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Stochastic Volatility and GARCH: Do Squared End-of-Day Returns Provide Similar Information?

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Abstract: The paper examines the relative performance of Stochastic Volatility (SV) and GARCH(1,1) models fitted to twenty plus years of daily data for three indices. As a benchmark, I use the realized volatility (RV) for the S&P 500, DOW JONES and STOXX50 indices, sampled at 5-minute intervals, taken from the Oxford Man Realised Library. Both models demonstrate comparable performance and are correlated to a similar extent with the RV estimates, when measured by OLS. However, a crude variant of Corsi's (2009) Heterogenous Auto-Regressive (HAR) model, applied to squared demeaned daily returns on the indices, appears to predict the daily RV of the series, better than either of the two base models. The base SV model was then enhanced by adding a regression matrix including the first and second moments of the demeaned return series. Similarly, the GARCH(1,1) model was augmented by adding a vector of demeaned squared returns to the mean equation. The augmented SV model showed a marginal improvement in explanatory power. This leads to the question of whether we need either of the two standard volatility models, if the simple expedient of using lagged squared demeaned daily returns provides a better RV predictor, at least in the context of the indices in the sample. The paper thus explores whether simple rules of thumb match the volatility forecasting capabilities of more sophisticated models.

1. Introduction

The paper explores the performance of Stochastic Volatility and GARCH(1,1) models as estimators of the volatility of the S&P 500, the DOW JONES and the STOXX50 indices. The volatilities estimated by these models are compared with realized volatility estimates for the three indices, obtained from the Oxford Man Realised Library, sampled at 5-minute intervals, as described in Heber et al. (2009). Their volatility forecasts are further compared with those derived from a simple historical volatility model. The data series features a 20 year sample window of daily prices taken from 3 January 2000 to 30 April 2020, initially comprising 5100 observations, also taken from the Oxford Man Realised Library. The study is a companion study to Allen and McAleer (2020).

The paper is motivated by Poon and Granger (2003, p. 507) who observed that: "as a rule of thumb, historical volatility methods work equally well compared with more sophisticated ARCH class and SV models." The paper features a continued exploration of this observation in the context of basic GARCH and Stochastic Volatility models as compared with a simple historical volatility model based on lags of squared demeaned daily returns.

The R packages, stochvol and factorstochvol, are used, which employ Markov chain Monte Carlo (MCMC) samplers to conduct inference by obtaining draws from the posterior distribution of parameters and latent variables, which can then be used for predicting future volatilities. This is done within the context of a fully Bayesian implementation of heteroskedasticity modelling within the framework of stochastic volatility. For more information, see the discussion of the methods by Kastner and Frühwirth-Schnatter (2014), Kastner et al. (2017), and of the stochvol and factorstochvol packages by Kastner (2016, 2019).

Taylor (1982) developed modelling volatility probabilistically, through a state-space model where the logarithm of the squared volatilities—the latent states—follow an autoregressive process of order one, which became known as the stochastic volatility (SV) model. Jacquier et al. (1994), Ghysels et al. (1996), and Kim et al. (1998) provided evidence in support of the application of stochastic volatility models, but their practical use has been infrequent. The shortage of empirical applications of the SV models has been limited by two major factors: the variety (and potential incompatibility) of estimation methods for SV models, plus the lack of standard software packages (see Bos (2012)). The situation for multivariate SV was even more problematic until Hosszejni and Kastner (2019) developed the R package 'factorstochvol'.

Taylor (1994) reviews the stochastic volatility and the ARCH/GARCH literature. Other reviews are by McAleer (2005) and Asai et al. (2006). Poon and Granger (2003, p. 485), noted the difficulties in the application of the SV model: "the SV model has no closed form, and hence cannot be estimated directly by maximum likelihood". The advantage of the stochvol and factorstochvol R packages is that they incorporate an efficient MCMC estimation scheme for SV models, as discussed by Kastner and Frühwirth-Schnatter (2014) and Kastner et al. (2017). These two R library packages facilitate the analysis in the paper, which features a direct comparison of the volatility predictions of a SV model, a GARCH (1,1) model, and a simple application of a historical volatility based estimation method, as applied to the the three indices.

The paper is divided into four sections: Section 2 reviews the literature and econometric method employed. Section 3 presents the results, and Section 4 presents conclusions.

2. Previous Work and Econometric Models

2.1. Stochastic Volatility

There have been numerous empirical studies of changes in volatility in various stock, currency, and commodities markets. The findings in volatility research have implications for option pricing, volatility estimation, and the degree to which volatility shocks persist. These research questions have been approached by means of different models and methodologies.

Taylor (1982) suggested a novel SV approach, and Taylor (1994) developed the SV model as follows: if P_t denotes the prices of an asset at time t, and assuming no dividend payments, the returns on the asset can be defined, in the context of discrete time periods, as:

$$X_t = ln[P_t/P_{t-1}]. \tag{1}$$

Volatility is customarily indicated by σ, and prices are described by a stochastic differential equation:

$$d(lnP) = \mu dt + \sigma dW, \tag{2}$$

with W a standard Weiner process. If μ and σ are constants, X_t has a normal distribution, and:

$$X_t = \mu + \sigma U_t, \tag{3}$$

with U_t independent and identically distributed (*i.i.d*).

Equation (3) can be generalised by replacing σ with a positive random variable σ_t, to give:

$$X_t = \mu + \sigma_t U_t, \tag{4}$$

where $U_t \sim N(0,1)$.

In circumstances where the returns process $\{X_t\}$ can be presented by Equation (4), Taylor (1994) calls σ_t the stochastic volatility for period t. His definition assumes that $(X_t - \mu)/\sigma_t$ follows a normal distribution.

The stochastic process $\{\sigma_t\}$ generates realised volatilities $\{\sigma_t^*\}$ which, in general, are not observable. For any realisation, σ_t^*:

$$X_t \mid \sigma_t = \sigma_{t*}(\mu, \sigma_{t*2}). \tag{5}$$

The mixture of these conditional normal distributions defines the unconditional distribution of X_t, which will have excess kurtosis whenever σ_t has positive variance and is independent of U_t.

In the empirical section which follows, I use the RV of the S&P500, DOW JONES and STOXX50 indices, sampled at 5-minute intervals provided by Oxford Man, as a proxy for the true realised volatility. I then compare the estimates of volatility obtained from SV and GARCH(1,1) models, using the RV estimates as a benchmark.

Taylor (1994, p. 3) suggests using "capital letters to represent random variables and lower case letters to represent outcomes". I shall follow that convention. Given observed returns of $I_{t-1} = \{x_1, x_2, ...x_{t-1}\}$ the conditional variance for period t is:

$$h_t = var(X_t \mid I_{t-1}). \tag{6}$$

Taylor (1994) notes that, in general, the random variable H_t, which generates the observed conditional variance h_t is not, in general, equal to σ_t^2. A convenient way to use economic theory to motivate changes in volatility is to assume that returns are generated by a number of intra-period price revisions, as in the manner of Clark (1973) and Tauchen and Pitts (1983).

It is assumed that there are N_t price revisions during trading day t, each caused by unpredictable information. Let event i on day t change the logarithmic price by ω_{it}, with:

$$X_t = \mu + \sum_{i=1}^{N_t} \omega_{it}. \tag{7}$$

If we assume that $\omega_{it} \sim i.i.d.$ and is independent of the random variable N_t, with $\omega_{it} \sim N(0, \sigma_\omega^2)$, then:

$$\sigma_t^2 = \sigma_\omega^2 N_t \ \ and \ \ H_t = \sigma_\omega^2 E[N_t \mid I_{t-1}]. \tag{8}$$

The above model suggests that squared volatility is proportional to the amount of price information.

The lack of standard software to estimate such a model is addressed by Kastner and Frühwirth-Schnatter (2014) who propose an efficient MCMC estimation scheme which is implemented in the R stochvol package, Kastner (2016). Kastner (2016, p. 2) proceeds by "letting $y = (y_1, y_2, ...y_n)^|$ be a vector of returns, with mean zero. An intrinsic feature of the SV model is that each observation y_t is assumed to have its 'own' contemporaneous variance, e^{h_t}, which relaxes the usual assumption of homoscedasticity". It is assumed that the logarithm of this variance follows an autoregressive process of order one. This assumption is fundamentally different to GARCH models, where the time-varying conditional volatility is assumed to follow a deterministic instead of a stochastic evolution.

The centered parameterization of the SV model can be given as:

$$y_t \mid h_t \sim N(0, exp^{h_t}), \tag{9}$$

$$h_t \mid h_{t-1}, \mu, \phi, \sigma_n \sim N(\mu, +\phi(h_{t-1} - \mu), \sigma_n^2), \tag{10}$$

$$h_0 \mid \mu, \phi, \sigma_n \sim N(\mu, \sigma_n^2 / (1 - \phi^2)), \tag{11}$$

where $N(\mu, \sigma_n^2)$ denotes the normal distribution with mean μ and variance σ_n^2. $\theta = (\mu, \phi, \sigma_n)^|$ is the vector of parameters which consists of the level of the log variance μ, the persistence of log variance, ϕ, and the volatility of log variance, σ_n. The process $h = (h_0, h_1, \ldots, h_n)^|$ which features in equations (10) and (11) is the unobserved or latent time-varying volatility process.

Kastner (2016, p. 4) remarks that: "A novel and crucial feature of the algorithm implemented in stochvol is the usage of a variant of the "ancillarity-sufficiency interweaving strategy" (ASIS) which was suggested in the general context of state-space models by Yu and Meng (2011). ASIS exploits the fact that, for certain parameter constellations, sampling efficiency improves substantially when considering a non-centered version of a state-space model".

Another key feature of the algorithm used in stochvol is the joint sampling of all instantaneous volatilities "all without a loop" (AWOL), a technique with links to Rue (2001) and as discussed in McCausland et al. (2011). The combination of these features enables the R package stochvol to estimate SV models efficiently even when large datasets are involved.

Kastner et al. (2017) suggest that the multivariate factor stochastic volatility (SV) model (Chib et al. 2006) provides a means of uniting simplicity with flexibility and robustness. It is simple in the sense that the potentially high-dimensional observation space is reduced to a lower-dimensional orthogonal latent factor space. It is flexible in the sense that these factors are allowed to exhibit volatility clustering, and it is robust in the sense that idiosyncratic deviations are themselves stochastic volatility processes, thereby allowing for the degree of volatility co-movement to be time-varying. Hosszejni and Kastner (2019) set up a factor SV model employed in the factorstochvol package in R, and the analysis also uses code from this package.

2.2. ARCH and GARCH

Engle (1982) developed the Autoregressive Conditional Heteroskedasticity (ARCH) model that incorporates all past error terms. It was generalised to GARCH by Bollerslev (1986) to include lagged term conditional volatility. GARCH predicts that the best indicator of future variance is a weighted average of long-run variance, the predicted variance for the current period, and any new information in this period, as captured by the squared residuals.

Consider a time series $y_t = E_{t-1}(y_t) + \varepsilon_t$, where $E_{t-1}(y_t)$ is the conditional expectation of y_t at time $t-1$ and ε_t is the error term. The basic GARCH model has the following specification:

$$\varepsilon_t = \sqrt{h_t \eta_t} \quad \eta_t (0, 1) \tag{12}$$

$$h_t = \omega + \sum_{j=1}^{p} \alpha_j \varepsilon_{t-j}^2 + \sum_{j=1}^{q} \beta_j h_{t-j} \tag{13}$$

in which $\omega > 0$, $\alpha_j \geq 0$ and $\beta_j \geq 0$ (usually a positive fraction), to ensure a positive conditional variance, $h_t \geq 0$ (see Tsay (1987)). The ARCH effect is captured by the parameter α_j, which represents the short-run persistence of shocks to returns, β_j captures the GARCH effect that contributes to long-run persistence, and $\alpha_j + \beta_j$ measures the persistence of the impact of shocks to returns to long-run persistence. A GARCH(1,1) process is weakly stationary if $\alpha_j + \beta_j \leq 1$. (See the discussion in Allen et al. (2013).

We contrast the estimates of volatility from the SV model with those from a GARCH(1,1) model, and assess which better explains the behaviour of the RV of FTSE sampled at 5-minute intervals.

2.3. Realised Volatility

Use was made of the RV 5-min estimates from Oxford Man for the three indices as the RV benchmark (see: https://realized.oxford-man.ox.ac.uk/data). This database contains "daily (close to close) financial returns , and a corresponding sequence of daily realised measures $rm_1, rm_2, ..., rm_T$. Realised measures are theoretically sound high frequency, nonparametric-based estimators of the variation of the price path of an asset during the times at which the asset trades frequently on an exchange. Realised measures ignore the variation of prices overnight and sometimes the variation in the first few minutes of the trading day when recorded prices may contain large errors". The metrics were developed by Andersen et al. (2001), Andersen et al. (2003), and Barndorff-Nielsen and Shephard (2002). Shephard and Sheppard (2010) provide an account of the RV measures used in the Oxford Man Realised Library.

The simplest realised metric is realised variance (RV):

$$RV_t = \sum x_{j,t}^2, \tag{14}$$

where $x_{j,t} = X_{t_{j,t}} - X_{t_{j-1,t}}$. The $t_{j,t}$ are the times of trades or quotes on the t-th day. The theoretical justification of this measure is that, if prices are observed without noise then, as $min\ j \mid t_{j,t} - t_{j-1,t} \mid \downarrow 0$, it consistently estimates the quadratic variation of the price process on the t-th day. If the sampling is reduced to very small intervals of time, market microstructure noise may become a contaminant. In order to avoid this issue, we use RV estimates from Oxford Man, sampled at 5-minute intervals, hereafter $RV5$.

2.4. Historical Volatility Model

Poon and Granger (2005) discuss various practical issues involved in forecasting volatility. They suggest that the HISVOL model has the following form:

$$\hat{\sigma}_t = \phi_1 \sigma_{t-1} + \phi_2 \sigma_{t-2} + .. + \phi_\tau \sigma_{t-\tau}, \tag{15}$$

where $\hat{\sigma}_t$ is the expected standard deviation at time t, ϕ is the weight parameter, and σ is the historical standard deviation for periods indicated by the subscripts. Poon and Granger (2005) suggest that this group of models include the random walk, historical averages, autoregressive (fractionally integrated) moving average, and various forms of exponential smoothing that depend on the weight parameter ϕ.

We use a simple form of this model in which the estimate of σ is the previous day's demeaned squared return. Poon and Granger (2005) review 66 previous studies, and suggest that implied standard deviations appear to perform best, followed by historical volatility and GARCH which have roughly equal performance. They also note that, at the time of writing, there were insufficient studies of SV models to come to any conclusions about this class of models. This observation provides the motivation for the current study which assesses the performance of all three classes of models. It also provides the motivation to use a crude rule of thumb in the form of 20 lags of daily squared demeaned returns. The choice of 20 lags is conditioned by Corsi's (2009) "Heterogeneous Autoregressive model of Realized Volatility" (HAR) model, as discussed in the next section. A feature of this model is that it includes estimates of daily, weekly, and monthly ex-post realised volatility. The crude HISVOL model adopted in the paper takes 20 lags as an approximation for this, as it roughly represents a month of trading days.

Barndorff-Nielsen and Shephard (2003) point out that taking the sums of squares of increments of log-prices has a long tradition in the financial economics literature. See, for example, Poterba and Summers (1986), Schwert (1989), Taylor and Xu (1997), Christensen and Prabhala (1998), Dacorogna et al. (1998), and Andersen et al. (2001). (Shephard and Sheppard (2010), p 200, footnote 4) note that: "Of course, the most basic realised measure is the squared daily return". We utilise this approach as the basis of our historical volatility model.

2.5. Heterogenous Autoregressive Model (HAR)

Corsi (2009, p. 174) suggests "an additive cascade model of volatility components defined over different time periods. The volatility cascade leads to a simple AR-type model in the realized volatility with the feature of considering different volatility components realized over different time horizons and which he termed as a "Heterogeneous Autoregressive model of Realized Volatility". Corsi (2009) suggests that the model successfully achieves the purpose of reproducing the main empirical features of financial returns (long memory, fat tails, and self-similarity) in a parsimonious way. He writes his model as:

$$\sigma_{t+1d}^{(d)} = c + \beta^{(d)} RV_t^{(d)} + \beta^{(w)} RV_t^{(w)} + \beta^{(m)} RV_t^{(m)} + \tilde{\omega}_{t+1d}^{(d)}, \tag{16}$$

where $\sigma^{(d)}$ is the daily integrated volatility, and $RV_t^{(d)}$, $RV_t^{(w)}$ and $RV_t^{(m)}$ are respectively the daily, weekly, and monthly (ex post) observed realized volatilities, and $w_{t+1d} = \tilde{\omega}_{t+1d}^{(d)} - \omega_{t+1d}^{(d)}$

Corsi (2009) inspires the HISVOL model adopted in the paper, which uses lags of historical RV estimates, but, in the current case, lags of squared demeaned daily close-to-close returns are employed.

A further justification for the approach adopted in the current study is provided by a recent publication by Perron and Shi (2020), who show that squared low-frequency daily returns can be expressed in terms of the temporal aggregation of a high-frequency series. They explore the links between the spectral density function of squared low-frequency and high-frequency returns. They analyze the properties of the spectral density function of realized volatility, constructed from squared returns with different frequencies under temporal aggregation. However, for the low frequency data on S&P 500 returns, they cannot infer whether the noise is stationary long memory but a long-memory process appears needed to explain the features related to high frequency S&P 500 futures. However, they caution that they cannot explain this difference and that it may also be related to the fact that they have used both spot and futures series for the S&P500 in the case of the high frequency data.

Perron and Shi (2020, p. 14), suggest that: "that both the realized volatility and the squared daily returns contain the same information about long memory. However, the squared daily returns contain a larger noise component than does the realized volatility". The current paper uses the long-memory component of low frequency squared daily demeaned returns to capture this long memory feature and to provide an approximation to a HAR model. The paper does not apply a full HAR model as the intention is to explore whether a simple HISVOL rule of thumb model, based on lagged squared demeaned returns, performs as well as standard GARCH or SV models. It is clear that a full HAR specification is likely to perform better, though Perron and Shi (2020) suggest that many of the HAR's long memory features will be captured by the approach adopted in the paper.

3. Results of the Analysis

3.1. Preliminary Analysis

The sample data set consists of 20 years and four months of daily data of adjusted continuously compounded close to close returns for S&P 500, DOW JONES and STOXX50 indices, taken from 3 January 2000 through to 30 April 2020. There are a matching set of daily RV5 estimates for the three indices obtained from the Oxford Man Realised library. These three indices are chosen because they constitute major components of global capital markets in the US and in Europe.

The S&P500 reflects the performance of the top 500 stocks of leading companies in the USA, and is listed on the NYSE (New York Stock Exchange) and (Nasdaq Exchange). The companies included comprise roughly 80% coverage of the available US market capitalisation. The DOW JONES index is a much narrower index that measures the daily price movements of 30 large American companies on the Nasdaq and the New York Stock Exchange and is comprised of blue-chip stocks. The EURO STOXX 50 is a stock index of Eurozone stocks designed by STOXX, an index provider owned by Deutsche

Börse Group. STOXX suggests that the aim is "to provide a blue-chip representation of Supersector leaders in the Eurozone". It is made up of fifty of the largest and most liquid stocks. These indices are representative of blue chip stocks in the USA and Eurozone, plus the broader US market. Thus, these indices are likely to be of great importance and appeal to investors given the nature of the markets that they cover.

　　　Summary statistics for the six series are provided in Table 1. The sample size varies for the three indices varies given a different incidence of holidays in the USA and in Europe. For example, in 2019, there were nine days of holiday related closures on the NYSE but only five in Europe. The total number of sample observations for the USA indices initially comprised 5100 data points whilst the total for the STOXX50 was 5180.

　　　Close to close returns were used because it was thought that these are likely to capture all the information released over a 24 h period and would provide more accurate measures of volatility for the demeaned squared return series used in the regression tests.

　　　The S&P500 has a mean daily return of 0.01361 per cent and a standard deviation of 1.248 per cent. Plots of the daily returns and RV5 estimates are provided in Figure 1. It had positive excess kurtosis and does not conform to a Gaussian distribution, as can be seen from the QQ plot in Figure 2. The S&P500 RV5 has a mean of 0.00011202 and a standard deviation of 0.00026873. However, Rv5 is measured as a variance, and, if we take the square root of its value and multiply it by 100, it will be on a common scale with the S&P500 returns. We undertake this transformation in some of the comparison plots in subsequent figures. It has very high skewness and kurtosis which is also evident in the QQ plots in Figure 2.

　　　The DOWJONES has a mean return of 0.01496 per cent and a standard deviation of 1.1946 per cent, while STOXX50 has a mean return of −0.0098108 per cent and a standard deviation of 1.4439. DOWJONES RV5 has a mean of 0.00011386 and a standard deviation of 0.00028661, while RV5 of DOWJONES has a mean of 0.00011386 and a standard deviation of 0.00028661. STOXX50 RV5 has a mean of 0.00016066 and a standard deviation of 0.00033556. All three RV5 series are skewed and have high excess kurtosis.

Table 1. Summary statistics.

|	Mean	Median	Minimum	Maximum	Standard Deviation	Skewness	Excess Kurtosis
SPRET	0.000136	0.000540	−0.126700	0.106420	0.012485	−0.360540	10.874
SPRV5	0.000112	0.000047	0.000001	0.007748	0.000269	10.6520	188.60
DJRET	0.000150	0.000488	−0.138070	0.107540	0.011946	−0.376640	13.380
DJRV5	0.000114	0.000049	0.000001	0.008624	0.000287	12.0870	238.02
STOXX50RET	−0.000098	0.000236	−0.120050	0.105540	0.014399	−0.225470	5.8411
STOXX50RV5	0.000161	0.000081	0.000000	0.010827	0.000336	12.0260	256.24
KEY:							
SPRET	Continuously compounded close to close return on the S&P500 Index						
SPRV5	Daily realised volatility on the S&P500 Index sampled at 5 min intervals provided by Oxford Man						
DJRET	Continuously compounded close to close return on the DOWJONES Index						
DJRV5	Daily realised volatility on the S&P500 Index sampled at 5 min intervals provided by Oxford Man						
STOXX50RET	Continuously compounded close to close return on the S&P500 Index						
STOXX50RV5	Daily realised volatility on the S&P500 Index sampled at 5 min intervals provided by Oxford Man						
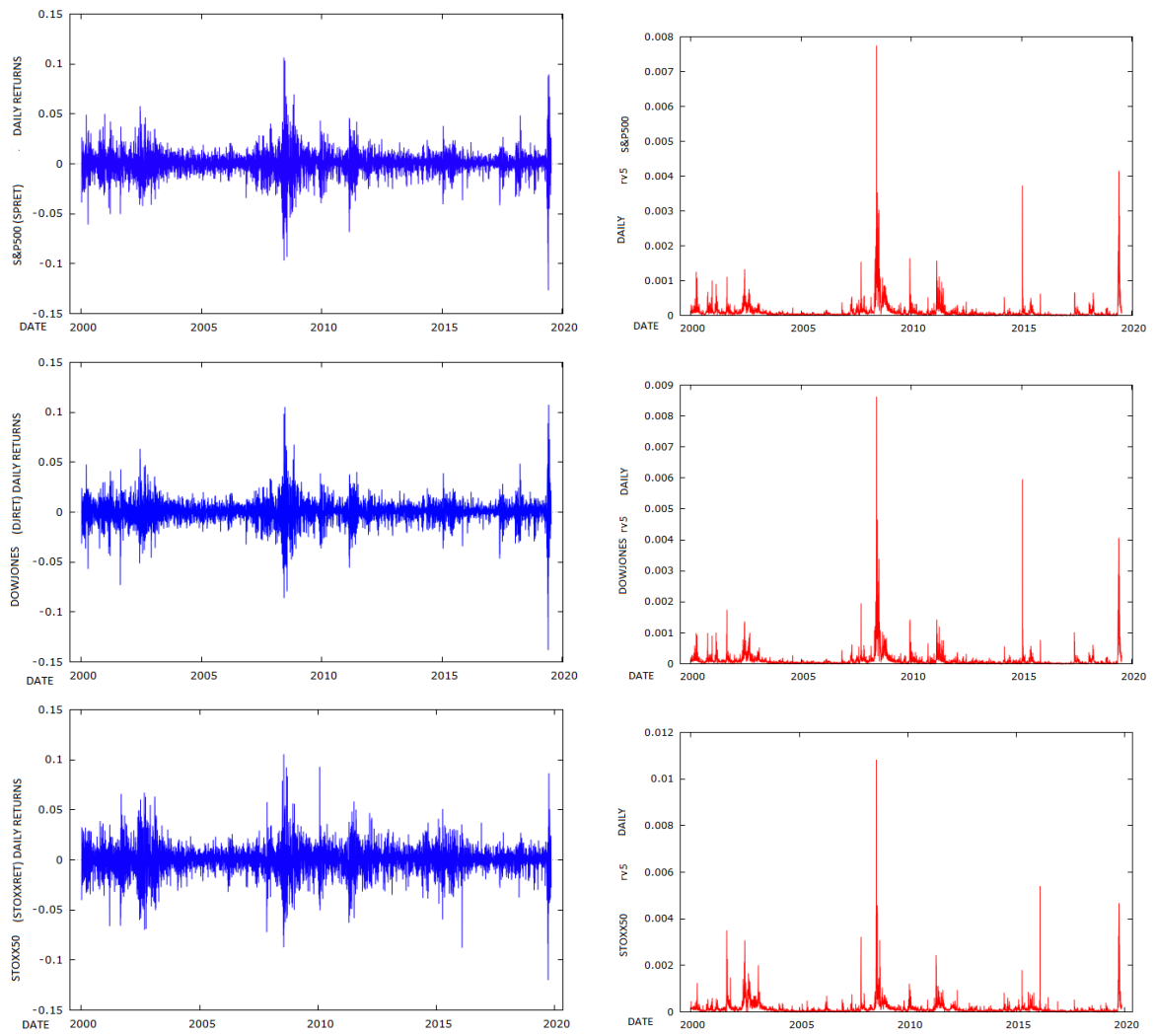
Figure 1. Series plots.

QQ PLOT SPRET and S&P500 RV5
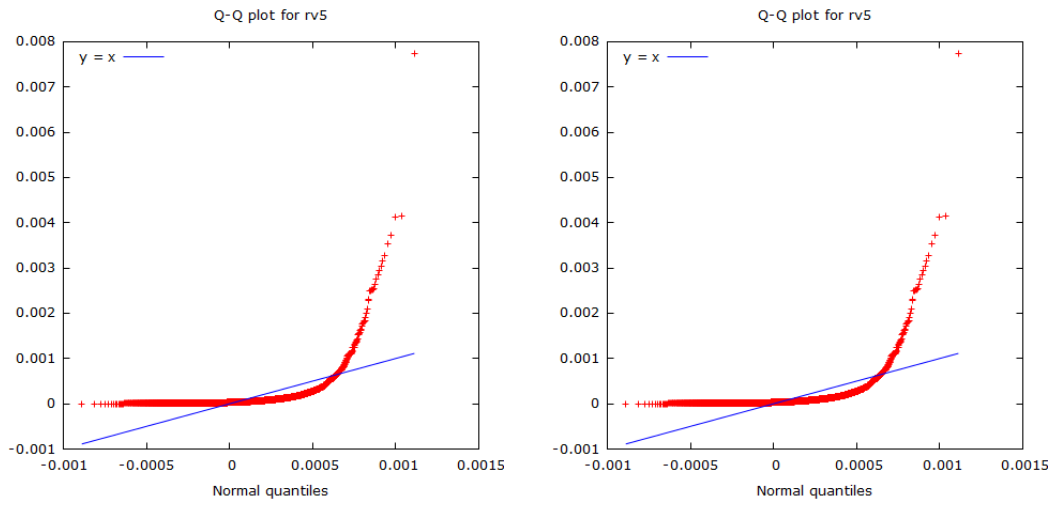
QQPLOTS DJRET and DJrv5
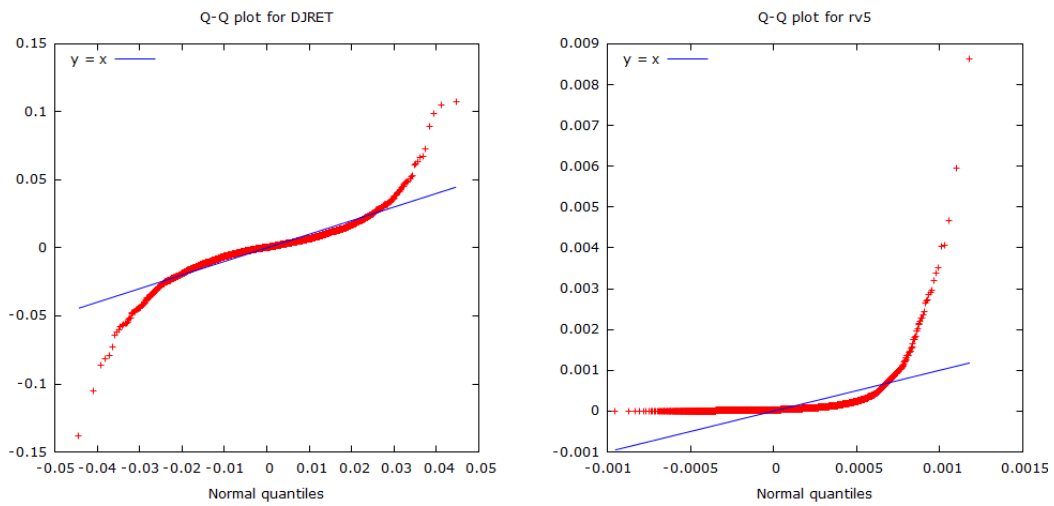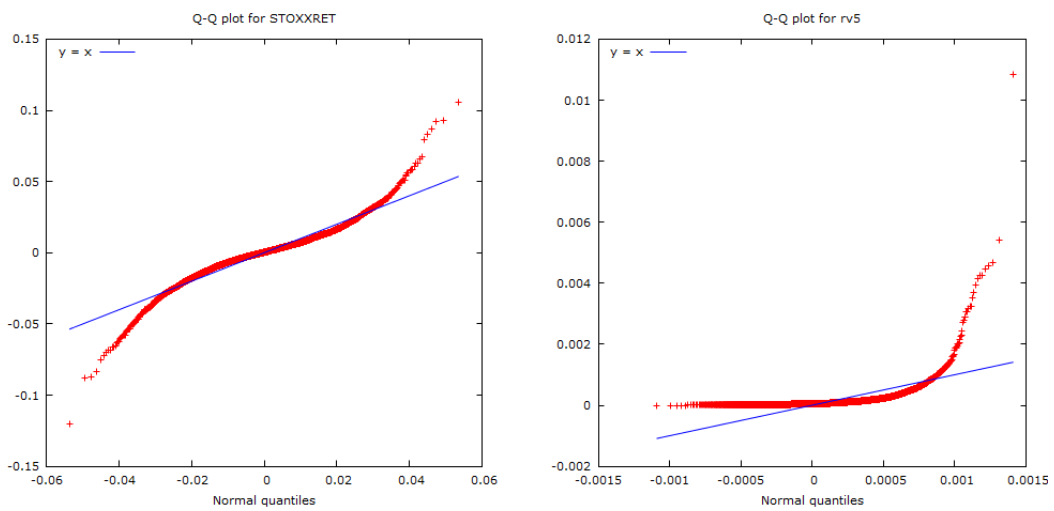

QQPLOT STOXX50RET AND STOXX50rv5

Figure 2. QQ plots.

3.2. SV and GARCH Estimates

The R library stochvol was used to fit a stochastic volatility model to the S&P500, DOWJONES and STOXX50 de-meaned return series, and Gaussian distributions applied to fit the SV model. Some of the initial parameters for the SV model estimation, as applied to the three series, are shown in Table 2. The SV model applied to the three series produces the volatility estimates shown in Figure 3, while Figure 4 displays a comparison of the volatility estimates for both a GARCH (1,1) model and the SV model, as applied to the three return series. The estimates of volatility are quite similar. The estimates for the GARCH(1,1) model used to obtain conditional volatilities are shown in Table 3. The diagnostic tests (not reported), suggested that the models are a satisfactory fit and that the volatility model parameter estimates are within stable limits.

Table 2. Stochastic volatility estimates.

Summary of 1000 MCMC Draws after Burn in of 1000					
Prior Distributions					
$\mu \sim normal$	mean = 0	S.D. = 100			
$(\phi + 1)/2 \sim \beta$	$a_0 = 5$	$b_0 = 1.5$			
$\sigma^2 \sim \chi^2(df = 1)$					
S&P500					
Posterior draws thinning = 1					
	Mean	S.D.	5%	50%	95%
μ	−9.4565	0.15973	−9.7172	−9.4562	−9.193
ϕ	0.9803	0.00377	0.9739	0.9804	0.986
σ	0.2154	0.01526	0.1902	0.2152	0.241
$exp(\mu/2)$	0.0089	0.00071	0.0078	0.0088	0.010
σ^2	0.0466	0.00659	0.0362	0.0463	0.058
DOWJONES					
Posterior draws thinning = 1					
	Mean	S.D.	5%	50%	95%
μ	−9.5491	0.14899	−9.7890	−9.7890	−9.3053
ϕ	0.9784	0.00393	0.9718	0.9785	0.9846
σ	0.2222	0.01509	0.1982	0.2214	0.2470
$exp(\mu/2)$	0.0085	0.00063	0.0075	0.0084	0.0095
σ^2	0.0496	0.00676	0.0393	0.0490	0.0610
STOXX50					
Posterior draws thinning = 1					
	Mean	S.D.	5%	50%	95%
μ	−8.969	0.15544	−9.2223	−8.969	−8.719
ϕ	0.983	0.00346	0.9774	0.983	−8.719
σ	0.177	0.01362	0.1556	0.176	0.200
$exp(\mu/2)$	0.011	0.00088	0.0099	0.011	0.200
σ^2	0.031	0.00484	0.0242	0.031	0.040
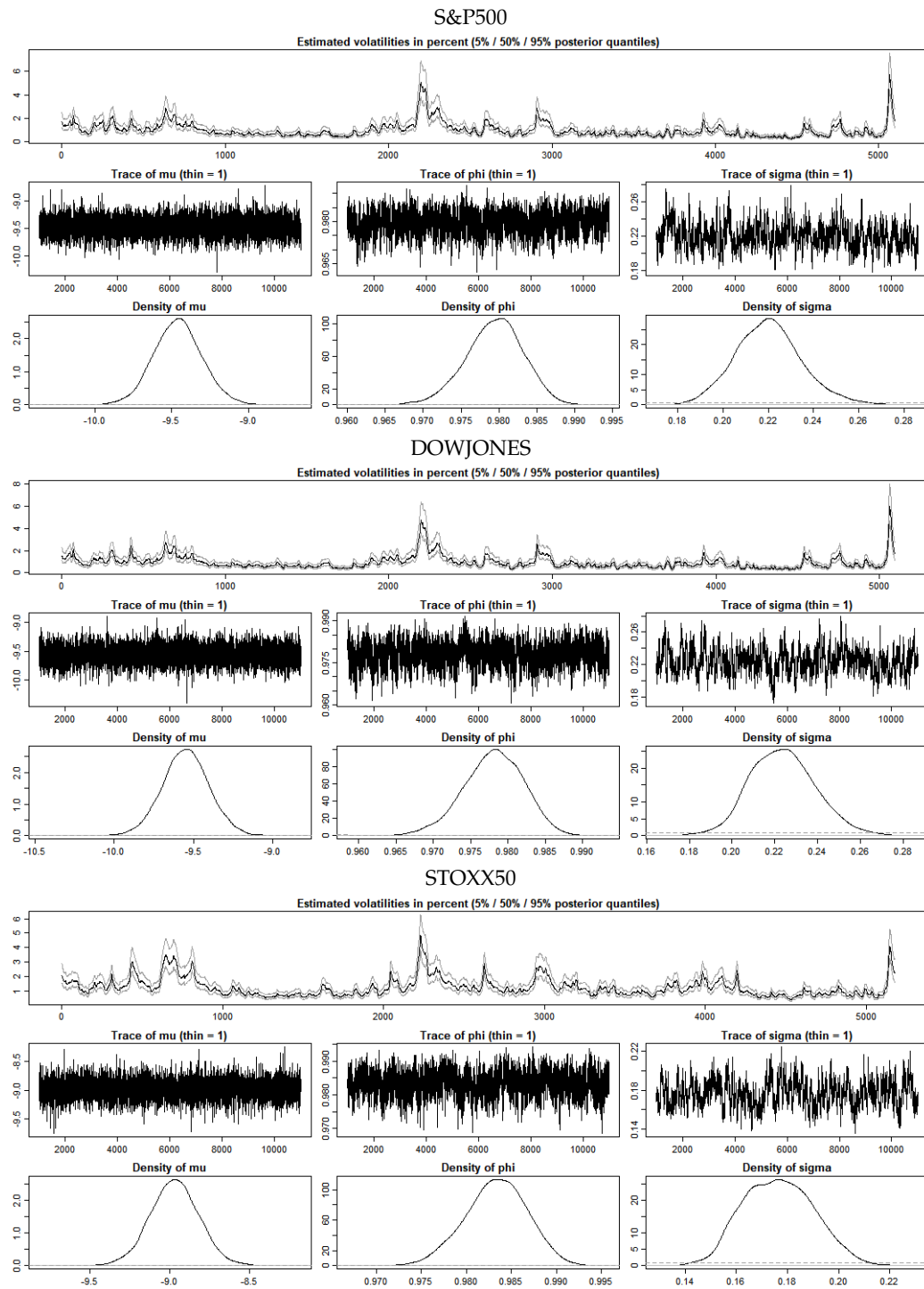
Figure 3. Posterior density plots of parameters in θ.

S&P500

DOWJONES

STOXX50
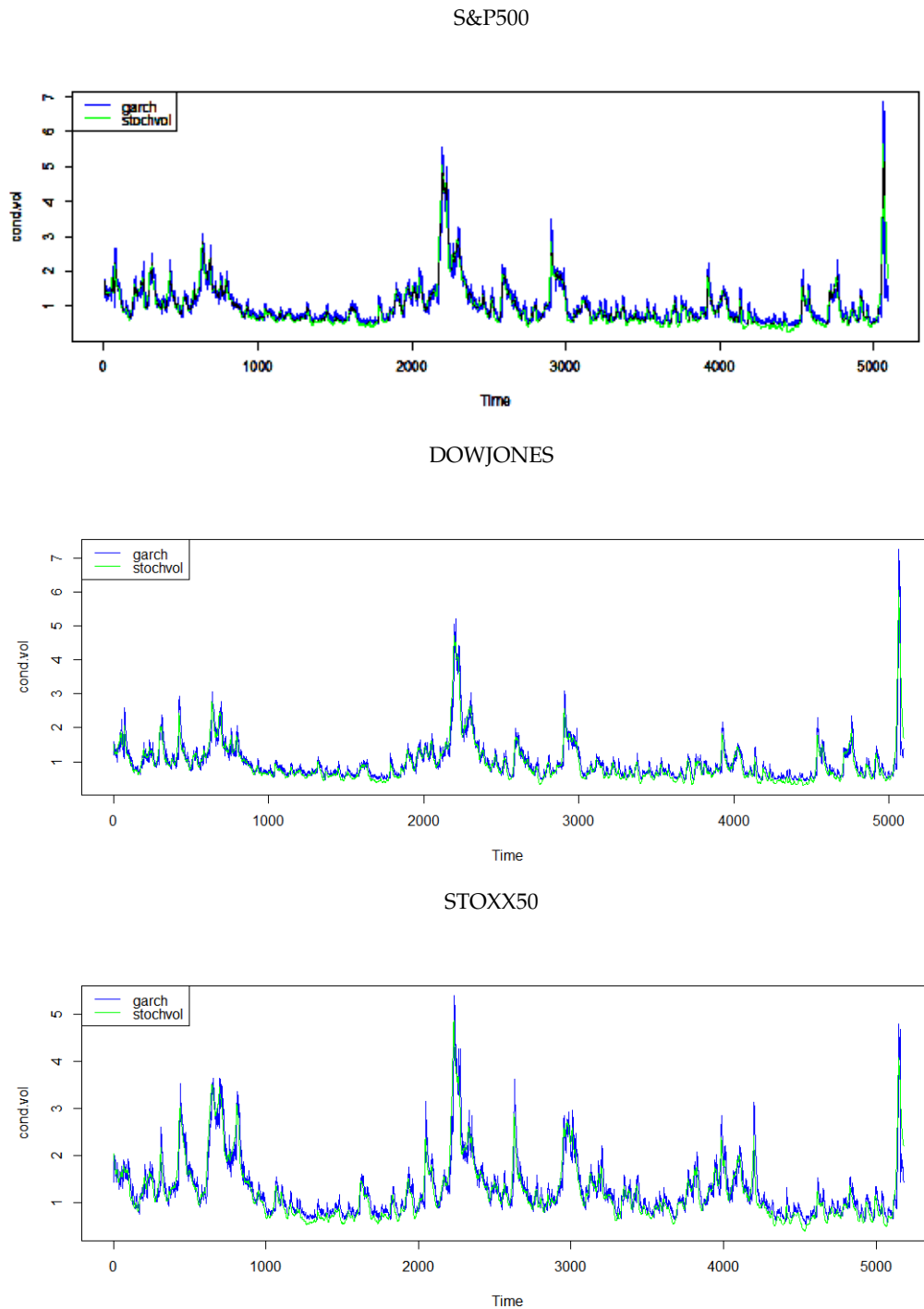
Figure 4. Comparison SV and GARCH(1,1) Volatility Estimates.

The benchmark used in this paper is the same as in Allen and McAleer (2020), namely estimates of realised volatility sampled at five-minute intervals, as obtained from Oxford Man. These are employed as the baseline in ordinary least squares regressions adopted to explore the linear correlations between the estimates of the two volatility models and the base RV5 estimates of volatility. The results are shown in panels A, B and C of Table 4.

Table 3. GARCH(1,1) fitted to INDICES RETURNS.

	Coefficients	Standard Error	T Statistic
		S&P500	
μ	0.00071466	0.0001000	7.144 ***
ω	0.0000013037	0.0000002911	4.479 ***
α_1	0.12429	0.0118	10.530 ***
β_1	0.87470	0.01081	80.918 ***
		DOWJONES	
μ	0.00028945	0.0001063	2.723 ***
ω	0.000001861	0.000000258	7.212 ***
α_1	0.12121	0.009328	12.995 ***
β_1	0.86583	000.9399	92.123 ***
		STOXX50	
μ	0.000093821	0.0001411	0.665
ω	0.0000024108	0.000000407	5.911 ***
α_1	0.099385	0.008582	11.580 ***
β_1	0.89020	0.009059	98.266 ***

Note: *** Indicates significance at the 1 per cent level.

　　All the regressions reported in Table 4 use RV5 values as the benchmark dependent variable, for the three index series. The first set of regressions in each panel reports the results of the regression of RV5 on the predictions of volatility obtained from the SV model, lagged by one day. Each of the three estimated coefficients on SV, for the three indices S&P500, DOW JONES and STOXX50, with respective values of 0.000305766, 0.000328020 and 0.000249773 are significant at the 1 per cent level. The respective adjusted R-Squares of these regressions are 0.528418, 0.483024 and 0.398682. The results suggest that the base SV model captures between 40 and 50 per cent of the volatilities of the three index series when the benchmark is RV5.

　　As a further cross-check of the effectiveness of the two models, we used a further crude estimate of volatility with the demeaned squared daily returns on the three series, in the context of a HISVOL model, which is motivated by Corsi's (2009) Heterogeneous Autoregressive model of Realized Volatility (HAR-RV).

　　This crude model produces better results than those using SV and GARCH(1,1) conditional volatilities as explanatory variables. The adjusted R-squared values are 0.540166 for the S&P500 Index in Panel A, 0.445781 in Panel B for DOWJONES, and 0.443043 in Panel C for STOXX50. The adjusted R-square values obtained by this crude approximation to a HAR-RV model all exceed the explanatory values produced by the SV and GARCH(1,1) models in two of the three cases with the exception of the DOWJONES. These results are largely consistent with Allen and McAleer (2020).

Table 4. Regression analysis of the three volatility models as Explanators of RV5.

S&P500				
OLS, using observations 2000-01-06–2020-04-30 (T = 5098)				
Dependent variable: rv5				
	Coefficient	Std. Error	t-ratio	p-value
const	−0.000193903	4.80241×10^{-6}	−40.38	0.0000
STVOL_1	0.000305766	4.04560×10^{-6}	75.58	0.0000
Mean dependent var	0.000112	S.D. dependent var	0.000269	
Sum squared resid	0.000174	S.E. of regression	0.000185	
R^2	0.528511	Adjusted R^2		0.528418
$F(1, 5096)$	5712.306	P-value(F)		0.000000
$\hat{\rho}$	0.355673	Durbin–Watson		1.288578
OLS, using observations 2000-01-06–2020-04-30 (T = 5098)				
Dependent variable: rv5				
	Coefficient	Std. Error	t-ratio	p-value
const	−0.000186320	5.58228×10^{-6}	−33.38	0.0000
garchh_t_1	0.000282384	4.54881×10^{-6}	62.08	0.0000
Mean dependent var	0.000112	S.D. dependent var		0.000269
Sum squared resid	0.000210	S.E. of regression		0.000203
R^2	0.430599	Adjusted R^2		0.430488
$F(1, 5096)$	3853.762	P-value(F)		0.000000
$\hat{\rho}$	0.432641	Durbin–Watson		1.134608
OLS, using observations 2000-02-02–2020-04-30 (T = 5079)				
Dependent variable: rv5				
	Coefficient	Std. Error	t-ratio	p-value
const	3.02687×10^{-5}	2.84918×10^{-6}	10.62	0.0000
sq_DMSPRET_1	0.174138	0.00556030	31.32	0.0000
sq_DMSPRET_2	0.0933966	0.00559548	16.69	0.0000
sq_DMSPRET_3	0.0527851	0.00588575	8.968	0.0000
sq_DMSPRET_4	0.0280498	0.00588689	4.765	0.0000
sq_DMSPRET_5	0.0591621	0.00588274	10.06	0.0000
sq_DMSPRET_6	0.0387770	0.00590617	6.565	0.0000
sq_DMSPRET_7	0.0127639	0.00593373	2.151	0.0315
sq_DMSPRET_8	0.0128494	0.00591714	2.172	0.0299
sq_DMSPRET_9	0.0460289	0.00591518	7.781	0.0000
sq_DMSPRET_10	0.0108100	0.00588990	1.835	0.0665
sq_DMSPRET_11	−0.0125045	0.00589057	−2.123	0.0338
sq_DMSPRET_12	0.00683578	0.00591643	1.155	0.2480
sq_DMSPRET_13	0.000837936	0.00591795	0.1416	0.8874
sq_DMSPRET_14	−0.00804767	0.00593584	−1.356	0.1752
sq_DMSPRET_15	−0.00239430	0.00590724	−0.4053	0.6853
sq_DMSPRET_16	−0.0108916	0.00588602	−1.850	0.0643
sq_DMSPRET_17	0.00482912	0.00588926	0.8200	0.4123
sq_DMSPRET_18	0.0125097	0.00593287	2.109	0.0350
sq_DMSPRET_19	0.0123520	0.00564065	2.190	0.0286
sq_DMSPRET_20	−0.00717624	0.00560040	−1.281	0.2001
Mean dependent var	0.000112	S.D. dependent var		0.000269
Sum squared resid	0.000169	S.E. of regression		0.000183
R^2	0.541977	Adjusted R^2		0.540166
$F(20, 5058)$	299.2563	P-value(F)		0.000000
$\hat{\rho}$	0.283092	Durbin–Watson		1.433782

Table 4. *Cont.*

DOWJONES				
OLS, using observations 2000-01-06–2020-04-30 ($T = 5094$)				
Dependent variable: rv5				
	Coefficient	Std. Error	t-ratio	p-value
const	−0.000198625	5.37151×10^{-6}	−36.98	0.0000
SVDJ_1	0.000328020	4.75466×10^{-6}	68.99	0.0000
Mean dependent var	0.000114	S.D. dependent var		0.000287
Sum squared resid	0.000216	S.E. of regression		0.000206
R^2	0.483125	Adjusted R^2		0.483024
$F(1, 5092)$	4759.515	P-value(F)		0.000000
$\hat{\rho}$	0.331936	Durbin–Watson		1.336075
OLS, using observations 2000-01-06–2020-04-30 ($T = 5094$)				
Dependent variable: rv5				
	Coefficient	Std. Error	t-ratio	p-value
const	−0.000170947	6.16017×10^{-6}	−27.75	0.0000
Djht_1	0.000280420	5.18265×10^{-6}	54.11	0.0000
Mean dependent var	0.000114	S.D. dependent var		0.000287
Sum squared resid	0.000266	S.E. of regression		0.000228
R^2	0.365056	Adjusted R^2		0.364932
$F(1, 5092)$	2927.610	p-value(F)		0.000000
$\hat{\rho}$	0.434841	Durbin–Watson		1.130290
OLS, using observations 2000-01-04–2020-04-30 ($T = 5075$)				
Dependent variable: lrv5				
	Coefficient	Std. Error	t-ratio	p-value
const	0.0405870	0.00329492	12.32	0.0000
SQDMDJRET_1	0.000165523	6.55973×10^{-6}	25.23	0.0000
SQDMDJRET_2	0.000101750	6.61717×10^{-6}	15.38	0.0000
SQDMDJRET_3	5.22487×10^{-5}	6.99793×10^{-6}	7.466	0.0000
SQDMDJRET_4	2.10587×10^{-5}	6.99864×10^{-6}	3.009	0.0026
SQDMDJRET_5	6.58804×10^{-5}	7.00022×10^{-6}	9.411	0.0000
SQDMDJRET_6	4.64120×10^{-5}	7.00145×10^{-6}	6.629	0.0000
SQDMDJRET_7	1.56715×10^{-6}	7.13269×10^{-6}	0.2197	0.8261
SQDMDJRET_8	1.54284×10^{-6}	7.11737×10^{-6}	0.2168	0.8284
SQDMDJRET_9	4.60489×10^{-5}	7.11644×10^{-6}	6.471	0.0000
SQDMDJRET_10	-2.98185×10^{-6}	7.06670×10^{-6}	−0.4220	0.6731
SQDMDJRET_11	-1.35137×10^{-5}	7.06710×10^{-6}	−1.912	0.0559
SQDMDJRET_12	-2.77575×10^{-6}	7.11818×10^{-6}	−0.3900	0.6966
SQDMDJRET_13	5.94860×10^{-6}	7.11930×10^{-6}	0.8356	0.4034
SQDMDJRET_14	-8.20421×10^{-6}	7.13518×10^{-6}	−1.150	0.2503
SQDMDJRET_15	-8.26171×10^{-6}	7.00344×10^{-6}	−1.180	0.2382
SQDMDJRET_16	-4.30980×10^{-6}	7.00422×10^{-6}	−0.6153	0.5384
SQDMDJRET_17	6.42163×10^{-6}	7.00281×10^{-6}	0.9170	0.3592
SQDMDJRET_18	2.34879×10^{-5}	7.08619×10^{-6}	3.315	0.0009
SQDMDJRET_19	2.02923×10^{-5}	6.69703×10^{-6}	3.030	0.0025
SQDMDJRET_20	-4.02896×10^{-6}	6.63951×10^{-6}	−0.6068	0.5440
Mean dependent var	0.113763	S.D. dependent var		0.287164
Sum squared resid	230.9808	S.E. of regression		0.213782
R^2	0.447966	Adjusted R^2		0.445781
$F(20, 5054)$	205.0615	p-value(F)		0.000000
$\hat{\rho}$	0.324610	Durbin–Watson		1.350773

Table 4. *Cont.*

STOXX50				

OLS, using observations 2000-01-06–2020-04-30 (T = 5179)				

Dependent variable: rv5				
	Coefficient	Std. Error	t-ratio	p-value
const	-0.000255606	7.97034×10^{-6}	-32.07	0.0000
STOXXSV_1	0.000249773	4.26226×10^{-6}	58.60	0.0000
Mean dependent var	0.000161	S.D. dependent var		0.000336
Sum squared resid	0.000351	S.E. of regression		0.000260
R^2	0.398798	Adjusted R^2		0.398682
$F(1, 5177)$	3434.088	p-value(F)		0.000000
$\hat{\rho}$	0.334086	Durbin–Watson		1.331768

OLS, using observations 2000-01-05–2020-04-30 (T = 5179)				

Dependent variable: rv5				
	Coefficient	Std. Error	t-ratio	p-value
const	-3.19171×10^{-6}	4.70679×10^{-6}	-0.6781	0.4977
h2STOXX50_1	0.783757	0.0140212	55.90	0.0000
Mean dependent var	0.000161	S.D. dependent var		0.000336
Sum squared resid	0.000364	S.E. of regression		0.000265
R^2	0.376384	Adjusted R^2		0.376264
$F(1, 5177)$	3124.586	p-value(F)		0.000000
$\hat{\rho}$	0.342843	Durbin–Watson		1.314043

OLS, using observations 2000-02-02–2020-04-30 (T = 5160)				

Dependent variable: rv5				
	Coefficient	Std. Error	ct-ratio	p-value
const	2.05927×10^{-5}	4.29557×10^{-6}	4.794	0.0000
sq_DMSTOXXRET_1	0.118968	0.00666728	17.84	0.0000
sq_DMSTOXXRET_2	0.120534	0.00666684	18.08	0.0000
sq_DMSTOXXRET_3	0.0933684	0.00669582	13.94	0.0000
sq_DMSTOXXRET_4	0.0896290	0.00678649	13.21	0.0000
sq_DMSTOXXRET_5	0.0342222	0.00682410	5.015	0.0000
sq_DMSTOXXRET_6	0.0105941	0.00684945	1.547	0.1220
sq_DMSTOXXRET_7	0.0193777	0.00684880	2.829	0.0047
sq_DMSTOXXRET_8	0.0415985	0.00685618	6.067	0.0000
sq_DMSTOXXRET_9	0.0676568	0.00686950	9.849	0.0000
sq_DMSTOXXRET_10	0.0648663	0.00687561	9.434	0.0000
sq_DMSTOXXRET_11	0.00745897	0.00687600	1.085	0.2781
sq_DMSTOXXRET_12	0.00363238	0.00687086	0.5287	0.5971
sq_DMSTOXXRET_13	0.00415015	0.00685846	0.6051	0.5451
sq_DMSTOXXRET_14	0.00309319	0.00685311	0.4514	0.6518
sq_DMSTOXXRET_15	0.0459385	0.00685415	6.702	0.0000
sq_DMSTOXXRET_16	-0.0110292	0.00682975	-1.615	0.1064
sq_DMSTOXXRET_17	3.58184×10^{-5}	0.00680192	0.005266	0.9958
sq_DMSTOXXRET_18	-0.0226702	0.00671209	-3.378	0.0007
sq_DMSTOXXRET_19	0.00604883	0.00668237	0.9052	0.3654
sq_DMSTOXXRET_20	-0.0196779	0.00668193	-2.945	0.0032
Mean dependent var	0.000161	S.D. dependent var		0.000336
Sum squared resid	0.000324	S.E. of regression		0.000251
R^2	0.445202	Adjusted R^2		0.443043
$F(20, 5139)$	206.1915	p-value(F)	0.000000	
$\hat{\rho}$	0.216381	Durbin–Watson		1.567234

Note: squared de-meaned return series used for the three index series.

3.3. Mincer–Zarnowitz Tests

As an additional test of the accuracy of the forecasts, I used some Mincer and Zarnowitz (1969) regression tests. (I am grateful to a reviewer for this suggestion). The test involves regressing realised values on the forecasts:

$$\sigma_{t+1} = \beta_0 + \beta_1 \hat{\sigma}_{t+1}.$$

The joint hypothesis tested is that $\beta_0 = 0$, and $\beta_1 = 1$. The results of the test on the full sample are shown in Table 5.

Table 5. Mincer–Zarnowitz regressions.

Method	Test Statistic	Probability
	S&P500	
STOCHVOL	1.06838×10^{11}	0.0
GARCH	1.01522×10^{11}	0.0
sq_DMSPRET-1	2205.89	0.0
	DOWJONES	
STOCHVOL	7.72384×10^{10}	0.0
GARCH	7.33155×10^{10}	0.0
sq_DMDJRET-1	1.84768×10^{16}	0.0
	STOXX50	
STOCHVOL	626.635	0.0
GARCH	2.7139×10^{7}	0.0
sq_DMSTOXRET-1	5449.58	0.0

The results in Table 5 uniformly and significantly reject the accuracy of the forecasts. However, the forecasts involve long time series with in excess of 5000 observations. As a further check, I ran some rolling Mincer–Zarnowitz regressions using a 100 observation window to explore how frequently the slope coefficient had a value of 1 bounded within two standard deviation intervals. This was done to check the relative frequency of periods in which the forecasts could not reject the null hypothesis. The results for the S&P500 index are shown in Figure 5 (I have omitted the results of the other series in the interests of brevity, but they are available from the author on request).

It can be seen in Figure 5 that there are extended periods of time in which the null hypothesis that the slope coefficient of the results of regressing the actual on the forecast is not significantly different from 1. This highlights the issues related to type one and type two errors. It also is consistent with the Mincer–Zarnowitz tests consistently rejecting the null hypothesis yet the adjusted Rsquares of the regressions of actuals on forecasts being consistently in the range of 40–50%
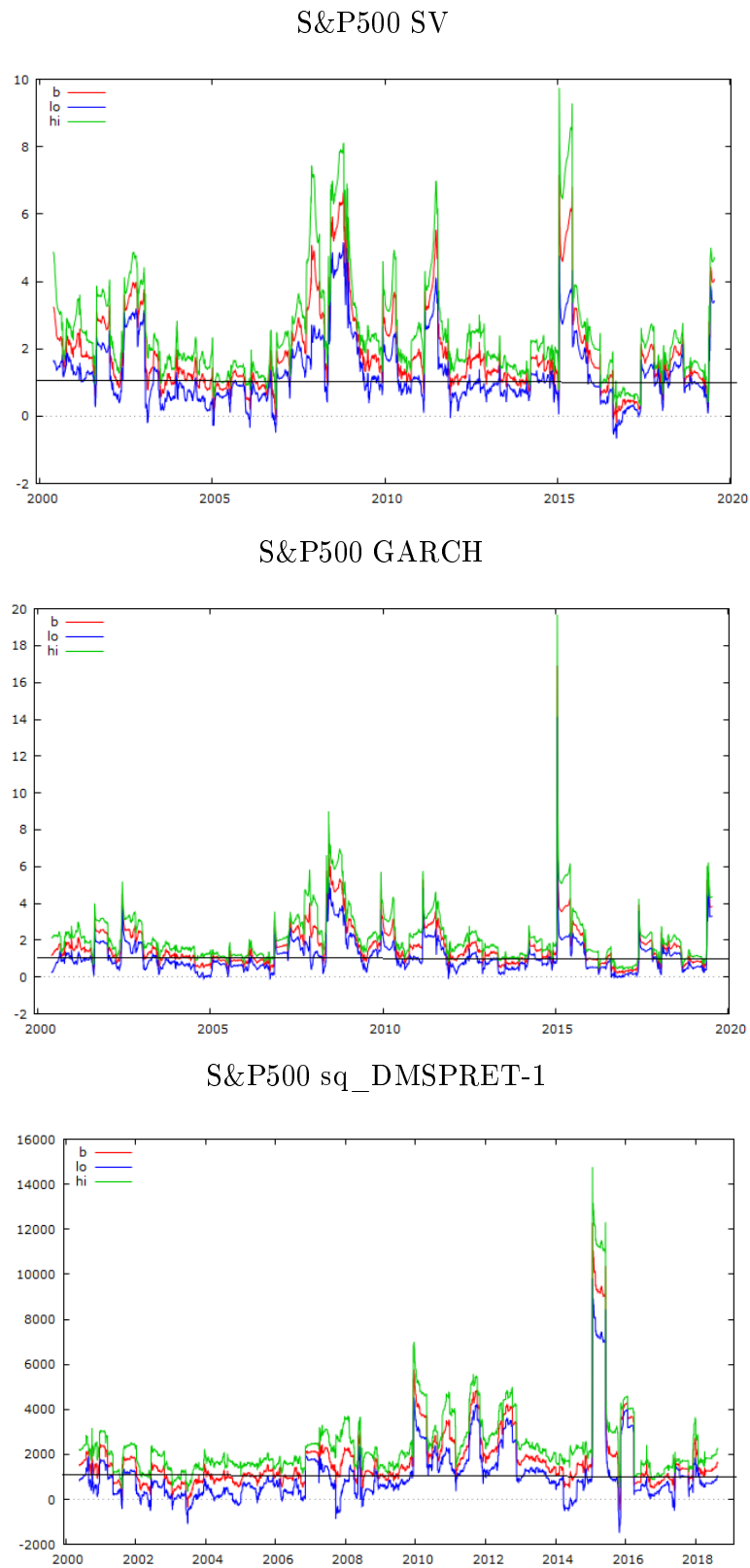
S&P500 SV

S&P500 GARCH

S&P500 sq_DMSPRET-1

Figure 5. Mincer–Zarnowitz rolling regression slope coefficients bounded by two standard deviations.

3.4. Further Analysis

Kastner (2019) in the factorstochvol R package, together with the Hosszejni and Kastner (2019) vignette, on factorstochvol, demonstrated how to expand the capabilities of the original stochvol package. In particular, they explained how it is possible to construct a multiple regression model with an intercept, two regressors, and SVl residuals using the following construction:

$$
\begin{pmatrix} y_t \\ h_{t+1} \end{pmatrix} \mid h_t, \varsigma, \begin{pmatrix} x_{t1} \\ x_{t2} \end{pmatrix}, \beta_0, \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} 2 \left(\begin{pmatrix} \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} \\ u + \varphi(h_t - \mu) \end{pmatrix}, \sum\nolimits^\rho \right), \tag{17}
$$

$$
\Sigma^\rho = \begin{pmatrix} exp(h_t) & \rho\sigma exp(h_t/2) \\ \rho\sigma exp(h_t/2 & \sigma^2 \end{pmatrix}.
$$

This approach was adopted to include the second and third moments of the index series in the form of de-meaned squared returns and de-meaned cubed returns as the two regressors in the regression model. The intention was to check whether these simple enhancements would improve the performance of the basic stochastic volatility model, in regression tests which used the estimated volatility, as reported in the previous section, as the explanatory variable regressed on the benchmark values of RV5 for the three index series. Table 6 presents the the stochastic volatility estimates for the augmented model and posterior density plots of parameters in θ are shown in Figure 6.

The values appear to be reasonably well-behaved and the density plots in Figure 6 are acceptable, although the density estimates for the STOXX50 are the weakest of the three with some evidence of bimodality in the mu estimation plot.

A similar simple enhancement was applied to the GARCH(1,1) conditional volatility model in which the mean equation featured the addition of a vector of squared de-meaned returns to assess whether this improved its explanatory performance. The results of the GARCH(1,1) models with enhanced mean equations are not reported in the paper in the interests of brevity, but are available on request. Suffice to say all the estimates appeared to be satisfactory.

Table 7 reports the results of the regression analyses featuring the volatility estimates from both the augmented stochastic volatility and GARCH models. The results are mixed. The benchmark in Table 4 is provided by the 'rule of thumb' historical volatility model in which 20 lags of de-meaned close-to-close returns were regressed on the Oxford Man RV5 estimates for the three indices. The adjusted R-squares in the cases of the S&P500, DOWJONES and STOXX50 were, respectively, 0.54, 0.54, and 0.44. The augmented SV model for the S&P500, as shown in the first regression in Table 6, has an adjusted R-square of 0.56, a marginal improvement. The original GARCH model for this index had an adjusted R-square of 0.36, which also improves marginally to 0.37.

The DOWJONES historical volatility model's adjusted R-square of 0.54 is not matched by that for the augmented SV model, which is 0.43 in Table 7. However, the adjusted R-square of the enhanced GARCH model has a value of 0.47, which is a considerable improvement over its previous value in Table 4 of 0.37. Finally, in the case of the STOXX50, the original historical volatility model had an adjusted R-square of 0.44. This is surpassed by that of the augmented SV model which has a value of 0.47. However, the augmented GARCH model shows a marked deterioration with an adjusted R-square of 0.17.

Thus, in two cases out of three, the augmented SV model does have higher explanatory power, when regressed on the RV5 estimates for these three indices. However, the improvement is by a couple of percent, and so it remains a moot point about whether it is of practical worth considering the difference in estimation complexity required to estimate an augmented SV model, as opposed to merely applying 20 lags of squared demeaned returns.

Table 6. Stochastic volatility estimates augmented SV model.

Summary of 1000 MCMC draws after burn in of 1000					
Prior Distributions					
$\mu \sim normal$	mean = 0	S.D. = 100			
$(\phi + 1)/2 \sim \beta$	$a_0 = 5$	$b_0 = 1.5$			
$\sigma^2 \sim \chi^2(df = 1)$					
S&P500					
Posterior draws thinning = 1					
	Mean	S.D.	5%	50%	95%
μ	−0.78	0.1562	−1.05	−0.78	−0.53
ϕ	0.97	0.0048	0.96	0.97	0.97
σ	0.36	0.0182	0.32	0.36	0.39
$exp(\mu/2)$	0.68	0.0529	0.59	0.68	0.77
σ^2	0.13	0.0130	0.11	0.13	0.15
DOWJONES					
Posterior draws thinning = 1					
	Mean	S.D.	5%	50%	95%
μ	−0.907	0.1447	−1.14	−0.913	−0.656
ϕ	0.957	0.0058	0.95	0.957	0.966
σ	0.437	0.0232	0.40	0.435	0.478
$exp(\mu/2)$	0.637	0.0462	0.56	0.634	0.720
σ^2	0.191	0.0204	0.16	0.189	0.229
STOXX50					
Posterior draws thinning = 1					
	Mean	S.D.	5%	50%	95%
μ	−0.37	0.1197	−0.57	−0.37	−0.18
ϕ	0.95	0.0062	0.94	0.95	0.96
σ	0.41	0.0211	0.37	0.41	0.44
$exp(\mu/2)$	0.83	0.0494	0.75	0.83	0.92
σ^2	0.17	0.0173	0.14	0.17	0.20

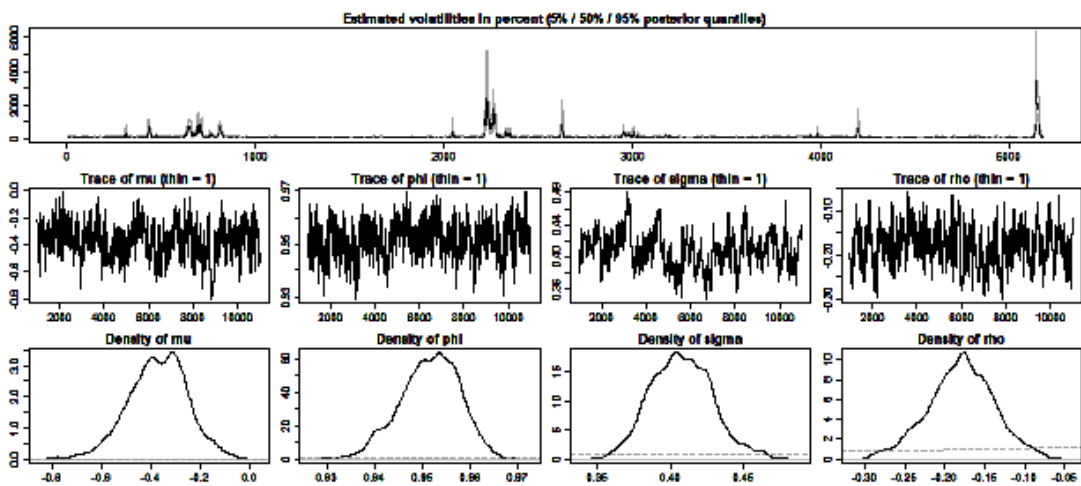S&P500
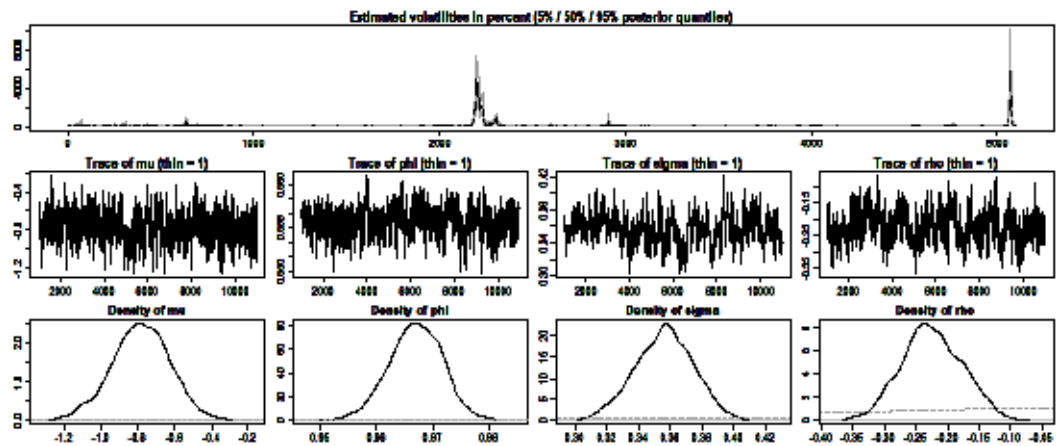
DOWJONES
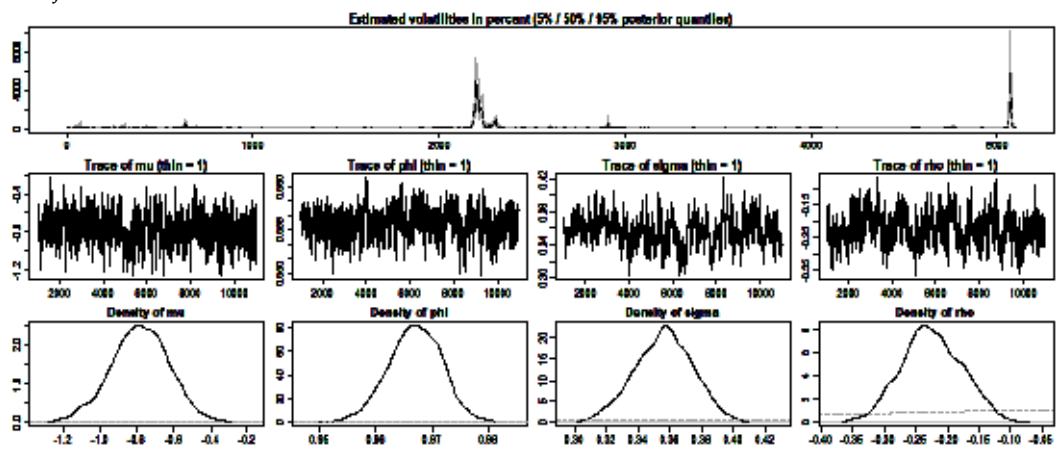
STOXX50

Figure 6. Posterior density plots of parameters in θ.

Table 7. Regression analysis of the three augmented stochastic volatility and GARCH models as explanators of RV5.

	S&P500			
	OLS, using observations 2000-01-05–2020-04-30 ($T = 5098$)			
	Dependent variable: rv5			
	Coefficient	Std. Error	*t*-ratio	*p*-value
const	4.80570×10^{-5}	2.59906×10^{-6}	18.49	0.0000
SVREGSPRET_1	5.61407×10^{-7}	6.87701×10^{-9}	81.64	0.0000
Mean dependent var	0.000112	S.D. dependent var		0.000269
Sum squared resid	0.000160	S.E. of regression		0.000177
R^2	0.566679	Adjusted R^2		0.566594
$F(1, 5096)$	6664.347	*p*-value(F)		0.000000
$\hat{\rho}$	0.288688	Durbin–Watson		1.422414
	OLS, using observations 2000-01-05–2020-04-30 ($T = 5098$)			
	Dependent variable: rv5			
	Coefficient	Std. Error	*t*-ratio	*p*-value
const	0.000136717	3.02817×10^{-6}	45.15	0.0000
yhat3_1	-0.0810177	0.00148855	-54.43	0.0000
Mean dependent var	0.000112	S.D. dependent var		0.000269
Sum squared resid	0.000233	S.E. of regression		0.000214
R^2	0.367610	Adjusted R^2		0.367486
$F(1, 5096)$	2962.324	*p*-value(F)		0.000000
$\hat{\rho}$	0.248686	Durbin–Watson		1.502451
	DOWJONES			
	OLS, using observations 2000-01-05–2020-04-30 ($T = 5094$)			
	Dependent variable: rv5			
	Coefficient	Std. Error	*t*-ratio	*p*-value
const	7.08759×10^{-5}	3.11272×10^{-6}	22.77	0.0000
DJSVREG_1	3.42022×10^{-7}	5.52569×10^{-9}	61.90	0.0000
Mean dependent var	0.000114	S.D. dependent var		0.000287
Sum squared resid	0.000239	S.E. of regression		0.000217
R^2	0.429352	Adjusted R^2		0.429240
$F(1, 5092)$	3831.187	*p*-value(F)	0.000000	
$\hat{\rho}$	0.382376	Durbin–Watson		1.235179
	OLS, using observations 2000-01-06–2020-04-30 ($T = 5093$)			
	Dependent variable: rv5			
	Coefficient	Std. Error	*t*-ratio	*p*-value
const	0.000144841	2.96368×10^{-6}	48.87	0.0000
yhat7_1	-0.00213366	3.18142×10^{-5}	-67.07	0.0000
Mean dependent var	0.000114	S.D. dependent var		0.000287
Sum squared resid	0.000222	S.E. of regression		0.000209
R^2	0.469072	Adjusted R^2		0.468968
$F(1, 5091)$	4497.872	*p*-value(F)		0.000000
$\hat{\rho}$	0.347045	Durbin–Watson		1.305745

Table 7. *Cont.*

STOXX50				
OLS, using observations 2000-01-05–2020-04-30 (T = 5179)				
Dependent variable: rv5				
	Coefficient	Std. Error	t-ratio	p-value
const	3.92593×10^{-5}	3.84434×10^{-6}	10.21	0.0000
STOXXREGFAC_1	1.04919×10^{-6}	1.55219×10^{-8}	67.59	0.0000
Mean dependent var	0.000161	S.D. dependent var		0.000336
Sum squared resid	0.000310	S.E. of regression		0.000245
R^2	0.468805	Adjusted R^2		0.468702
$F(1, 5177)$	4568.947	p-value(F)		0.000000
$\hat{\rho}$	0.212647	Durbin–Watson		1.574424
OLS, using observations 2000-01-05–2020-04-30 (T = 5179)				
Dependent variable: rv5				
	Coefficient	Std. Error	t-ratio	p-value
const	0.000191985	4.32834×10^{-6}	44.36	0.0000
yhat2_1	-0.111512	0.00332039	-33.58	0.0000
Mean dependent var	0.000161	S.D. dependent var		0.000336
Sum squared resid	0.000479	S.E. of regression		0.000304
R^2	0.178889	Adjusted R^2		0.178731
$F(1, 5177)$	1127.876	p-value(F)	7.0×10^{-224}	
$\hat{\rho}$	0.258795	Durbin–Watson		1.482403

4. Conclusions

The paper featured a further examination of the effectiveness of SV and GARCH(1,1) models, as explanators of model-free estimates of the volatility of S&P500, DOWJONES and STOXX50, using RV samples at 5-minute intervals, as provided by Oxford Man Institute's Realised Library, as a benchmark. In order to provide further contrast, 1 also used lags of squared demeaned daily returns on FTSE to provide a simple alternative estimate of daily volatility. The effectiveness of these three methods was explored via the application of ordinary least squares (OLS) regression analysis. Poon and Granger (2005) provided motivation in their analysis of 66 studies of this topic in which they noted that, at that time, there were an insufficient number of SV studies to provide a comparison between GARCH and HISVOL models. My intention in the paper was to further address this sparsity in the literature.

The enhanced estimates of SV were obtained by means of the R package factorstockvol and the addition of a regression matrix which included vectors of the second and third moments of the demeaned return series for the three indices. The enhanced GARCH(1,1) model was obtained by adding the squared demeaned return series to the mean equation.

The results were consistent with those of an earlier companion study by Allen and McAleer (2020) which featured the FTSE. In all the three base cases, the simple expedient of adopting 20 lags of squared demeaned returns in the regression model, out-performing the volatility estimates from those of the base SV model and the GARCH(1,1) models.

The enhanced SV model did show higher explanatory power than the base models in the cases of both the S&P500 and the STOXX50. The results from the enhanced GARCH model were also variable, but in no case matched those of the simple HISVOL model.

However, both performed relatively poorly as compared with the simple expedient of using squared demeaned daily returns on the three indices, in order to predict RV5 volatility. The results support Poon and Granger (2005), in that neither GARCH or SV models outperform a simple form of a HISVOL model in this sample when RV sampled at 5-minute intervals are used as a benchmark. In the case of the enhanced SV model, there was evidence of a marginal increase in adjusted R-squares in

two cases out of three. It is a moot point as to whether the difficulty of the estimation of an enhanced SV model justifies the marginal gain in explanatory power.

The results are also consistent with Perron and Shi (2020) who demonstrate that squared low-frequency returns can be expressed in terms of the temporal aggregation of a high-frequency series.

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