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Existence of Classical Solutions for Nonlinear Elliptic Equations with Gradient Terms

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Abstract: This paper deals with the existence of solutions of the elliptic equation with nonlinear gradient term $-\Delta u = f(x, u, \nabla u)$ on Ω restricted by the boundary condition $u|_{\partial\Omega} = 0$, where Ω is a bounded domain in \mathbb{R}^N with sufficiently smooth boundary $\partial\Omega$, $N \geq 2$, and $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is continuous. The existence results of classical solutions and positive solutions are obtained under some inequality conditions on the nonlinearity $f(x, \xi, \eta)$ when $|(\xi, \eta)|$ is small or large enough.

Keywords: elliptic equation; gradient term; classical solution; positive solution

MSC: 5J25; 35J60; 47H10

1. Introduction and Main Results

Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 2$) whose boundary $\partial\Omega$ is $C^{2+\mu}$ -smooth for given $\mu \in (0, 1)$. In this paper, we discuss the existence of solutions of the elliptic boundary value problem (BVP) with gradient term

$$\begin{cases} -\Delta u = f(x, u, \nabla u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (1)$$

where $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is the nonlinearity. This problem arises in many different areas of applied mathematics. Due to the appearance of the gradient term in the nonlinearity, BVP(1) has no variational structure, and the variational method and critical point theory cannot be applied to it directly. The authors of [1,2] proposed a method combining the mountain-pass theorem with an approximation technique to solve BVP(1). Firstly, for any given $w \in H_0^1(\Omega)$, they considered the boundary value problem

$$\begin{cases} -\Delta u = f(x, u, \nabla w), & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \quad (2)$$

Note that BVP(2) has the variational structure. They established the existence of a solution u_w of BVP(2) by using the mountain-pass theorem. Then, they constructed a sequence $\{u_n\} \subset H_0^1(\Omega)$ by the iterative equation

$$\begin{cases} -\Delta u_n = f(x, u_n, \nabla u_{n-1}), & x \in \Omega, \\ u_n|_{\partial\Omega} = 0 \end{cases} \quad (3)$$

starting with an arbitrary $u_0 \in H_0^1(\Omega) \cap C^1(\bar{\Omega})$, and they proved that $\{u_n\}$ converges to a solution of BVP(1) in that $f(x, \xi, \eta)$ satisfies Lipschitz conditions on (ξ, η) in the neighborhood of $(0, 0)$ with appropriately small coefficients and certain growth conditions on ξ . Later, this iterative method based on the mountain-pass theorem was applied to many semilinear and quasilinear elliptic equations; see [3–7]. In [8], Ruiz obtained the existence of a positive solution for BVP(1) by combining Krasnoselskii's fixed-point theorem in cones



Citation: Li, Y.; Ma, W. Existence of Classical Solutions for Nonlinear Elliptic Equations with Gradient Terms. *Entropy* **2022**, *24*, 1829. <https://doi.org/10.3390/e24121829>

Academic Editors: Ravi P. Agarwal and Maria Alessandra Ragusa

Received: 26 October 2022

Accepted: 30 November 2022

Published: 15 December 2022

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with blow-up techniques when $f(x, \zeta, \eta)$ is a nonnegative function and satisfies a suitable growth condition on ζ and η . When Ω is a ball, annulus, or exterior domain of a ball, and $f(x, \zeta, \eta)$ is radially symmetric on x , the authors of [9–13] obtained the existence of positive radial solutions of BVP(1) by discussing the corresponding boundary value problem of second-order ordinary differential equations.

On the other hand, the lower- and upper-solutions method is an effective way to obtain the existence of solutions of BVP(1). In [14], Amann built a lower- and upper-solution theorem of BVP(1) in $C^{2+\mu}(\overline{\Omega})$ in that $f(x, \zeta, \eta)$ has a continuous partial derivative with respect to ζ and η , and there is, at most, quadratic growth on η . He assumed BVP(1) has pair of ordered lower and upper solutions and proved the existence of a solution between the lower and upper solutions. In [15], Amann and Crandall slightly generalized the results of [14] by a more-direct argument. In [16], Pohozaev obtained the existence results for BVP(1) via the method of lower and upper solutions in the Sobolev space $W^{2,p}(\Omega)$ with $p > N$ when $f(x, \zeta, \eta)$ is Lipschitzian with respect to η . In [17–22], the authors obtained the existence of solutions or positive solutions by using the lower- and upper-solutions method and fixed-point theorem under some growth condition of the nonlinearity.

In this paper, we apply the upper- and lower-solution method and the Leray–Schauder fixed-point theory to obtain new existence results. In the following, we denote a generic point of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ by (x, ζ, η) with $x \in \overline{\Omega}$, $\zeta \in \mathbb{R}$, and $\eta = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N$. To obtain the classical solution of BVP(1), we assume the nonlinearity $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies the following conditions:

(F1) The partial derivatives $f'_\zeta, f'_{\eta_i}, i = 1, \dots, N$, exist and are continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, and for every $\rho > 0$, there exists $L := L(\rho) > 0$ such that

$$|f(x_2, \zeta, \eta) - f(x_1, \zeta, \eta)| \leq L|x_2 - x_1|^\mu, \tag{4}$$

for any $x_1, x_2 \in \overline{\Omega}$, $\zeta \in [-\rho, \rho]$ and $\eta_2 \in \overline{B}_\rho := \{\eta \in \mathbb{R}^N \mid |\eta| \leq \rho\}$.

(F2) For every $\rho > 0$, there exists $C := C(\rho) > 0$ such that

$$|f(x, \zeta, \eta)| \leq C(1 + |\eta|^2), \tag{5}$$

for any $(x, \zeta, \eta) \in \overline{\Omega} \times [-\rho, \rho] \times \mathbb{R}^N$.

Condition (F1) implies that f is continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ and is a stronger regularity condition. Condition (F2) restricts f to at most quadratic growth with respect to η . If f grows at most like $|\eta|^{2-\varepsilon}$ for some $\varepsilon \in (0, 1)$, the regularity condition (F1) can be weakened as

(F1)' For every $\rho > 0$, there exists $L := L(\rho) > 0$ such that

$$|f(x_2, \zeta_2, \eta_2) - f(x_1, \zeta_1, \eta_1)| \leq L(|x_2 - x_1|^\mu + |\zeta_2 - \zeta_1| + |\eta_2 - \eta_1|),$$

for any $(x_1, \zeta_1, \eta_1), (x_2, \zeta_2, \eta_2) \in \overline{\Omega} \times [-\rho, \rho] \times \overline{B}_\rho$.

See [3,14].

Our existence results are related to the principle eigenvalue λ_1 of Laplace operator $-\Delta$ on the boundary condition $u|_{\partial\Omega} = 0$, which is given by

$$\lambda_1 = \inf \left\{ \frac{\|\nabla u\|_2}{\|u\|_2} \mid u \in H_0^1(\Omega), \|u\|_2 \neq 0 \right\}. \tag{6}$$

Theorem 1. Let $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy (F1) and (F2). If there exist constants $a, b \geq 0$ satisfying

$$\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} < 1 \tag{7}$$

and $H > 0$ such that

$$f(x, \zeta, \eta) \leq a\zeta + b|\eta|, \quad x \in \overline{\Omega}, \zeta \geq 0, |(\zeta, \eta)| \geq H \tag{8}$$

and

$$f(x, -\xi, -\eta) \geq -a\xi - b|\eta|, \quad x \in \bar{\Omega}, \quad \xi \geq 0, \quad |(\xi, \eta)| \geq H, \tag{9}$$

then, BVP(1) has at least one classical solution $u \in C^{2+\mu}(\bar{\Omega})$.

In Theorem 1, if $b = 0$, the result is known (see [1, Theorem 1.2]), and if $b \neq 0$, the result is new.

Theorem 2. Let $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy (F1) and (F2). If there exist constants $a, b \geq 0$ satisfying (7) and $H > 0$ such that f satisfies (8), and there exists a positive constant δ such that

$$f(x, \xi, \eta) \geq \lambda_1 \xi, \quad x \in \bar{\Omega}, \quad \xi \geq 0, \quad |(\xi, \eta)| \leq \delta, \tag{10}$$

then, BVP(1) has at least one classical positive solution $u \in C^{2+\mu}(\bar{\Omega})$.

If f satisfies the condition of Theorem 1, but assume that

$$f(x, 0, 0) \geq 0, \quad x \in \bar{\Omega} \tag{11}$$

instead of (9), then $v_0 \equiv 0$ is a lower solution of BVP(1), and BVP(1) has at least one nonnegative solution, see [1, Theorem 1.3]. Theorem 2 is an addition of this result and uses (10) instead of (11) to obtain a positive solution of BVP(1).

Theorem 3. Let the conditions of Theorem 1 be satisfied, and there exists a positive constant δ such that (10) and

$$f(x, -\xi, -\eta) \leq -\lambda_1 \xi, \quad x \in \bar{\Omega}, \quad \xi \geq 0, \quad |(\xi, \eta)| \leq \delta, \tag{12}$$

hold. Then, BVP(1) has at least one positive solution $u_1 \in C^{2+\mu}(\bar{\Omega})$ and one negative solution $u_2 \in C^{2+\mu}(\bar{\Omega})$.

In Theorem 3, from (10) and (12), it follows that $f(x, 0, 0) \equiv 0$ by letting $|(\xi, \eta)| \rightarrow 0$. Hence, $u_3 \equiv 0$ is a trivial solution. This means that BVP(1) has at least three distinct solutions.

The proofs of Theorems 1–3 are based on the method of lower and upper solutions built by Amann [14]. A lower solution v of BVP(1) means that $v \in C^{2+\mu}(\bar{\Omega})$ and satisfies

$$\begin{cases} -\Delta v \leq f(x, v, \nabla v), & x \in \Omega, \\ v|_{\partial\Omega} \leq 0, \end{cases}$$

and an upper solution w of BVP(1) means that $w \in C^{2+\mu}(\bar{\Omega})$ and satisfies

$$\begin{cases} -\Delta w \geq f(x, w, \nabla w), & x \in \Omega, \\ w|_{\partial\Omega} \geq 0. \end{cases}$$

By [1, Theorem 1.1], we have the following existence result:

Theorem 4. Let $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy (F1) and (F2). If BVP(1) has a lower solution v_0 and an upper solution w_0 such that $v_0 \leq w_0$, then BVP(1) has at least one solution $u_2 \in C^{2+\mu}(\bar{\Omega})$ between v_0 and w_0 .

Theorem 4 is a special case of [1, Theorem 1.1]. In Section 3, we use Theorem 4 to prove Theorems 1–3. Some preliminaries to discuss BVP(1) are presented in Section 2.

2. Preliminaries

Let $W^{m,p}(\Omega)$ be the usual Sobolev space on domain Ω and $H^m(\Omega) := W^{m,2}(\Omega)$. To discuss BVP(1), we first consider the corresponding linear elliptic boundary value problem (LBVP)

$$\begin{cases} -\Delta u = h(x), & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{13}$$

where $h \in L^p(\Omega)$ ($1 < p < +\infty$) is a given function. Since the boundary $\partial\Omega$ of Ω is $C^{2+\mu}$ -smooth, by the L^p theory of linear elliptic equations (see [23]), for every $h \in L^p(\Omega)$, LBVP(13) has a unique strong solution $u := Sh \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, and the solution operator $S : L^p(\Omega) \rightarrow W^{2,p}(\Omega)$ is a linear bounded operator. Especially when $p = 2$, the solution $u = Sh$ of LBVP(13) satisfies

$$\|u\|_2 \leq \frac{1}{\sqrt{\lambda_1}} \|\nabla u\|_2, \quad \|\nabla u\|_2 \leq \frac{1}{\sqrt{\lambda_1}} \|\Delta u\|_2. \tag{14}$$

In fact, since the Laplace operator $-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ is a positive definite operator in $L^2(\Omega)$,

$$\|\nabla u\|_2^2 = (-\Delta u, u) \geq \lambda_1(u, u) = \lambda_1 \|u\|_2^2.$$

Hence, the first inequality of (14) holds. Noting $\|\Delta u\|_2 = \|h\|_2$, from Equation (13), it follows that

$$\|\nabla u\|_2^2 = (-\Delta u, u) = (h, u) \leq \|h\|_2 \|u\|_2 = \|\Delta u\|_2 \|u\|_2 \leq \frac{1}{\sqrt{\lambda_1}} \|\Delta u\|_2 \|\nabla u\|_2.$$

Hence, the second inequality of (14) holds.

When $h \in C^v(\bar{\Omega})$ for some $v \in (0, \mu]$, by the Schauder theory of linear elliptic equations (see [23,24]), the solution of LBVP(13) $u = Sh \in C^{2+v}(\bar{\Omega})$ is a classical solution.

Next, consider the nonlinear elliptic equation BVP(1). We have the following existence result of the classical solution:

Theorem 5. *Let $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy (F1)' and in the following growth condition*

(F3) *let there exist constants $a, b \geq 0$ satisfying (7) and $c > 0$ such that*

$$|f(x, \xi, \eta)| \leq a|\xi| + b|\eta| + c, \quad (x, \xi, \eta) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N.$$

Then, BVP(1) has a unique classical solution $u \in C^{2+\mu}(\bar{\Omega})$.

Proof. We first show that BVP(1) has an L^2 solution $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Since the solution operator of LBVP(13) $S : L^2(\Omega) \rightarrow H^2(\Omega) \cap H_0^1(\Omega)$ is a linear bounded operator, by the compactness of the Sobolev embedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$, $S : L^2(\Omega) \rightarrow H_0^1(\Omega)$ is completely continuous. Define a mapping F on $H_0^1(\Omega)$ by

$$F(u)(x) = f(x, u(x), \nabla u(x)), \quad u \in H_0^1(\Omega), \quad x \in \Omega. \tag{15}$$

By Condition (F3), $F : H_0^1(\Omega) \rightarrow L^2(\Omega)$ is continuous, and it maps every bounded set of $H_0^1(\Omega)$ into a bounded set of $L^2(\Omega)$. Hence, the composite mapping

$$A = S \circ F : H_0^1(\Omega) \rightarrow H_0^1(\Omega) \tag{16}$$

is completely continuous. By the definition of S , the strong L^2 solution of LBVP(13) is equivalent to the fixed point of A . We use the Leray–Schauder fixed point theorem [25] to show that A has a fixed point. For this, we consider the equation family

$$u = \lambda Au, \quad 0 < \lambda < 1, \tag{17}$$

and show that the set of their solutions is bounded in $H_0^1(\Omega)$.

Let $u \in H_0^1(\Omega)$ be a solution of (17) for $\lambda \in (0, 1)$. Set $h = \lambda F(u)$. Since $h \in L^2(\Omega)$, by the definition of S , $u = Sh \in H^2(\Omega) \cap H_0^1(\Omega)$ is the unique solution of LBVP(13). Hence, u satisfies the differential equation

$$\begin{cases} -\Delta u = \lambda f(x, u, \nabla u), & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{18}$$

By this equation and Condition (F3), we have

$$|\Delta u(x)| = |\lambda f(x, u(x), \nabla u(x))| \leq a|u(x)| + b|\nabla u(x)| + c, \quad x \in \Omega.$$

By this inequality and (14), we obtain that

$$\begin{aligned} \sqrt{\lambda_1}\|\nabla u\|_2 &\leq \|\Delta u\|_2 \leq a\|u\|_2 + b\|\nabla u\|_2 + c\sqrt{|\Omega|} \\ &\leq \left(\frac{a}{\sqrt{\lambda_1}} + b\right)\|\nabla u\|_2 + c\sqrt{|\Omega|}. \end{aligned}$$

From this, it follows that

$$\|\nabla u\|_2 \leq \frac{c\sqrt{|\Omega|}}{\sqrt{\lambda_1}\left(1 - \left(\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}}\right)\right)} := C_0.$$

Hence, the set of the solutions of Equation Family (17) is bounded in $H^1_0(\Omega)$. By the Leray–Schauder fixed-point theorem, $A = S \circ F$ has a fixed point $u_0 \in H^1_0(\Omega)$, which belongs to $H^2(\Omega)$ and is an L^2 solution of BVP(1).

Next, we show that $u_0 \in C^{2+\mu}(\bar{\Omega})$, and it is a classical solution of BVP(1). Set

$$h_0(x) = f(x, u_0(x), \nabla u_0(x)), \quad x \in \Omega. \tag{19}$$

Then, $u_0 = Sh_0$ is the solution of LBVP(13) for $h = h_0$.

If $p_0 := 2 < N$, choose $p_1 = \frac{Np_0}{N-p_0} (> p_0)$; then, by the Sobolev embedding theorem, $H^2(\Omega) \hookrightarrow W^{1,p_1}(\Omega)$. Since $u_0 \in H^2(\Omega)$, it follows that $u_0 \in W^{1,p_1}(\Omega)$. By Condition (F3) and (19), we see that $h_0 \in L^{p_1}(\Omega)$. Hence, by the existence and uniqueness of the L^p solution of LBVP(13), $u_0 = Sh_0 \in W^{2,p_1}(\Omega)$.

If $p_1 < N$, choose $p_2 = \frac{Np_1}{N-p_1} (> p_1)$; then, by the Sobolev embedding theorem, $W^{2,p_1}(\Omega) \hookrightarrow W^{1,p_2}(\Omega)$. Hence, $u_0 \in W^{1,p_2}(\Omega)$. By Condition (F3) and (19), we obtain that $h_0 \in L^{p_2}(\Omega)$. Hence, $u_0 = Sh_0 \in W^{1,p_2}(\Omega)$.

To continue, since the step length $p_k - p_{k-1} (k = 1, 2, \dots)$ is increasing, we can choose $p > N$ such that $h_0 \in L^p(\Omega)$. Thus, $u_0 = Sh_0 \in W^{2,p}(\Omega)$.

Choose $\sigma = \min\{1 - \frac{N}{p}, \mu\}$. By the Sobolev embedding theorem, $W^{2,p}(\Omega) \hookrightarrow C^{1+\sigma}(\bar{\Omega})$. Hence, $u_0 \in C^{1+\sigma}(\bar{\Omega})$. By Assumption (F1)' and (19), $h_0 \in C^\sigma(\bar{\Omega})$. Hence, by the Schauder theory of linear elliptic equations, the solution of LBVP(2.1) $u_0 = Sh_0 \in C^{2+\sigma}(\bar{\Omega})$. By this and Assumption (F1)', $h_0 \in C^\mu(\bar{\Omega})$. Hence, $u_0 = Sh_0 \in C^{2+\mu}(\bar{\Omega})$. Clearly, u_0 is a classical solution of BVP(1). \square

Strengthen Condition (F3) of Theorem 5; we have following existence and uniqueness result.

Theorem 6. Let $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy (F1)', and for Following Condition

(F4), there exist constants $a, b \geq 0$ satisfying (19) such that

$$\begin{aligned} |f(x, \xi_2, \eta_2) - f(x, \xi_1, \eta_1)| &\leq a|\xi_2 - \xi_1| + b|\eta_2 - \eta_1|, \\ x \in \bar{\Omega}, (\xi_1, \eta_1), (\xi_2, \eta_2) &\in \times \mathbb{R} \times \mathbb{R}^N. \end{aligned}$$

Then, BVP(1) has at least one classical solution $u \in C^{2+\mu}(\bar{\Omega})$.

Proof. We show that (F4) \implies (F3). Set $c = \max\{|f(x, 0, \mathbf{0})| : x \in \overline{\Omega}\} + 1$. For every $(x, \zeta, \eta) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, by Condition (F4), we have

$$\begin{aligned} |f(x, \zeta, \eta)| &\leq |f(x, \zeta, \eta) - f(x, 0, \mathbf{0})| + |f(x, 0, \mathbf{0})| \\ &\leq a|\zeta| + b|\eta| + c. \end{aligned}$$

Hence, f satisfies Condition (F3). By Theorem 5, BVP(1) has at least one classical solution in $C^{2+\mu}(\overline{\Omega})$.

Let $u_1, u_2 \in C^{2+\mu}(\overline{\Omega})$ be the solutions of BVP(1). Set $u = u_2 - u_1$ and $h = F(u_2) - F(u_1)$. By Assumption (F1)', $h \in C^\mu(\overline{\Omega})$. Since $u = u_2 - u_1 = Au_2 - Au_1 = S(F(u_2)) - S(F(u_1)) = Sh$, it follows that u is the classical solution of LBVP(13). By (14) and Condition (F4), we have

$$\begin{aligned} \|\nabla u\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \|\Delta u\|_2 = \frac{1}{\sqrt{\lambda_1}} \|h\|_2 = \frac{1}{\sqrt{\lambda_1}} \|F(u_2) - F(u_1)\|_2 \\ &\leq \frac{1}{\sqrt{\lambda_1}} (a\|u_2 - u_1\|_2 + b\|\nabla u_2 - \nabla u_1\|_2) \\ &= \frac{a}{\sqrt{\lambda_1}} \|u\|_2 + \frac{b}{\sqrt{\lambda_1}} \|\nabla u\|_2 \\ &\leq \left(\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} \right) \|\nabla u\|_2. \end{aligned}$$

Since $\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} < 1$, from this inequality, it follows that $\|\nabla u\|_2 = 0$. By (14), $\|u\|_2 = 0$, and hence, $u_1 = u_2$. This implies that BVP(1) has only one solution.

The proof of Theorem 6 is completed. \square

Theorem 7. Let $a, b \geq 0$ and $c > 0$ be constants, and $\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} < 1$. Then, the elliptic boundary value

$$\begin{cases} -\Delta u = a u + b |\nabla u| + c, & x \in \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{20}$$

has a unique positive classical solution $u \in C^{2+\mu}(\overline{\Omega}) \cap C^+(\overline{\Omega})$.

Proof. Consider the elliptic boundary value

$$\begin{cases} -\Delta u = a |u| + b |\nabla u| + c, & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{21}$$

Corresponding to BVP(1), the nonlinearity f of BVP(21) is given by

$$f(x, \zeta, \eta) := a |\zeta| + b |\eta| + c, \quad (x, \zeta, \eta) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N. \tag{22}$$

It is easy to verify that the function f defined by (22) satisfies Conditions (F1)' and (F4). Hence, by Theorem 6, BVP(21) has a unique solution $w_0 \in C^{2+\mu}(\overline{\Omega})$. Set

$$h(x) = f(x, w_0(x), \nabla w_0(x)), \quad x \in \overline{\Omega}; \tag{23}$$

then, w_0 is the classical solution of LBVP(13). Since $-\Delta w_0 = h > 0$, by the maximum principle of the elliptic operators, $w_0(x) > 0$ for every $x \in \Omega$. Hence, w_0 is a positive classical solution of BVP(20). On the other hand, the positive solution of BVP(20) is also a solution of BVP(21). By the uniqueness of the solution of BVP(21), w_0 is the unique positive classical solution of BVP(20). \square

3. Proofs of the Main Results

Proof of Theorem 1. Let a, b, H be the constants in the condition of Theorem 1. Choose a positive constant by

$$c = \max\{|f(x, \zeta, \eta)| : (x, \zeta, \eta) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N, |(\zeta, \eta)| \leq H\} + 1; \tag{24}$$

then, from Conditions (8) and (9), it follows that

$$f(x, \zeta, \eta) \leq a\zeta + b|\eta| + c, \quad (x, \zeta, \eta) \in \bar{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^N, \tag{25}$$

and

$$f(x, -\zeta, -\eta) \geq -a\zeta - b|\eta| - c, \quad (x, \zeta, \eta) \in \bar{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^N, \tag{26}$$

respectively. By Theorem 7, BVP(20) has a unique positive solution $w_0 \in C^{2+\mu}(\bar{\Omega})$. By Equation (20) and Inequality (25), we easily see that w_0 is an upper solution of BVP(1), and by (20) and (26), $-w_0$ is a lower solution of BVP(1). Since $-w_0 \leq w_0$, by Theorem 4, BVP(1) has at least one solution $u \in C^{2+\mu}(\bar{\Omega})$ between $-w_0$ and w_0 . \square

Proof of Theorem 2. Let c be the positive constant defined by (24) and w_0 be the unique positive solution of BVP(20). Then by the proof of Theorem 1, $w_0 \in C^{2+\mu}(\bar{\Omega})$ is a upper solution of BVP(1).

It is well-known that the minimum positive real eigenvalue λ_1 of the elliptic eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{27}$$

has a positive unit eigenfunction; that is, there exists a function $\varphi_1 \in C^2(\bar{\Omega}) \cap C^+(\bar{\Omega})$ with $\|\varphi_1\|_{C(\bar{\Omega})} = 1$ that satisfies the equation

$$\begin{cases} -\Delta\varphi_1 = \lambda_1 \varphi_1, & x \in \Omega, \\ \varphi_1|_{\partial\Omega} = 0. \end{cases} \tag{28}$$

Let δ be the constant in (10), and choose

$$\delta_0 = \min\{\delta/(1 + \|\nabla\varphi_1\|_{C^2})^{1/2}, c/\lambda_1\}. \tag{29}$$

Set $v_0 = \delta_0\varphi_1$. By the regularity of the solution of the linear equation LBVP(13), $v_0 \in C^{2+\mu}(\bar{\Omega})$. For every $x \in \bar{\Omega}$, since $v_0(x) \geq 0$ and

$$|(v_0(x), \nabla v_0(x))| \leq \delta_0 (|\varphi_1(x)|^2 + |\nabla\varphi_1(x)|^2)^{1/2} \leq \delta_0 (1 + \|\nabla\varphi_1\|_{C(\bar{\Omega})}^2)^{1/2} \leq \delta, \tag{30}$$

by the inequality (10) and Equation (28), we have

$$f(x, v_0(x), \nabla v_0(x)) \geq \lambda_1 v_0(x) = -\Delta v_0(x).$$

Hence, v_0 is a lower solution of BVP(1). We show that $v_0 \leq w_0$.

Consider the function $u = w_0 - v_0$. Since w_0 satisfies Equation (20) and v_0 satisfies Equation (28), it follows that

$$-\Delta u(x) = a w_0(x) + b|\nabla w_0(x)| + c - \lambda_1 v_0(x) \geq c - \lambda_1 \delta_0 \geq 0, \quad x \in \Omega.$$

Since $u|_{\partial\Omega} = 0$, by the maximum principle of the elliptic operators, $u(x) > 0$ for every $x \in \Omega$. Hence, $v_0 \leq w_0$.

Therefore by Theorem 4, BVP(1) has at least one solution $u_0 \in C^{2+\mu}(\bar{\Omega})$ between v_0 and w_0 that is a positive solution of BVP(1). \square

Proof of Theorem 3. Let c be the positive constant defined by (24) and w_0 be the unique positive solution of BVP(20). Then by the proof of Theorem 1, w_0 is an upper solution and $-w_0$ is a lower solution of BVP(1).

Let δ_0 be the positive constant defined by (29) and $v_0 = \delta_0 \varphi_1$. By the proof of Theorem 2, v_0 satisfies (30) and is a positive lower solution of BVP(1), and $v_0 \leq w_0$. By the inequalities (30) and (12), we have

$$f(x, -v_0(x), -\nabla v_0(x)) \leq -\lambda_1 v_0(x) = \Delta v_0(x) = -\Delta(-v_0(x)).$$

That is, $-v_0$ is an upper solution of BVP(1). Hence, we obtain two lower and upper solution pairs: (v_0, w_0) and $(-w_0, -v_0)$, respectively. By Theorem 4, BVP(1) has one solution, u_1 , between v_0 and w_0 , and has another solution, u_2 , between $-w_0$ and $-v_0$. Obviously, $u_1 \geq v_0$ is positive and $u_2 \leq -v_0$ is negative. \square

Example 1. Consider the elliptic boundary value problem

$$\begin{cases} -\Delta u = a u - u^2 + b |\nabla u| - u |\nabla u|^2, & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{31}$$

Clearly, 0 is a trivial solution of BVP(31). Let $a > \lambda_1$ and $b \leq \sqrt{\lambda_1}$. We use Theorem 2 to prove that BVP(31) has a positive classical solution.

Corresponding to BVP(1), the nonlinearity of BVP(31) is

$$f(x, \xi, \eta) = a\xi - \xi^2 + b|\eta| - \xi|\eta|^2. \tag{32}$$

Obviously, f satisfies (F1) and (F2). Choose

$$\varepsilon = \frac{1}{2} \frac{\lambda_1 - \sqrt{\lambda_1} b}{1 + \sqrt{\lambda_1}}, \quad a_1 = \varepsilon, \quad b_1 = b + \varepsilon, \quad H = \frac{a^2}{4\varepsilon}.$$

Then,

$$\frac{a_1}{\lambda_1} + \frac{b_1}{\sqrt{\lambda_1}} = \left(\frac{1}{\lambda_1} + \frac{1}{\sqrt{\lambda_1}}\right) \varepsilon + \frac{b}{\sqrt{\lambda_1}} < \left(\frac{1}{\lambda_1} + \frac{1}{\sqrt{\lambda_1}}\right) \frac{\lambda_1 - \sqrt{\lambda_1} b}{1 + \sqrt{\lambda_1}} + \frac{b}{\sqrt{\lambda_1}} = 1;$$

when $|(\xi, \eta)| > H$, by (32), we have

$$\begin{aligned} f(x, \xi, \eta) &\leq a\xi - \xi^2 + b|\eta| \leq \frac{a^2}{4} + b|\eta| = H\varepsilon + b|\eta| \\ &\leq \varepsilon|(\xi, \eta)| + b|\eta| \leq \varepsilon|\xi| + (\varepsilon + b)|\eta| \\ &= a_1|\xi| + b_1|\eta|. \end{aligned}$$

Hence, f satisfies (8) for a_1 and b_1 restricted by (7). On the other hand, choose

$$\delta = \min\left\{1, \frac{a - \lambda_1}{2}\right\}.$$

Then, when $\xi \geq 0$ and $|(\xi, \eta)| \leq \delta$, by (32)

$$\begin{aligned} f(x, \xi, \eta) &\geq \lambda_1 \xi + \xi(a - \lambda_1 - \xi - |\eta|^2) \\ &\geq \lambda_1 \xi + \xi(a - \lambda_1 - |(\xi, \eta)| - |(\xi, \eta)|^2) \\ &\geq \lambda_1 \xi + \xi(a - \lambda_1 - 2|(\xi, \eta)|) \\ &\geq \lambda_1 \xi. \end{aligned}$$

Hence, f satisfies (10). By Theorem 2, BVP(31) has at least one positive solution. This result cannot be obtained from [14,15].

Author Contributions: Y.L. formulated the methodology; Y.L. and W.M. carried out the first draft of this manuscript; Y.L. prepared the final version of the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundations of China under grant numbers 12061062 and 11661071.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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