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Unanimity, Coexistence, and Rigidity: Three Sides of Polarization

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Abstract: Political polarization is perceived as a threat to democracies. Using the Galam model of opinion dynamics deployed in a five-dimensional parameter space, I show that polarization is the byproduct of an essential hallmark of a vibrant democratic society, namely open and informal discussions among agents. Indeed, within a homogeneous social community with floaters, the dynamics lead gradually toward unanimity (zero entropy). Polarization can eventually appear as the juxtaposition of non-mixing social groups sharing different prejudices about the issue at stake. On the other hand, the inclusion of contrarian agents produces a polarization within a community that mixes when their proportion x is beyond a critical value $x_c = \frac{1}{6} \approx 0.167$ for discussing groups of size three and four. Similarly, the presence of stubborn agents also produces a polarization of a community that mixes when the proportion of stubborn agents is greater than some critical value. For equal proportions of stubborn agents a along each opinion, $a_c = \frac{2}{9} \approx 0.22$ for group size four against $a_c = \frac{1}{4} = 0.25$ for group size three. However, the evaluation of the proportion of individual opinion shifts at the attractor $\frac{1}{2}$ and indicates that the polarization produced by contrarians is fluid with a good deal of agents who keep shifting between the two opposed blocks (high entropy). That favors a coexistence of opposite opinions in a divided community. In contrast, the polarization created by stubborn agents is found to be frozen with very few individuals shifting opinion between the two opinions (low entropy). That yields a basis for the emergence of hate between the frozen opposed blocks.

Keywords: sociophysics; polarization; opinion dynamics; prejudices; stubbornness; contrarians

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Citation: Galam, S. Unanimity, Coexistence, and Rigidity: Three Sides of Polarization. *Entropy* **2023**, *25*, 622. https://doi.org/10.3390/e25040622

Academic Editor: Francisco W. De Sousa Lima

Received: 6 March 2023
Revised: 2 April 2023
Accepted: 4 April 2023
Published: 6 April 2023

1. Introduction

In the last years, the phenomenon of political polarization has become an issue of major concern among scholars, pundits, journalists, and politicians [1–7]. Indeed, the current polarization of modern societies is often perceived as a direct and immediate threat to the stability of democratic societies. Both the 2021 American [8] and 2023 Brazilian [9] elections as well as the Israeli 2023 crisis related to the will of reforming the judicial system [10] have enlightened the reality of this fear by exhibiting countries divided into two almost equal, opposite parts, irreconcilable and hating each other. When the hateful trait is present, the polarization is often referred to as affective polarization [3,11]. Polarization also emerges on non-political but societal issues or challenges such as global warming, Brexit, nuclear energy, and secularism.

While the issue of understanding the phenomenon of polarization has attracted a great deal of work, there is still no consensus among the researchers working on the topic as to what causes polarization to occur in a given population [12–14]. A good part of the work addresses the issue from the perspective of the dynamics of opinion within sociophysics [15–18]. Most related papers consider binary variables [19–42], with a few having three discrete opinions [43,44]. Among the numerous models is the Galam model, which has been deployed in a multi-dimensional space of parameters to study the competition between discrete opinions [45–54].

In this paper, I review and extend the Galam model of opinion dynamics in connection to the emergence of polarization. Introducing a novel quantifier which calculates the proportions of agents shifting opinion at a given moment, I found that polarization is a threefold feature, which I denote unanimity, coexistence, and rigidity.

The origin of each side of the opinion update scheme is found to stem from heterogeneity among the psychological characters of the agents composing the social group when engaged in a debate about a collective issue. To date, three particular psychological traits have been included with floaters, contrarians, and the stubborn.

- **The floaters**: floaters are agents who have an opinion and argue and vote for it, but eventually they may shift to the opposite one when being in the minority in a local discussion group [47–50].
- **The contrarians:** contrarians are agents who have an opinion and argue and vote for it but eventually shift to the opposite one when being in the majority in their local discussion group. The shift is independent of the opinions themselves [51].
- **The stubborn:** the stubborn are agents who do have an opinion and argue and vote for it, but contrary to other agents, they stick to their initial opinions whatever the composition of their local discussion group [52–54].

The various associated effects on the opinion dynamics can be investigated thanks to a universal formula for the updates of opinions. The formula has been derived within a parameter space of five dimensions which are the size of the discussion groups, the proportion of contrarians, the two proportions of stubborn agents for respective opinions, and the distribution of prejudices in favor of each opinion [45].

The corresponding multi-dimensional phase diagram is rather rich with a combination of both tipping point dynamics and single attractor dynamics as a function of the values of the five parameters. Polarization, unanimity, and coexistence are then recovered as respective deviations from one another as a function of the respective proportions of each psychological subgroup. Those deviations are pinned by the psychological traits which deviate from the associated dynamics towards unanimity.

New results are obtained with respect to the nature of a polarized stable state. In particular, floaters are found to produce segregated polarization (zero entropy), contrarians produce a fluid polarization (high entropy), and the stubborn produce a frozen polarization (low entropy).

The rest of the paper is organized as follows: Section 2 sets the ground of opinion dynamics with the various ingredients used in the paper. The spontaneous drive towards democratic unanimity is outlined in Section 3. The prejudices breaking a perfect democratic dynamics are reviewed in Section 4, while Section 5 investigates the effect of having contrarian agents on the related dynamics of opinion. The stubborn agents are introduced in Section 6 followed by some conclusions.

2. Opinion Dynamics, Definitions, and Reality

The phenomenon of polarization is mainly used to describe large communities of agents which, over some period of time, are divided into two opposite groups, each considering the other as extreme. These extreme societal visions are generally deployed along a global societal project.

To address this phenomenon and account for the above definition, I consider a community of people who have to decide between two opposite choices denoted A and B. Before the launching of the collective campaign, each agent has reached a choice, either A or B according to its own values, experiences, and visions. These individual alignments yield initial proportions p_0 and $(1 - p_0)$ of agents holding, respectively, opinions A and B. What made each agent's initial choice is out of the scope of the present work. I only assume that agents are aware of what motivates their respective choices and will argue to promote them among those who have made the opposite choice when debating the topic in an informal social meeting.

Moreover, the model assumes that people discuss informally the issue at stake in small groups during social events, such as dinners, lunches, coffees, drinks, commuting, and more. Even during large gatherings of people, such as, for instance, at a wedding, when the assembly divides into small groups.These ongoing and repeated encounters in small groups shape individual opinions, which end up, over time, as aggregated collective opinions.

The debate will eventually modify the proportions p_0 and $(1 - p_0)$ to new values p_T and $(1 - p_T)$, where T is a function of the time at which either a vote is taking place or people stop debating the topic to eventually start on a new one. On this basis, I define three states of polarization.

1. Unanimity: when p_T is either equal to 1 or very large, around 0.80, as well as equal to 0 or very low, around 0.20, I define the associated state as unanimity. Most agents share opinion A in the first case and opinion B in the second one. The values 0.80 and 0.20 are chosen arbitrarily to set a boundary beyond which one opinion overwhelms the other. In reality, these values fluctuate but preserve the feeling of a landslide victory. Having an overwhelming majority of agents who share the same opinion against a small minority holding the other opinion makes the related entropy small and even zero in cases $p_T = 1$ and $p_T = 0$.
2. Coexistence: when p_T is of the order of 0.50 ± 0.03, i.e., 0.53 and 0.47, I define the associated state as coexistence if and only if a substantial part of the population keeps shifting opinion without modifying the overall proportions p_T and $(1 - p_T)$. It means that the global opinion has reached an attractor located around 0.50, but individual choices are not frozen with noticeable parts of the agents who keep shifting opinions. The value ± 0.03 is chosen arbitrarily to set a fuzzy boundary around 0.50. In reality, these values fluctuate a bit but preserve the feeling of a hung outcome in the case of an election. The high level of ongoing shift of individual opinion puts the associated entropy at a high value.
3. Rigidity: when p_T is of the order of 0.50 ± 0.03, i.e., 0.53 and 0.47, I define the associated state as rigidity if and only if the stable global opinion around 0.50 is frozen at individual choices. No noticeable part of the agents keeps shifting opinions. The choices of ± 0.03 are chosen arbitrarily to set a boundary around 0.50. In reality, these values fluctuate a little while preserving the feeling that the winner has stolen its victory from the competitor in the event of an election. In this case, the low level of individual opinion shifts produces a low entropy.

At this stage, to avoid any misunderstanding, I would like to emphasize that the results obtained from the model should not be taken literally. They are intended to be indicators of the hidden trends which drive the dynamics of the social and political reality the model aims to describe.

3. The Spontaneous Drive towards Democratic Unanimity

The Galam model of opinion dynamics operates in three successive steps, which are repeated a number, T, of times. First, agents are distributed randomly in small groups. Second, majority rules are applied simultaneously in each group to update locally the opinions of agents. Third, all agents are reshuffled.

When the population is homogeneous, being composed of only floaters, all agents who are a minority in a group shift opinion to adopt the one having gained the vote majority [47–49].

For the sake of readiness and analytical solving, I restrain the review of update groups of size $r = 4$. A random distribution of A and B agents in a group of 4 leads to $2^4 = 16$ possible configurations; 5 configurations have a majority of A, 5 have a majority of B, and 6 have 2 A and 2 B. Majority rule attributes the first five to A and the last five to B. The 6 configurations with a tie do not have a majority, and a physicist would assume that in this case no update occurs, keeping 2 A and 2 B.

Accordingly, starting with an initial proportions p_0 and $(1 - p_0)$ of agents holding respectively opinions A and B, one update cycle leads to new proportions p_1 and $(1 - p_1)$, where p_1 is given by

$$p_1 = p_0^4 + 4p_0^3(1 - p_0) + 3p_0^2(1 - p_0)^2. \quad (1)$$

The first two terms of Equation (1) account for a majority of A among the 4 agents yielding 4 A, and the last term accounts for the unchanged tie configurations 2 A and 2 B.

After one cycle of updates, agents are reshuffled and distributed again randomly in groups of size four, leading to a proportion p_2 obtained from Equation (1) applied to p_1. Repeating the process yields a tipping point dynamic, as seen by solving the fixed point equation $p_1 = p_0$, which yields two attractors $p_A = 1$ and $p_B = 0$ separated by a tipping point in between located at $p_c = \frac{1}{2}$.

At this stage, it is worth noting that expanding Equation (1) yields $p_1 = -2p^3 + 3p^2$, which is equal to $p_1 = p^3 + 3p^2(1 - p)$, which turns out to be the update Equation for groups of size three. Keeping invariant the tie configuration makes update groups of four agents identical to the update groups of three.

The tipping point $p_c = \frac{1}{2}$ makes an iteration of Equation (1) to produce a series $p_0 < p_1 < \cdots < p_T \rightarrow 1$ when $p_0 > 0.50$. For $p_0 < 0.50$, the series is $p_0 > p_1 > \cdots > p_T \rightarrow 0$. Therefore, the initial majority of aggregated opinions convinces the minority through local and repeated discussions leading towards unanimity, with all agents sharing the same opinion provided enough updates have been made. In this case, at the attractor, the related entropy is thus zero.

Provided that people keep discussing for a sufficient time, I obtain a representation of an ideal perfect democracy. Through informal and open-minded discussions, conflicts between opinions have disappeared with no more difference among the opinions of agents. No additional arguing is required, with everyone sharing the same opinion via a rational process, which made the opinion initially supported by the majority of the agents prevail. The unanimity which has emerged is thus democratic.

It is of importance to underline that in connection to reality, one cycle of opinion update driven by Equation (1) is the equivalent of an average of several local discussions in the real world. The number of these real discussions is a matter of intensity of the ongoing campaign.

4. Prejudices Unconsciously Break the Perfect Democratic Dynamics

However, even within an ideal society, perfection does not exist. Paradoxically, rationality may produce local collective doubts by gathering arguments. The result is two different choices that seem equally valid given the arguments for and against each. Indeed, such a situation appears quite naturally within the model in the tie configurations 2A–2B.

In case of a tie, the agents thus decide either A or B, by chance, as in a coin toss. No rational argument is evoked in the selected choice. They could have equally selected the other one. Mathematically, this tie breaking yields an identical contribution $3p_0^2(1 - p_0)^2$ to Equation (1). However, now at a tie, instead of no update implemented with 2A–2B, all 4 agents choose either A (4A) or B (4B) with equal probabilities.

At this point, I introduce a fundamental assumption to incorporate the human character of agents as opposed to the processing of atoms. I assume that in the event of a tie, all four agents doubt and that this state of doubt unconsciously opens the door to an invisible bias that guides their choice. Therefore, when the group selects "by chance" one choice over the other, the "chance" is being biased along the leading prejudice, which is activated by the issue at stake.

I account for that effect by allocating the group choice to A with probability k and to B with probability $(1 - k)$ [50]. The value of k is a function of the distribution of prejudices, which are in tune with opinion A among the agents. The associated update Equation (1) becomes

$$p_{1,k} = p_0^4 + 4p_0^3(1 - p_0) + 6kp_0^2(1 - p_0)^2, \quad (2)$$

which yields the same two attractors $p_{A,k} = 1$ and $p_{B,k} = 0$ as above, but now the tipping point is located at

$$p_{c,k} = \frac{(6k-5) + \sqrt{13 - 36k + 36k^2}}{6(2k-1)}. \quad (3)$$

Equation (3) gives $p_{c,0} = \frac{5-\sqrt{13}}{6} \approx 0.23$, $p_{c,\frac{1}{2}} = \frac{1}{2}$ and $p_{c,1} = \frac{1+\sqrt{13}}{6} \approx 0.77$ for respectively $k = 0, \frac{1}{2}, 1$. Therefore, $0 \leq k \leq \frac{1}{2} \Rightarrow 0.23 \ \leq p_{c,k} \leq \frac{1}{2}$ and $\frac{1}{2} \leq k \leq 1 \Rightarrow \frac{1}{2} \leq p_{c,k} \leq 0.77$.

The case $k = 1$ illustrates the phenomenon of minority spreading. Opinion A, being favored by the group prejudice, needs to gather an initial minority proportion of only 0.23 to convince the initial majority of agents who share opinion B to adopt instead opinion A via local and open mind discussions. The previous democratic character of the dynamics has been broken naturally and unconsciously without notice. No coercion has been used.

4.1. Segregated Polarization

While the prejudice effect preserves the dynamic of unanimity, it produces a democratic breaking at the advantage of the opinion in tune with the leading prejudice of a social community. This opinion is either initially held by a minority or a majority of agents.

However, within the same country, different social communities are spread over with their respective members not mixing together in social meetings where informal discussions are being held, and it often happens that in these different communities different prejudices are activated by the same issue at stake. As a result, opposite opinions may end up to spread along opposite unanimities in adjacent communities, creating a de facto stable polarization at a higher aggregated level.

Nevertheless, this segregated frozen polarization is not perceived as a threat or a problem since it subscribes to the specific features which make what differentiates those communities. Moreover, it is usually not used as a background to reach power at global levels, which overrules all communities.

4.2. Combining Groups of Different Sizes

I have restricted the update Equations to groups of sizes three and four, but in practice people discuss in groups of different sizes s. On this basis, it is possible to consider a distribution of group sizes from $s = 1$ to $s = L$, where size 1 accounts for agents who do not discuss during one update, and L is the larger size of group discussion. L is rarely larger than five or six since informal larger groups always split spontaneously into smaller groups.

For a distribution of size s with respective proportions g_s, Equation (1) becomes

$$p_{L,1} = \sum_{s=1}^{L} \left\{ g_s \left[\sum_{l=\bar{s}+1}^{s} \binom{s}{l} p_0^l (1-p_0)^{s-l} + \delta[\bar{s} - \frac{s}{2}] k \binom{s}{s/2} p_0^{\frac{s}{2}} (1-p_0)^{\frac{s}{2}} \right] \right\}, \quad (4)$$

where $\bar{s} \equiv [\frac{s}{2}]$ with $[\dots]$ meaning the integer part, $\delta[\dots]$ is the Kronecker function and

$$\sum_{s=1}^{s=L} g_s = 1. \quad (5)$$

The two attractors $p_{A,k} = 1$ and $p_{B,k=0}$ obtained from Equation (2) are also the attractors associated with Equation (4). However, the value of the tipping point $p_{c,k}$ is now modified as expected. On the one hand, larger even sizes reduce the probability of occurrence of a tie shifting $p_{c,k}$ closer to $\frac{1}{2}$, but on the other hand, for size two, the tipping point is located at 0 for $k = 1$ and at 1 for $k = 0$. These two opposite effects show that having a combination of sizes for small groups will not modify qualitatively the results obtained for sizes " and four. For instance, choosing $g_1 = 0.20, g_2 = 0.30, g_3 = 0.20, g_4 = 0.20, g_5 = 0.10$ yields $p_{c,0} = 0.85$ and $p_{c,1} = 0.15$ instead of $p_{c,0} = 0.77$ and $p_{c,1} = 0.23$ for size four.

5. Contrarians Fuel Coexistence

In the case of no prejudice effect ($k = \frac{1}{2}$), the inclusion of a proportion x of contrarians makes the equation (2) written as [51]

$$p_{1,x} = (1-2x)\left[p_0^4 + 4p_0^3(1-p_0) + 3p_0^2(1-p_0)^2\right] + x, \tag{6}$$

which yields three fixed points, $p_{c,x} = \frac{1}{2}$ and

$$p_{A,x;B,x} = \frac{-1+2x \pm \sqrt{1-8x+12x^2}}{2(-1+2x)}. \tag{7}$$

with the last two being valid only in the range $0 \leq x < x_c = \frac{1}{6} \approx 0.167$ and $x \geq \frac{1}{2}$. However, for $x \geq \frac{1}{2}$, the two values are not valid since there $p_{A,x} > 1$ and $p_{B,x} < 0$.

The associated dynamics are identified using the parameter $\lambda_x = \frac{p_{1,x}}{dp_0}|_{\frac{1}{2}} = \frac{3}{2}(1-2x)$ to determine the stability of $p_{c,x}$. The fixed point $p_{c,x} = \frac{1}{2}$ is thus a tipping point when $\lambda_x < -1$ and $\lambda_x > 1$, making $p_{A,x;B,x}$ the two attractors of the dynamics. For $-1 < \lambda_x < 1$, $p_{c,x}$ is an attractor, which implies that in this range $p_{A,x;B,x}$ does not exist. Accordingly, to obtain the associated phase diagram, I solve the inequality which makes $p_{c,x}$ an attractor,

$$-1 < \frac{3}{2}(1-2x) < 1, \tag{8}$$

which is identical to

$$\frac{1}{6} < x < \frac{5}{6}. \tag{9}$$

Four distinct regions with different behaviors are obtained from Equation (9), as shown in Figures 1 and 2.

- **Region 1** lies within the range $0 \leq x < x_c$, featuring tipping point dynamics with $p_{c,x} = \frac{1}{2}$ being the tipping point. The initial majority is increased by the repeated cycles of local discussions with a monotonic convergence towards the relevant attractor, either $p_{A,x}$ when $p_0 > \frac{1}{2}$ or $p_{B,x}$ when $p_0 < \frac{1}{2}$. In the first case, A wins the public debate or the related vote but loses in the second case. In both cases, a core minority B (A) subsists against the majority A (B). The two attractors move towards each other with increasing x towards x_c.
- **Region 2** starts at x_c where the two attractors $p_{A,x}$ and $p_{B,x}$ merge and disappear at $p_{c,x}$, turning the tipping point $p_{c,x}$ into an attractor. The dynamics shifts suddenly from a tipping point one to a single attractor dynamics. In the range $x_c \leq x \leq \frac{1}{2}$, any initial proportion p_0 is moved monotonously by the update dynamics towards $\frac{1}{2}$, i.e., an equal proportion of agents hold opinions A and B. We have a perfect stable coexistence of both competing opinions in the range $x_c \leq x \leq \frac{1}{2}$.
- **Region 3** marks the transition to a situation where contrarians are more numerous than floaters with $x > \frac{1}{2}$. Due to this fact, while $p_{c,x} = \frac{1}{2}$ remains an attractor, reaching it follows an oscillatory convergence. The oscillatory convergence holds in the range $\frac{1}{2} < x < \frac{5}{6}$. Once the attractor has been reached, the two competing opinions coexist at equal proportions as Region 2.
- **Region 4** is the counter part of region 1 where $p_{c,x} = \frac{1}{2}$ is again a tipping point, but now, the very high proportion of contrarians turn the dynamics into an oscillating divergence from the tipping point instead of a monotonic divergence. In addition, once an attractor has been reached, the dynamics become oscillating between $p_{A,x}$ and $p_{B,x}$. Region 4 extends in the range $\frac{5}{6} < x \leq 1$.

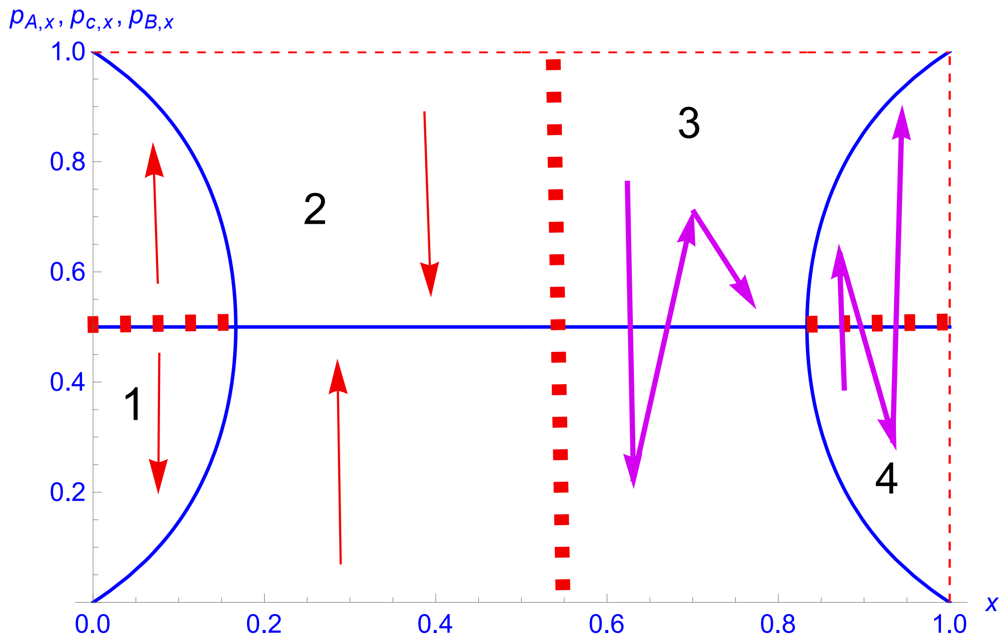

Figure 1. Contrarians produce four distinct regions with different behaviors as a function of their proportion. A tipping point dynamic with $p_{c,x} = \frac{1}{2}$ prevails in the range $0 \leq x < x_c$ (region 1). The two associated attractors feature a stable coexistence of a majority and a minority. In the range $x_c \leq x \leq \frac{1}{2}$, the dynamics turn into a one-attractor dynamic located at $\frac{1}{2}$. Any initial support p_0 moves monotonously towards $\frac{1}{2}$ with repeating local updates (region 2). There, both opinions coexist in a perfect overall balance. When $\frac{1}{2} < x < \frac{5}{6}$ the dynamics are still monitored by one attractor at $\frac{1}{2}$, but the convergence towards it becomes oscillatory (region 3). The fourth region extends in the range $\frac{5}{6} < x \leq 1$. The dynamic returns to a tipping point one but with oscillatory dynamics between the two attractors (region 4).

5.1. *The Polarization at* $p_{c,x} = \frac{1}{2}$ *Is Fluid*

In Regions 2 and 3, the stable state is a perfect equality between the respective numbers of agents holding opinions A and B. The community is thus divided into opposite parts, which in turn could lead to feature it as a polarized community. However, that could be misleading since the two opposite parts are not two frozen opposite parts.

Indeed, contrarians make the division fluid with a good number of agents constantly moving from one side to the other but in equal proportions. I thus denote that fluid polarized state as coexistence.

To quantify the degree of fluidity of coexistence, I introduce four new quantities

$$\begin{aligned}
S_{A,p,x_-} &= (1-x)\left[p^3(1-p) + \frac{3}{2}p^2(1-p)^2\right], \\
S_{A,p,x_+} &= x\left[(1-p)^4 + 3p(1-p)^3 + \frac{3}{2}p^2(1-p)^2\right], \\
S_{B,p,x_-} &= (1-x)\left[p(1-p)^3 + \frac{3}{2}p^2(1-p)^2\right], \\
S_{B,p,x_+} &= x\left[p^4 + 3p^3(1-p) + \frac{3}{2}p^2(1-p)^2\right],
\end{aligned} \tag{10}$$

which are, given proportions p and x, the proportions of agents shifting opinions from B to A and from A to B. These shifts are triggered respectively by local majorities (S_{A,p,x_-}, S_{B,p,x_-}) and contrarians (S_{A,p,x_+}, S_{B,p,x_+}) during one update.

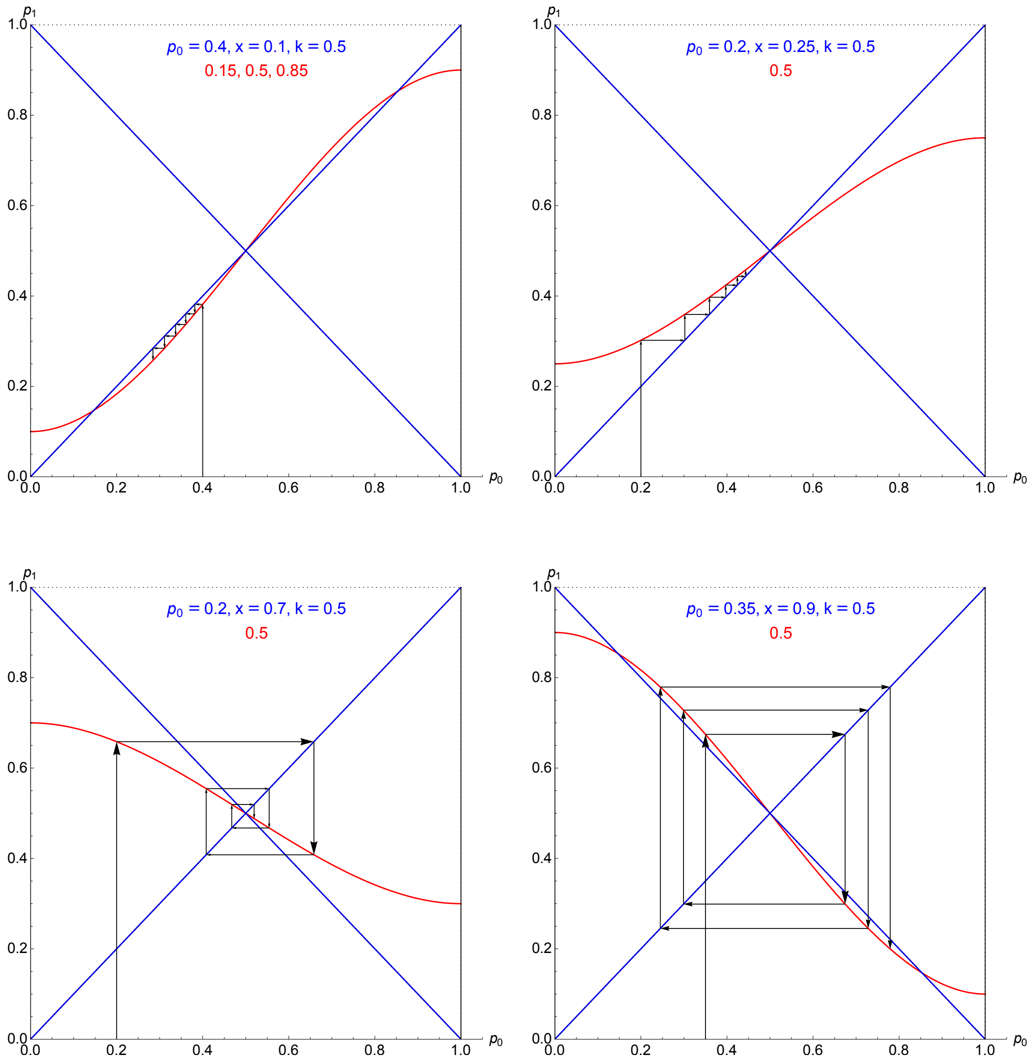

Figure 2. Illustration of the dynamics of opinions in each one of the four regions produced by contrarians with respectively $x = 0.10, p_0 = 0.40$ (**upper left**), $x = 0.25, p_0 = 0.20$ (**upper right**), $x = 0.70, p_0 = 0.20$ (**lower left**), $x = 0.90, p_0 = 0.35$ (**lower right**). All cases have $k = \frac{1}{2}$. The respective values of the attractors and tipping points are indicated (in red).

The total proportion of agents shifting opinion from B to A during one update is thus $S_{A,p,x} = S_{A,p,x_-} + S_{A,p,x_+}$ and $S_{B,p,x} = S_{B,p,x_-} + S_{B,p,x_+}$ from A to B. The resulting total proportion of shifts is $S_{T,p,x} = S_{A,p,x} + S_{A,p,x}$. With only floaters (no contrarians), the associated values are $S_{A,p,0}, S_{A,p,0}, S_{T,p,0}$ at $k = \frac{1}{2}$.

Figure 3 shows the variations of theses quantities as a function of p for $x = 0.20$ and $x = 0.65$, which are located respectively in Regions 2 and 3, where $p_{c,x} = \frac{1}{2}$ is the attractor of the dynamics.

- The top parts exhibit the magnitudes of the shifts with respect to opinion A where S_{A,p,x_-} (in red) is the gain from local majority rule diminished by the contrarians. The gain from the loss of local majorities favorable to B is S_{A,p,x_+} (in blue). The net gain for A is $S_{A,p,x}$ (in green). When only floaters are discussing, $S_{A,p,0}$ (in red dashes) is the gain for A.
- The middle part exhibits the magnitudes of all shifts at the benefit of A ($S_{A,p,x}$ in red) and B ($S_{B,p,x}$ in blue) as well as the total shifts accounting for both A and B ($S_{T,p,x} = S_{A,p,x} + S_{A,p,x}$ in green). This total is also shown in the absence of contrarians with $S_{T,p,0}$ (in red dashes).

- The bottom part exhibits the magnitude of the difference $S_{A,p,x} - S_{A,p,x}$ (in red) in proportions of shifts at the benefit of respectively A and B. The equivalent $S_{A,p,0} - S_{A,p,0}$ (in blue) with only floaters is also shown.

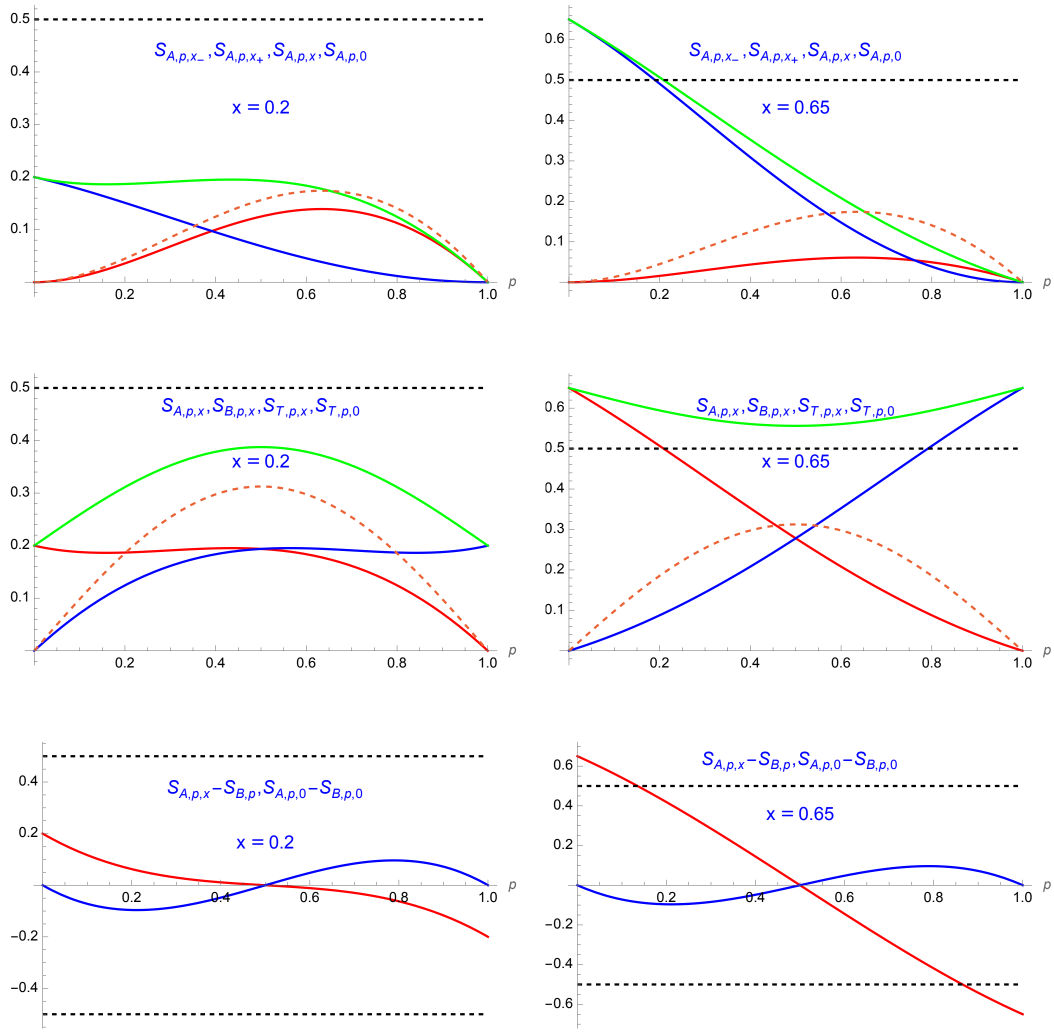

Figure 3. Given $x = 0.20$ (region 2, **left** part) and $x = 0.65$ (region 3, **right** part), the Figure shows the variations of $S_{A,p,x_-}, S_{A,p,x_+}, S_{A,p,x}, S_{A,p,0}$ (**upper** part), $S_{A,p,x}, S_{B,p,x}, S_{T,p,x}, S_{T,p,0}$ (**middle** part), $S_{A,p,x} - S_{B,p,x}, S_{A,p,0} - S_{B,p,0}$ (**lower** part), as a function of p.

To label the nature of the polarized state at the attractor, I evaluate the total proportion of shifts at $p = p_{c,x} = \frac{1}{2}$ as a function of x. From Equation (10), the associated proportions of individual shifts are $S_{A,\frac{1}{2},x} = S_{B,\frac{1}{2},x} = \frac{5+6x}{32}$ and $S_{T,\frac{1}{2},x} = \frac{5+6x}{16}$. It yields 0.388 and 0.556 for $x = 0.20$ and $x = 0.65$, respectively. The attractor is thus marked by a significant proportion of individual shifts between A and B. Despite the division of the community into two opposite halves, the high fluidity between the groups prevents the setting of hate between them. On this basis, I label that polarization as coexistence. The associated entropy is high.

It is worth noting that, having $S_{T,\frac{1}{2},0.65} = 0.556$ points, above fifty percent of contrarians, the magnitude of individual shifts becomes smaller than the proportion of contrarians as expected with more contrarians than floaters and shown in the upper part of Figure 4.

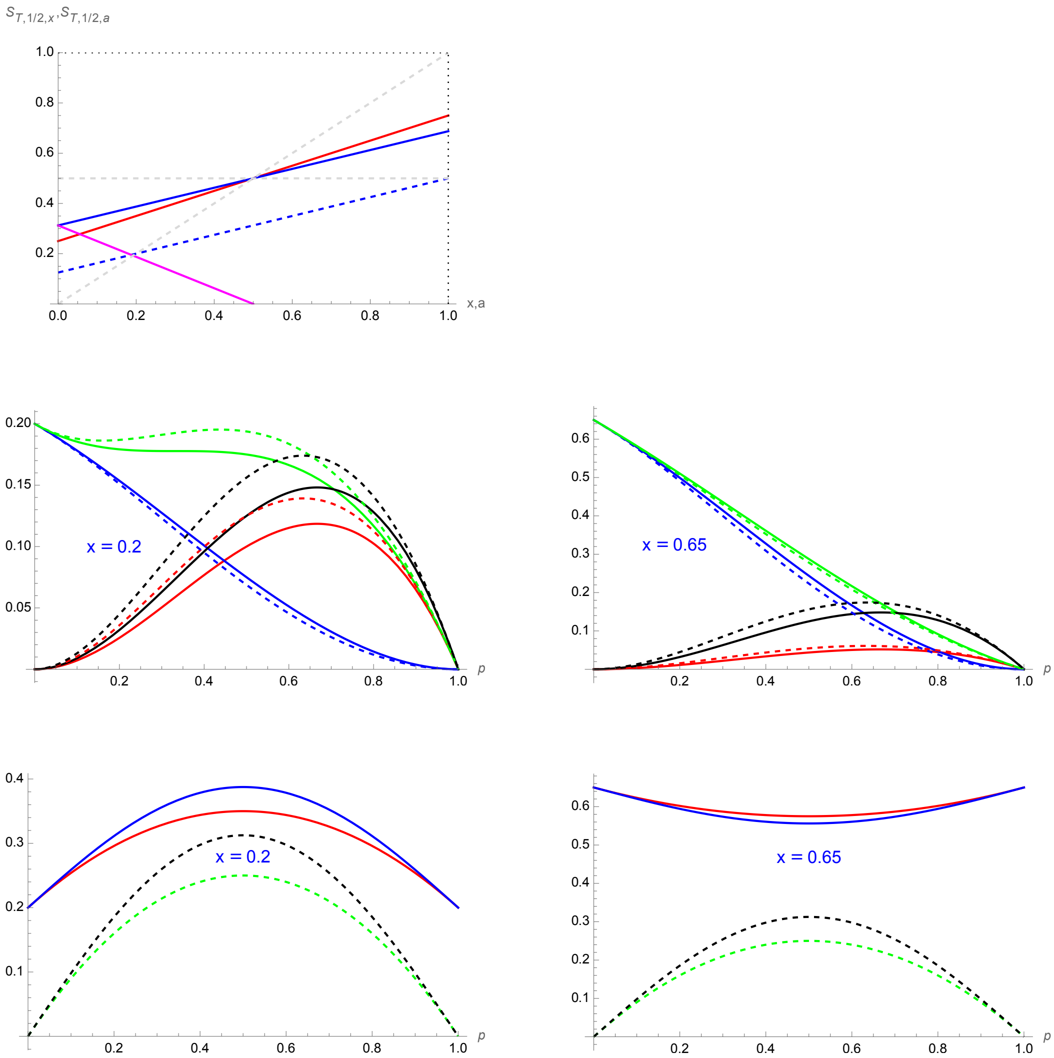

Figure 4. The upper part shows $S_{T,\frac{1}{2},x}$ with $\frac{5+6x}{16}$ for size 4 (in blue), $\frac{1+3x}{8}$ for size 4 with no change at ties (in dotted blue), $\frac{1+2x}{4}$ for size 3 (in red) and $S_{T,\frac{1}{2},a} = \frac{5}{8}(\frac{1}{2} - x)$ for stubborn agents studied below (in magenta). The middle part shows $S_{A,p,x_-}, S_{A,p,x_+}, S_{A,p,x}$, for size 3 (in red, blue, green) and 4 (in dashed red, dashed blue, dashed green) at respectively $x = 0.20$ and $x = 0.65$. The quantities at $x = 0$ are added in both cases (in black for size 3 and dashed black for size 4).

5.2. The Magnitude of Fluidity Is a Function of the Discussion Group Size

I showed in Section 3 that keeping invariant the tie configuration makes the update rule identical for both groups of four and three agents. However this equality does not hold with respect to the proportions of individual opinion shifts at the attractor $p_{c,x} = \frac{1}{2}$. Indeed, for groups of size three, Equation (10) is

$$\begin{aligned} S_{A,p,x_-} &= (1-x)p^2(1-p), \\ S_{A,p,x_+} &= x\Big[(1-p)^3 + 2p(1-p)^2\Big], \\ S_{B,p,x_-} &= (1-x)p(1-p)^2, \\ S_{B,p,x_+} &= x\Big[p^3 + 2p^2(1-p)\Big], \end{aligned} \tag{11}$$

yielding $S_{T,\frac{1}{2},x} = \frac{1+2x}{4}$, giving 0.350 and 0.575 for $x = 0.20$ and $x = 0.65$, respectively.

Middle and lower parts of Figure 4 exhibit the proportions of individual shifts as a function of p for groups of size three and four with $x = 0$, $x = 0.20$, and $x = 0.65$.

The results indicate that larger groups of discussion increase the magnitude of individual shifts and thus enlarge the coexistence part of the divided community.

However, it is worth stressing that in the case of size four, keeping the tie 2A2B unchanged implies a weaker magnitude of individual shifts against turning the tie 2A2B into either 4A or 4B with equal probabilities $\frac{1}{2}$. In the first case, at the attractor, the magnitude of shifts is $\frac{1+3x}{8}$ versus $\frac{5+6x}{16}$ for the second case. For groups of size three, the magnitude is $\frac{1+2x}{4}$. Table 1 shows the respective magnitudes as a function of x and for $x = 0.20$ and $x = 0.65$.

Table 1. Total proportion of individual shits $S_{T,\frac{1}{2},x}$ at the attractor $p_{c,x} = \frac{1}{2}$ for sizes 3, 4 with tie-breaking and 4 with unchanged tie. The related values are also shown for $x = 0.20$ and $x = 0.65$.

Size	3	4	4 Unchanged
$S_{T,\frac{1}{2},x}$	$\frac{1+2x}{4}$	$\frac{5+6x}{16}$	$\frac{1+3x}{8}$
$S_{T,\frac{1}{2},0.20}$	0.35	0.388	0.20
$S_{T,\frac{1}{2},0.65}$	0.575	0.556	0.369

Thus, when nothing happens at a tie, groups of size four involve less shifts than groups of size three. Indeed, in the unchanged tie case, contrarians are neutralized at a tie by the absence of local majority. The proportion of groups at a tie being $6p^2(1-p)^2$, its value at the attractor $p = \frac{1}{2}$ is significant with $\frac{3}{8} = 0.375$.

Variations of the different magnitudes of individual shifts as a function of x are shown in the upper part of Figure 4. All three cases exhibit a good deal of fluidity between the two opposite halves of the community. Contrarians add to the fluidity of the polarized state by their own shifts of opinion.

6. Stubbornness Produces Polarization

Denoting a and b the respective proportions of stubborn agents along opinions A and B and keeping a balanced prejudice effect with $k = \frac{1}{2}$, update Equation (1) is written [52],

$$\begin{aligned} p_{1,a,b} =& p_0^4 + 4p_0^3(1-p_0) + 3p_0^2(1-p_0)^2 \\ &+ \frac{a}{2}(1-p_0)^2(2+p_0) - \frac{b}{2}p_0^2([3-p_0). \end{aligned} \tag{12}$$

While Equations (1) and (6) have one parameter each, k and x, Equation (12) has two parameters, a and b, which makes it richer with respect to its dynamics. Figure 5 illustrates the various associated regimes exhibiting the interplay of attractors and tipping points as a function of a for $b = 0, 0.15, 0.20, 0.25$. Two regimes are found.

- **Regime 1** The first regime is shown in Figure 5 for $b = 0, 0.15, 0.20$, where two dynamics are taking place. For small values of a, the dynamic is a tipping point dynamic, but the associated region shrinks with increasing values of b. When a becomes a bit large, above about 0.20, the dynamics become a single attractor dynamic with A always winning over B.
- **Regime 2** The second regime shown in Figure 5 for $b = 0.20$ has a unique type of dynamics with single attractor dynamics. The opinion having more stubborn agents on its side eventually wins over the other.

At this stage, for the sake of simplicity and without loss of generality, I investigate further the case $a = b$, which allows an analytical solution. In this case, Equation (12) becomes

$$p_{1,a,b} = -2p_0^3 + 3p_0^2 + \frac{a}{2}(2p_0^3 - 3p_0^2 - 3p + 2), \tag{13}$$

which yields three fixed points, $p_{c,a} = \frac{1}{2}$ and

$$p_{A,a;B,b} = \frac{-2 + a \pm \sqrt{4 - 20a + 9a^2}}{2(-2 + a)}. \quad (14)$$

with last two being valid only in the range $0 \leq a < a_c = \frac{2}{9} \approx 0.22$.

Therefore, regime 1 prevails in the range $0 \leq a < a_c = \frac{2}{9}$ with $p_{c,a} = \frac{1}{2}$ acting as a tipping point between the two attractors $p_{A,a}$ and $p_{B,a}$. On the other hand, regime 2 prevails for $\frac{2}{9} \leq a \leq \frac{1}{2}$ with $p_{c,a} = \frac{1}{2}$ being the unique attractor, as seen in Figure 6.

The polarization arises in regime 2 monitored by the attractor $p_{c,a} = \frac{1}{2}$. It is worth to mention that in case $a \neq b$ with a small difference between a and b the polarization still occurs but then the two opposite parts are unequal. Either A ($a > b$) or B ($a < b$) has a numerical advantage at the attractor.

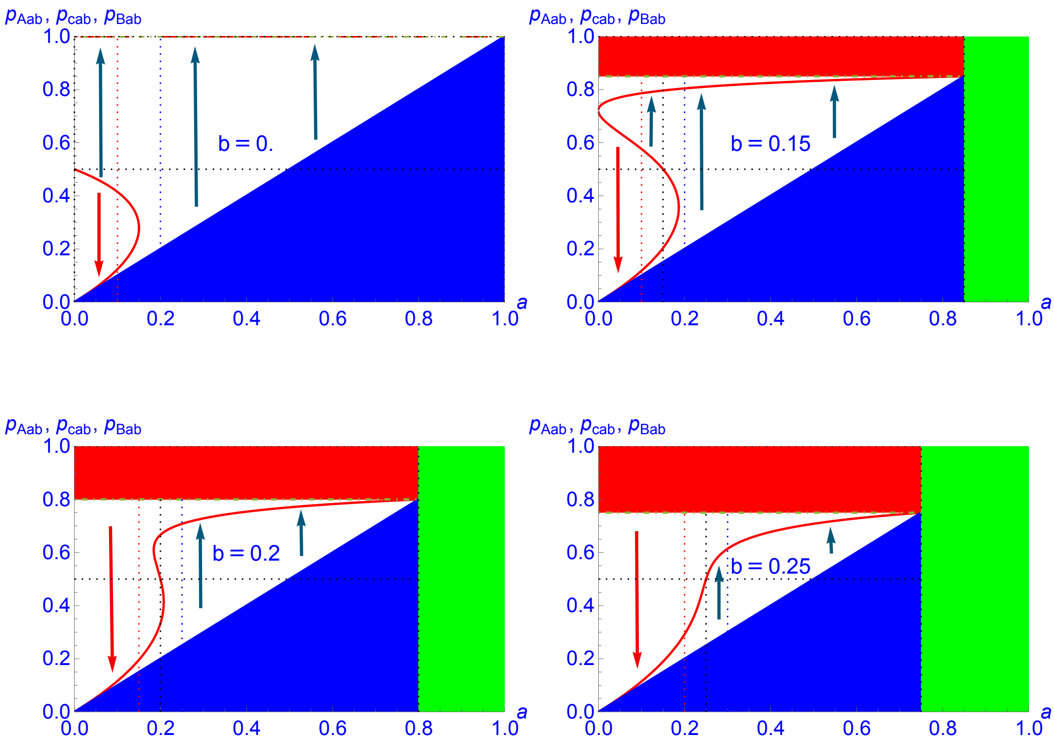

Figure 5. Attractors and tipping points as a function of a for a given b from Equation (12). Four cases are exhibited with $b = 0$, $b = 0.15$, $b = 0.20$, and $b = 0.25$.

6.1. Size Three for the Discussion Group

Earlier study of the case of size three with stubborn agents has the update Equation

$$p_{1,a,b} = p_0^3 + 3p_0^2(1 - p_0 - \frac{b}{3}) + a(1 - p_0)^2 \quad (15)$$

instead of Equation (12). With $a = b$, the associated fixed points are $p_{c,a} = \frac{1}{2}$ (as for size 4) and,

$$p_{A,a;B,b} = \frac{1}{2}(1 \pm \sqrt{1 - 4a}). \quad (16)$$

with last two being valid only in the range $0 \leq a < a_c = \frac{1}{4} = 0.25$ instead of $a_c = \frac{2}{9} \approx 0.22$ for size four.

It is worth notiing that while contrarians do not modify the value x_c when moving from size three to four, stubborn agents do modify a_c from size three to four.

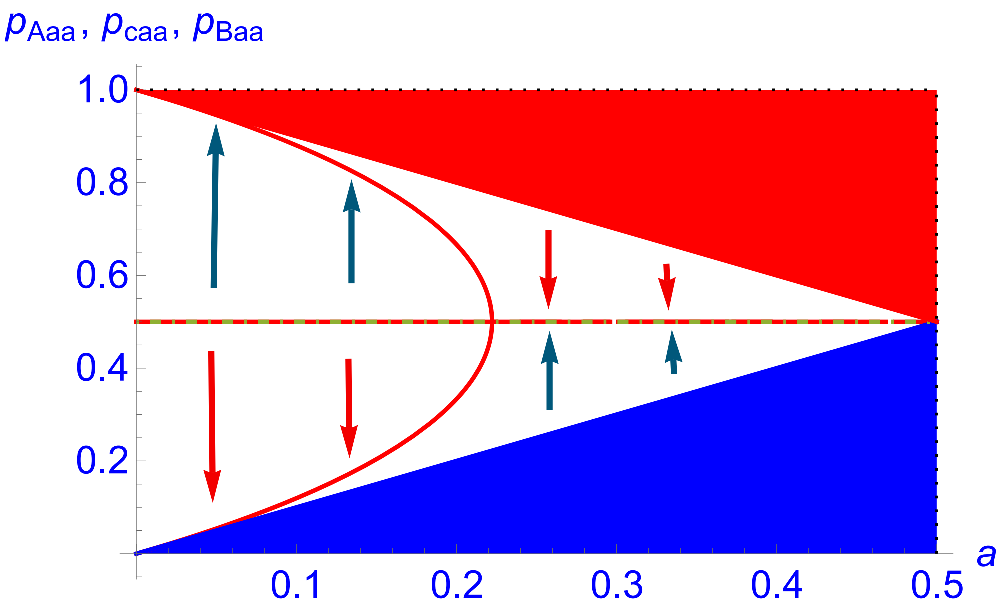

Figure 6. The two distinct regimes produced by stubborn agents as a function of their proportion for $a = b$. A tipping point dynamic with $p_{c,x} = \frac{1}{2}$ prevails in the range $0 \leq a < a_c$. The two associated attractors feature a stable coexistence of a majority and a minority. In the range $a_c \leq a \, \frac{1}{2}$, the dynamic is driven by one single attractor located at $\frac{1}{2}$. Any initial support $p_0 < 1 - a$ moves monotonously towards $\frac{1}{2}$ with repeating local updates. There, both opinions coexist in a perfect balance.

6.2. The Rigidity of the Stubborn Made Polarization

Along the analysis of the fluidity of polarization produced by contrarians, I now evaluate the extent of freezing produced by stubborn agents within the two opposite halves of the population. Equations (10) and (11) are deeply modified since stubbornness reduces only the shifting of agents from each opinion. The related shifts from B to A and from A to B are written, respectively,

$$\begin{aligned} S_{A,p,a} &= \frac{1}{2}p^2(1-p-b)(3-p), \\ S_{B,p,b} &= \frac{1}{2}(1-p)^2(p-a)(2+p), \end{aligned} \tag{17}$$

with by symmetry $S_{A,p,a} = S_{B,1-p,b}$.

Setting $a = b$, when $a > a_c = \frac{2}{9}$, the unique attractor of the dynamics is $p_{c,a} = \frac{1}{2}$, as seen in Figure 6. The associated total proportion of shifts is $S_{T,p,a} = \frac{5}{8}(\frac{1}{2} - b)$ (Figure 4). It yields 0.188 for $a = 0.20$, which is quite low. Accordingly, at the attractor, the community is mostly frozen between two opposite halves. With $a = 0.25$ the shifts amount to only 0.156.

7. Conclusions

I have addressed the issue of polarization using the Galam model of opinion dynamics to find that within its frame, polarization arises spontaneously by the dynamics of opinions in the presence of either contrarians or stubborn agents.

Moreover, by calculating the proportion of individual opinion shifts at the attractor $p_{c,a} = \frac{1}{2}$, I unveiled three types of polarization, which all arise from the same local majority update dynamics.

- **The fluid polarization** is produced by contrarian agents with a good deal of agents who keep shifting opinion between the two opposite parts of the community. This polarization favors a coexistence between the group with a related high entropy.
- **The frozen polarization** is produced by stubborn agents, which in turn provides a social and psychological basis for hate between the two split parts of a community

with a kind of inside ignorance of the other side, with a low value entropy. The community is trapped in a rigid distribution of opinions.

- **The segregated polarization** is produced by floaters In the absence of contrarians and stubborn agents. The dynamics lead toward unanimity within a connected social subgroup and to segregated polarization between adjacent non-mixing sub-communities. Associated entropies are zero.

In a future work, I intend to investigate the combined effect of mixing together the three kinds of agents, the floaters, the contrarians, and the stubborn.

Funding: This research received no external funding.

Data Availability Statement: No data has been used. No data has been created.

Conflicts of Interest: The author declares no conflict of interest.

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