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Modeling Portfolio Credit Risk Taking into Account the Default Correlations Using a Copula Approach: Implementation to an Italian Loan Portfolio

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Abstract: This work aims to illustrate an advanced quantitative methodology for measuring the credit risk of a loan portfolio allowing for diversification effects. Also, this methodology can allocate the credit capital coherently to each counterparty in the portfolio. The analytical approach used for estimating the portfolio credit risk is a binomial type based on a Monte Carlo Simulation. This method takes into account the default correlations among the credit counterparties in the portfolio by following a copula approach and utilizing the asset return correlations of the obligors, as estimated by rigorous statistical methods. Moreover, this model considers the recovery rates as stochastic and dependent on each other and on the time until defaults. The methodology utilized for coherently allocating credit capital in the portfolio estimates the marginal contributions of each obligor to the overall risk of the loan portfolio in terms of Expected Shortfall (ES), a risk measure more coherent and conservative than the traditional measure of Value-at-Risk (VaR). Finally, this advanced analytical structure is implemented to a hypothetical, but typical, loan portfolio of an Italian commercial bank operating across the overall national country. The national loan portfolio is composed of 17 sub-portfolios, or geographic clusters of credit exposures to 10,500 non-financial firms (or corporates) belonging to each geo-cluster or sub-portfolio. The outcomes, in terms of correlations, portfolio risk measures and capital allocations obtained from this advanced analytical framework, are compared with the results found by implementing the Internal Rating Based (IRB) approach of Basel II and III. Our chief conclusion is that the IRB model is unable to capture the real credit risk of loan portfolios because it does not take into account the actual dependence structure among the default events, and between the recovery rates and the default events. We underline that the adoption of this regulatory model can produce a dangerous underestimation of the portfolio credit risk, especially when the economic uncertainty and the volatility of the financial markets increase.

Keywords: portfolio credit risk; asset correlation; coherent capital allocation; copula function; Monte Carlo simulation; time until default

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1. Introduction

Currently, the financial system has become more volatile due to increasing globalization and financial integration. In such a context, financial intermediaries bear a greater amount of risk because of the growing interconnectedness of financial markets. Various kinds of risks affect the balance-sheets of financial intermediaries, but the most important is credit risk, particularly for commercial banks. For most banks, the loans to the real economy are the largest source of credit risk. For these reasons, banks need to adopt accurate methodologies for consistently assessing the credit risk of their loan portfolio and for allocating economic capital coherently to each counterparty in the portfolio. Bank capital plays a fundamental role in the safety and soundness of the banking system, since the main role of bank capital is to absorb large unexpected losses.

It is well known that the worth of a portfolio model lies in its ability to take into account the effects of diversification, namely the default correlations among the credit assets in a portfolio (Crouhy et al. 2000, 2014; Gordy 2000).

On the other hand, the regulatory approach based on internal ratings, namely the IRB-model proposed by Basel II and III, founded on the hypothesis of the "portfolio invariant" model (Gordy 2000, 2003), assumes that banking portfolios are perfectly fine grained; namely, that idiosyncratic risks have been diversified away. Consequently, the IRB model calculates the banking capital requirements to cover the unexpected credit losses on banking loans as a function of the characteristics of the borrower and the credit line only, ignoring the empirical portfolio composition and, in particular, the real level of diversification among the credit assets in the portfolio.

More specifically, the two theoretical and restrictive hypotheses underlying Basel's IRB model are the following:

(i) the infinite granularity of the credit portfolio and, therefore, the asymptotic approximation of the overall portfolio risk to the only non-diversifiable risk. In other words, the loan portfolio is highly diversified.

(ii) the existence of only a single systematic risk factor, and the subsequent quantification of this risk using the correlation between the economic assets of each counterparty in the portfolio and the index of the general economic condition.

The adoption of a portfolio invariant model by Basel II and III offers obvious analytical advantages to regulatory authorities. In particular, this approach permits the calculation of regulatory capital requirements analytically, without considering the real composition and the granularity of the empirical loan portfolios.

On the other hand, real-world portfolios are not perfectly fine grained. The asymptotic assumption might be approximately valid for some large bank portfolios, but could be much less acceptable for portfolios of smaller or more specialized institutions (Gordy and Lütkebohmert 2013).

The IRB model can underestimate the real credit risk amount of undiversified loan portfolios, omitting the contribution of the idiosyncratic risks in the portfolio.

Moreover, Basel sets for the category "corporate exposures" a regulatory value of the correlation between 12% and 24%. The values of the correlation obtained utilizing the regulatory formula, although inclosing an adjustment for small and medium enterprises, appear empirically conservative; that is, too high (De Servigny and Renault 2002; Duellmann and Scheule 2003; Duellmann and Koziol 2014; Hamerle and Rösch 2006; Sironi and Zazzara 2003; Dietsch and Petey 2004; Lopez 2004; Kitano 2007).

It is well known that asset correlations play a critical role in measuring portfolio credit risk, and in determining both economic and regulatory capital. In a credit portfolio, having many components does not assure good diversification, because the components may be highly correlated to each other, and the default of one may lead to default of the rest of the portfolio. This concept is called concentration risk in credit risk management.

Another reason is the incremental risk. Incremental risk measures the portfolio's risk sensibility to any changes in the portfolio's components. Therefore, correlation indicates the movement direction of the portfolio's assets with each other and with economic events.

An additional purpose for studying the correlations is to achieve a better allocation of assets in the portfolio. Optimal allocation means the minimization of the volatility of the portfolio, which depends on correlations (Mausser and Rosen 2008). Any change in the correlations of the portfolio changes the optimal asset allocation (Mizgier and Pasia 2015). The empirical results (Zhou 2001; De Servigny and Renault 2002) show that the default correlation between the components of the portfolio increases when the market does not perform well, or when there is an event that affects the market adversely. Also, De Servigny and Renault (2002) examined the effect of the time horizon on the default correlation, and showed that the correlation increases with time.

The standard approach for allocating capital in terms of Value-at-Risk (VaR) is founded on the traditional mean-variance approach (Markowitz 1952). However, a lot of recent academic studies

(Artzner et al. 1999; Acerbi and Tasche 2002) proved that the mean-variance capital allocation presents many shortcomings. The most important drawback is that the capital amount allocated to the whole portfolio may be greater than the capital amounts allocated to the individual sub-portfolios when the return distributions are not Gaussian (this is the case in credit assets). These problems can be overcome by allocating the capital in terms of Expected Shortfall (ES) as described, for instance, in Kalkbrener et al. (2004). While VaR (or more exactly, the Maximum Loss, ML) can be considered as a quantile of the portfolio loss distribution, the Expected Shortfall is approximately the conditional mean of losses exceeding VaR.

In light of all these considerations, this work aims to illustrate an advanced quantitative methodology capable of measuring the credit risk of the loan portfolio adequately, allowing for diversification effects, and suitable for allocating credit capital coherently (in the sense of Artzner et al. 1999) to each counterparty in the portfolio.

The added value of this research is, therefore, to introduce a new quantitative methodology capable of overcoming the weaknesses of Basel's IRB model. In order to achieve this goal, the principal tasks of this research are:

(i) Emphasizing that the main disadvantages of the Basel model are its underlying restrictive assumptions.
(ii) Comparing the results in terms of the portfolio's credit risk measures derived from the new methodology and from the IRB model when we assume both diversified and concentrated loan portfolios.
(iii) Introducing two sound statistical methodologies for estimating the asset return correlations between the obligors in the portfolio.
(iv) Improving the methodological framework of the portfolio credit risk model by assuming a dependence structure between the default events and the recovery rates.
(v) Introducing a coherent methodology for capital allocation that takes into account the non-normality of the portfolio credit loss distribution.

In particular, the quantitative approach used in this paper for estimating the portfolio's credit risk is of the binomial type, and is based on Monte Carlo Simulation. This methodology takes into account the default correlations among the credit counterparties in the portfolio—following the idea of the copula approach, first developed in Li (2000)—and utilizes the asset return correlations of the obligors, as estimated by means of rigorous statistical methods (Lucas 1995; De Servigny and Renault 2002; Frye 2000; Hamerle and Rösch 2006). Li (2000) first introduced a random variable called time-until-default, which measures the length of time from today until default time, to indicate the survival time of each defaultable obligor. Then, Li (2000) defined the default correlation between two obligors by the correlation between their survival times. Following this original idea, we construct the dependence structure of defaults by means of a one-factor model, generating the scenarios of the times until default's random vector for the N exposures in the portfolio from the Gaussian copula (Gregory and Laurent 2004). The concept of copula goes back to Sklar (1959). Copula is a function of several variables, and describes, in a powerful way, how joint distribution is linked to its univariate margins. Copula functions are used to combine marginal distributions into a multivariate distribution. They are unique: for any given multivariate distribution (with continuous marginal distributions) there is a unique copula function that represents it. They are also invariant under strictly increasing transformations of the marginal distributions. Moreover, copula functions have long been recognized as a powerful tool for modelling dependence between random variables (Nelsen 1999). The basic idea behind copulas is to separate dependence and the marginal behavior of the univariates. Also, our methodology can consider the recovery rates as stochastic and dependent on each other and on the time until defaults, following the examples of Pykhtin (2003); Tasche (2004); Emmer and Tasche (2004); Gregory and Laurent (2004); and Chabaane et al. (2004).

Moreover, the approach utilized for allocating credit capital coherently (Overbeck 2000; Denault 2001; Kalkbrener et al. 2004; Kalkbrener 2005) estimates the marginal contributions of each obligor to the overall risk of the loan portfolio in terms of Expected Shortfall (ES), a risk measure that is coherent (in sense of Artzner et al. 1999; Tasche 2002; Acerbi and Tasche 2002) and more conservative than the traditional measure of Value-at-Risk (VaR).

Finally, this advanced analytical structure is implemented to a hypothetical but typical loan portfolio of an Italian commercial bank operating across the country. The national loan portfolio is structured with 17 sub-portfolios or regional clusters of credit exposures to 10,500 non-financial firms (or corporates) belonging to each geo-cluster or sub-portfolio.

The outcomes in term of correlations, portfolio risk measures, and capital allocations obtained from this advanced analytical framework are compared with the results found by implementing the internal rating based approach of Basel II and III.

In particular, the contributions of each geographical sub-portfolio to the credit risk of the overall Italian loan portfolio have been estimated in terms of Value-at-Risk (VaR), Maximum Loss (ML) and Expected Shortfall (ES), calculated for a confidence level of 99.9% over an annual time horizon.

Our chief conclusion is that the IRB model is unable to capture the real credit risk of loan portfolios because it does not take into account the actual dependence structure among the default events, and between the recovery rates and the default events. For this reason, we underline that the adoption of this regulatory model can produce a dangerous underestimation of the portfolio credit risk, especially when the economic uncertainty and volatility of the financial markets increase. The whole paper is structured as follows. Section 2, and Sections 2.1 and 2.2, illustrate the quantitative characteristics of the advanced approach for estimating the credit risk of the loan portfolio consistently. In particular, this binomial (default/non-default) model is based on Monte Carlo Simulation, and takes into account the default correlations among obligors in the portfolio, following the idea of the copula approach first developed by Li (2000). Section 3 describes two sound statistical methods for estimating the asset correlations of obligors in the portfolio consistently. First, we follow Frye (2000) for calculating the asset correlation coefficients by factor loadings (namely, the sensitivity coefficients of the obligor asset returns to the changes in the systematic factor), estimated through the maximum likelihood method (MLH). Secondly, we implement an alternative methodology for estimating the correlations, first presented by Lucas (1995) (De Servigny and Renault 2002; Hamerle and Rösch 2006). For comparison purposes, we apply these two robust statistical methodologies to the Italian loan portfolio utilizing the historical time series of default numbers and default rates from 2006 to 2019. These input data can be freely downloaded from the website of the Bank of Italy. Section 4 explains how a dependence structure between recovery rates and default events can be introduced into this credit portfolio model. Section 5 describes a coherent capital allocation technique, emphasizing its peculiarities with reference to the traditional capital allocation scheme founded on Markowitz (1952) portfolio theory. In Section 6, we implement the advanced analytical framework to a hypothetical but typical Italian loan portfolio composed of banking credit exposures to 10,500 non-financial Italian firms (or corporates) residing in each of the 17 Italian geographic clusters (regions). Comments and conclusions are reported in Section 7.

2. Credit Portfolio Model and Credit Risk Measures

We assume a loan portfolio with N obligors and a time horizon equal to the longest maturity among the credit assets in the portfolio[1]. The random variable (r.v.) L, representing the portfolio loss, is defined following the notation used in Jouanin et al. (2004):

1 Suppose we are in time 0, if the longest maturity is M_{max}, the time horizon is $[0, M_{max}]$.

$$L = \sum_{i=1}^{N} L_i = \sum_{i=1}^{N} EaD_i \cdot (1 - R_i) \cdot 1\{\tau_i \leq M_i\} \quad (1)$$

Equation (1) denotes a default/non-default model, where L_i is the r.v. loss for each obligor i, EaD_i is the exposure at default of obligor i, R_i is its recovery rate, M_i is the maturity of debt of obligor i, τ_i is the r.v. time until the default of obligor i[2] and 1 is a function that assumes a value equal to 1 if $\tau_i \leq M_i$ and a value of 0 otherwise.

The recovery rate R_i may be assumed to be deterministic[3] or stochastic with mean m_i and standard deviation s_i, and independent[4] of each other and of their respective times until default τ_i. The most common assumption about the distributional form of R_i is the Beta (a_i,b_i) distribution, with the parameters a_i and b_i estimated by the method of moments, knowing the values of m_i and s_i analytically:

$$a_i = \frac{m_i^2(1 - m_i)}{s_i^2} - m_i,\ b_i = \frac{m_i^2(1 - m_i)^2}{m_i s_i^2} - (1 - m_i) \quad (2)$$

The stochastic vector of the times until default ($\tau_1, \ldots, \tau_N$) has a multivariate cumulative distribution function (c.d.f.), F. This may be written by the following copula representation:

$$F(t_1, \ldots, t_N) = \Pr\{\tau_1 \leq t_1, \ldots, \tau_N \leq t_N\} = C(F_1(t_1), \ldots, F_N(t_N)) \quad (3)$$

In Equation (3), F_i is the marginal c.d.f. of τ_i, and C is the copula function that determines the dependence structure of the multivariate c.d.f. of the times until default vector.

The concept of copula goes back to Sklar (1959). Copula is a function of several variables which describes, in a powerful way, how joint distribution is linked to its univariate margins. A n-dimensional copula function is a multivariate c.d.f., C, with margins uniformly distributed on [0,1], and with the following properties:

1. C is grounded and n-increasing.
2. C has margins C_i which satisfy $C_i(u) = C(1, \ldots, 1, u, 1, \ldots, 1)$ = u for all $u \in [0,1]$.
3. C: $[0,1]^n \rightarrow [0,1]$.

Copula functions are used to combine marginal distributions into a multivariate distribution. They are unique: for any given multivariate distribution (with continuous marginal distributions) there is a unique copula that represents it. They are also invariant under strictly increasing transformations of the marginal distributions. Moreover, copulas have long been recognized as a powerful tool for modelling dependence between random variables (Nelsen 1999). The basic idea behind copulas is to separate the dependence and marginal behavior of the univariates. The most known copulas are the elliptical or standard ones. Important examples in this family of distributions are the Gaussian and Student's t examples. The unknown c.d.f. G of the r.v. L (portfolio loss) may be estimated by Monte Carlo simulation using the following algorithm:

(1) Generate a determination of N random variables uniformly distributed on [0,1], ($u_1, \ldots, u_N$) from the copula C.
(2) Determine a scenario for the times until default by inverting ($u_1, \ldots, u_N$) using the margins: $t_i = F_i^{-1}(u_i), i = 1, \ldots, N$.

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2 It is assumed that M_i and τ_i have been expressed in years.
3 This is the case of the Internal Rating Based (IRB) approach as formulated by Basel Committee for calculating banking capital requirement.
4 In a following section we will see how this hypothesis may be relaxed.

(3) For every obligor $i = 1, \ldots, N$, if $t_i \le M_i$ we then obtain a loss scenario equal to $EaD_i(1-R_i)$, or equal to 0 otherwise. In the case of stochastic recovery rates, the determination of R_i is generated from a Beta (a_i, b_i) c.d.f.
(4) Add up the losses of the N obligors, obtaining a scenario of the portfolio loss, L_j.
(5) Steps from 1 to 4 are repeated a great number of times, s.

From the distribution of the portfolio losses obtained by the simulation, we may estimate different risk measures for the loan portfolio, such as the expected loss, *EL*; the maximum loss, *ML*; the Value at Risk, *VaR*; and the Expected Shortfall, *ES*. In particular, portfolio *EL* is calculated as the mean of the portfolio losses for all s scenarios. Analytically:

$$EL = \frac{\sum_{j=1}^{s} L_j}{s} \tag{4}$$

The portfolio *ML* at the probability level α, ML_α, may be calculated by ordering the s scenarios of portfolio loss in non-decreasing order and cutting the obtained distribution at the α-th percentile.

Portfolio Credit *VaR*, at the probability level α, is calculated as the difference between the *ML* at the same probability level α and the *EL* of the portfolio. Analytically:

$$VaR_\alpha = ML_\alpha - EL \tag{5}$$

The portfolio *ES*, at the probability level α, ES_α, is calculated as the conditional mean of the portfolio losses exceeding the *ML*. Analytically:

$$ES_\alpha = ML_\alpha + \frac{1}{(1-\alpha)\cdot s}\sum_{j=1}^{s}\left(L_j - ML_\alpha\right)^+ \tag{6}$$

where $\left(L_j - ML_\alpha\right)^+ = L_j - ML_\alpha$ if $L_j - ML_\alpha > 0$; $\left(L_j - ML_\alpha\right)^+ = 0$ if $L_j - ML_\alpha \le 0$.

2.1. Determining the Marginal Distributions for the Times Until Default

In order to apply the algorithm described in the previous Section 2, it is necessary to give a functional form to the marginal cumulative distribution functions, F_i, for the random variables' times until default, and to estimate their parameters. In order to do this, we have to introduce the hazard rate function, $h_i(t)$, defined as follows:

$$h_i(t) = \lim_{\Delta t \to 0} \frac{\Pr\{\, t < \tau_i \le t + \Delta t | \tau_i > t\,\}}{\Delta t} \tag{7}$$

By extending Equation (7), the following is obtained:

$$h_i(t) = \lim_{\Delta t \to 0} \frac{\Pr\{\, t < \tau_i \le t + \Delta t\}}{\Delta t \Pr\{\tau_i > t\}} = \frac{\frac{\partial}{\partial t} F_i(t)}{1 - F_i(t)} = -\frac{\partial}{\partial t} \ln(1 - F_i(t)) \tag{8}$$

By solving the differential Equation (8), we obtain

$$F_i(t) = P_r[\tau_i \le t] = 1 - \exp\left(-\int_0^t h_i(u)du\right) \tag{9}$$

If we assume that the time structure of the hazard rate function is flat, that is $h_i(t) = h_i$ for each t, we can rewrite Equation (9) as follows:

$$F_i(t) = P_r[\tau_i \leq t] = 1 - e^{-h_i t} \quad (10)$$

The hazard rate function completely characterizes the distribution of the random variable T_i. Therefore, the calibration of $h_i(t)$ from real data is a core issue. In pricing applications, the hazard rates are usually calibrated using market data such as the quotations of defaultable bonds, asset swap spreads, or Credit Default Swaps[5]. Conversely, for risk management applications, the hazard rates may be calibrated using the probability of default provided by an internal or external[6] credit rating assessment. For instance, if $q_i(0, t)$ is the average cumulative default rate over the time horizon $[0, t]$[7], from Equation (10), we obtain:

$$1 - e^{-h_i(t)t} = q_i(0,t) \Rightarrow h_i(t) = -\ln(1 - q_i(0,t))/t \quad (11)$$

If we dispose of a term structure of the default probabilities[8], a piecewise constant functional form for the hazard rates may be assumed. If $T_1, T_2, \ldots, T_m$ are the nodes of the term structure of the default probabilities (for years), then the hazard rate function may be written in the following way:

$$h_i(t) = \sum_{j=1}^{m} h_{i,j} \mathbf{1}_{(T_{j-1},T_j]}(t) \quad (12)$$

In Equation (12), $h_{i,j}$ are positive constants, $j = 1, \ldots, m$ and $\mathbf{1}_{(T_{j-1},T_j]}(t) = 1$ if $t \in (T_{j-1}, T_j]$. This hypothesis implies that the c.d.f. $F_i(t)$ may be written as follows:

$$F_i(t) = 1 - \exp\left\{-\sum_{j=1}^{k} h_{i,j}(T_j - T_{j-1})\right\}, k = \begin{cases} 1 \text{ if } t \leq T_1 \\ 2 \text{ if } T_1 < t \leq T_2 \\ \ldots \\ m \text{ if } t > T_{m-1} \end{cases} \quad (13)$$

From Equation (13), $h_{i,1}$ may be estimated using the probability of default over the maturity T_1; $h_{i,2}$ may be calibrated using the default probability over the maturity T_2, known $h_{i,1}$, and so on. The remaining $h_{i,j}$ may be calibrated up to time T_m.

2.2. A One-Factor Model for Generating Scenarios from the Gaussian Copula

To apply the algorithm of Section 2, it is necessary to generate scenarios $(u_1, \ldots, u_N)$ from the generic copula, C. For this purpose, in the academic literature and in practical industry applications, the most utilized copulas are the Gaussian and the Student's t examples. These copulas are easy to implement and, furthermore, are endowed with a sufficient parameter number for describing the portfolio's dependence structure effectively. The most important parameter to calibrate is the correlation matrix[9].

Li (2000) demonstrated that, when we model the c.d.f. in Equation (3) using the Gaussian copula, in the bivariate case, the correlation parameter is equal to the asset correlation between the two counterparties. This result may be extended to the case of the t-copula (see Mashal and Naldi 2002;

5 In this case, a risk neutral measure of h_i is obtained.
6 i.e., rating agencies.
7 Usually, since we dispose of the one-year default probabilities, $t = 1$.
8 e.g., we dispose of the probabilities of default over the time horizons 1, 2, 5 and 10 years.
9 We use the term correlation matrix even if it is not completely appropriate in the case of the Student's t-copula.

Meneguzzo and Vecchiato 2004). Therefore, the elements of a correlation matrix with dimensions *NxN* are the asset correlations among the *N* obligors in the portfolio.

Nevertheless, the number of obligors for a typical Italian commercial bank is so high as to require high costs in terms of memory space and computational time to implement the Monte Carlo methodology. For this reason, it is convenient to utilize a factorial model with J clusters[10] in order to simulate scenarios from the Gaussian or the Student's t copula (Gregory and Laurent (2004). Since J (the number of clusters) is much lower than N (the number of obligors), the number of parameters to be estimated and the computational costs will also be much smaller.

We suppose, for instance, to generate scenarios from the Gaussian copula simply by a one-factor model[11] (Merton 1974) which represents the asset return of obligor i, Y_i, for $i = 1, \ldots, N$. Analytically:

$$Y_i = b_{m(i)}X + \sqrt{1 - b_{m(i)}^2}\, e_i \tag{14}$$

In Equation (14), X and e_i are independent standard normal random variables[12], $b_{m(i)}$ is the weight of the systematic component X, $m(i)$ is the relation linking obligor i to his cluster $j = m(i)$, $j = 1, \ldots, J$; moreover $e_i, \ldots, e_n$ are independent. In this setting, Y_{i} is a standard normal r.v., too.

The weights $b_{m(i)}$ have been assumed equal for each obligor who belongs to the same cluster. They are calibrated using the asset return correlation intra-cluster. In fact, the asset return correlation between two obligors i and j belonging to the same cluster k is the following:

$$\rho_k = E\left[Y_i Y_j\right] = b_{m(i)} b_{m(j)} = b_k^2 \tag{15}$$

In Equation (15), $b_k = \sqrt{\rho_k}$. Therefore, the asset return correlation between two obligors i and j belonging to two different clusters, respectively k and l, is the following:

$$\rho_{kl} = E\left[Y_i Y_j\right] = b_{m(i)} b_{m(j)} = b_k b_l = \sqrt{\rho_k \rho_l} \tag{16}$$

To generate a scenario from the Gaussian copula, the following algorithm may be applied:

(1) Generate $N + 1$ independent random variates from the standard normal distribution (they are the determinations of $X, e_1, \ldots, e_N$);
(2) Calculate a scenario y_i of Y_i, $i = 1, \ldots, N$;
(3) The scenario $u_i = \Phi(y_i)$, $i = 1, \ldots, N$, where Φ is the standard normal c.d.f., is generated from the Gaussian copula.

To generate a scenario from the Student's t copula with ν degrees of freedom by a one-factor model (see Frey and McNeil 2003; Wehrspohn 2003), it is sufficient to transform Equation (14) as follows:

$$Y_i = \sqrt{\frac{\nu}{W}}\left(b_{m(i)}X + \sqrt{1 - b_{m(i)}^2}\, e_i\right) \tag{17}$$

In Equation (17) $X, e_1, \ldots, e_N$ are independent standard normal random variables, and W is a chi-square r.v. with ν degrees of freedom, independent of $X, e_1, \ldots, e_N$. In this case, the algorithm to apply is the following:

[10] The clusters may be industrial sectors or geographical areas.
[11] The Merton Model is used in Basel's IRB model.
[12] X may be seen as the return of the macroeconomic factor or the global market index common to the all obligors in the portfolio, representing the systematic factor, Y_i, while e_i may be interpreted as the portion of the asset return which is not explained by the systematic factor (that is the specific or idiosyncratic factor).

(1) Generate $N + 1$ independent random variates from the standard normal distribution (they are the determinations of $X, e_1, \ldots, e_N$), and a determination from the chi-square r.v. with ν degrees of freedom, W, independent of $X, e_1, \ldots, e_N$;
(2) Calculate a scenario y_i of Y_i, $i = 1, \ldots, N$ using Equation (17);
(3) The scenario $u_i = T_\nu(y_i)$, $i = 1, \ldots, N$, where T_ν is the standardized Student's t c.d.f. with ν degrees of freedom, is generated from the Student's t-copula with ν degrees of freedom.

The correlation structure implicit in the one-factor model (Equation (14))) is very restrictive. In fact, the correlations of assets belonging to different clusters k and l (namely the inter-cluster asset correlations) are implicitly determined by the intra-cluster asset correlations by Equation (16). To get a more complete correlation structure, the following factorial model (Gregory and Laurent 2004) may be used[13]:

$$Y_i = b_{m(i)} X_{m(i)} + \sqrt{1 - b_{m(i)}^2}\, e_i \tag{18}$$

For Equation (18), the same consideration made for Equation (14) holds; moreover, the systematic risk factor, $X_{m(i)}$, is expressed by a second one-factor model:

$$X_j = a_j X + \sqrt{1 - a_j^2}\, e'_j \,,\; j = 1, \ldots, J \tag{19}$$

In Equation (19) X, e_j', Y_j are independent standard normal random variables. Therefore, by substituting Equation (19) into Equation (18), we get to:

$$Y_i = b_{m(i)} a_{m(i)} X + b_{m(i)} \sqrt{1 - a_j^2}\, e'_{m(i)} + \sqrt{1 - b_{m(i)}^2}\, e_i \tag{20}$$

Therefore, by the factorial model described in Equation (20), the asset correlation between two obligors belonging to the same cluster $m(i) = j$, $j = 1, \ldots, J$, is $b_{m(i)}^2$. On the contrary, the correlation between two different clusters $m(i)$ and $m(j)$, with con $m(i) \neq m(j)$, is $b_{m(i)} b_{m(j)} a_{m(i)} a_{m(j)}$. Let us assume, for instance, a correlation structure where the intra-cluster correlations are equal to ρ_j, $j = 1, \ldots, J$, while all the inter-cluster correlations are equal to ρ. In order to get to this kind of dependence structure, it is sufficient to calibrate the model in Equation (20) in the following way: $b_j = \sqrt{\rho_j}$ e $a_j = \sqrt{\frac{\rho}{\rho_j}}, j = 1, \ldots, J$.

3. Estimating Asset Correlations

The simplest methodology for estimating the asset return correlation for corporates is to assume them all equal; for instance, to 0.20[14]. A second methodology, following the last version of Basel's IRB approach[15], determines the asset correlation for each cluster by implementing the regulatory formula in Equation (21):

$$\rho_i = 0.12 \cdot \frac{\left(1 - e^{(-50 \cdot P(i))}\right)}{\left(1 - e^{(-50)}\right)} + 0.24 \cdot \left[1 - \frac{\left(1 - e^{(-50 \cdot P(i))}\right)}{\left(1 - e^{(-50)}\right)}\right] \tag{21}$$

In Equation (21), ρ_i is the intra-cluster asset return correlation for cluster i, $i = 1, \ldots, J$. The asset return correlations between clusters i and j are implicitly calculated as $\rho_{i,j} = \sqrt{\rho_i \rho_j}$.

An alternative methodology for estimating the correlations, adopted first in Lucas (1995), utilizes the historical yearly time series of the default number and of the obligor number for each geo-sectorial cluster[16] as input data.

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13 In order to get a model with an even less restricted dependence structure, see Jouanin et al. (2004).
14 This is the solution adopted in the first version of the IRB model by the Basel Committee.
15 See Basel Committee on Banking Supervision (2003, 2004).
16 These data can be downloaded freely from the web site of Bank of Italy: www.bancaditalia.it/statistiche/index.html.

Let $N_j(i)$ be the number of obligors at the beginning of year j in cluster i, $j = 1, \ldots, n$, $i = 1, \ldots, J$; let $S_j(i)$ be the number of defaults, proceeding from the $N_j(i)$ obligors, over year j in cluster i.

The probability of k defaults in cluster i may be assessed as follows:

$$P_k(i) = \frac{1}{n}\sum_{j=1}^{n} \frac{\binom{S_j(i)}{k}}{\binom{N_j(i)}{k}}, \text{ otherwise}$$

$$P_k(i) = \sum_{j=1}^{n} \frac{N_j(i)}{\sum_{s=1}^{n} N_j(i)} \cdot \frac{\binom{S_j(i)}{k}}{\binom{N_j(i)}{k}} \tag{22}$$

The probability of two defaults, the first in cluster i and the second in cluster k, is the following:

$$P_2(i,k) = \frac{1}{n}\sum_{j=1}^{n} \frac{S_j(i)S_j(k)}{N_j(i)N_j(k)} \tag{23}$$

The intra-cluster default correlation for cluster i is:

$$\rho_D(i) = \frac{P_2(i) - (P_1(i))^2}{P_1(i) - (P_1(i))^2} \tag{24}$$

The default correlation between clusters i and k is:

$$\rho_D(i,k) = \frac{P_2(i,k) - P_1(i)P_1(k)}{\sqrt{\left[P_1(i) - (P_1(i))^2\right] \cdot \left[P_1(k) - (P_1(k))^2\right]}} \tag{25}$$

The corresponding asset return correlation may be obtained by solving the following Equation (26)[17] for ρ:

$$P_2(i,k) = \int_{-\infty}^{\Phi^{-1}(P_1(i))} \int_{-\infty}^{\Phi^{-1}(P_1(k))} \Phi_2(x, y; \rho_{i,k}) dxdy \tag{26}$$

A further methodology (Frye 2000) for estimating the asset return correlation intra-cluster is presented in the following. For application to an Italian loan portfolio, the input data may be represented by the historical time series of the decay or deterioration rates[18], $TdD_{t,j}$, downloaded from the website of the Bank of Italy for each year t, $t = 1, \ldots, T$, concerning different geo-sectorial clusters j, $j = 1, \ldots, J$ and counterparty categories.

The standardized asset return for a generic obligor belonging to cluster j is assumed to be represented by the one-factor model in Equation (14).

Therefore, estimating the intra-cluster asset return correlation ρ_j is equivalent to estimating the weight (factor loading) b_j. In fact, it is easy to demonstrate[19] that $\rho_j = {b_j}^2$. In order to estimate the weights b_j, it is assumed that the number of obligors into each cluster is very high, and that these obligors are homogeneous; that is, all obligors in a cluster get the same downgrading rate and the same factor loading. Therefore, by the Law of Large Numbers (LLN), it is assumed that the downgrading

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17 Assuming $P_2(i,i) = P_2(i)$ if $i = k$.
18 They may be considered as estimates of the default probabilities.
19 See Equation (15).

rate observed in year t is equal to the default probability conditional to the value x_t of the systematic risk factor X observed in year t, that is:

$$TdD_{t,j} = \Pr\left[Y_j < \Phi^{-1}(TdD_j)|X = x_t\right] = \Phi\left[\frac{\Phi^{-1}(TdD_j) - b_j x_t}{\sqrt{1-b_j^2}}\right] = g_j(x_t) \quad (27)$$

In Equation (27), Φ is the standard normal p.d.f. and $TdDj$ is the mean decay rate over the long period in cluster j. Since $g_j(x_t)$ in Equation (27) is a monotonic function of the systematic risk factor X, which is standard normal distributed, the probability density function (p.d.f.) of $g_j(X)$ may be written following the Vasicek (2015a, 2015b) formula in this way (Finger 1999, 2001):

$$f_j(TdD_{t,j}) = \frac{\sqrt{1-b_j^2}}{b_j} \cdot \frac{\varphi\left(\frac{\sqrt{1-b_j^2}\cdot\Phi^{-1}(TdD_{t,j})-\Phi^{-1}(TdD_j)}{b_j}\right)}{\varphi\left(\Phi^{-1}(TdD_{t,j})\right)} \quad (28)$$

In Equation (28), ϕ is the standard normal p.d.f.

To estimate b_j, it is necessary to maximizing the log-likelihood function in Equation (29):

$$\hat{b}_j = \max_{b_j} \sum_{t=1}^{T} \ln f_j(TdD_{t,j}) \quad (29)$$

As a consequence of the one-factor model, it is simple to demonstrate that the asset correlation between two obligors belonging to two different clusters i and j is: $\rho_{i,j} = b_i b_j = \sqrt{\rho_i \rho_j}$.

4. Introducing a Dependence Structure between Recovery Rates and Default Events

So far, we have always supposed the independence among the recovery rates themselves and the times until default. However, in this section, we assume that the recovery rates are correlated to each other and to the times until default by a factorial model. In particular, we follow the approach described in Pykhtin (2003); Tasche (2004); and Gregory and Laurent (2004).

In this context, the portfolio loss, calculated as the sum of the losses of all obligors i in the portfolio, is driven by two random variables: Y_{i} linking up with the times until default, and V_{i} driving the recovery rate and hence the loss amount. Both these two random variables may be interpreted, according to the Merton model, as the asset return for obligor i, respectively before, Y_{i}, and immediately after default, V_{i}. The economic intuition is that, at the time of default, the recovery rate will be as low as the return on assets is lower immediately after default.

Both the two random variables, Y_{i} and V_{i}, may be expressed through the following factorial model, assuming the correlation among recovery rates and between them and the times until default is constant in each cluster:

$$\begin{aligned} Y_i &= b_{m(i)}X + \sqrt{1-b_{m(i)}^2}e_i \\ V_i &= c_{m(i)}X + \sqrt{1-c_{m(i)}^2}e_i' \end{aligned} \quad (30)$$

In the model in Equation (30), X, e_i and $e_i{'}$, $i = 1, \ldots, N$, are independent standard normal random variables; the asset correlations are functions of the factor loadings $b_{m(i)} > 0$ and $c_{m(i)} > 0$. Due to calibration problems, it is convenient to assume $b_{m(i)} = c_{m(i)}$, as in Tasche (2004).

The r.v. Y_i drives the time until default of each obligor I, as described in the previous sections. On the contrary, the value of the recovery rate R_i proceeds from the value assumed by V_i. If the obligor i defaults in scenario j, then a determination of the recovery rate is generated; otherwise, the loss in this scenario is 0. In particular, the obligor i defaults if $Y_i < \Phi^{-1}(q_{m(i)}(0,M_i))$, where Φ is the standard normal c.d.f. and $q_{m(i)}(0,M_i)$ is the default probability of cluster $m(i)$ for the maturity M_i. This kind of event is equivalent to the event $\tau_i = F_i^{-1}(\Phi(Y_i)) < M_i$, since for Equation (10): $q_{m(i)}(0,M_i) = F_i(M_i) = \Pr\{\tau_i \le M_i\}$. If $G(x;a,b)$ is the Beta c.d.f., with parameters a and b estimated using Equation (2), then the determination of the recovery rate in the default case is the following:

$$R_i = G^{-1}\Big(\Phi(c_{m(i)}X + \sqrt{1 - c_{m(i)}^2}e'_i); a, b\Big) \tag{31}$$

In Equation (31), the value of the recovery rate is correlated to the default event through the systematic factor X. The factor X may represent, for instance, a proxy of the general economic condition. In case of a negative economic condition, X assumes low values; hence, a higher number of defaults (driven by the random variables Y_i) may happen and the values of the recovery rates (driven by the random variables V_i) are expected to be lower. The contrary may happen in case of a positive economic condition.

5. Capital Allocation

After calculating the credit portfolio risk measures by the methodology described in the previous sections, it is necessary to allocate the estimated capital among the different obligors or sub-portfolios (clusters).

The typical industry standard solution (Litterman 1996; Overbeck 2000) is to allocate the portfolio VaR among all obligors, or sub-portfolios, proportionally to their covariance: $\mathrm{Cov}(L_1,L), \ldots, \mathrm{Cov}(L_N,L)$, where L_i is the r.v. loss of the generic obligor i ($i = 1, \ldots N$) and L is the r.v. loss of the overall credit portfolio. This capital allocation technique, known as volatility allocation, is the natural choice in the bounds of classical portfolio theory, where risk is measured by the standard deviation. The use of this technique for VaR allocation is correct when all the marginal loss distributions are normal. Unfortunately, this is not the case, mainly for the credit asset portfolios. The capital allocated to a sub-portfolio P^* might be greater than the risk capital of P^* considered as a stand-alone portfolio (discouraging portfolio diversification). In other words, the capital requirement of a single loan might be greater than its exposure value. On the contrary, a coherent capital allocation scheme has to satisfy the following three properties (Kalkbrener et al. 2004; Kalkbrener 2005):

- The capital allocated to a union of sub-portfolios has to be equal to the sum of the capital amounts allocated to the single sub-portfolios. In particular, the whole portfolio risk capital is the sum of the risk capitals of its sub-portfolios.
- The capital allocated to a sub-portfolio X belonging to a larger portfolio Y never has to exceed the risk capital of X considered as a stand-alone portfolio.
- A small increase of exposition value has to produce a small effect on the risk capital allocated to that exposition.

The capital allocation performed using the Expected Shortfall as a risk measure satisfies the three previous requirements. In other words, the capital allocation by ES is coherent. Analytically, the capital amount allocated to the obligor (or cluster) i through the ES measure, calculated at the probability level α, is the following:

$$E\Big(L_i \Big| L > ML_\alpha(L)\Big) = \frac{1}{1-\alpha}E\Big(L_i \cdot \mathbf{1}_{\{L > ML_\alpha(L)\}}\Big) \tag{32}$$

In Equation (32), L_i is the r.v. loss for obligor i, L is the r.v. portfolio loss, $ML_\alpha(L)$ is the portfolio Maximum Loss, and $\mathbf{1}_E$ is an r.v., assuming the value 1 if event E is true and 0 if it is false.

Equation (32) represents the mean contribution of the obligor i to the portfolio losses exceeding ML_α. It may be easily computed by Monte Carlo simulation: first by storing the losses $L_{i,j}$ occurred in the j scenarios when the portfolio loss L_j is greater than ML_α, and secondly by calculating their conditional mean.

6. Implementation to a Typical Italian Loan Portfolio

In this section, we implement the overall methodology described previously to a typical, but hypothetical, loan portfolio of an Italian commercial bank. The results, in terms of the portfolio credit risk measures and capital allocations, are compared to the ones obtained by the Internal Rating Based (IRB) approach developed by the Basel Committee[20]. According to Basel's IRB approach, given a portfolio composed of corporate exposures, the minimum banking capital requirement for a generic obligor i is the following:

$$\mathbf{K_i} = EaD_i \times LGD_i \times (\Phi\left[\frac{1}{\sqrt{1-\rho_i}}\Phi^{-1}(PD_i) + \sqrt{\frac{\rho_i}{1-\rho_i}}\Phi^{-1}(0.999)\right] - PD_i) \times \frac{1}{(1-1.5\times b(PD_i))} \times (1 + (M_i - 2.5) \times b(PD_i))$$

In the regulatory formula above, PD_i is the one-year default probability of obligor i, EaD_i is the banking exposure at default i, $LGD_i = (1 - R_i)$ is the loss given default, M_i is the maturity of the loan given to the generic obligor i and ρ_i is the asset return correlation estimated through the regulatory formula (see Equation (21)).

The minimum capital requirement, K_i, is the contribution of the exposure of obligor i to the credit risk of the loan portfolio; namely, the one-year Credit VaR, adjusted over a time horizon [0, M], at the probability level of 99.9% for a homogeneous portfolio with infinite granularity[21].

The hypothetical Italian banking portfolio is composed of credit exposures to 10,500 Italian non-financial firms (corporates) residing in each of the 17 different Italian regions. We assume that all obligors belonging to the same Italian region share the same probability of default (that is, the *PD* of that region or cluster). As a proxy of the long period *PD* for each cluster, we have utilized the annualized quarterly credit decay rates provided by the Statistic Report of the Bank of Italy, from 06/30/2006 to 09/30/2019, for each Italian regional cluster and the category of non-financial firms[22]. The asset return correlations are assumed to be constant for each cluster. We calculate the correlations by both the Basel regulatory formula (Equation (21)) and the two robust statistical methodologies described in Section 3. All the maturities are assumed to be equal to one year. Initially, all the *LGD*s are assumed to be non-stochastic and constant to 50%.

In Table 1, we show the main characteristics of the hypothetical Italian loan portfolio. It is composed of 17 clusters (or sub-portfolios) representing the 17 different Italian regions in which the obligors (namely, non-financial firms) reside. In Table 1, we report for each regional cluster i ($i = 1, \ldots, 17$), the value of the credit exposure in Euros, EaD_i; the number of the obligors belonging to each regional cluster, N_i; the probability of default of the cluster, PD_i; and the asset return correlation of each cluster, calculated by both Basel's formula and by the two selected statistical methods.

20 See Basel Committee on Banking Supervision (2003, 2004).
21 i.e., all the obligors in the portfolio have the same exposure, the number of obligors is very high, each exposure is very low compared to the total portfolio exposure, EaD, the VaR is estimated by a one-factor version of the Merton (1974) model.
22 These data can be freely downloaded by the web site: www.bancaditalia.it.

Table 1. Composition and characteristics of the Italian loan portfolio composed of 17 regional clusters.

Cluster	EaD	N	PD	rho (Lucas)	rho (MLH)	rho (Basel)
LIGURIA	102,000	510	3.43%	1.85%	1.87%	14.16%
LOMBARDIA	252,000	1260	3.22%	1.97%	2.05%	14.40%
TRENTINO-ALTO ADIGE	72,000	360	2.71%	1.69%	2.03%	15.10%
VENETO	142,000	710	3.09%	1.77%	1.84%	14.56%
FRIULI-VENEZIA GIULIA	64,000	320	3.19%	1.58%	1.62%	14.43%
EMILIA-ROMAGNA	183,000	915	3.01%	1.90%	1.98%	14.66%
MARCHE	94,000	470	3.80%	2.68%	2.56%	13.79%
TOSCANA	128,000	640	3.80%	1.92%	2.00%	13.80%
UMBRIA	76,000	380	4.01%	2.45%	2.45%	13.62%
LAZIO	231,000	1155	4.89%	1.68%	1.68%	13.04%
CAMPANIA	132,000	660	5.17%	1.81%	1.72%	12.91%
CALABRIA	54,000	270	5.82%	2.20%	2.17%	12.65%
SICILIA	174,000	870	5.39%	1.91%	1.77%	12.81%
SARDEGNA	68,000	340	4.93%	1.69%	1.72%	13.02%
PIEMONTE E VALLE D'AOSTA	153,000	765	3.06%	1.59%	1.60%	14.60%
ABRUZZO E MOLISE	84,000	420	4.83%	2.55%	2.58%	13.07%
PUGLIA E BASILICATA	91,000	455	4.35%	2.05%	1.96%	13.36%
TOTAL	2,100,000	10,500				

Source: our elaboration.

We may note that the asset return correlations estimated by the maximum likelihood method and by Lucas's approach are remarkably lower than the ones calculated by Basel's formula.

We underline that this outcome is coherent with the results of other empirical studies on the Italian, French and German markets (Dietsch and Petey 2004; Duellmann and Scheule 2003; Hamerle and Rösch 2006; Sironi and Zazzara 2003).

The majority of Italian companies are small and medium enterprises (SMEs). It is well known that SMEs are more affected by their specific or idiosyncratic risks than by systematic risks. On the contrary, large firms are much more sensitive to the fluctuations of systematic risk factors such as, for example, the general economic condition.

Moreover, from our empirical results, we do not observe a negative relation between the estimated correlations and the PDs. Conversely, Basel's correlation formula assumes that the correlation decreases when the PD increases (and vice versa).

In Table 2, we collect the results in terms of capital requirements, K, or portfolio risk contributions for each cluster and for the total portfolio, obtained by implementing Basel's IRB approach and utilizing both the regulatory correlations and the correlations estimated by maximum likelihood method alternately. We can immediately observe that the values of capital requirements (expressed in both percentages and monetary terms) collapse when we use the estimated correlations (rho MLH) instead of the regulatory ones (rho Basel). Basel's IRB model is, therefore, extremely sensitive to the value of the asset correlations.

Subsequently, we have calculated different portfolio risk measures, namely VaR, ML and ES, for a confidence level of 99.9% for each cluster and the total loan portfolio by implementing the MC simulation model described in Section 2. We have expressed the portfolio losses in monetary terms and in the percentage of the exposure value of the total portfolio and of each cluster. In order to estimate the portfolio loss distribution, we have generated 100,000 MC scenarios. First, we have supposed that the credit exposure to each obligor is homogenous and equal to 200 euros; in this way, the hypothesis of the infinite granularity of the portfolio is approximated.

The outcomes, in terms of different portfolio risk metrics obtained by utilizing the rho estimated by MLH method, are reported in Table 3.

Table 2. Capital requirements, K, for each regional cluster, calculated by the IRB model and by utilizing the regulatory correlations (rho Basel) and the correlations estimated by MLH method (rho MLH) alternatively.

Cluster	K% (rho Basel)	K (rho Basel)	K% (rho MLH)	K (rho MLH)
LIGURIA	10.75%	10,962	3.56%	3627
LOMBARDIA	10.45%	26,340	3.49%	8806
TRENTINO-ALTO ADIGE	9.71%	6990	3.00%	2163
VENETO	10.27%	14,582	3.23%	4592
FRIULI-VENEZIA GIULIA	10.41%	6661	3.17%	2030
EMILIA-ROMAGNA	10.16%	18,590	3.26%	5959
MARCHE	11.25%	10,573	4.38%	4121
TOSCANA	11.24%	14,388	3.97%	5086
UMBRIA	11.52%	8755	4.50%	3419
LAZIO	12.66%	29,239	4.62%	10,678
CAMPANIA	13.01%	17,175	4.88%	6444
CALABRIA	13.82%	7464	5.84%	3155
SICILIA	13.29%	23,126	5.11%	8895
SARDEGNA	12.71%	8645	4.69%	3192
PIEMONTE E VALLE D’AOSTA	10.23%	15,646	3.05%	4666
ABRUZZO E MOLISE	12.59%	10,577	5.35%	4498
PUGLIA E BASILICATA	11.97%	10,896	4.43%	4028
TOTAL	11.46%	240,610	4.06%	85,359

Source: our elaboration.

Table 3. Portfolio risk measures calculated by the MC simulation model, assuming the granularity of the loan portfolio and correlations estimated by maximum likelihood method.

Cluster	VaR 99.9%	%	ML 99.9%	%	ES 99.9%	%
LIGURIA	3215	3.15%	4791	4.70%	5184	5.08%
LOMBARDIA	8312	3.30%	11,964	4.75%	13,418	5.32%
TRENTINO-ALTO ADIGE	2415	3.35%	3293	4.57%	3365	4.67%
VENETO	4841	3.41%	6816	4.80%	7430	5.23%
FRIULI-VENEZIA GIULIA	2365	3.70%	3284	5.13%	3412	5.33%
EMILIA-ROMAGNA	6440	3.52%	8922	4.88%	10,012	5.47%
MARCHE	4444	4.73%	6052	6.44%	6436	6.85%
TOSCANA	5449	4.26%	7637	5.97%	8284	6.47%
UMBRIA	3254	4.28%	4625	6.09%	5513	7.25%
LAZIO	9371	4.06%	14,450	6.26%	16,431	7.11%
CAMPANIA	6406	4.85%	9475	7.18%	10,206	7.73%
CALABRIA	2765	5.12%	4179	7.74%	4998	9.26%
SICILIA	10,266	5.90%	14,486	8.33%	14,568	8.37%
SARDEGNA	2586	3.80%	4095	6.02%	4684	6.89%
PIEMONTE E VALLE D’AOSTA	4158	2.72%	6265	4.09%	7482	4.89%
ABRUZZO E MOLISE	4198	5.00%	6026	7.17%	6480	7.71%
PUGLIA E BASILICATA	4613	5.07%	6396	7.03%	6519	7.16%
TOTAL	85,098	4.05%	122,755	5.85%	134,422	6.40%

Source: our elaboration.

As expected, the results, in terms of the VaR in Table 3, are not too different from the ones obtained by Basel’s IRB model (see Table 2) and by utilizing the rho estimated by the MLH method.

In fact, from the simulative model we obtain, for example, a value of VaR for the total portfolio equal to 85,098 euros (or 4.05 percent of the total exposure) versus a capital requirement K from the Basel IRB model equal to 85,359 euros (or 4.06 percent of the total exposure).

Also, when we calculate the capital requirements by the two different models, assuming again the granularity of the loan portfolio, but utilizing the regulatory correlations, we find similar outcomes from the two approaches (see Table 4).

Table 4. 99.9% VaR, 99.9% ML and 99.9% ES calculated from the simulation model by assuming the granularity of the portfolio and utilizing the regulatory correlations.

Cluster	VaR 99.9%	%	ML 99.9%	%	ES 99.9%	%
LIGURIA	11,251	11.03%	12,826	12.57%	28,598	28.04%
LOMBARDIA	25,709	10.20%	29,362	11.65%	83,373	33.08%
TRENTINO-ALTO ADIGE	6185	8.59%	7062	9.81%	18,757	26.05%
VENETO	13,920	9.80%	15,895	11.19%	41,601	29.30%
FRIULI-VENEZIA GIULIA	6679	10.44%	7598	11.87%	11,975	18.71%
EMILIA-ROMAGNA	17,576	9.60%	20,058	10.96%	53,219	29.08%
MARCHE	10,412	11.08%	12,021	12.79%	29,338	31.21%
TOSCANA	14,484	11.32%	16,671	13.02%	38,106	29.77%
UMBRIA	8963	11.79%	10,333	13.60%	25,755	33.89%
LAZIO	33,447	14.48%	38,526	16.68%	43,155	18.68%
CAMPANIA	19,514	14.78%	22,583	17.11%	36,561	27.70%
CALABRIA	8788	16.27%	10,202	18.89%	16,884	31.27%
SICILIA	26,525	15.24%	30,745	17.67%	39,369	22.63%
SARDEGNA	9604	14.12%	11,112	16.34%	19,470	28.63%
PIEMONTE E VALLE D'AOSTA	15,288	9.99%	17,396	11.37%	43,516	28.44%
ABRUZZO E MOLISE	11,534	13.73%	13,361	15.91%	29,008	34.53%
PUGLIA E BASILICATA	12,334	13.55%	14,117	15.51%	18,726	20.58%
TOTAL	252,211	12.01%	289,868	13.80%	577,410	27.50%

Source: our elaboration.

For example, we obtain a total portfolio VaR of 252,211 euros, equal to 12.01 percent of the total exposure from the simulative model (Table 4), versus a capital requirement value K of 240,610 euros for the total portfolio, or 11.46% of the total exposure from the IRB model (Table 2). It is obvious that the capital requirements grow as correlations increase, other conditions being equal.

Successively, we relax the hypotheses of infinite granularity, or the absence of undiversified idiosyncratic risks in the portfolio, for calculating the capital requirements of a loan portfolio with the same characteristics but with much more concentrated credit exposures. Precisely, we assume that, in each cluster, half of the banking exposure is concentrated towards a single counterparty. For instance, in the case of cluster "Lazio", 115,500 euros of the total exposure of 231,000 euros are concentrated in a single obligor, while the remaining 115,500 euros are homogeneously distributed among the remaining 1154 obligors in this cluster.

The outcomes, in terms of different portfolio risk measures estimated by the MC simulation model, are reported in Table 5 for the case of asset correlations calculated by the maximum likelihood method, and in Table 6 for the case of asset correlations calculated by Basel's formula.

Dropping the hypothesis of infinite granularity for the loan portfolio, the results in terms of VaR obtained from the two different models differ strongly, particularly when the correlations are low. In fact, as we have already said, when the asset correlations are close to 1, the portfolio may be thought of as being composed of only one obligor. Therefore, when the correlations are high, a portfolio with high granularity exhibits similar results in terms of VaR to a portfolio with low granularity.

Table 5. 99.9% VaR, 99.9% ML and 99.9% ES estimated by the MC simulation model, assuming a concentrated portfolio and asset correlations estimated by the maximum likelihood method.

Cluster	VaR 99.9%	%	ML 99.9%	%	ES 99.9%	%
LIGURIA	7475	7.33%	9051	8.87%	12,496	12.25%
LOMBARDIA	49,148	19.50%	52,801	20.95%	105,417	41.83%
TRENTINO-ALTO ADIGE	1830	2.54%	2707	3.76%	730	1.01%
VENETO	12,998	9.15%	14,973	10.54%	2104	1.48%
FRIULI-VENEZIA GIULIA	1491	2.33%	2410	3.77%	796	1.24%
EMILIA-ROMAGNA	22,352	12.21%	24,835	13.57%	24,690	13.49%
MARCHE	7402	7.87%	9011	9.59%	7508	7.99%
TOSCANA	14,848	11.60%	17,035	13.31%	9980	7.80%
UMBRIA	4426	5.82%	5796	7.63%	1448	1.90%
LAZIO	50,678	21.94%	55,757	24.14%	70,900	30.69%
CAMPANIA	18,102	13.71%	21,171	16.04%	25,979	19.68%
CALABRIA	3300	6.11%	4714	8.73%	3873	7.17%
SICILIA	27,839	16.00%	32,058	18.42%	34,769	19.98%
SARDEGNA	4493	6.61%	6002	8.83%	1601	2.35%
PIEMONTE E VALLE D'AOSTA	15,680	10.25%	17,788	11.63%	2332	1.52%
ABRUZZO E MOLISE	7765	9.24%	9593	11.42%	7311	8.70%
PUGLIA E BASILICATA	6896	7.58%	8679	9.54%	13,678	15.03%
TOTAL	256,723	12.22%	294,380	14.02%	325,613	15.51%

Source: our elaboration.

Table 6. 99.9% VaR, 99.9% ML and 99.9% ES, estimated by MC simulation model, assuming a concentrated portfolio and asset correlations calculated by regulatory formula.

Cluster	VaR 99.9%	%	ML 99.9%	%	ES 99.9%	%
LIGURIA	21,027	20.62%	22,778	22.33%	32,008	31.38%
LOMBARDIA	84,547	33.55%	88,606	35.16%	146,614	58.18%
TRENTINO-ALTO ADIGE	10,166	14.12%	11,141	15.47%	11,003	15.28%
VENETO	26,553	18.70%	28,748	20.25%	50,064	35.26%
FRIULI-VENEZIA GIULIA	6915	10.81%	7936	12.40%	6364	9.94%
EMILIA-ROMAGNA	43,261	23.64%	46,019	25.15%	85,417	46.68%
MARCHE	17,663	18.79%	19,450	20.69%	21,906	23.30%
TOSCANA	27,615	21.57%	30,045	23.47%	21,644	16.91%
UMBRIA	17,732	23.33%	19,255	25.34%	29,415	38.70%
LAZIO	70,481	30.51%	76,124	32.95%	79,155	34.27%
CAMPANIA	34,217	25.92%	37,627	28.51%	30,717	23.27%
CALABRIA	11,771	21.80%	13,341	24.71%	8766	16.23%
SICILIA	42,209	24.26%	46,898	26.95%	46,402	26.67%
SARDEGNA	14,430	21.22%	16,106	23.69%	17,411	25.60%
PIEMONTE E VALLE D'AOSTA	33,115	21.64%	35,457	23.17%	25,110	16.41%
ABRUZZO E MOLISE	20,262	24.12%	22,293	26.54%	25,770	30.68%
PUGLIA E BASILICATA	15,460	16.99%	17,441	19.17%	22,168	24.36%
TOTAL	497,427	23.69%	539,268	25.68%	659,934	31.43%

Source: our elaboration.

The capital requirements estimated by the simulative model are always greater than those calculated by the IRB model; they are approximately double. In particular, we find a value of VaR equal to 23.69% (see Table 6) for the simulation model versus a value of 11.46% for the Basel model (see Table 2) when we utilize the regulatory correlations, which are typically much higher than the correlations estimated by MLH method.

Utilizing low correlations, that is, the correlations obtained by the statistical methods (MLH and Lucas's model, in our case), the capital requirements from the MC simulative model are always more severe than those attained from the IRB model; they are approximately triple. For example, we find by the simulative model a VaR of 12.22% (see Table 5) versus a capital requirement of 4.06% from the IRB model (see Table 2).

The underestimation of risk and capital is evident when we drop the strong hypothesis of a highly diversified portfolio. For this reason, mostly in the case of undiversified portfolios, coherent capital allocation is the appropriate choice for the purpose of risk management.

In Table 7, we report the capital allocated to each cluster as a percentage of the total capital, in order to underline the differences between the asset allocations in terms of ML (or, equivalently, VaR) and in terms of ES. In this case, the results in terms of mean-variance (Maximum Loss, ML) and coherent asset allocation (Expected Shortfall, ES) are remarkably different.

Table 7. Results of capital allocation, expressed in terms of ML (traditional allocation) and ES (coherent allocation), assuming concentrated clusters and correlations calculated by both MLH method and Basel's formula.

Cluster	ML (rho MLH)	ES (rho MLH)	ML (rho Basel)	ES (rho Basel)
LIGURIA	3.07%	3.84%	4.22%	4.85%
LOMBARDIA	17.94%	32.37%	16.43%	22.22%
TRENTINO-ALTO ADIGE	0.92%	0.22%	2.07%	1.67%
VENETO	5.09%	0.65%	5.33%	7.59%
FRIULI-VENEZIA GIULIA	0.82%	0.24%	1.47%	0.96%
EMILIA-ROMAGNA	8.44%	7.58%	8.53%	12.94%
MARCHE	3.06%	2.31%	3.61%	3.32%
TOSCANA	5.79%	3.07%	5.57%	3.28%
UMBRIA	1.97%	0.44%	3.57%	4.46%
LAZIO	18.94%	21.77%	14.12%	11.99%
CAMPANIA	7.19%	7.98%	6.98%	4.65%
CALABRIA	1.60%	1.19%	2.47%	1.33%
SICILIA	10.89%	10.68%	8.70%	7.03%
SARDEGNA	2.04%	0.49%	2.99%	2.64%
PIEMONTE E VALLE D'AOSTA	6.04%	0.72%	6.58%	3.81%
ABRUZZO E MOLISE	3.26%	2.25%	4.13%	3.90%
PUGLIA E BASILICATA	2.95%	4.20%	3.23%	3.36%

Source: our elaboration.

Typically for concentrated portfolios, a coherent capital allocation is advisable, given the strong differences deriving from the two different techniques, the traditional (ML) and the coherent (ES). In particular, the capital allocated to the clusters with greater concentration (such as Lombardia, Lazio, and Sicilia) has a greater increase.

Therefore, mostly for concentrated portfolios, the utilization of a coherent capital allocation technique is particularly justified.

Conversely, in the case of granular or diversified portfolios, the outcomes in terms of capital allocation derived from the two different techniques, the traditional (ML) and the coherent (ES), seem very similar, especially when the correlations are low (see Table 8). The reason for this is the adoption of a one-factor model alongside the assumption of high granularity for the loan portfolio. In order to compare the differences between the asset allocation in terms of ML (or, equivalently, VaR), calculated by the mean-variance approach, and in terms of ES, we collect in Table 8 the capital allocated to every cluster in the portfolio as a percentage of the total capital.

Table 8. Results of capital allocation, expressed in terms of ML (traditional allocation) and ES (coherent allocation), assuming diversified clusters and correlations calculated by both MLH method and Basel's formula.

Cluster	ML (rho MLH)	ES (rho MLH)	ML (rho Basel)	ES (rho Basel)
LIGURIA	3.90%	3.86%	4.42%	4.95%
LOMBARDIA	9.75%	9.98%	10.13%	14.44%
TRENTINO-ALTO ADIGE	2.68%	2.50%	2.44%	3.25%
VENETO	5.55%	5.53%	5.48%	7.20%
FRIULI-VENEZIA GIULIA	2.68%	2.54%	2.62%	2.07%
EMILIA-ROMAGNA	7.27%	7.45%	6.92%	9.22%
MARCHE	4.93%	4.79%	4.15%	5.08%
TOSCANA	6.22%	6.16%	5.75%	6.60%
UMBRIA	3.77%	4.10%	3.56%	4.46%
LAZIO	11.77%	12.22%	13.29%	7.47%
CAMPANIA	7.72%	7.59%	7.79%	6.33%
CALABRIA	3.40%	3.72%	3.52%	2.92%
SICILIA	11.80%	10.84%	10.61%	6.82%
SARDEGNA	3.34%	3.48%	3.83%	3.37%
PIEMONTE E VALLE D'AOSTA	5.10%	5.57%	6.00%	7.54%
ABRUZZO E MOLISE	4.91%	4.82%	4.61%	5.02%
PUGLIA E BASILICATA	5.21%	4.85%	4.87%	3.24%

Source: our elaboration.

From Table 8, we can see that, when the asset correlations are lower, the capital allocation performed by the mean-variance approach (Maximum Loss, ML) is very close to the coherent capital allocation (Expected Shortfall, ES). In fact, if the asset correlation is close to zero, the loss distributions of the single clusters and the whole portfolio distribution converge rapidly towards the Normal distribution. In this case, the mean-variance capital allocation is equivalent to the coherent capital allocation (see, for example, Rockafellar and Uryasev 2000; 2002).

Indeed, we find some differences when the asset correlations are higher. In this case, the loss distributions for each cluster have a slower convergence towards the Normal distribution. It is well known that, when the correlations go to one, all the obligors in the portfolio may be considered as a single obligor. Consequently, when the correlations increase, the hypothesis of infinite granularity tends to fail. In particular, greater differences can be observed in those clusters with higher correlations and with a lower number of obligors.

Now, we relax the hypothesis of deterministic and constant recovery rates. Specifically, we implement the simulative model to the original loan portfolio, but assume the recovery rates to be stochastic and related to each other and with the default event, following the approach of Pykhtin (2003) and Tasche (2004), previously described in Section 4.

In this context, we assume $b_{m(i)} = c_{m(i)}$, with homogenous credit exposures $i = 1, \ldots, N$, and with all the maturities equal to one year. For each obligor, we adopt a mean recovery rate $m = 0.5$ and a standard deviation $s = 0.2$. For each cluster in the portfolio, we estimate its own loss distribution, the respective risk measures, and the capital allocations utilizing both the correlations estimated by the MLH method and the correlations derived from Basel's formula.

The results are reported in Table 9. Comparing the results in Table 9 with those described previously in Tables 3 and 4, we find that all the credit risk measures are more severe when we drop the hypothesis of deterministic and constant recovery rates to assume stochastic recovery rates related to each other and to the default event. This is particularly true when the correlations are calculated by Basel's formula. In fact, we have assumed $b_{m(i)} = c_{m(i)}$, for $i = 1, \ldots, N$, and, therefore, the correlations among the recovery rates and between the default events and the recovery rates themselves are higher, as the asset return correlations are also higher. In other words, we obtain greater values of risk measures (ML and ES) when we utilize the correlations calculated by Basel's formula.

Table 9. 99.9% ML and 99.9% ES, estimated by an MC simulation model, assuming the recovery rates as stochastic and related with the default event and homogeneous portfolios.

Cluster	ML 99.9% (rho MLH)	%	ES 99.9% (rho MLH)	%	ML 99.9% (rho Basel)	%	ES 99.9% (rho Basel)	%
LIGURIA	5375	5.27%	5433	5.33%	29,499	28.92%	37,557	36.82%
LOMBARDIA	13,337	5.29%	14,609	5.80%	78,158	31.02%	108,433	43.03%
TRENTINO-ALTO ADIGE	3679	5.11%	3451	4.79%	16,266	22.59%	24,167	33.56%
VENETO	7598	5.35%	7946	5.60%	38,587	27.17%	55,300	38.94%
FRIULI-VENEZIA GIULIA	3672	5.74%	4142	6.47%	11,506	17.98%	15,987	24.98%
EMILIA-ROMAGNA	10,139	5.54%	10,888	5.95%	47,863	26.15%	68,440	37.40%
MARCHE	6931	7.37%	7599	8.08%	28,131	29.93%	39,004	41.49%
TOSCANA	8584	6.71%	8900	6.95%	39,448	30.82%	53,269	41.62%
UMBRIA	5202	6.84%	6272	8.25%	26,564	34.95%	34,676	45.63%
LAZIO	16,500	7.14%	18,319	7.93%	71,067	30.76%	80,739	34.95%
CAMPANIA	10,972	8.31%	12,151	9.21%	42,368	32.10%	48,838	37.00%
CALABRIA	4699	8.70%	5582	10.34%	19,865	36.79%	21,415	39.66%
SICILIA	16,529	9.50%	17,136	9.85%	47,665	27.39%	55,082	31.66%
SARDEGNA	4692	6.90%	4916	7.23%	23,125	34.01%	27,819	40.91%
PIEMONTE E VALLE D'AOSTA	7075	4.62%	7490	4.90%	42,939	28.06%	58,622	38.32%
ABRUZZO E MOLISE	6816	8.11%	7405	8.82%	31,623	37.65%	38,294	45.59%
PUGLIA E BASILICATA	7297	8.02%	7290	8.01%	21,555	23.69%	24,633	27.07%
TOTAL	139,096	6.62%	149,530	7.12%	616,230	29.34%	792,277	37.73%

Source: our elaboration.

7. Conclusions

From the outcomes of this work, the strong underestimation of portfolio credit risk produced by the IRB model is evident, given its restrictive underlying hypotheses. First, when we drop the assumption of highly diversified portfolios, the estimates of the portfolio risk measures obtained by implementing the advanced quantitative methodology (described in Sections 2 and 3) increase significantly. Also, the results in terms of coherent capital allocation rise considerably. For this reason, mostly in the case of undiversified portfolios, coherent capital allocation is the appropriate choice for efficient credit risk management. Secondly, when we lower the hypothesis of constant and independent recovery rates, we obtain capital requirements more severe than those from the IRB model. Our main conclusion is that the IRB model is unable to capture the real credit risk of a loan portfolio because it does not take into account the real dependence structure among the default events and between the recovery rates and the default events. The adoption of this regulatory model can produce a dangerous underestimation of the portfolio credit risk, particularly when the economic uncertainty and the volatility of the financial markets increase. In summary, the most original findings of this research are the following:

- For Italian SMEs, the asset return correlations estimated by the maximum likelihood method and by Lucas' approach are remarkably lower than those calculated by Basel's formula.
- Contrary to the regulatory hypothesis, a negative relation between the estimated correlations and the PDs is not found for Italian SMEs.
- The Basel IRB model, all things being equal, is very and positively influenced by the value of the correlations.
- The credit capital requirements calculated by the IRB model and by the simulative model are quite similar if we maintain the restrictive hypotheses of the regulatory approach.
- After removing the hypothesis of infinite granularity for the loan portfolios, the results in terms of VaR obtained from the two different models differ strongly. The capital requirements estimated by the simulative model are always greater than those calculated by the IRB model, mostly when the correlations are low.
- The underestimation of risk and capital is evident when we drop the strong hypothesis of a highly diversified portfolio. For this reason, mostly in the case of undiversified portfolios, coherent capital allocation is the appropriate choice for purposes of risk management.
- Typically, for concentrated portfolios, coherent capital allocation is advisable, given the strong differences deriving from the two different techniques, the traditional (ML) and the coherent (ES). In particular, the capital allocated to the clusters with greater concentration (such as Lombardia, Lazio, and Sicilia) has a greater increase.
- On the other hand, in the case of granular or diversified portfolios, the outcomes in terms of the capital allocation derived from the two different techniques, ML and ES, seem very similar, especially when the correlations are low. In other words, the mean-variance capital allocation is equivalent to coherent capital allocation.
- The values of the portfolio's credit risk measures (ML and ES) become more severe when the hypotheses of deterministic and constant recovery rates are dropped. This is particularly true when we utilize the high correlations calculated utilizing Basel's formula.

One limitation of this advanced methodology is the use of a "pure" or standard Monte Carlo simulation for computing the risk contributions in terms of a coherent risk metric as the Expected Shortfall. These estimates present a very slow convergence towards their true values when traditional Monte Carlo algorithms are used. This problem underlines the necessity of utilizing importance sampling (IS) techniques for estimating the coherent risk contributions efficiently, instead of a simple Monte Carlo simulation model. IS permits us to artificially increase the number of scenarios in the tail of the probability distribution, thus reducing considerably the volatility of the estimate of the tail risk measures. Additional extensions are the possibility of introducing country risk and contagion risk.

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