, B \in \mathbb{GL}_N(\mathbb{C})$.

(ii) Relation (32) follows from (31) and vice versa. To see this we use (10). For example, let us get (32) from (31). We replace all Schur functions in (31) with power sums in accordance with (7). Then we multiply the both sides of relation (31) by $\varphi_\lambda(\Delta) \dim \lambda$, then we sum over λ . Then the first orthogonality relation (10) and (27) results in (32).

(iii) The alternative way to prove the Lemma is to start with the left-hand side of (32), where $A, B \in \mathbb{GL}_N(\mathbb{C})$. Such integrals were considered in [31] in the context of generating of Hurwitz numbers (see Appendix C). Using the results of [31] we state that it is equal to the right-hand side of (32), where $H_2(\Delta, \Delta^a, \Delta^b)$ is the three-point Hurwitz number given by (27), namely, to right-hand side of (33). Then with the help of orthogonality relation (10), we derive (31), where $A, B \in \mathbb{GL}_N(\mathbb{C})$. \square

Remark 6. Regarding the last point (iii): The volume of the article does not allow describing the construction of work [31,32]. In short: The derivation of the formula (33), and the derivation of more general formulas (68) below, are based on the application of Wick’s theorem and the realization of the fact that each Wick pairing corresponds to a certain gluing of surfaces from polygons: the surface obtained in the first order of the perturbation theory ($|\Delta| = d = 1$) is basic. (For instance, the torus can be obtained from a rectangular by gluing the opposite sides;

see (a) and (b) in Figure 3.) The surfaces obtained in the following orders ($d > 1$) are the surfaces which cover the basic one. In this case, the Hurwitz numbers simply give a weighted number of possible covering surfaces, where the Young diagrams correspond to the so-called branching profiles. The basic surface in case (33) is a sphere, glued from the 2-gon; see Figure 2b below where $X_1 = Z, X_{-1} = Z^\dagger, C_1 = A, C_{-1} = B$.

Examples of (33). Suppose $\Delta = (2)$ on the left-hand side of (33). It is known that $H_2((2), \Delta^a, \Delta^b) = 1/2$ in two cases—where $\Delta^a = (2), \Delta^b = (1, 1)$ and where $\Delta^a = (1, 1), \Delta^b = (2)$ —and it is zero otherwise. Additionally $z_{(2)} = 2$, see (1). Thus,

$$\mathbb{E}_{1,0} \left\{ \text{tr} \left((ZAZ^\dagger B)^2 \right) \right\} = \hbar^2 \text{tr} \left(A^2 \right) (\text{tr} B)^2 + \hbar^2 (\text{tr} A)^2 \text{tr} \left(B^2 \right).$$

Next, suppose $\Delta = (1, 1)$. As we know, $H_2((1, 1), (2), (2)) = 1/2$ and vanishes for different choices of Δ^a, Δ^b , and $z_{(1,1)} = 2$; see (1); therefore,

$$\mathbb{E}_{1,0} \left\{ \left(\text{tr} ZAZ^\dagger B \right)^2 \right\} = \hbar^2 \text{tr} \left(A^2 \right) \text{tr} \left(B^2 \right).$$

Corollary 1. Suppose $\det A \neq 0, \det B \neq 0$, and $\lambda = (\lambda_1, \dots, \lambda_N), \ell(\lambda) \leq N$. We get

$$\mathbb{E}_{1,0} \left\{ s_\lambda(ZAZ^\dagger B) \det \left((Z^\dagger Z)^{p_0} \right) \right\} = \hbar^{d+p_0N} \frac{s_\lambda(A) s_\lambda(B)}{s_\lambda(\mathbf{p}_\infty)} \frac{(N+p_0)_\lambda}{(N)_\lambda} \tag{34}$$

In particular,

$$\mathbb{E}_{1,0} \left\{ s_\lambda(ZAZ^\dagger) \det \left((Z^\dagger Z)^{p_0} \right) \right\} = \hbar^{d+p_0N} s_\lambda(A) (N+p_0)_\lambda \tag{35}$$

Proof. In case p_0 is a natural number the proof is as follows:

$$s_\lambda(ZAZ^\dagger B) \det \left((ZAZ^\dagger B)^{p_0} \right) = s_\lambda(ZAZ^\dagger B) \det \left((ZZ^\dagger)^{p_0} \right) \det \left((A)^{p_0} \right) \det \left((B)^{p_0} \right) = s_{\lambda+p_0}(ZAZ^\dagger B)$$

and from Lemma 1: where $\lambda + p_0 = (\lambda_1 + p_0, \lambda_2 + p_0, \dots, \lambda_N + p_0), \ell(\lambda') = N$. We have

$$\frac{s_\lambda(\mathbf{p}_\infty)}{s_{\lambda'}(\mathbf{p}_\infty)} = \prod_{i=1}^N \frac{\Gamma(h_i + 1 + p_0)}{\Gamma(h_i + 1)} = \frac{(N+p_0)_\lambda}{(N)_\lambda} \tag{36}$$

As we see the right-hand side can be analytically continued as the function of p_0 . \square

Lemma 3. (I) Suppose $\lambda \in Y_d$, and let ν be any partition. We have

$$\mathbb{E}_{1,0} \left\{ s_\lambda(ZA) s_\nu(Z^\dagger B) \right\} = \hbar^d \delta_{\lambda\nu} \frac{s_\lambda(AB)}{s_\lambda(\mathbf{p}_\infty)}. \tag{37}$$

where $s_\lambda(\mathbf{p}_\infty)$ is given in (8).

(II) Equivalently, let $\Delta^a \in Y_d$ and $\Delta^b \in Y_{d'}$. Then

$$\mathbb{E}_{1,0} \left\{ \mathbf{p}_{\Delta^a}(ZA) \mathbf{p}_{\Delta^b}(Z^\dagger B) \right\} = z_{\Delta^a} z_{\Delta^b} \delta_{d,d'} \hbar^d \sum_{\Delta \in Y_d} H_2(\Delta^a, \Delta^b, \Delta) \mathbf{p}_\Delta(AB), \tag{38}$$

where $H_{\mathbb{C}\mathbb{P}^1}(\Delta, \Delta^a, \Delta^b)$ is given by (33).

The identity (37) is well-known; for instance, see [23]. Relation (38) was proven independently in [31]. To obtain (37) from (38) we replace power sums under the integral with the left-hand side of (6) and use (27) and the orthogonality relations (10) and (11).

Denote $N \times N$ identity matrix \mathbb{I}_N . The following equality is known (see [23]: combine Examples 4 and 5 of Section 3 and Example 1 from Section I of Chapter I):

$$s_\lambda(\mathbb{I}_N) = (N)_\lambda s_\lambda(\mathbf{p}_\infty). \tag{39}$$

Here

$$(N)_\lambda := (N)_{\lambda_1} (N - 1)_{\lambda_2} \cdots (N - \ell + 1)_{\lambda_\ell}, \tag{40}$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\lambda_\ell \neq 0$ and where $(a)_k := a(a + 1) \cdots (a + k - 1) = \frac{\Gamma(a+k)}{\Gamma(a)}$ is the Pochhammer symbol.

Lemma 4. For any $\lambda \in Y_d$, we have

$$\mathbb{E}_{0,1} \left\{ s_\lambda(UAU^{-1}B) \right\} = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(\mathbb{I}_N)}. \tag{41}$$

The proof is contained in [23] Chap VII, Section 5, Example 3.

Lemma 5. Suppose $\lambda \in Y_d$. For any μ , we get

$$\mathbb{E}_{0,1} \left\{ s_\mu(UA)s_\lambda(U^{-1}B) \right\} = \frac{s_\lambda(AB)}{s_\lambda(\mathbb{I}_N)} \delta_{\mu,\lambda}. \tag{42}$$

The proof follows from relations (1) and (3) Chap VII, Section 5, Example 1 in [23].

Remark 7. Formula (41) gives the fastest derivation of the famous formula HCIZ:

$$\int e^{\alpha \text{tr} UAU^\dagger B} d_* U = c \frac{\det [e^{a_i b_j}]_{ij}}{\prod_{i>j} (a_i - a_j)(b_i - b_j)}$$

Indeed, we use (19) where $\mathbf{p}^1 = (1, 0, 0, \dots) =: \mathbf{p}_\infty$ and $\mathbf{p}^2 = (p_1^{(2)}, p_2^{(2)}, \dots)$ with $p_m^{(2)} = \text{tr}((UAU^\dagger B)^m)$; thus,

$$e^{\alpha \text{tr} UAU^\dagger B} = \sum_\lambda \alpha^{|\lambda|} s_\lambda(\mathbf{p}_\infty) s_\lambda(UAU^\dagger B)$$

which gives

$$\int e^{\alpha \text{tr} UAU^\dagger B} d_* U = \sum_\lambda \alpha^{|\lambda|} \frac{s_\lambda(\mathbf{p}_\infty) s_\lambda(A) s_\lambda(B)}{s_\lambda(\mathbb{I}_N)} = \sum_\lambda \frac{\alpha^{|\lambda|}}{(N)_\lambda} s_\lambda(A) s_\lambda(B)$$

which has the form (15). In a similar way one evaluates $\int (1 - \alpha UAU^\dagger B)^{-a} d_* U$ and a number of integrals; see, for instance [26].

The expectation values $\mathbb{E}_{0,1} \{ \mathbf{p}_\Delta(AUBU^{-1}) \}$ and $\mathbb{E}_{0,1} \{ \mathbf{p}_{\Delta^a}(AU) \mathbf{p}_{\Delta^b}(BU^\dagger) \}$ can also be expressed in terms of Hurwitz numbers, but this requires a lot of space, and we will not do this; see the last section [31] for details.

3. Our Models. Products of Random Matrices We Choose

3.1. Preliminary. On the Products of Random Matrices

In a number of articles the spectral correlation functions of certain products of were studied. These are products

$$Z_1 Z_1^\dagger \cdots Z_n Z_n^\dagger, \tag{43}$$

$$Z_1 \cdots Z_n (Z_1 \cdots Z_n)^\dagger, \tag{44}$$

or, a pair of products

$$Z_1 \cdots Z_n, \quad (Z_1 \cdots Z_n)^\dagger; \tag{45}$$

see [10–12].

Similar products in which complex matrices are replaced by unitary and certain generalizations have also been considered [16,21,22,33–35].

Below we suggest a generalization of these models (see Examples 18 and 16 respectively for (43)–(45)). We call it matrix models related to dessin d’enfants, more precisely, to the so-called clean dessin d’enfants (see [36]).

This is a child’s drawing of a “constellation”, like the Greek constellation, which is painted in the sky with stars, and the sky is a surface with a chosen Euler characteristic.

See below for a more accurate description.

As a part of dessin d’enfants we will need the following modification:

(1) We shall consider mixed ensembles, and in what follows, $X_{\pm i}$ denotes either $Z_{\pm i}$, where $Z_{-i} = Z_i^\dagger$, or $U_{\pm i}$, where $U_{-i} = U_i^\dagger$.

(2) We need additional data, which we call source matrices $C_{\pm 1}, \dots, C_{\pm n} \in \mathbb{GL}_N(\mathbb{C})$. Each random matrix X_i enters only in combination $X_i C_i$:

$$X_i \rightarrow X_i C_i \tag{46}$$

Such combinations are very helpful (in particular, this makes it possible to consider rectangular random matrices). We shall use the notion of the *dressed* source matrices $C_i, i = \pm 1, \dots, \pm n$ with the notation $\mathcal{L}_X[*]$:

$$\mathcal{L}_X[f(C_1, C_{-1}, \dots, C_n, C_{-n})] := f(X_1 C_1, X_{-1} C_{-1}, \dots, X_n C_n, X_{-n} C_{-n}) \tag{47}$$

where f is any function. (The letter \mathcal{L} is used to remember that it is the dressing from the left side.)

The matrices $\{C_i\}$ we call the source matrices which play the role of coupling constants in the matrix models below.

3.2. Models Obtained from Graphs (Geometrical View)

Consider a connected graph Γ on an orientable connected surface Σ without boundary with Euler characteristic E . We require such properties of the graph:

(1) its edges do not intersect. For example, the edges of the graph in Figure 1a do not intersect: the fact is that the graph is drawn on a torus, and not on a piece of paper.

(2) if we cut the surface Σ along the edges of the graph, then the surface will decompose into disks (more precisely: into pieces homeomorphic to disks) (see Figure 3a,b as an example).

As an example, see Figure 1, which contains all such graphs with two edges.

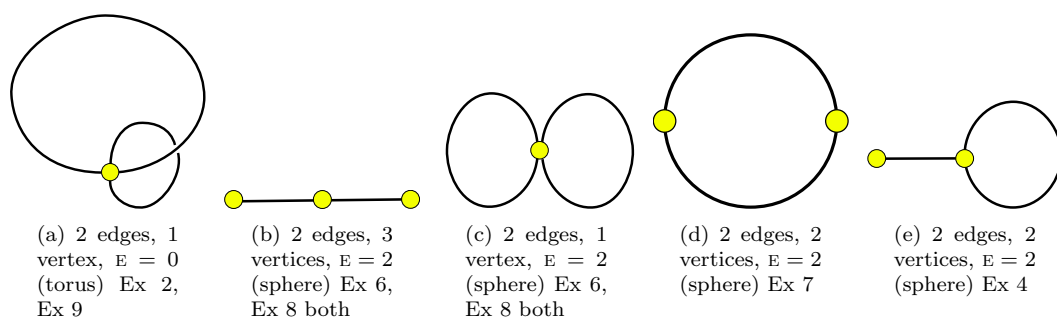


Figure 1. All possible graphs with 2 edges (two matrix models). Graph (b) is dual to (c). Ex means an example below.

Such a graph is sometimes called a (clean) dessin d'enfants (the term dessin d'enfants without the additional "clean" serves for such a graph with a bipartite structure), sometimes—a map [28].

Let our graph have F faces, n edges and V vertices; then $E = V - n + F$.

We number all stars (i.e., all vertices of the graph Γ) with numbers from 1 to V and all faces of Γ with numbers from 1 to F and all edges of Γ with numbers from 1 to n in any way.

We will slightly expand the vertices of the graph and turn them into the *small disk*, which sometimes for the sake of visual clarity we will call *stars*.

In this case, the edges coming out of the vertex will divide the border of a small disk into segments. We orient the boundary of each such segment clockwise, and the segment can be represented by an arrow that goes from one edge to another; see the Figures 2a,b,d and 3a,c as examples. Our graph will have a total of $2n$ arrow segments.

It is more correct to assume that the edges of the graph are very thin ribbons; that is, they have finite thickness. That is, the edge number $|i|$ has two sides, one of which we number with the number i , and the other with the number $-i$ (this choice is arbitrary but fixed). It would be more correct to depict each edge of Γ numbered by $|i|$ in the form of a ribbon, the sides of which are two oppositely directed arrows; one arrow has a number i , and the second $-i$. However, with the exception of Figure 3, we did not do that so as not to clutter up the drawings.

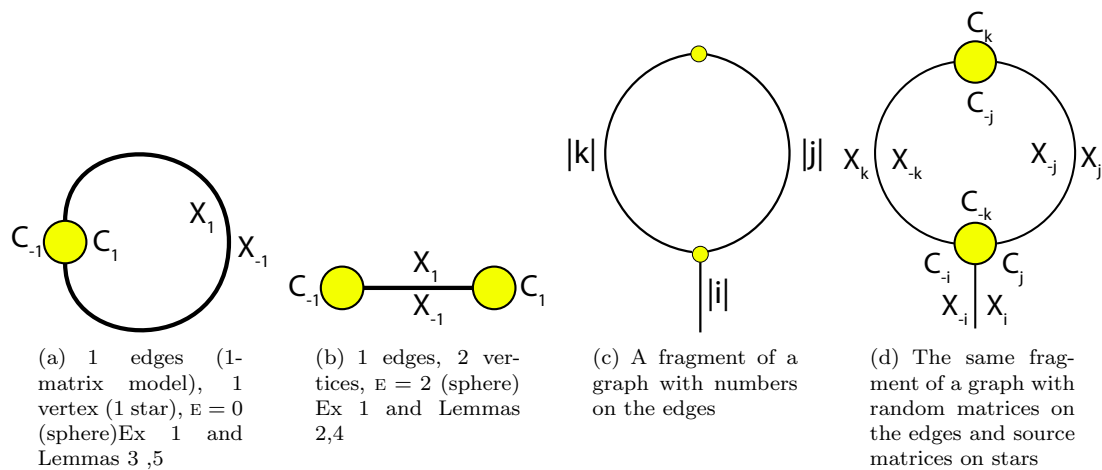


Figure 2. Decorated graphs.

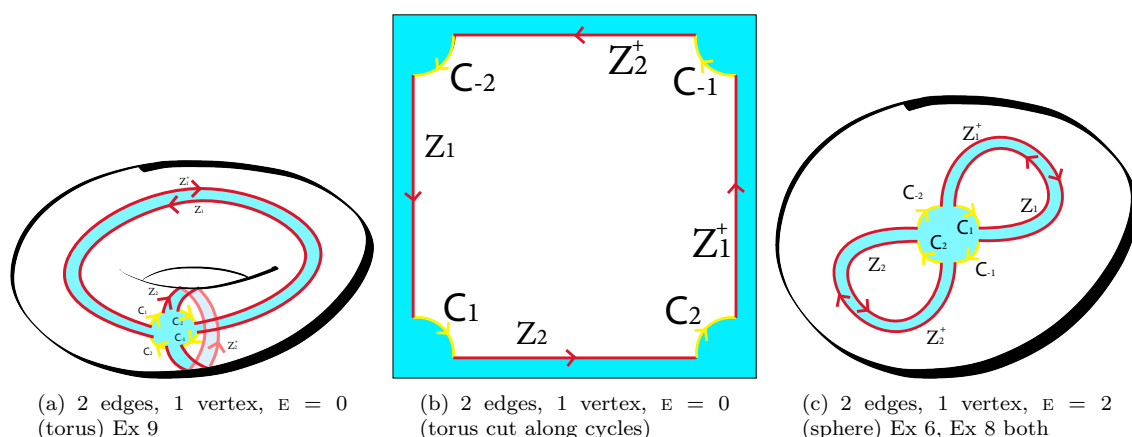


Figure 3. Graphs with 2 edges (two matrix models) where (a) is the zoom of (a) in Figure 1 and (c) is the zoom of (c) in Figure 1.

Now, each arrow of the side of the edge rests with its end against the arrow-segment—as if continued by the arrow-segment. We assign the same number i ($i = \pm 1, \dots, \pm n$) to the side of the

edge Γ and to the segment of the small disk, which continues this side while traversing the face in the positive direction.

Additionally, to the number i ($i = \pm 1, \dots, \pm n$) we attribute the product of the $N \times N$ matrices

$$i \rightarrow X_i C_i$$

where C_i is assigned to the i segment of a small disk and will be called *source matrix*, and X_i is assigned to the arrow i , which is the side of the edge $|i|$ of the graph Γ ; see Figure 2 for an illustration.

Let us use the following numbering. If one side of the edge of the tape $|i|$ is labeled i , then the other side of the same edges is labeled $-i$. (It does not matter which side of the edge $|i|$ we assign the number i to, and which we assign the number $-i$ to, but we should fix the numbering we have chosen). The two sides of an edge of the graph Γ are actually arrows pointing in opposite directions. (In order not to complicate the drawing, we do not depict these arrows in our figures, except for Figure 3a,b). We draw the arrows so that they point in the positive direction when bypassing the boundary of the face (that is, when bypassing the face in the counterclockwise direction). Each arrow, say i (which either a positive or a negative number), ends at the boundary of the small disk (star) and with further bypassing of the face we pass along the segment of the boundary of the star. We assign the same number i to this segment. Acting in this way, we give numbers to all segments of all small disks. It is easy to understand that all the arrows on the segments of the boundary of any small disc are directed clockwise (i.e., in the negative direction) when passing along the boundary of each small disk.

We attribute a sequential set of numbers to each star as follows. (this set is defined up to a cyclic permutation, and we will call it a cycle associated with the star). Examples: These are numbers $(1, -1)$ assigned to the star (yellow small disk) in Figure 2a. These are the numbers $(k, -j)$ that we attribute to the star on the top in the Figure 2c and the numbers $(-k, j, -i)$ that we attribute to the lower star in the same figure. Additionally, to each cycle we ascribe the related cyclic products:

$$(1, -1) \leftrightarrow C_1 C_{-1},$$

to the star (yellow small disk) in Figure 2a. As for the stars in Figure 2c we obtain

$$(k, -j) \leftrightarrow C_k C_{-j} \quad (-k, j, -i) \leftrightarrow C_{-k} C_j C_{-i}$$

Each cycle product we will call the *star's monodromy*. As a result, a star's monodromy is a product of matrices that are attributed to arrow segments, taken in the same sequence in which arrows follow each other when moving around a small disk clockwise. We number the stars in any way by numbers from 1 to v . The monodromy of a star i will be denoted by the letter W_i^* .

In addition to the edges and in addition to the vertices, we number the faces of the graph Γ and the corresponding face monodromies with numbers from 1 to F . The cycles corresponding to the face i will be denoted by f_i and defined as follows. When going around the face border in the positive direction, namely, counterclockwise for an ordinary face (or counterclockwise if the face contains infinity), we collect segment numbers of small disks; for a face with a number i , this ordered collection of ordered numbers is f_i . As in the case of stars, we build cyclic products $f_i \leftrightarrow W_i$. Examples:

$$f_1 = (1) \leftrightarrow C_1 = W_1 \quad \text{and} \quad f_2 = (-1) \leftrightarrow C_{-1} = W_2$$

for two faces in Figure 2a.

We also introduce *dressed monodromies*:

$$\mathcal{L}_X(W_1) = X_1 C_1, \quad \mathcal{L}_X(W_{-1}) = X_{-1} C_{-1}$$

Additionally, for the face in Figure 2c:

$$f_1 = (-j, -k) \leftrightarrow C_{-j}C_{-k} = W_1 \leftrightarrow \mathcal{L}_X(W_1) = X_{-j}C_{-j}X_{-k}C_{-k}$$

Thus, we have two sets of cycles and two sets of monodromies: vertex cycles and vertex monodromies

$$\sigma_i^{-1} \leftrightarrow W_i^*, \quad i = 1, \dots, V \tag{48}$$

and face cycles and face monodromies: The cycles corresponding to the face i will be denoted by f_i ; we have

$$f_i \leftrightarrow W_i \leftrightarrow \mathcal{L}_X[W_i], \quad i = 1, \dots, F \tag{49}$$

Let us note that both cycles and monodromies are defined up to the cyclic permutation.

Remark 8. *Important remark. Please note that each of the matrices $C_i, i = \pm 1, \dots, \pm n$ enters the set of monodromies W_1, \dots, W_f once and only once. Accordingly, each random matrix $X_i, i = \pm 1, \dots, \pm n$ is included once and only once in the set of dressed monodromies $\mathcal{L}_X[W_1], \dots, \mathcal{L}_X[W_f]$. This determines the class of matrix models that we will consider and which we will call matrix models of dessin d'enfants.*

Remark 9. *In the description of maps (the same, of clean dessin d'enfants) the following combinatorial relation is well-known; see the wonderful textbook [28], Remark 1.3.19:*

$$\left(\prod_{i=1}^n \alpha_i \right) \left(\prod_{i=1}^F f_i \right) = \left(\prod_{i=1}^V \sigma_i^{-1} \right) \tag{50}$$

where each of α_i, f_i and σ_i belongs to the permutation group S_{2n} . Here f_i are face cycles, σ_i are vertex cycles and $(\prod_{i=1}^n \alpha_i)$ is an involution without fixed points; each α_i transposes i and $-i$. Since $\alpha_i^2 = 1$ we can also write

$$\left(\prod_{i=1}^n \alpha_i \right) \left(\prod_{i=1}^V \sigma_i^{-1} \right) = \left(\prod_{i=1}^F f_i \right) \tag{51}$$

The reader can check this relation for the pair of the simplest examples where $n = 1$.

For any given set of the face cycles f_1, \dots, f_F , the relation (50) allows you to uniquely reconstruct the set of the vertex cycles $\sigma_1^{-1}, \dots, \sigma_V^{-1}$ (the power -1 is related to the negative (clockwise) counting of numbers).

We get

Proposition 1.

$$\hbar^{-n_1} \mathbb{E}_{n_1, n_2} \left\{ \prod_{i=1}^F \mathcal{L}_X [\text{tr} W_i] \right\} = N^{-n_2} \prod_{i=1}^V \text{tr} W_i^* \tag{52}$$

and

$$\hbar^{-n_1} \mathbb{E}_{n_1, n_2} \left\{ \prod_{i=1}^V \mathcal{L}_X [\text{tr} W_i^*] \right\} = N^{-n_2} \prod_{i=1}^F \text{tr} W_i, \tag{53}$$

where we use the notation (47) and where the parameters n_1, n_2, \hbar were defined in Section 2.2.

We note that each trace is a sum of monomials. Let us note that each of $X_i, i = \pm 1, \dots, \pm n$ enters only once in each monomials term insider of the integral in the left and side of (52) and of (53).

There are two ways to prove these relations: algebraic and geometrical ones. In short, the sketches of the proof are as follows. We should notice that $\mathbf{p}_{(1)}(*) = s_{(1)}(*)$ and apply Lemmas 2–5 while having in mind that this is actually the cut-or-join procedure (57) stated below in Section 3.3. In this case the difference between Lemmas 2, 3 and Lemmas 4, 5 is only in the power of the factor N , since $(N)_\lambda = N$ in case $\lambda = (1)$.

Geometric way: We use the relation

$$\hbar^{-1} \mathbb{E}_{n_1,0} \left\{ (Z_i)_{a,b} (Z_j^\dagger)_{b',a'} \right\} = \delta_{i,j} \delta_{a,a'} \delta_{b,b'}$$

and calculate monodromies along faces of the graph Γ as described in the beginning of this section.

Remark 10. From geometrical point of view in this way we create a surface by gluing the polygons related to the dressed face monodromies; see [31].

Remark 11. You may notice some similarities between formulas (52), (53) and formulas (50), (51). Additionally, there is. One can say that the role of involution α_i is played by the integration over the matrix X_i (by the Wick coupling of X_i and X_{-i}).

We draw attention to two facts.

- The answer (i.e., the left-hand side of (52)) depends only on the spectrum of star monodromies

$$\text{Spect}(W_1^*), \dots, \text{Spect}(W_\nu^*).$$

- The answer does not depend on how exactly in our model we distribute the matrices $\{Z\}$ and $\{U\}$ to dress the source matrices—only two numbers n_1 and n_2 are important. For example, it does not matter, in the right-hand side of (52), whether we dress $\mathcal{L}_X[C_i, C_{-i}] = U_i C_i, U_{-i} C_{-i}$ and $\mathcal{L}_X[C_j, C_{-j}] = Z_j C_j, Z_{-j} C_{-j}$ or $\mathcal{L}_X[C_i, C_{-i}] = Z_i C_i, Z_{-i} C_{-i}$ and $\mathcal{L}_X[C_j, C_{-j}] = U_j C_j, U_{-j} C_{-j}$.

3.3. Our Models. Algebraic View

You can forget about graphs and dessin d'enfants and set them out differently.

We have a set of random matrices

$$X_{\pm 1}, \dots, X_{\pm n}, \quad X_{-i} = X_i^\dagger$$

and there are as many source matrices

$$C_{\pm 1}, \dots, C_{\pm n}, \tag{54}$$

and we want to consider expectation value of the products of these matrices. We want each matrix X_i to come in combination with the source matrix labeled with the same i ; see (46) and (47).

The set (54) we will call the alphabet of pairs and the matrices $\{C_i\}$ can be considered letters of the alphabet.

The products of these matrices can be considered as words constructed from letters of the alphabet of pairs. Consider a group of words

$$W_1, \dots, W_F$$

with the following condition: each of matrices $C_i, i = \pm 1, \dots, \pm n$ is included *once and only once* in one of the words of this group.

In addition, we ask the following property: In this group of words there is no such subset of words that could be constructed from the alphabet of pairs with a smaller set of pairs of letters (in other words, we will consider only connected groups of words. The connected group of words will be related to the connected graphs Γ).

Each word W_i can be associated with an ordered set of numbers f_i consisting of the numbers of the matrices included in the word: $f_i \leftrightarrow W_i$. Recall that each word W_i (and respectively f_i) is defined up to a cyclic permutation.

There is a one-to-one correspondence between dessin d'enfants with n edges and F faces and word sets constructed in this way. Based on the set W_1, \dots, W_F , the surface Σ is built uniquely. To do this, the procedure is as follows. First, for each word, say W_i , we associate the polygon with the edges,

numbered by the numbers of the matrices in the product, namely, by the set f_i . By going around the boundary of the polygon counterclockwise, we assign the numbers of the matrices to the edges if we write the word from left to right. We get a set of F polygons. After that, we glue these polygons so that the side with the number i sticks together with the side with the number $-i$. We assign an orientation to each polygon, considering its edges arrows, showing the direction of counterclockwise. We glue the edges so that the beginning of the arrow i sticks together with the end of the arrow $-i$. We get the surface Σ and on it is the graph Γ , whose edges are glued from two oppositely directed arrows (ribbon edge).

To determine the Euler characteristic of Σ we need to know the number of vertices of the graph Γ . Cut-or-join procedure. Purely algebraically, one should act like this.

Consider $W_1 \otimes W_2 \otimes \dots \otimes W_F$ (the order in this tensor product is not important) and the set of involutions $T_i, i = 1, \dots, n$, which act on this tensor product as follows. Each involution of T_i does not affect those W_a that contain neither C_i nor C_{-i} . Two situations are possible. (I) The matrices C_i and C_{-i} are in the same word, say, the word W_a . How then can we to rearrange the matrices with the word cyclically, to bring it to the form $C_iXC_{-i}Y$, where X and Y are some matrices. (II) The matrices C_i and C_{-i} are included in different words; in this case we will write these two words as C_iX and $C_{-i}Y$. As you can see, the action of involutions T_i corresponds to taking the integral in Lemmas 2–5. Then

$$T_i [\dots \otimes C_iXC_{-i}Y \otimes \dots] = \dots \otimes C_iX \otimes C_iY \otimes \dots \tag{55}$$

$$T_i [\dots \otimes C_iX \otimes C_iY \otimes \dots] = \dots \otimes C_iXC_{-i}Y \otimes \dots \tag{56}$$

It is easy to see that involutions commute: $T_i[T_j[*]] = T_j[T_i[*]]$.

The transformation

$$W_1, \dots, W_F \leftrightarrow W_1^*, \dots, W_V^* \tag{57}$$

can be obtained purely algebraically in n steps. We call these two sets of matrices *dual sets*.

Proposition:

$$\prod_{i=1}^n T_i [W_1 \otimes W_2 \otimes \dots \otimes W_F] = W_1^* \otimes \dots \otimes W_V^* \tag{58}$$

$$\prod_{i=1}^n T_i [W_1^* \otimes \dots \otimes W_V^*] = W_1 \otimes W_2 \otimes \dots \otimes W_F \tag{59}$$

Remark 12. Actually this is a manifestation of the Equation (50) where α_i is related to T_i . An involution without fixed points $\prod_{i=1}^n T_i$ takes the graph Γ to the graph dual to it.

Here are some examples (57) with explanations about the geometric representation in Figures 1–5 and in some other graphs:

Example 1. Suppose $W_1 = C_1, W_2 = C_{-1}$. In 1 step we get:

$$W_1 \otimes W_2 = C_1 \otimes C_{-1} \leftrightarrow C_1C_{-1} = W_1^*$$

Therefore $v = 1$ and as we have $F = 2, n = 1$; then the Euler characteristic is $E = F - n + v = 2$. The right hand side can be obtained from Figure 2a as the face monodromies while the left-hand side can be obtained as star monodromies there. The left-hand side contains also the face monodromies in Figure 2b; and the right-hand side can be obtained as star monodromies there. (This is because graphs (a) and (b) are dual ones.)

Example 2. In 2 steps:

$$W_1 = C_1C_2C_{-1}C_{-2} \leftrightarrow C_1C_{-2}C_{-1}C_2 = W_1^*$$

Therefore $V = 1$, and as we have $F = 1, n = 2$, then the Euler characteristic is $E = F - n + V = 0$ (torus). The right-hand side can be obtained from Figure 1a as the face monodromies while the left-hand side can be obtained as star monodromies there; see the more detailed Figure 3a. The dual graph looks like the same as the graph on the torus: there is 1 vertex and 1 face; the left-hand side plays the role of the face monodromy for the dual graph. This a particular case of Ex 9.

Example 3. In two steps:

$$W_1 \otimes W_2 \otimes W_3 = C_{-1}C_{-2} \otimes C_1 \otimes C_2 \leftrightarrow C_1C_{-1}C_2C_{-2} = W_1^*$$

Therefore $V = 1$ and as we have $F = 3, n = 2$, the Euler characteristic is $E = F - n + V = 2$. The right-hand side can be obtained from Figure 1b as the face monodromies while the left-hand side can be obtained as star monodromies there. The left-hand side contains also the face monodromies in Figure 1c (see Figure 3c for more details); the right-hand side can be obtained as star monodromies there. (This is because Figure 1b and Figure 1c are dual ones).

Example 4. In 2 steps:

$$C_1C_2C_{-1} \otimes C_{-2} \leftrightarrow C_1C_{-2}C_2 \otimes C_{-1}$$

Therefore $V = 2$ and as we have $F = 2, n = 2$, the Euler characteristic is $E = F - n + V = 2$. The right-hand side can be obtained from Figure 1e as the face monodromies while the left-hand side can be obtained as star monodromies there. The left-hand side contains also the face monodromies of the dual graph which is the graph of the same type (the loop from which the segment sticks out).

Example 5. In 5 steps:

$$W_1 \otimes W_2 \otimes W_3 = C_1C_2C_3C_4 \otimes C_{-3}C_{-2}C_5 \otimes C_{-5}C_{-1}C_{-4} \leftrightarrow C_2C_{-2}C_{-5} \otimes C_2C_{-3} \otimes C_5C_3C_{-4} \otimes C_4C_{-1} = W_1^* \otimes W_2^* \otimes W_3^* \otimes W_4^*$$

Therefore $V = 4$ and as we have $F = 3, n = 5$, the Euler characteristic is $E = F - n + V = 2$ (the same is obtained from Figure 5d).

Example 6. In n steps:

$$C_1C_2 \cdots C_nC_{-n} \cdots C_{-1} \leftrightarrow C_1C_{-2} \otimes C_2C_{-3} \otimes \cdots \otimes C_{n-1}C_{-n} \otimes C_n \otimes C_{-1}$$

Therefore $V = n + 1$ and as we have $F = 1$, the Euler characteristic is $E = F - n + V = 2$. The left-hand side is related to Γ in form of a chain with $n + 1$ vertices, n edges drawn on the sphere (see Figure 1b for the $n = 2$ chain and see Figure 4a for the $n = 3$ chain). In case each source matrix is the identity ones, this is related to (44) which is a rather popular product. The right side is represented by such a graph: there are n circles; each subsequent one decreases and is inside the previous one. They all touch at one point (the vertex); see Figure 4b.

Example 7. In n steps:

$$C_1C_2 \cdots C_n \otimes C_{-n}C_{1-n} \cdots C_{-1} \leftrightarrow C_1C_{-2} \otimes C_2C_{-3} \otimes \cdots \otimes C_{n-1}C_{-n} \otimes C_nC_{-1}$$

Therefore $V = n$, and as we have $F = 2$, then the Euler characteristic is $E = F - n + V = 2$. (the case $n = 2$ is obtained from Figure 1d). The left-hand side is related to Γ in form of the chain from the previous case where we replace n by $n - 1$ and connect its ends by n -th edge. Namely, it is n -gon drawn on the sphere. The right-hand side is related to the graph where two vertices are connected by n edges. In case $n = 4$, the right-hand side can be obtained from Figure 4d as the face monodromies while the left-hand side can be obtained as star monodromies there. The left-hand side contains also the face monodromies in Figure 4c; the right-hand side can be obtained as star monodromies there. (Indeed, graphs in Figure 4d and in Figure 4c are dual ones.)

Example 8. In n steps:

$$C_1 C_{-1} C_2 C_{-2} \cdots C_n C_{-n} \leftrightarrow C_{-1} C_{-2} \cdots C_{-n} \otimes C_1 \otimes C_2 \otimes \cdots \otimes C_n$$

Therefore $V = n + 1$, and as we have $F = 1$, then the Euler characteristic is $E = F - n + V = 2$. The left-hand side is related to a star-graph Γ ; the right-hand side—to a petal graph (as an example take $n = 3$ and look at Figure 5a and dual Figure 5b).

After the dressing procedure, in case all source matrices are chosen to be identical ones we get a popular product (43).

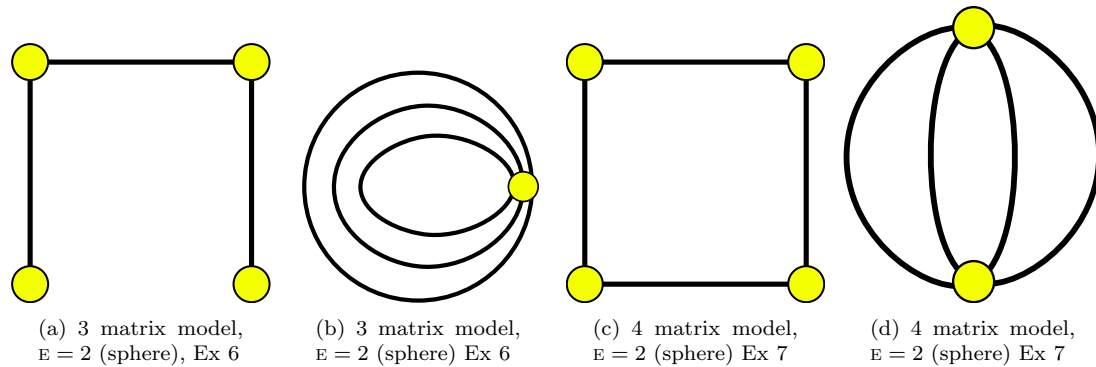


Figure 4. Graphs drawn without decoration. Graph (a) is dual to (b). Graph (c) is dual to (d).

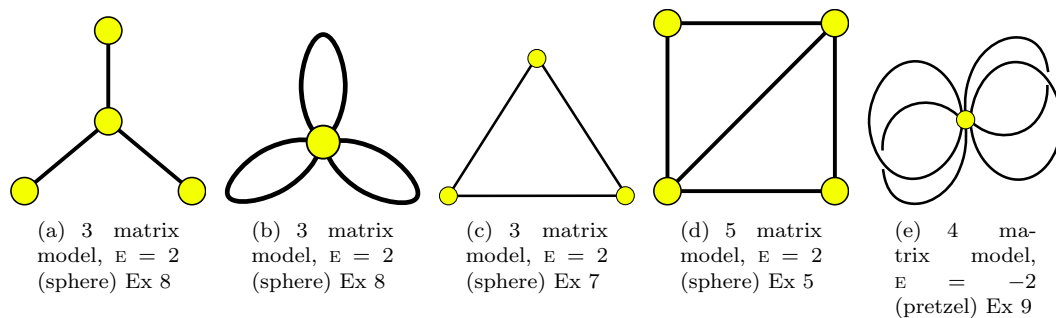


Figure 5. Graphs drawn without decoration. Graph (a) is dual to (b).

Example 9. Suppose $W_1 = C_{a_1} C_{b_1} C_{-a_1} C_{-b_1} \cdots C_{a_g} C_{b_g} C_{-a_g} C_{-b_g}$, where we number source matrices according a_i, b_i -cycles structure of a Riemann surface of genus g . In $n = 2g$ steps we obtain

$$C_{a_1} C_{b_1} C_{-a_1} C_{-b_1} \cdots C_{a_g} C_{b_g} C_{-a_g} C_{-b_g} \leftrightarrow C_{-a_1} C_{b_1} C_{a_1} C_{-b_1} \cdots C_{-a_g} C_{b_g} C_{a_g} C_{-b_g}$$

Therefore $V = 1$, and as we have $F = 1, n = 2g$, then the Euler characteristic is $E = F - n + V = 2 - 2g$.

The case $n = 2$ yields a torus and was considered in Example 2. The case $n = 4$ can be related to a graph drawn on a pretzel; see Figure 5e. The left-hand side and the right-hand sides are related to dual graphs each of which has one face and one vertex and is drawn on a surface with genus g whose edges are a_i, b_i cycles.

Example 10. In $n = 6$ steps we obtain

$$C_{-1} C_{-2} C_{-3} \otimes C_{-5} C_{-4} C_1 \otimes C_{-6} C_5 C_2 \otimes C_6 C_4 C_3 \leftrightarrow C_1 C_4 C_{-3} \otimes C_{-5} C_{-1} C_2 \otimes C_{-6} C_{-2} C_3 \otimes C_{-4} C_5 C_6$$

Therefore $V = 4$, and as we have $F = 4, n = 6$, then the Euler characteristic is $E = F - n + V = 2$. This can be represented as a tetrahedron inscribed in a sphere. The tetrahedron graph is self-dual.

4. Expectation Values of Matrix Products

Lemmas 2–5 are generalized to the case of mixed ensembles; see Propositions 2 and 4 below. Propositions 2–4 are extended versions of statements studied in [31,32,37].

Proposition 2. Consider ensemble (29). Consider dual sets W_1, \dots, W_F and W_1^*, \dots, W_V^* of (57). For any given set of partitions $\lambda^1, \lambda^2, \dots, \lambda^F = \lambda \in Y_d$, we have

$$\mathbb{E}_{n_1, n_2} \{ \mathcal{L}_X [s_{\lambda^1}(W_1) \cdots s_{\lambda^F}(W_F)] \} = \delta_{\lambda^1, \lambda^2, \dots, \lambda^F} \hbar^{n_1 d} (s_\lambda(\mathbf{p}_\infty))^{-n_1} (s_\lambda(\mathbb{I}_N))^{-n_2} s_\lambda(W_1^*) \cdots s_\lambda(W_V^*), \tag{60}$$

where $\delta_{\lambda^1, \lambda^2, \dots, \lambda^F}$ is equal to 1 in case $\lambda^1 = \lambda^2 = \dots = \lambda^F$ and to 0 otherwise.

Similarly, for any set of partitions $\lambda^1, \lambda^2, \dots, \lambda^V = \lambda \in Y_d$, we get

$$\mathbb{E}_{n_1, n_2} \{ \mathcal{L}_X [s_{\lambda^1}(W_1^*) \cdots s_{\lambda^F}(W_V^*)] \} = \delta_{\lambda^1, \lambda^2, \dots, \lambda^V} \hbar^{n_1 d} (s_\lambda(\mathbf{p}_\infty))^{-n_1} (s_\lambda(\mathbb{I}_N))^{-n_2} s_\lambda(W_1) \cdots s_\lambda(W_F), \tag{61}$$

where $\delta_{\lambda^1, \lambda^2, \dots, \lambda^V}$ is equal to 1 in case $\lambda^1 = \lambda^2 = \dots = \lambda^V$ and to 0 otherwise.

The sketch of proof. The proof is based on the cut-or-join procedure (57) of Section 3.3. which is the result of the step-by-step application of Lemmas 2–5. The different (geometrical) proof is based on the treating of the Wick rule as a way to glue surfaces from polygons and the use of (11) and of (27), (28).

Corollary 2. Consider ensemble (29). Consider dual sets W_1, \dots, W_F and W_1^*, \dots, W_V^* of (57), such that $\det W_i, \det W_j^* \neq 0$ for $i = 1, \dots, F, j = 1, \dots, V$. Suppose that $\alpha_m = \max(\alpha^1, \dots, \alpha^F) =: \alpha$, where $\alpha^1, \dots, \alpha^F$ is a given set of nonnegative integers. Consider a given set of partitions $\lambda^i, i = 1, \dots, F$ and denote $\lambda = \lambda^{(m)}$. We have

$$\mathbb{E}_{n_1, n_2} \left\{ \mathcal{L}_X \left[\prod_{i=1}^F s_{\lambda^i}(W_i) \det((W_i)^{\alpha_i}) \right] \right\} = \tag{62}$$

$$\delta_{\lambda^{1+\alpha_1}, \lambda^{2+\alpha_2}, \dots, \lambda^{F+\alpha_F}} \left(\hbar^{-(|\lambda|+\alpha N)} s_\lambda(\mathbf{p}_\infty) \frac{(N)_\lambda}{(N+\alpha)_\lambda} \right)^{-n_1} (s_\lambda(\mathbb{I}_N))^{-n_2} \prod_{i=1}^V s_\lambda(W_i^*) \det((W_i^*)^\alpha) \tag{63}$$

where $\delta_{\lambda^{1+\alpha_1}, \lambda^{2+\alpha_2}, \dots, \lambda^{F+\alpha_F}}$ is equal to 1 in case for each $k = 1, \dots, N$ we have $\lambda_k^{(1)} + \alpha_1 = \lambda_k^{(2)} + \alpha_2 = \dots = \lambda_k^{(F)} + \alpha_F$, and it is 0 otherwise.

Proposition 3 ([32]). let $\Delta^i = (\Delta_1^i, \Delta_2^i, \dots), i = 1, \dots, k$ be a set of partitions of the weights $d_1, \dots, d_k = d$. Let $\mu^{k+1} = (\mu_1^{(k+1)}, \mu_2^{(k+1)}, \dots), \dots, \mu^{(F)} = \mu = (\mu_1, \mu_2, \dots)$, where $1 < k < F$, be a set of partitions. We get

$$\mathbb{E}_{n_1, n_2} \left\{ \mathcal{L}_X \left[\prod_{i=1}^k \mathbf{p}_{\Delta^i}(W_i) \prod_{i=k+1}^F s_{\mu^i}(W_i) \right] \right\} = \tag{64}$$

$$\delta_{|\mu|, d} \delta_{d_1, \dots, d_k} \delta_{\mu^{k+1}, \dots, \mu^{F-k}} \hbar^{n_1 d} ((N)_\mu)^{-n_2} \left(\frac{\dim \mu}{d!} \right)^{-n_1 - n_2} \chi_\mu(\Delta^1) \cdots \chi_\mu(\Delta^k) \prod_{i=1}^V s_\lambda(W_i^*) \tag{65}$$

where $(N)_\mu$ is given by (40) and $\chi_\mu(\Delta)$ is the character of the symmetric group; see Remark (5). The symbol $\delta_{\mu^{k+1}, \dots, \mu^{F-k}}$ is equal to 1 in case $\mu^{k+1} = \dots = \mu^{F-k}$ and is equal to 0 otherwise.

Similarly, let $\mu^{k+1}, \dots, \mu^V = \mu$, where $1 < k < V$, be a set of partitions. Then

$$\mathbb{E}_{n_1, n_2} \left\{ \mathcal{L}_X \left[\prod_{i=1}^k \mathbf{p}_{\Delta^i}(W_i^*) \prod_{i=k+1}^V s_{\mu^i}(W_i^*) \right] \right\} = \tag{66}$$

$$\delta_{|\mu|, d} \delta_{d_1, \dots, d_k} \delta_{\mu^{k+1}, \dots, \mu^{V-k}} \hbar^{n_1 d} ((N)_\mu)^{-n_2} \left(\frac{\dim \mu}{d!} \right)^{-n_1 - n_2} \chi_\mu(\Delta^1) \cdots \chi_\mu(\Delta^k) \prod_{i=1}^F s_\lambda(W_i) \tag{67}$$

The Proposition is derived from the previous one using (5) and (11).

Proposition 4. Let $\Delta^i = (\Delta_1^i, \Delta_2^i, \dots)$, $i = 1, \dots, F$ be a set of partitions of weights $d_1, \dots, d_F = d$. Then

$$\mathbb{E}_{n_1, n_2} \{ \mathcal{L}_X [\mathbf{p}_{\Delta^1} (W_1) \cdots \mathbf{p}_{\Delta^F} (W_F)] \} \prod_{i=1}^F \frac{1}{z_{\Delta^i}} = \tag{68}$$

$$\delta_{d_1, \dots, d_F} \hbar^{n_1 d} \sum_{\tilde{\Delta}^1, \dots, \tilde{\Delta}^V \in Y_d} \mathbf{p}_{\tilde{\Delta}^1} (W_1^*) \cdots \mathbf{p}_{\tilde{\Delta}^V} (W_V^*) H_E(\tilde{\Delta}^1, \dots, \tilde{\Delta}^V, \Delta^1, \dots, \Delta^F | n_2), \tag{69}$$

where $\delta_{d_1, \dots, d_F} = 1$ in case $d_1 = \dots = d_F$ and 0 otherwise, and Y_d is the set of all partitions of weight d . The prefactor $H_E(\tilde{\Delta}^1, \dots, \tilde{\Delta}^V, \Delta^1, \dots, \Delta^F | n_2)$ in (69) is the weighted Hurwitz number (28) with $k = F + V$ and $E = F - n + V$.

Similarly, for a given set of partitions $\tilde{\Delta}^i = (\tilde{\Delta}_1^i, \tilde{\Delta}_2^i, \dots)$, $i = 1, \dots, V$ with weights $d_1, \dots, d_V = d$ respectively, we get

$$\mathbb{E}_{n_1, n_2} \{ \mathcal{L}_X [\mathbf{p}_{\tilde{\Delta}^1} (W_1^*) \cdots \mathbf{p}_{\tilde{\Delta}^V} (W_V^*)] \} = \tag{70}$$

$$\delta_{d_1, \dots, d_V} \hbar^{n_1 d} \sum_{\tilde{\Delta}^1, \dots, \tilde{\Delta}^F \in Y_d} \mathbf{p}_{\tilde{\Delta}^1} (W_1) \cdots \mathbf{p}_{\tilde{\Delta}^V} (W_V) H_E(\tilde{\Delta}^1, \dots, \tilde{\Delta}^V, \Delta^1, \dots, \Delta^F | n_2), \tag{71}$$

where $H_E(\tilde{\Delta}^1, \dots, \tilde{\Delta}^V, \Delta^1, \dots, \Delta^F)$ is exactly the same as in (69).

Propositions 2–4 and are equivalent. This can be proven with the help of (10), (11) and (28).

Remark 13. Proposition 4 was proved in [31] using a geometrical construction of Hurwitz numbers as a number of ways to glue polygons. Each matrix entry, say $(X_i)_{a,b}$, may be drawn as an arrow with labels a and b at the startpoint and the endpoint respectively. We draw solid arrow for an entry of a random matrix and a dashed arrow for an entry of a source matrix. The product of matrices we draw as arrows sequentially assigned to each other. The trace of a product is drawn as a polygon. Now each $L_X[\text{tr}W_c]$ is a polygon with alternating solid and dashed-edge arrows; we orient the edges counterclockwise. In [31] we named such polygons countries. Thus, we relate each dressed word to a country. It may be shown that the expectation

$$\mathbb{E}_{n,0} \{ L_X [\text{tr} (W_1) \cdots \text{tr} (W_1)] \}$$

may be viewed as the result of gluing of the net of countries into a surface, say Σ , and $E = F - n + V$ being the Euler characteristic of this surface. This is a result of the Gaussian integration over matrices Z_i . Oppositely directed solid arrows (corresponding to Z_i and Z_i^\dagger) form sides of ribbon edges. These edges end at a boundary of disks (inflated vertices) with dashed boundaries (source matrices are attached to the segments of dashed boundaries). In [31] we named this disk watchtowers. There are n ribbon edges (the borders of countries) and $2n$ dashed edges (segments of boundaries of disks—of the borders of the watchtowers); there are V watchtowers; and there are F countries with alternating (solid-dashed) edges. There are $2n$ 3-valent vertices (ends of ribbons): there are two dashed arrows (one is outgoing; another is incoming) and one ribbon (one side is the solid outgoing arrow; the other side is a incoming solid arrow) attached to each vertex. This is a graph Γ drawn on Σ . This graph is related to

$$\mathbb{E}_{n,0} \{ L_X [\text{tr} (W_1) \cdots \text{tr} (W_1)] \} = \hbar^n \text{tr} (W_1^*) \cdots \text{tr} (W_V^*), \tag{72}$$

which is (68) for $d = 1$ (in this case, all Δ^i has weight 1, and $p_{(1)}(W) = \text{tr}(W)$). In case $d > 1$, instead of each $\text{tr} (W_i)$ we have the product

$$\text{tr} \left((W_i)^{\Delta_1^i} \right) \cdots \text{tr} \left((W_i)^{\Delta_{\ell_i}^i} \right) \rightarrow \text{tr} (W_i).$$

One may interpret it as a projection of ℓ_i polygons to the country (the polygon) labeled by i .

Remark 14. Notice that the answers for the expectation values which were considered above depend only on eigenvalues of W_i^* or W_i $i = 1, \dots, v$.

5. Examples of Matrix Models

Recall that in Section 2.1 we introduced the function $\tau_r(n, \mathbf{p}^{(1)}, \mathbf{p}^{(2)})$. In what follows we use the conventions:

$$\tau_r(\mathbf{p}^{(1)}, \mathbf{p}^{(2)}) := \tau_r(0, \mathbf{p}^{(1)}, \mathbf{p}^{(2)}) = \sum_{\lambda} r_{\lambda} s_{\lambda}(\mathbf{p}^{(1)}) s_{\lambda}(\mathbf{p}^{(2)}), \quad r_{\lambda} = r_{\lambda}(0) \tag{73}$$

It depends on two sets $\mathbf{p}^i = (p_1^{(i)}, p_2^{(i)}, \dots)$, $i = 1, 2$, and on the choice of an arbitrary function of the variable r . (This is an example of the so-called tau function, but we will not use this fact.) As one of their sets, we will choose $\mathbf{p}^2 = \mathbf{p}(X)$ like in (21), and the second set will be the set of arbitrary parameters. With $r = 1$ we get

$$\tau_1(\mathbf{p}, X) = e^{\sum_{m>0} \frac{1}{m} p_m \text{tr}(X^m)} \tag{74}$$

For example, if we take

$$r(x) = \frac{\prod_i^p (a_i + x)}{\prod_i^q (b_i + x)},$$

and in addition to $\mathbf{p}^1 = (1, 0, 0, \dots)$ we get the so-called hypergeometric function of the matrix argument:

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| X \right) = \sum_{\lambda} \frac{\dim \lambda}{|\lambda|!} s_{\lambda}(X) \frac{\prod_i^p (a_i + x)_{\lambda}}{\prod_i^q (b_i + x)_{\lambda}} \tag{75}$$

Special cases:

$$e^{\text{tr}X} = \sum_{\lambda} s_{\lambda}(X) \frac{\dim \lambda}{|\lambda|!}, \tag{76}$$

$$\det(1 - zX)^{-a} = \sum_{\lambda} z^{|\lambda|} (a)_{\lambda} s_{\lambda}(X) \frac{\dim \lambda}{|\lambda|!} \tag{77}$$

Integrals. Using Proposition 2 we obtain

Theorem 1. Suppose W_1, \dots, W_F and W_1^*, \dots, W_v^* are dual sets (57). Let sets $\mathbf{p}^i = (p_1^{(i)}, p_2^{(i)}, p_3^{(i)}, \dots)$, $i = 1, \dots, \max(F, v)$ be independent complex parameters and $r(i)$, $i = 1, \dots, \max(F, v)$ be a set of given functions in one variable.

$$\mathbb{E}_{n_1, n_2} \left\{ \mathcal{L}_X \left[\tau_{r(1)}(\mathbf{p}^1, W_1) \cdots \tau_{r(F)}(\mathbf{p}^F, W_F) \right] \right\} \tag{78}$$

$$= \sum_{\lambda} r_{\lambda} \hbar^{n_1 |\lambda|} \left(\frac{\dim \lambda}{|\lambda|!} \right)^{-n} \prod_{i=1}^v s_{\lambda}(W_i^*) \prod_{i=1}^F s_{\lambda}(\mathbf{p}^i), \tag{79}$$

where each $\tau_{r(i)}(\mathbf{p}^i, W_i)$ is defined by (21)

$$r_{\lambda} = ((N)_{\lambda})^{-n_2} \prod_{i=1}^F r_{\lambda}^{(i)}(n)$$

Similarly

$$\mathbb{E}_{n_1, n_2} \left\{ \mathcal{L}_X \left[\tau_{r(1)}(\mathbf{p}^1, W_1^*) \cdots \tau_{r(v)}(\mathbf{p}^v, W_v^*) \right] \right\} \tag{80}$$

$$= \sum_{\lambda} r_{\lambda} \hbar^{n_1 |\lambda|} \left(\frac{\dim \lambda}{|\lambda|!} \right)^{-n} \prod_{i=1}^F s_{\lambda}(W_i) \prod_{i=1}^v s_{\lambda}(\mathbf{p}^i), \tag{81}$$

Remark 15. We recall the convention (73) In (79) r_λ is the content product (16)

$$r_\lambda = r_\lambda(0) = \prod_{(i,j) \in \lambda} r(j - i)$$

where

$$r(x) = (N + x)^{-n_2} \prod_{i=1}^F r^{(i)}(x)$$

To get examples we choose

- Dual sets $W_1, \dots, W_F \leftrightarrow W_1^*, \dots, W_V^*$;
- The fraction of unitary matrices given by n_2 ;
- The set of functions $r^{(i)}, i = 1, \dots, F$;
- The sets $\mathbf{p}^{(i)}, i = 1, \dots, F$.

Remark 16. Answers in some cases are further simplified. Let us mark two cases

(i) Firstly, this is the case when the spectrum of the stars has the form

$$\text{Spect } W_i^* = \text{Spect } \mathbb{I}_{N,k_i} = \text{diag}\{1, 1, \dots, 1, 0, 0, \dots, 0\}, \quad i = 1, \dots, V \tag{82}$$

where \mathbb{I}_{N,k_i} is the matrix with k_i units of the main diagonal. Such star monodromies obtained in case source matrices have a rank smaller than N . Insertion of such matrices in the left-hand sides of (79) and (81) corresponds to the integration over rectangular random matrices. One should take into account that

$$s_\lambda(\mathbb{I}_{N,k}) = (k)_\lambda s_\lambda(\mathbf{p}_\infty),$$

where we recall the notation

$$(a)_\lambda := (a)_{\lambda_1} (a - 1)_{\lambda_2} \cdots (a - \ell + 1)_{\lambda_\ell}, \tag{83}$$

(ii) The case is the specification of the sets $\mathbf{p}^i, i = 1, \dots, F$ according to the following

Lemma 6. Denote

$$\mathbf{p}_\infty = (1, 0, 0, \dots) \tag{84}$$

$$\mathbf{p}(a) = (a, a, a, \dots) \tag{85}$$

$$\mathbf{p}(\mathbf{q}, t) = (p_1(\mathbf{q}, t), p_2(\mathbf{q}, t), \dots), \quad p_m(\mathbf{q}, t) = \frac{1 - q^m}{1 - t^m} \tag{86}$$

Then

$$\frac{s_\lambda(\mathbf{p}(a))}{s_\lambda(\mathbf{p}_\infty)} = (a)_\lambda, \quad \mathbf{p}(a) = (a, a, a, \dots) \tag{87}$$

where $(a)_\lambda := (a)_{\lambda_1} (a - 1)_{\lambda_2} \cdots (a - \ell + 1)_{\lambda_\ell}$, $(a)_n := a(a + 1) \cdots (a + n - 1)$, where $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a partition. More generally

$$\frac{s_\lambda(\mathbf{p}(\mathbf{q}, t))}{s_\lambda(\mathbf{p}(0, t))} = (\mathbf{q}; t)_\lambda, \tag{88}$$

where $(\mathbf{q}; t)_\lambda = (\mathbf{q}; t)_{\lambda_1} (q t^{-1}; t)_{\lambda_2} \cdots (q t^{1-\ell}; t)_{\lambda_\ell}$ where $(\mathbf{q}; t)_k = (1 - q)(1 - q t) \cdots (1 - q t^{k-1})$ is t -deformed Pochhammer symbol. $(\mathbf{q}; t)_0 = 1$ is implied.

For such specifications the right-hand side of (79) can be

With such specifications, one can diminish the number of the Schur functions in the right-hand side of (79) (or of (81)) and the right-hand side can take one of the forms:

$$\sum_\lambda r_\lambda s_\lambda(A) s_\lambda(B) \tag{89}$$

$$\sum_{\lambda} r_{\lambda} s_{\lambda}(A) \tag{90}$$

$$\sum_{\lambda} r_{\lambda} \tag{91}$$

For (89) there is a determinant representation; for (90) there is a Pfaffian representation and (91) can be rewritten as a sum of products. Indeed if we introduce $r(x) = e^{T_{x-1}-T_x}$, and $\lambda = (\lambda_1, \dots, \lambda_N)$, then

$$r_{\lambda}(m) = \prod_{(i,j) \in \lambda} r(m+j-i) = e^{T_m+\dots+T_{m-N}} \prod_{i=1}^N e^{T_{\lambda_i-i+m}} \tag{92}$$

For instance, one can take $r(x) = a + x$ and get

$$(a)_{\lambda} = \frac{\Gamma(a + \lambda_1 - 1)\Gamma(a + \lambda_2 - 2) \cdots \Gamma(a + \lambda_N - N)}{\Gamma(a)\Gamma(a - 1) \cdots \Gamma(a - N + 1)} \tag{93}$$

Then we introduce $h_i = \lambda_i - i + N$ and write

$$\sum_{\lambda} \frac{(a)_{\lambda}}{(b)_{\lambda}} = \sum_{h_1 > \dots > h_N \geq 0} \prod_{i=1}^N \frac{\Gamma(b - i + 1) \Gamma(h_i + a - N)}{\Gamma(a - i + 1) \Gamma(h_i + b - N)} \tag{94}$$

where Γ is the gamma-function. (In case the argument of gamma-function turns out to be a nonpositive integer one should keep in mind both the numerator and denominator.) See examples below.

Example 11. See Example 1 and Figure 2a. Take $X_1 = Z$ and r given by (13). The example of (79) can be chosen as follows

$$\mathbb{E}_{1,0} \left\{ {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| Z C_1 \right) {}_{p'}F_{q'} \left(\begin{matrix} a'_1, \dots, a'_{p'} \\ b'_1, \dots, b'_{q'} \end{matrix} \middle| Z^{\dagger} C_{-1} \right) \det \left(Z Z^{\dagger} \right)^{\alpha} \right\} = \tag{95}$$

$${}_{p'}F_{q'} \left(\begin{matrix} a_1, \dots, a_p, a'_1, \dots, a'_{p'}, N + \alpha \\ b_1, \dots, b_q, b'_1, \dots, b'_{q'}, N \end{matrix} \middle| C_1 C_{-1} \right), \tag{96}$$

corresponding determinantal representation see a (21).

See and Figure 2b which is dual to Figure 2a. An example of (81) can be chosen as

$$\mathbb{E}_{1,0} \left\{ {}_{p'}F_{q'} \left(\begin{matrix} a_1, \dots, a_p, a'_1, \dots, a'_{p'}, N + \alpha \\ b_1, \dots, b_q, b'_1, \dots, b'_{q'}, N \end{matrix} \middle| Z C_1 Z^{\dagger} C_{-1} \right) \det \left(Z Z^{\dagger} \right)^{\beta} \right\} = \tag{97}$$

$$\sum_{\lambda} s_{\lambda}(C_1) s_{\lambda}(C_{-1}) \frac{(N + \alpha)_{\lambda} (N + \beta)_{\lambda}}{((N)_{\lambda})^2} \frac{\prod_i^p (a_i)_{\lambda} \prod_i^{p'} (a_i)_{\lambda}}{\prod_i^q (b_i)_{\lambda} \prod_i^{q'} (b_i)_{\lambda}}$$

The determinantal representation of the left-hand side is given by (15).

Example 12. See Example 2 and Figure 3a. Take $X_1 = U_1, X_2 = U_2$.

$$\mathbb{E}_{0,2} \left\{ e^{\sum_{m>0} \frac{1}{m} p_m \text{tr}(U_1 C_1 U_2 C_2 U_1^{\dagger} C_{-1} U_2^{\dagger} C_{-2})^m} \right\} \tag{98}$$

$$= \sum_{\lambda} \frac{1}{((N)_{\lambda})^2} \frac{s_{\lambda}(\mathbf{p}) s_{\lambda}(C_1 C_{-2} C_{-1} C_2)}{(s_{\lambda}(\mathbf{p}_{\infty}))^2}$$

Let us take $\mathbf{p} = (az, az^2, az^3, \dots)$ and $W_1^* = \mathbb{I}_{N,k}$; see (85) and (82). We obtain the left-hand side as

$$\mathbb{E}_{0,2} \left\{ \det \left(1 - zU_1C_1U_2C_2U_1^\dagger C_{-1}U_2^\dagger C_{-2} \right)^{-a} \right\} = \sum_{\lambda} z^{|\lambda|} \frac{(a)_{\lambda}(k)_{\lambda}}{((N)_{\lambda})^2} =$$

$$z^{-\frac{1}{2}N(N-1)} \sum_{h_1 > \dots > h_N \geq 0} \prod_{i=1}^N z^{h_i} \frac{(\Gamma(N-i+1))^2}{\Gamma(a-i+1)\Gamma(k-i+1)} \frac{\Gamma(h_i+a-N)\Gamma(h_i+k-N)}{(\Gamma(h_i))^2}$$

where $\text{Spect } C_1C_{-2}C_{-1}C_2 = \text{Spect } \mathbb{I}_{N,k}$ see (94).

Example 13. See Example 3 and Figure 3c.

$$\mathbb{E}_{2,0} \left\{ e^{\sum_{m>0} \frac{1}{m} p_m^{(1)} \text{tr}(Z_1C_1)^m + \sum_{m>0} \frac{1}{m} p_m^{(2)} \text{tr}(Z_2C_2)^m} \det \left(1 - zZ_1^\dagger C_{-1}Z_2^\dagger C_{-2} \right)^{-a} \prod_{i=1}^3 \det \left(Z_iZ_i^\dagger \right)^\alpha \right\}$$

$$= \sum_{\lambda} z^{|\lambda|} s_{\lambda}(\mathbf{p}^1) s_{\lambda}(\mathbf{p}^2) (a)_{\lambda} \left(\frac{(N+\alpha)_{\lambda}}{(N)_{\lambda}} \right)^3$$

For a determinant representation see (20)

Example 14. For decoration of Figure 1e we put $X_i = Z_i$.

$$\mathbb{E}_{2,0} \left\{ \mathbf{p}_{\Delta}(Z_1^\dagger C_{-1}) s_{\lambda}(Z_1C_1Z_2C_2Z_2^\dagger C_{-2}) \right\} = \delta_{|\lambda|,|\Delta|} \frac{\dim \lambda}{|\lambda|!} \chi_{\lambda}(\Delta) s_{\lambda}(C_2) s_{\lambda}(C_1C_{-2}C_{-1})$$

see (66).

Example 15. Figure 5d in particular yields

$$\mathbb{E}_{2,0} \left\{ s_{\lambda}(Z_1C_1Z_2C_2Z_3C_3Z_4C_4) s_{\lambda}(Z_3^\dagger C_{-3}Z_2^\dagger C_{-2}Z_5C_5) s_{\lambda}(Z_5^\dagger C_{-5}Z_1^\dagger C_{-1}Z_4^\dagger C_{-4}) \right\} =$$

$$\left(\frac{\dim \lambda}{|\lambda|!} \right)^{-5} s_{\lambda}(C_1C_{-2}C_{-5}) s_{\lambda}(C_2C_{-3}) s_{\lambda}(C_5C_3C_{-4}) s_{\lambda}(C_4C_{-1}).$$

Example 16. In the case below we use an open chain with n edges as in Figures 1b, 2b and 4a.

$$\mathbb{E}_{n,0} \left\{ e^{\sum_{m>0} \frac{1}{m} p_m^{(1)} \text{tr}((Z_1C_1Z_2C_2 \dots Z_nC_nZ_n^\dagger C_{-n} \dots Z_1^\dagger C_{-1})^m)} \right\} = \sum_{\lambda} s_{\lambda}(C_n) s_{\lambda}(C_{-1}) \frac{s_{\lambda}(\mathbf{p})}{s_{\lambda}(\mathbf{p}_{\infty})} \prod_{i=1}^{n-1} \frac{s_{\lambda}(C_iC_{-i-1})}{s_{\lambda}(\mathbf{p}_{\infty})} \tag{99}$$

Graphs dual to the chain look like in Figure 4b.

$$\mathbb{E}_{n,0} \left\{ e^{\sum_{m>0} \frac{1}{m} p_m \text{tr}((Z_1^\dagger C_{-1})^m) + \sum_{m>0} \frac{1}{m} p_m^{(n)} \text{tr}((Z_nC_n)^m)} \prod_{i=1}^{n-1} e^{\sum_{m>0} \frac{1}{m} p_m^{(i)} \text{tr}((Z_iC_iZ_{i+1}^\dagger C_{-i-1})^m)} \right\}$$

$$= \sum_{\lambda} s_{\lambda}(\mathbf{p}) s_{\lambda}(C_1C_2 \dots C_nC_{-n} \dots C_{-1}) \prod_{i=1}^n \frac{s_{\lambda}(\mathbf{p}^i)}{s_{\lambda}(\mathbf{p}_{\infty})}$$

These relations generalize product (44) and (45).

Example 17. Our graph is a polygon with n edges and n vertices (stars); see Figures 1d, 2a and 4c for examples.

$$\mathbb{E}_{n,0} \left\{ e^{\sum_{m>0} \frac{1}{m} p_m^{(1)} \text{tr}(Z_1C_1Z_2C_2 \dots Z_nC_n)^m + \sum_{m>0} \frac{1}{m} p_m^{(2)} \text{tr}(Z_n^\dagger C_{-n}Z_{n-1}^\dagger C_{-n-1} \dots Z_1^\dagger C_{-1})^m} \prod_{i=1}^n \det \left(Z_iZ_i^\dagger \right)^\alpha \right\} \tag{100}$$

$$= \sum_{\lambda} s_{\lambda}(\mathbf{p}^1) s_{\lambda}(\mathbf{p}^2) \prod_{i=1}^n \frac{(N + \alpha)_{\lambda} s_{\lambda}(C_i C_{-i-1})}{(N)_{\lambda} s_{\lambda}(\mathbf{p}_{\infty})}$$

(where we put $C_{-n-1} = C_{-1}$).

A graph dual to the polygon can be viewed as two-stars graph with n edges which connect stars; see Figures 1d and 4d as examples.

$$\mathbb{E}_{n,0} \left\{ \prod_{i=1}^n e^{\sum_{m>0} \frac{1}{m} p_m^{(i)} \text{tr}(Z_i C_i Z_{i+1}^{\dagger} C_{-i-1})^m} \right\} = \sum_{\lambda} s_{\lambda}(C_1 C_2 \cdots C_n) s_{\lambda}(C_{-n} C_{1-n} \cdots C_{-1}) \prod_{i=1}^n \frac{s_{\lambda}(\mathbf{p}^i)}{s_{\lambda}(\mathbf{p}_{\infty})} \quad (101)$$

To apply determinantal formulas one should use Remark 16.

Example 18. Consider the star-graph with n -rays which end at other stars (see Figure 5a where $n = 3$). This situation corresponds to (43).

$$\mathbb{E}_{n,0} \left\{ e^{\sum_{m>0} \frac{1}{m} p_m \text{tr}(Z_1 C_1 Z_1^{\dagger} C_{-1} \cdots Z_n C_n Z_n^{\dagger} C_{-n})} \prod_{i=1}^n \det(Z_i Z_i^{\dagger})^{\alpha} \right\} = \quad (102)$$

$$\sum_{\lambda} s_{\lambda}(\mathbf{p}) s_{\lambda}(C_{-1} C_{-2} \cdots C_{-n}) \prod_{i=1}^n \frac{(N + \alpha)_{\lambda} s_{\lambda}(C_i)}{(N)_{\lambda} s_{\lambda}(\mathbf{p}_{\infty})}$$

A similar model was studied in [16,22]. It has the determinantal representation (21) in case all W_i^* except one are of form (82). There is the determinantal representation (15) in case we specialize the set \mathbf{p} according to Lemma 6 and choose each W_i^* except two be in form (82).

Now, let us choose the dual graph (this is petel graph. (see Figure 5b where $n = 3$)) and consider

$$\mathbb{E}_{n,0} \left\{ \det(1 - z Z_1^{\dagger} C_{-1} Z_2^{\dagger} C_{-2} \cdots Z_n^{\dagger} C_{-n})^{-a} \prod_{i=1}^n \det(Z_i Z_i^{\dagger})^{\alpha} e^{\sum_{m>0} \frac{1}{m} p_m^{(i)} \text{tr}((Z_i C_i)^m)} \right\} = \quad (103)$$

$$\sum_{\lambda} z^{|\lambda|} (a)_{\lambda} s_{\lambda}(C_1 C_{-1} C_2 C_{-2} \cdots C_n C_{-n}) \prod_{i=1}^n \frac{(N + \alpha)_{\lambda} s_{\lambda}(\mathbf{p}^i)}{(N)_{\lambda} s_{\lambda}(\mathbf{p}_{\infty})}$$

By Remark 16 we find all cases where the determinantal representations (20) or (21) exist.

Remark 17. Notice the following symmetry: the left-hand side produces the same right-hand side if we permute the set of exponents $\alpha_1, \dots, \alpha_n, a - N$.

Example 19. Below $g = 1, 2, \dots$ (For the case $g = 1$ see Figure 1a and zoomed Figure 3a; for $g = 2$ see Figure 5e).

$$\mathbb{E}_{0,2g} \left\{ e^{\sum_{m>0} \frac{1}{m} p_m \text{tr}(U_{a_1} C_{a_1} U_{b_1} C_{b_1} U_{a_1}^{\dagger} C_{-a_1} U_{b_1}^{\dagger} C_{-b_1} \cdots U_{a_g} C_{a_g} U_{b_g} C_{b_g} U_{a_g}^{\dagger} C_{-a_g} U_{b_g}^{\dagger} C_{-b_g})^m} \right\}$$

$$= \sum_{\lambda} \left(\frac{\dim \lambda}{|\lambda|!} \right)^{-2g} s_{\lambda}(W^*) s_{\lambda}(\mathbf{p})$$

where

$$W^* = C_{-a_1} C_{b_1} C_{a_1} C_{-b_1} \cdots C_{-a_g} C_{b_g} C_{a_g} C_{-b_g}$$

In particular, if $\text{Spect } W^* = \mathbb{I}_{N,k}$, then

$$\mathbb{E}_{0,2g} \left\{ \det(\mathbb{I}_N - z U_{a_1} C_{a_1} U_{b_1} C_{b_1} U_{a_1}^{\dagger} C_{-a_1} U_{b_1}^{\dagger} C_{-b_1} \cdots U_{a_g} C_{a_g} U_{b_g} C_{b_g} U_{a_g}^{\dagger} C_{-a_g} U_{b_g}^{\dagger} C_{-b_g})^{-a} \right\}$$

$$= \sum_{\lambda} z^{|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!} \right)^{2-2g} \frac{(a)_{\lambda} (k)_{\lambda}}{((N)_{\lambda})^{2g}} \tag{104}$$

Taking into account that

$$\dim \lambda = \frac{\prod_{i < j} (h_i - h_j)}{\prod_{i=1}^N \Gamma(h_i + 1)}$$

we can interpret that (104) is a discrete beta-ensemble where $\beta = 2 - 2g$.

Exotic models. An example. There are some more tricky problems which can be solved which can be solved in steps. Let me consider the simplest example. Look at Lemma 3. Suppose A and B depend in any way on an additional matrix Z_1 ; in any case, however, their product has a familiar form:

$$A = A(Z_1, Z_1^{\dagger}), \quad B = B(Z_1, Z_1^{\dagger}), \quad AB = Z_1 C_1 Z_1^{\dagger} C_{-1} \tag{105}$$

Say, $A = Z_1^a e^{-Z_1^{\dagger}}$, $B = e^{Z_1^{\dagger}} Z_1^{\dagger} Z_1^{1-a}$ which looks horrible. However, applying sequentially the series ${}_0F_0$, then (37), where $Z = Z_1$, and then (31), where $Z = Z_2$ one obtains

$$\mathbb{E}_{2,0} \left\{ e^{\text{tr} \left(Z_1 Z_2^a e^{-Z_2^{\dagger}} \right) + \text{tr} \left(Z_1^{\dagger} e^{Z_2^{\dagger}} Z_2^{\dagger} Z_1^{1-a} \right)} \right\} = \sum_{d \geq 0} \sum_{\lambda \in Y_d} s_{\lambda}(\mathbb{I}_N) s_{\lambda}(C) = \det(\mathbb{I}_N - C)^{-1} \tag{106}$$

It will be interesting to do the same with other ensembles of random matrices; Ginibre ensembles of real and quaternionic matrices; and ensembles of Hermitian matrices: complex, real and quaternionic.

6. Discussion

In this article, we examined matrix integrals of a certain type. We called them matrix models associated with children’s drawings—the so-called dessin d’enfants. They include some well-known models that have found applications in the theory of information transfer and the theory of quantum chaos. We hope that our matrix integrals will be in demand. We think that these models are related to quantum integrable systems [38–40], but this topic is waiting for its development; we expect connections with [41–49].

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Appendix A. Partitions and Schur Functions

Let us recall that the characters of the unitary group $\mathbb{U}(N)$ are labeled by partitions and coincide with the so-called Schur functions [23]. A partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is a set of nonnegative integers λ_i which are called parts of λ and which are ordered as $\lambda_i \geq \lambda_{i+1}$. The number of non-vanishing parts of λ is called the length of the partition λ , and will be denoted by $\ell(\lambda)$. The number $|\lambda| = \sum_i \lambda_i$ is called the weight of λ . The set of all partitions will be denoted by \mathbb{P} .

The Schur function labelled by λ may be defined as the following function in variables $x = (x_1, \dots, x_N)$:

$$s_\lambda(x) = \frac{\det [x_j^{\lambda_i - i + N}]_{i,j}}{\det [x_j^{-i + N}]_{i,j}} \tag{A1}$$

in case $\ell(\lambda) \leq N$ and vanishes otherwise. One can see that $s_\lambda(x)$ is a symmetric homogeneous polynomial of degree $|\lambda|$ in the variables x_1, \dots, x_N , and $\deg x_i = 1, i = 1, \dots, N$.

Remark A1. In case the set x is the set of eigenvalues of a matrix X , we also write $s_\lambda(X)$ instead of $s_\lambda(x)$.

There is a different definition of the Schur function as a quasi-homogeneous, non-symmetric polynomial of degree $|\lambda|$ in other variables, the so-called power sums, $\mathbf{p} = (p_1, p_2, \dots)$, where $\deg p_m = m$.

For this purpose let us introduce

$$s_{\{h\}}(\mathbf{p}) = \det[s_{(h_i + j - N)}(\mathbf{p})]_{i,j},$$

where $\{h\}$ is any set of N integers, and where the Schur functions $s_{(i)}$ are defined by $e^{\sum_{m>0} \frac{1}{m} p_m z^m} = \sum_{m \geq 0} s_{(i)}(\mathbf{p}) z^i$. If we put $h_i = \lambda_i - i + N$, where N is not less than the length of the partition λ ; then

$$s_\lambda(\mathbf{p}) = s_{\{h\}}(\mathbf{p}). \tag{A2}$$

The Schur functions defined by (5) and by (A2) are equal, $s_\lambda(\mathbf{p}) = s_\lambda(x)$, provided the variables \mathbf{p} and x are related by the power sums relation

$$p_m = \sum_i x_i^m \tag{A3}$$

In case the argument of s_λ is written as a non-capital fat letter the definition (A2), and we imply the definition (5) in case the argument is not fat and non-capital letter, and in case the argument is capital letter which denotes a matrix, then it implies the definition (5) with $x = (x_1, \dots, x_N)$ being the eigenvalues.

It may be easily checked that

$$s_\lambda(\mathbf{p}) = (-1)^{|\lambda|} s_{\lambda^{\text{tr}}}(-\mathbf{p}) \tag{A4}$$

where λ^{tr} is the partition conjugated to λ (in [23] it is denoted by λ^*). The Young diagram of the conjugated partition is obtained by the transposition of the Young diagram of λ with respect to its main diagonal. One gets $\lambda_1 = \ell(\lambda^{\text{tr}})$.

Appendix B. Integrals over the Unitary Group

Consider the following integral over the unitary group which depends on two semi-infinite sets of parameters $\mathbf{p} = (p_1, p_2, \dots)$ and $\bar{\mathbf{p}} = (p_1^*, p_2^*, \dots)$:

$$I_{\mathbb{U}(N)}(\mathbf{p}, \bar{\mathbf{p}}) := \int_{\mathbb{U}(N)} e^{\text{tr}V(\mathbf{p}, U) + \text{tr}V(\bar{\mathbf{p}}^*, U^{-1})} d_* U = \tag{A5}$$

$$\frac{1}{(2\pi)^N} \int_{0 \leq \theta_1 \leq \dots \leq \theta_N \leq 2\pi} \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{-i\theta_k}|^2 \prod_{j=1}^N e^{\sum_{m>0} \frac{1}{m} (p_m e^{im\theta_j} + p_m^* e^{-im\theta_j})} d\theta_j \tag{A6}$$

$$V(\mathbf{p}, x) := \sum_{n>0} \frac{1}{n} p_n x^n \tag{A7}$$

Here d_*U is the Haar measure of the group $\mathbb{U}(N)$:

$$d_*U = \frac{1}{(2\pi)^N} \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{-i\theta_k}|^2 \prod_{j=1}^N d\theta_j, \quad -\pi \leq \theta_1 < \dots < \theta_N \leq \pi \tag{A8}$$

and $e^{i\theta_1}, \dots, e^{i\theta_N}$ are the eigenvalues of $U \in \mathbb{U}(N)$. The exponential factors inside the integral may be treated as a perturbation of the Haar measure and parameters \mathbf{p}, \mathbf{p}^* are called coupling constants by the analogy with quantum field theory problems.

Using the Cauchy-Littlewood identity

$$\tau(\mathbf{p}|\mathbf{p}^*) := e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m^* p_m} = \sum_{\lambda \in \mathbb{P}} s_\lambda(\mathbf{p}^*) s_\lambda(\mathbf{p}) \tag{A9}$$

and the orthogonality of the irreducible characters of the unitary group

$$\int s_\lambda(U) s_\mu(U^{-1}) d_*U = \delta_{\lambda, \mu} \tag{A10}$$

we obtain that

$$I_{\mathbb{U}(n)}(\mathbf{p}, \bar{\mathbf{p}}) = \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq n}} s_\lambda(\mathbf{p}) s_\lambda(\bar{\mathbf{p}}) \tag{A11}$$

which express the integral over unitary matrices as the ‘‘perturbation series in coupling constants’’.

The formula (A11) first appeared in [50] in the context of the study of Brezin–Gross–Witten model. It was shown there that the integral $I_{\mathbb{U}(n)}(\mathbf{p}, \bar{\mathbf{p}})$ may be related to the Toda lattice tau function of [51,52] under certain restriction. Then, the series in the Schur functions (A11) may be related to the double Schur functions series found in [53–55].

Appendix C. Geometrical Definition of Hurwitz Numbers

In this presentation, we follow article [31].

The Hurwitz number is a characterisation of the branched covering of a surface with critical values of a prescribed topological type. Hurwitz numbers of oriented surfaces without boundaries were introduced by Hurwitz at the end of the 19th century. Later it turned out that they are closely related to the study of moduli spaces of Riemann surfaces [56], to integrable systems [57], to modern models of mathematical physics (matrix models) and to closed topological field theories [30]. In this paper we consider only Hurwitz numbers of compact surfaces without boundary.

Consider a branched covering $f : P \rightarrow \Sigma$ of degree d of a compact surface without boundary. In the neighborhood of each point $z \in P$, the map f is topologically equivalent to the complex map $u \mapsto u^p$, defined on a neighborhood $u \sim 0$ in \mathbb{C} . The number $p = p(z)$ is called the degree of the covering f at the point z . The point $z \in P$ is said to be a *branch point* or *critical point* if $p(z) \neq 1$. There are only a finite number of critical points. The image $f(z)$ of a critical point z is called the *critical value* of f at z .

Let us associate with a point $s \in \Sigma$ all points $z_1, \dots, z_\ell \in P$ for which $f(z_i) = s$. Let p_1, \dots, p_ℓ be the degrees of the map f at these points. Their sum $d = p_1 + \dots + p_\ell$ is equal to the degree d of f . Thus, to each point $s \in S$ there corresponds a partition $d = p_1 + \dots + p_\ell$ of the number d . Having ordered the degrees $p_1 \geq \dots \geq p_\ell > 0$ at each point $s \in \Sigma$, we introduce the Young diagram $\Delta^s = [p_1, \dots, p_\ell]$ of weight d with $\ell = \ell(\Delta^s)$ rows of length p_1, \dots, p_ℓ : Δ^s is called the *topological type* of the value s , and s is a critical value of f if and only if at least one of the row-lengths p_i is greater than 1.)

Let us note that the Euler characteristics $E(P)$ and $E(\Sigma)$ of the surfaces P and Σ are related via the Riemann–Hurwitz relation:

$$E(P) = E(\Sigma)d + \sum_{z \in P} (p(z) - 1)$$

or, equivalently,

$$E(P) = E(\Sigma)d + \sum_{i=1}^F (\ell(\Delta^{s_i}) - d). \tag{A12}$$

where s_1, \dots, s_F are critical values.

We say that coverings $f_1 : P_1 \rightarrow \Sigma$ and $f_2 : P_2 \rightarrow \Sigma$ are *equivalent* if there exists a homeomorphism $F : P_1 \rightarrow P_2$ such that $f_1 = f_2F$; in case $P_1 = P_2$ and $f_1 = f_2$ the homeomorphism F is called an *automorphism of the covering*. The set of all automorphisms of a covering f form the group $\text{Aut}(f)$ of finite order $|\text{Aut}(f)|$. Equivalent coverings have isomorphic automorphism groups.

We present two illustrative examples.

Example 1. Let $\Sigma = \overline{\mathbb{C}} = \{z \in \mathbb{C}\} \cup \infty, P = P_1 = P_2 = \overline{\mathbb{C}} = \{u \in \mathbb{C}\} \cup \infty$ be Riemann spheres. Consider the branched covering $z(u) = f(u) = f_1(u) = f_2(u) = u^3$. This covering $f : P \rightarrow \Sigma$ has 2 critical values 0 and ∞ with Young diagrams from one row of length 3. Automorphisms of the covering have the form $F(u) = u^{\sqrt[3]{1}}$. The group $\text{Aut}(f)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

Example 2. Let $\Sigma = \overline{\mathbb{C}} = \{z \in \mathbb{C}\} \cup \infty$ and $P = P_1 = P_2$ —this is a pair of Riemann spheres; that is $P = P' \cup P''$, where $P' = \{u' \in \mathbb{C}\} \cup \infty$ and $P'' = \{u'' \in \mathbb{C}\} \cup \infty$. Consider the branched covering $z(u') = f(u') = f_1(u') = f_2(u') = (u')^3, z(u'') = f(u'') = f_1(u'') = f_2(u'') = (u'')^3$. This covering $f : P \rightarrow \Sigma$ has two critical values 0 and ∞ with Young diagrams of two rows of length 3. Automorphisms of the covering are generated by the following mappings:

1. $F(u') = (u')^{\sqrt[3]{1}}, F(u'') = u''$.
2. $F(u'') = (u'')^{\sqrt[3]{1}}, F(u') = u'$.
3. $F(u') = (u''), F(u'') = u'$.

The group $\text{Aut}(f_i)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z}) \otimes (\mathbb{Z}/3\mathbb{Z}) \otimes (\mathbb{Z}/2\mathbb{Z})$.

From now on, unless indicated otherwise, we will assume that the surface Σ is connected. Let us choose points $s_1, \dots, s_F \in \Sigma$ and corresponding Young diagrams $\Delta^1, \dots, \Delta^F$ of weight d . Let Φ be the set of equivalence classes of the coverings for which s_1, \dots, s_F is the set of all critical values, and $\Delta^1, \dots, \Delta^F$ are the topological types of these critical values. The *Hurwitz number* is the number

$$H_{\Sigma}^d(\Delta^1, \dots, \Delta^F) = \sum_{f \in \Phi} \frac{1}{|\text{Aut}(f)|}. \tag{A13}$$

It is easy to prove that the Hurwitz number is independent of the positions of the points s_1, \dots, s_F on Σ . One can show that the right-hand side of (A13) depends only on the Young diagrams of $\Delta^1, \dots, \Delta^F$ and the Euler characteristic $E = E(\Sigma)$. Because of this sometimes we write $H_{E(\Sigma)}^d(\Delta^1, \dots, \Delta^F)$ instead of $H_{\Sigma}^d(\Delta^1, \dots, \Delta^F)$.

If $F = 0$ we get an unbranched covering. We denote such Hurwitz number $H_E((1^d))$.

Example 3. Let $f : \Sigma \rightarrow \mathbb{RP}^2$ be a covering without critical points. Then, if Σ is connected, then $\Sigma = \mathbb{RP}^2, \deg f = 1$ or $\Sigma = S^2, \deg f = 2$. Therefore if $d = 3$, then $\Sigma = \mathbb{RP}^2 \coprod \mathbb{RP}^2 \coprod \mathbb{RP}^2$ or $\Sigma = \mathbb{RP}^2 \coprod S^2$. Thus $H_1((1^3)) = \frac{1}{3!} + \frac{1}{2!} = \frac{2}{3}$.

Appendix D. Combinatorial Definition of Hurwitz Numbers

Consider the symmetric group (equivalently, the permutation group) S_d and the equation

$$\sigma_1 \cdots \sigma_F \rho_1^2 \cdots \rho_M^2 \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_H \beta_H \alpha_H^{-1} \beta_H^{-1} = 1, \tag{A14}$$

where $\sigma_1, \dots, \sigma_F, \rho_1, \dots, \rho_M, \alpha_1, \beta_1, \dots, \alpha_H, \beta_H \in S_d$, and moreover $\sigma_i \in C_{\Delta^i}, i = 1, \dots, F$, where C_{Δ^i} is the conjugacy class labeled by a partition $\Delta^i = (\Delta_1^i, \Delta_2^i, \dots)$. The Hurwitz number is the number of solutions of Equation (A14) divided by $d!$ (by the order of S_d).

It can be proved that so introduced the (combinatorial) Hurwitz number coincides with the (geometric) Hurwitz number $H_E(\Delta^1, \dots, \Delta^F)$ introduced in Appendix C where $E = 2 - 2H - M$. (One can look at the base surface Σ as a result of gluing H handles and M Möbius stripes to a sphere.

Consider the simplest example: $H = 0$ and $M = 1$; that is $\Sigma = \mathbb{R}P^2$ (real projective plane). Suppose $F = 0$; that is we deal with an unbranched covering. Suppose $d = 3$; that is we consider 3-sheeted covering. Let us solve $\rho^2 = 1$, where $\rho \in S_3$. One gets 4 solutions: 3 transpositions of the set 1, 2, 3 and one identity permutation. There are $3!$ permutations in S_3 . As a result we get $H_1((1^3)) = 4/3! = 2/3$ as we got in the last example of the previous section.

In the same way one can consider Example 1 of the previous section. In this case $H = M = 0$; that is $E = 2$; one gets the Riemann sphere with two branch points ($F = 2$) and 3-sheeted covering with profiles $\Delta^1 = \Delta^2 = (3)$. We solve the equation $\sigma_1\sigma_2 = 1$, where both $\sigma_{1,2}$ consist of a single cycle of length 3. There are two solutions $\sigma_1 = \sigma_2^{-1}$: one sends 1, 2, 3 to 3, 1, 2, the other sends 1, 2, 3 to 2, 3, 1. We get $H_2((3), (3)) = 2/3! = 1/3$.

Example 2 corresponds to $H_2((3,3), (3,3))$, $d = 6$. One can complete the exercise and get an answer $H_2((3,3)) = 1/z_{(3,3)} = 1/18$, where z_λ is given by (1). Actually, for any d and for any pair of profiles one gets $H_2(\Delta^1, \Delta) = \delta_{\Delta^1, \Delta} 1/z_\Delta$.

In [58,59] (and also in [60]) it was found that $H_E(\Delta^1, \dots, \Delta^F)$ is given by formula (27).

Appendix E. Differential Operators

In [32] we develop (see also [32]) the work [61], which offers a beautiful generalization of the cut-and-join formula (MMN formula):

$$\mathcal{W}^\Delta(\mathbf{p}) \cdot s_\mu(\mathbf{p}) = \varphi_\mu(\Delta) s_\mu(\mathbf{p}), \tag{A15}$$

which describes the merging of pairs of branch points in the covering problem. Here $\mu = (\mu_1, \mu_2, \dots)$ and $\Delta = (\Delta_1, \dots, \Delta_\ell)$ are Young diagrams (Δ is the ramification profile of one of branch points; for simplicity, we consider the case where $|\mu| = |\Delta|$), s_μ is the Schur function. Differential operators $\mathcal{W}^\Delta(\mathbf{p})$ generalize the operators of “additional symmetries” [62] in the theory of solitons and commute with each other for different Δ . In the work ([63]), it was noted that if they are written in the so-called Miwa variables, that is, in terms of the eigenvalues of the matrix X such that $p_m = \text{tr}(X^m)$; then the generalized cut-and-join formula is written very compactly and beautifully:

$$\mathcal{W}^\Delta \cdot s_\mu(X) = \varphi_\mu(\Delta) s_\mu(X) \tag{A16}$$

where

$$\mathcal{W}^\Delta = \frac{1}{z_\Delta} \text{tr}(D^{\Delta_1}) \cdots \text{tr}(D^{\Delta_\ell}), \tag{A17}$$

and the factor z_Δ is given by (1), D is

$$D_{a,b} = \sum_{c=1}^N X_{a,c} \frac{\partial}{\partial X_{b,c}} \tag{A18}$$

As G.I. Olshansky pointed out to us, this type of formula appeared in the works of Perelomov and Popov [64–66] and describe the actions of the Casimir operators in the representation λ ; see also [67], Section 9.

We propose a generalization of this relation, which in our case is constructed using a child’s drawing of a constellation (dessin d’enfants, or a map in terminology [28]). In fact, we are considering a modification in which the vertices are replaced by small disks—“stars”). This topic will be studied in more detail in the next article. Here we restrict ourselves only to a reference to important beautiful works [41–45].

(i) One can interpret the Gaussian integral as the integral of n -component two-dimensional charged bosonic fields Z_i and Z_i^\dagger :

$$\int (Z_i^\dagger)_{a,b} (Z_j)_{b',a'} d\Omega = \langle (Z_i^\dagger)_{a,b} (Z_j)_{b',a'} \rangle = \frac{1}{N} \delta_{a,a'} \delta_{b,b'} \delta_{i,j},$$

for $i, j = 1, \dots, n, \quad a, b = 1, \dots, N$.

The Fock space of these fields is all possible polynomials from the matrix elements of the matrices Z_1, \dots, Z_n .

The operators $(Z_i)_{ab}$ can be considered creation operators, and the operators

$$(Z_i^\dagger) \rightarrow \frac{1}{N} \partial_{i'}, \quad (\partial_i)_{a,b} = \frac{1}{N} \frac{\partial}{\partial Z_{b,a}} \tag{A19}$$

annihilation operators that act in this space. The integrands in our integrals should be considered anti-ordered, that is, all annihilation operators (all derivatives) considered to be moved to the left, while the matrix structure is considered to be preserved. We will denote this is anti-ordering of some A by the symbol $:: A ::$, where A is a polynomial of matrix elements of the matrices

From this point of view, on different sides of the ribbon of $\tilde{\Gamma}$ with the number i we place the canonically conjugated coordinates Z_i and momenta ∂_{Z_i} .

We recall that all partitions through the paper have the same weight d .

(ii) Then, for example, the relation we get

$$:: \tau_g(M_1, \dots, M_F) :: = \sum_{d=0}^{\infty} \sum_{\substack{\Delta^1, \dots, \Delta^V \\ |\Delta^1| = \dots = |\Delta^V| = d}} H_{\Sigma'}(g|\Delta^1, \dots, \Delta^V) C(\Delta^1, \dots, \Delta^V), \tag{A20}$$

We give another relation:

$$\left(:: \prod_{i=1}^F s_{\lambda^i}(M_i) :: \right) \cdot 1 = \delta_{\lambda^1, \dots, \lambda^F} \left(\frac{\dim \lambda}{d!} \right)^{-n} \prod_{i=1}^V s_{\lambda}(W_i^*) \tag{A21}$$

where $\lambda^1, \dots, \lambda^F = \lambda$ is a set of Young diagrams, and where $\delta_{\lambda^1, \dots, \lambda^F}$ is equal 1, if $\lambda^1 = \dots = \lambda^F = \lambda$ and is equal to 0 otherwise.

It looks like a simple rewrite, but can be helpfully used. Let us derive a beautiful formula (Theorem 5.1 in [63]), namely (A16) (and see also articles [41–46]).

In order to do this we should use the freedom to choose the source matrices:

(iii) For a partition $\Delta = (\Delta_1, \dots, \Delta_\ell)$ and a face monodromy M_i and a star monodromy W_i^* , let us introduce notations

$$\mathcal{M}_i^{\Delta_i} = \text{tr} \left((M_i)^{\Delta_1} \right) \cdots \text{tr} \left((M_i)^{\Delta_\ell} \right) \tag{A22}$$

$$\mathcal{C}_i^{\Delta_i} = \text{tr} \left((W_i^*)^{\Delta_1} \right) \cdots \text{tr} \left((W_i^*)^{\Delta_\ell} \right) \tag{A23}$$

We can write

$$N^{nd} \int \left[\prod_{i=1}^F \frac{\mathcal{M}_i^{\Delta_i}}{z_{\tilde{\Delta}^i}} \right] d\Omega \tag{A24}$$

$$= \sum_{\Delta^1, \dots, \Delta^V} H_{\Sigma}(\tilde{\Delta}^1, \dots, \tilde{\Delta}^F, \Delta^1, \dots, \Delta^V) \left[\prod_{i=1}^V \mathcal{C}_i^{\Delta_i} \right] \tag{A25}$$

Let us write the most general generating function for Hurwitz numbers which was obtained in [31]:

$$N^{nd} \int \left[\prod_{i=1}^{F_1} \frac{\mathcal{M}_i^{\tilde{\Delta}^i}}{z_{\tilde{\Delta}^i}} \right] \left(\prod_{i=F_1+1}^{F_2} \mathfrak{M}(M_i) \right) \left(\prod_{i=F_2+2, F_2+4, \dots}^F \mathfrak{H}(M_{i-1}, M_i) \right) d\Omega \tag{A26}$$

$$= \sum_{\substack{\Delta^1, \dots, \Delta^V \\ |\Delta^1| = \dots = |\Delta^V| = d}} H_{\tilde{\Sigma}}(\tilde{\Delta}^1, \dots, \tilde{\Delta}^{F_1}, \Delta^1, \dots, \Delta^V) C(\Delta^1, \dots, \Delta^V), \tag{A27}$$

where

$$H_{\tilde{\Sigma}}(\tilde{\Delta}^1, \dots, \tilde{\Delta}^{F_1}, \Delta^1, \dots, \Delta^V) = \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \mu}{d!} \right)^{F-n+V-2h-m} \varphi_{\mu}(\tilde{\Delta}^1) \cdots \varphi_{\mu}(\tilde{\Delta}^{F_1}) \varphi_{\mu}(\Delta^1) \cdots \varphi_{\mu}(\Delta^V) \tag{A28}$$

where $h = \frac{1}{2}(F - F_2)$ is the number of handles and $m = F_2 - F_1$ is the number of Moebius stripes glued to Σ where the graph $\tilde{\Gamma}$ (modified dessin d'enfants) was drawn. The Euler characteristic of $\tilde{\Sigma}$ is $F - n + V - 2h - m = F_1 - n + V$. Hurwitz number (A28) counts the coverings of $\tilde{\Sigma}$ with branching profiles $\tilde{\Delta}^1, \dots, \tilde{\Delta}^{F_1}, \Delta^1, \dots, \Delta^V$.

Let us multiply the both sides of (A26) by

$$\prod_{i=k+1}^{F_1} \frac{\dim \mu^i}{d!} \varphi_{\mu^i}(\tilde{\Delta}^i) z_{\tilde{\Delta}^i}$$

(where $k \leq F$), and then sum the both sides (A26) and (A27) over $\tilde{\Delta}^i, i = k + 1, \dots, F_1$, taking into account

$$s_{\mu}(X) = \frac{\dim \mu}{d!} \sum_{\Delta} \varphi_{\mu}(\Delta) \mathcal{X}^{\Delta}, \tag{A29}$$

when evaluating (A26), where

$$\mathcal{X}^{\Delta} = \text{tr} \left((X)^{\Delta_1} \right) \cdots \text{tr} \left((X)^{\Delta_{\ell}} \right) \tag{A30}$$

and the orthogonality relation (11) when evaluating (A27). We obtain

$$\int N^{nd} \left[\prod_{i=1}^k \frac{\mathcal{M}_{z_{\tilde{\Delta}^i}}^{\tilde{\Delta}^i}}{\mathcal{Z}_{\tilde{\Delta}^i}} \right] \left(\prod_{i=k+M+1}^{k+M+m} \mathfrak{M}(M_i) \right) \left(\prod_{i=k+M+m+2, k+M+m+4, \dots}^F \mathfrak{H}(M_{i-1}, M_i) \right) \prod_{i=k+1}^{k+M} s_{\mu^i}(M_i) d\Omega \tag{A31}$$

$$= \delta_{\mu^{k+1}, \dots, \mu^{F-k}} \left(\frac{\dim \mu}{d!} \right)^{k-n-2h-m} \varphi_{\mu}(\tilde{\Delta}^1) \cdots \varphi_{\mu}(\tilde{\Delta}^k) \left[\prod_{i=1}^V s_{\lambda}(W_i^*) \right] \tag{A32}$$

where $2h = F - F_1 - m$

Remark A2. Suppose that the edges of the graph Γ can be painted like a chessboard in black and white faces so that the face of one color borders only the faces of a different color. Then the matrices from the set $\{Z^{\dagger}\}$ (i.e., differential operators) can be assigned to the sides of the edges of white faces, that is, the matrices from the set $\{Z\}$ to the sides of black faces. In this case, the monodromies of the white faces will be those differential operators which will act on the monodromy of black faces.

The most natural and simple case is the following ‘‘polarization’’: Suppose that $M_i, i = k + 1, \dots, F_1$ are black faces and the rest part of the face monodromies M_i are white faces (see Remark A2).

(I) Let $m = h = 0$. Take as a graph $\tilde{\Gamma}$ a child’s drawing - sunflower with n white petals drawn on the background of black night sky. See (b) in the Figure 5 for Γ with 3 petals as an example. There is 1 vertex of Γ which inflated and we get a small disk as the center of sunflower. We have $n + 1$ faces of Γ : n petals, and the big and a big face, embracing all the petals and containing infinity. Then we place all ‘‘momentums’’ inside the petals:

$$M_i = C_{-i} Z_i^{\dagger}, i = 1, \dots, n.$$

Then all “coordinates” (the collections of $\{(Z_i)_a, b\}$) are placed on the other side of the ribbons, they are places along the boundary of the big embracing black face:

$$M_{n+1} = Z_1 C_1 \cdots Z_n C_n$$

See Figure 5b as an example.

Let remove the sign tilde above Young diagrams, then,

$$\int N^{nd} \left[\prod_{i=1}^n \frac{\mathcal{M}_i^{\Delta_i}}{z_{\Delta_i}} \right] s_{\mu}(Z_1 C_1 \cdots Z_n C_n) d\Omega = \varphi_{\mu}(\Delta^1) \cdots \varphi_{\mu}(\Delta^n) s_{\mu}(C_{-1} C_1 \cdots C_{-n} C_n) \tag{A33}$$

It is equivalent to

$$\mathcal{W}_{C_{-1}}^{\Delta^1} \cdots \mathcal{W}_{C_{-n}}^{\Delta^n} \cdot s_{\mu}(Z_1 C_1 \cdots Z_n C_n) = \varphi_{\mu}(\Delta^1) \cdots \varphi_{\mu}(\Delta^n) s_{\mu}(C_{-1}(Z_1) C_1 \cdots C_{-n}(Z_n) C_n), \tag{A34}$$

where each $N \times N$ matrix C_{-i} can now depend, for example, polynomially on $Z_i, i = 1, \dots, n$ and where

$$\mathcal{W}_{C_{-i}}^{\Delta_i} =: \text{tr} \left((C_{-i}(Z_i) \partial_i)^{\Delta_i^1} \right) \cdots \text{tr} \left((C_{-i}(Z_i) \partial_i)^{\Delta_i^{\ell}} \right) :, \tag{A35}$$

where each $C_{-i} \partial_i$ is a matrix whose entries are differential operators, more precisely, are the following vector fields:

$$(C_{-i} \partial_i)_{a,b} := \sum_{c=1}^N (C_{-i})_{a,c} \frac{\partial}{\partial (Z_i)_{b,c}} \tag{A36}$$

The normal ordering indicated by two dots is the same here as in [63]—that is, while maintaining the matrix structure, the derivative operators do not act on $C_{-i} = C_{-i}(Z_i)$. Note that the normal ordering procedure is necessary in order the Equation (A34) was equivalent to the equality (A33)!

The ordering is the same as in [63]: keeping the matrix structure the derivatives do not act on $C_{-i} = C_{-i}(Z_i)$. Notice that the ordering is necessary to relate (A34) to (A33)!

If we now take the case $n = 1$ (one petal), and in addition, $C_{-i} = Z_i$, then we get the desired formula (A16).

Take another example with the same graph $\tilde{\Gamma}$ with the same monodromies. However, let $k = 0$. In this case the integral (A33) can be re-written as the relation

$$\hat{\mathfrak{M}}_1 \cdots \hat{\mathfrak{M}}_m \hat{\mathfrak{S}}_1 \cdots \hat{\mathfrak{S}}_h \cdot s_{\mu}(Z_1 C_1 \cdots Z_n C_n) = \left(\frac{\dim \mu}{|\mu|!} \right)^{-2h-m-n} s_{\mu}(C_{-1}(Z_1) C_1 \cdots C_{-n}(Z_n) C_n), \tag{A37}$$

where

$$\hat{\mathfrak{M}}_i =: e^{\frac{1}{2} \sum_{j>0} \frac{1}{j} \left(\text{tr}(C_{-i} \partial_i)^j \right)^2 + \sum_{j>0, \text{odd}} \frac{1}{j} \text{tr} \left((C_{-i} \partial_i)^j \right)} :, \quad i = 1, \dots, m, \tag{A38}$$

$$\hat{\mathfrak{S}}_i =: e^{\sum_{j>0} \frac{1}{j} \text{tr} \left((C_{1-m-2i} \partial_{m+2i-1})^j \right) \text{tr} \left((C_{-m-2i} \partial_{m+2i})^j \right)} :, \quad i = 1, \dots, h. \tag{A39}$$

Remark A3. When $C_{-i} = Z_i, i = 1, \dots, n$ both equalities (A34) and (A37) describe eigenvalue problems for the corresponding Hamiltonians in the two-dimensional bosonic theory. Perhaps a comparison with the case analyzed by Dubrovin is appropriate. This is the case $n = 1, F = 1, \mathcal{W}^{(n)}$. In this case, the operators $\mathcal{W}^{(n)}, n = 1, 2, \dots$ are the dispersionless Hamiltonians KdV equations [68].

Remark A4. The case C_{-i} does not depend is also interesting in case the monodromies of the stars are degenerate matrices, then the whole integral is related to the integration over rectangular matrices. As an example one can choose $C_{-1} C_1 \cdots C_{-n} C_n$ in (A33) as $\text{diag}(1, 1, \dots, 1, 0, 0, \dots, 0)$. Then we get the Pochhammer symbol in the right-hand side which allows one to related the whole integral to the hypergeometric tau function [24]. It will be discussed in a more detailed text where we plan to relate out topic to certain topics in [41–46].

Another example. Γ has 2 vertices which are connected by 4 edges; see Figure 4d. We

$$\begin{aligned} &: \mathbf{p}_{\Delta^1}(\partial_1 C_{-1} \partial_2 C_{-2}) \mathbf{p}_{\Delta^2}(\partial_3 C_{-3} \partial_4 C_{-4}) : (s_\lambda(Z_1 C_1 Z_4 C_4) s_\mu(Z_2 C_2 Z_3 C_3)) \\ &= \delta_{\lambda, \mu} \left(\frac{\dim \mu}{d!} \right)^{-2} \varphi_\mu(\Delta^1) \varphi_\mu(\Delta^2) s_\mu(C_1 C_{-4} C_3 C_{-2}) s_\mu(C_4 C_{-1} C_2 C_{-3}) \end{aligned}$$

In particular, if one takes $C_3 = C_4 = C$ and $C_{-1} = C_{-1}(Z_2), C_{-2} = C_{-2}(Z_1), C_{-3} = C_{-3}(Z_3), C_{-4} = C_{-4}(Z_4)$ he gets

$$\begin{aligned} &: \mathbf{p}_{\Delta^1}(C_{-2}(Z_1) \partial_1 C_{-1}(Z_2) \partial_2) \mathbf{p}_{\Delta^2}(\partial_3 C_{-3}(Z_3) \partial_4 C_{-4}(Z_4)) : s_\lambda(Z_1 C_1 Z_4 C) s_\mu(Z_2 C_2 Z_3 C) \\ &= \delta_{\lambda, \mu} \left(\frac{\dim \mu}{d!} \right)^{-2} \varphi_\mu(\Delta^1) \varphi_\mu(\Delta^2) s_\mu(C_{-2}(Z_1) C_1 C_{-4}(Z_4) C) s_\mu(C_{-1}(Z_2) C_2 C_{-3}(Z_3) C) \end{aligned}$$

In particular, if one takes $C_3 = C_4 = C$ and $C_{-1} = Z_2, C_{-2} = Z_1, C_{-3} = Z_3, C_{-4} = Z_4$ (Euler fields) he gets an eigenvalue problem:

$$\begin{aligned} &: \mathbf{p}_{\Delta^1}(Z_1 \partial_1 Z_2 \partial_2) \mathbf{p}_{\Delta^2}(\partial_3 Z_3 \partial_4 Z_4) : (s_\lambda(Z_1 C_1 Z_4 C) s_\mu(Z_2 C_2 Z_3 C)) \\ &= \delta_{\lambda, \mu} \left(\frac{\dim \mu}{d!} \right)^{-2} \varphi_\mu(\Delta^1) \varphi_\mu(\Delta^2) s_\mu(Z_1 C_1 Z_4 C) s_\mu(Z_2 C_2 Z_3 C) \end{aligned}$$

In case $C_{-1} = C_{-2} = C_{-3} = C_{-4} = \mathbb{I}_N$ we get

$$\begin{aligned} &: \mathbf{p}_{\Delta^1}(\partial_1 \partial_2) \mathbf{p}_{\Delta^2}(\partial_3 \partial_4) : (s_\lambda(Z_1 C_1 Z_4 C_4) s_\mu(Z_2 C_2 Z_3 C_3)) \\ &= \delta_{\lambda, \mu} \varphi_\mu(\Delta^1) \varphi_\mu(\Delta^2) s_\mu(C_1 C_3) s_\mu(C_2 C_4) \end{aligned}$$

Now we consider another example $n = 4$ with the graph obtained from the graph (a) in the Figures 1 and 3 drawn on the torus by doubling the edges: instead of each edge we draw two ones. We have Γ with one vertex, four edges and three faces and obtain

$$\begin{aligned} &: \mathbf{p}_{\Delta^1}(\partial_1 C_{-1} \partial_3 C_{-3}) \mathbf{p}_{\Delta^1}(\partial_2 C_{-2} \partial_4 C_{-4}) : s_\lambda(Z_1 C_1 Z_2 C_2 Z_3 C_3 Z_4 C_4) \\ &= s_\lambda(C_1 C_{-2} C_4 C_{-1} C_3 C_{-4} C_2 C_{-3}) \end{aligned}$$

Take $C_{-1} = C_{-1}(Z_3), C_{-2} = C_{-2}(Z_2), C_{-3} = C_{-3}(Z_1), C_{-4} = C_{-4}(Z_4)$ and $C_2 = C_4 = C$. As an example we obtain

$$\begin{aligned} &: \mathbf{p}_{\Delta^1}(Z_1 \partial_1 Z_3 \partial_3) \mathbf{p}_{\Delta^1}(\partial_2 Z_2 \partial_4 Z_4) : s_\lambda(Z_1 C_1 Z_2 C Z_3 C_3 Z_4 C) \\ &= \left(\frac{\dim \lambda}{d!} \right)^{-2} \varphi_\lambda(\Delta^1) \varphi_\lambda(\Delta^2) s_\lambda(Z_1 C_1 Z_2 C Z_3 C_3 Z_4 C) \end{aligned}$$

(iv) Let us notice that if we take a dual graph to the sunflower graph with $n = 1$ (dual to one petal Γ , which is just a line segment; see Figure 2a,b), in this case we have one face and two vertices, we get a version of the Capelli-type relation. Then it is a task to compare explicitly such relations with beautiful results [43–46].

(v) There are several allusions to the existence of interesting structures related to quantum integrability. First, as noted in [31] by this appearance 2D Yang-Mills theory [40]. See also possible connection to [48]. Then the appearance of the Yangians in works [41,42] which, we hope, can be related to our subject. And finally, the work [68].

(vi) There is a direct similarity between integrals over complex matrices and integrals over unitary matrices. However, from our point of view direct analogues of the relations in the present

paper are more involved in the case of unitary matrices. In particular, Hurwitz numbers are replaced by a special combination of these numbers.

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