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Spurious Relationships for Nearly Non-Stationary Series

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Abstract: Literature shows that the regression of independent and (nearly) nonstationary time series could result in spurious outcomes. In this paper, we conjecture that under some situations, the regression of two independent and nearly non-stationary series does not have any spurious problem at all. To check whether our conjecture holds, we set up several situations and conduct simulations to justify our conjecture. Our simulations show that under some situations, the chance that the regressions being spurious is very high for all the cases simulated in our paper. Nonetheless, under some other situations, our simulation shows that the rejection rates are much smaller than the 5% level of significance for all the cases simulated in our paper, implying that our conjecture could hold under some situations that regression of two independent and nearly non-stationary series does not have any spurious problem at all.

Keywords: cointegration; stationarity; non-stationarity; spurious problem; nearly non-stationarity

JEL Classification: C01; C15; C22; C58; C60

1. Introduction

Granger and Newbold (1974) and others show that regression of independent (nearly) nonstationary time series could result in spurious outcomes, Pesaran et al. (1999) and others find that a mixed integration of orders; that is, I(0) or I(1), could be cointegrated and the residual is stationary, and Westerlund (2008) documents that many studies commit a Type 1 error by failing to reject the no-cointegration hypothesis. On the other hand, Engle and Granger (1987) establish the relationship between cointegration and error correction models that first suggested in Granger (1981) and develop estimation procedures and tests for the cointegration model. In addition, Phillips (1986) develops an asymptotic theory for regressions of integrated random processes, including the spurious regressions discovered by Granger and Newbold (1974) and the cointegrating regressions developed by Engle and Granger (1987). Entorf (1997) analyses the regression of two independent random walks with drifts and shows that the convergence to pseudo true values applies to the estimation of spurious fixed-effects models. Readers may refer to Ventosa-Santaulária (2009) for an overview of spurious regression.

Is it possible that the regression of two independent and nearly non-stationary series does not have any spurious problem? In this paper, we explore the issue. To explore the problem, we first conjecture that under some situations, regression of two independent and nearly non-stationary series does not have any spurious problem at all. To check whether the conjecture we set holds, we first generate two independent and nearly nonstationary AR(1) processes, $X_t = \alpha_1 X_{t-1} + \varepsilon_t$ and $Y_t = \alpha_2 Y_{t-1} + e_t$ with $0.9 < |\alpha_1|, |\alpha_2| < 1$. We then regress Y_t on the independent X_t to get $Y_t = \alpha + \beta X_t + u_t$ and check the proportion of rejecting the null hypothesis that the beta (β) is zero. We first find that under some situations, consistent with the literature, regressing two independent and (nearly) nonstationary time series could be spurious. Nonetheless, we also find that under some other situations, different from the literature, our results show that the rejection rates are much smaller than the 5% level of significance for all the cases simulated in our paper, implying that under some other situations, regressing nearly nonstationary Y_t on independent and nearly nonstationary X_t will not get any spurious problem at all as shown in all the cases being simulated in our paper.

The rest of the paper is organized as follows. In Section 2, we state the basic models for the regression and the regression with a spurious problem. In Section 3, we state our model setup and construct the algorithm for the simulation. In Section 4, we discuss our findings from our simulation and the last section concludes.

2. The Model

In this section, we state the basic models for a simple regression and the regression with a spurious problem. We first state the basic simple regression model.

2.1. Linear Regression Model

In this paper, we consider the following simple regression model:

$$Y_t = \alpha + \beta X_t + u_t, \tag{1}$$

where u_t is a random component denoted as error term assumed to be independent and identically distributed (iid) with mean 0 and variance σ_u^2, $t = 1, \cdots, T$ in which T is the sample size, α is the intercept parameter, and β is the slope parameter.

The most important hypothesis for the simple linear regression stated in (1) is

$$H_0 \ : \ \beta = 0 \quad \text{versus} \quad H_1 \ : \ \beta \neq 0 \,, \tag{2}$$

which is used to detect the linear relationship between two variables Y and X. If the null hypothesis H_0 in (2) is true, then for the model in (1), the population mean of Y_t is always equal to α for any value of X_t, concluding that Y_t does not depend on the value of X_t and there is no linear relationship between X_t and Y_t. On the other hand, if the alternative hypothesis H_1 is true, then one could conclude that a change in X_t is associated with a change in Y_t linearly.

To test whether the null hypothesis H_0 in (2) is true, one could use the following T test:

$$T = \frac{\hat{\beta} - \beta}{SE(\hat{\beta})} = \frac{\hat{\beta}}{SE(\hat{\beta})} \,, \tag{3}$$

where

$$\hat{\beta} = \frac{\sum_{t=1}^{N} (X_t - \bar{X})(Y_t - \bar{Y})}{\sum_{t=1}^{N} (X_t - \bar{X})^2}, \tag{4}$$

is the estimate of β in which $\bar{X} = \sum_{t=1}^{T} X_t / T$, $\bar{Y} = \sum_{t=1}^{T} Y_t / T$, and

$$SE(\hat{\beta}) = \sqrt{\frac{\sum_{t=1}^{T} (\hat{u}_t^2) / (T-2)}{\sum_{t=1}^{T} (X_t - \bar{X})^2}} \tag{5}$$

is the standard error of the estimate measuring the accuracy of prediction with $\hat{u}_t = (Y_t - \hat{\alpha} - \hat{\beta} X_t)^2$. It is well known that the test statistic T follows a t-distribution with $T-2$ degrees of freedom if the null hypothesis H_0 is true. We note that Goodness of fit can refer to specific individual factors (or variables) in the regression equation, whereas R-squared refers to a specific numerical value of the entire regression model.

2.2. Spurious Regression

A spurious relationship is a relationship that does not make sense. In this situation, two or more independent variables could appear to be correlated with the effect of an unseen factor ("confounding factor" or "lurking variable"). That is, goodness-of-fit indicators from regression such as R^2 are likely to be large, implying a valid fit even if the underlying variables are not truly related.

A variable, say, Y_t, is said to be integrated of order d, and denoted as $Y_t = I(d)$, if it has a stationary, invertible, and stochastic ARMA representation after differencing d times, it is stationary if $d < 1$, is non-stationary if $d \geq 1$, and is nearly non-stationary if $d < 1$ but close to 1. Granger and Newbold (1974) and others find that if

$$Y_t \sim I(1) \quad \text{and} \quad X_t \sim I(1)\,, \tag{6}$$

then regression in (1) could be spurious. It means that even when Y_t and X_t are known to be independent, applying the test statistic T in (3) to an ordinary least square regression could misleadingly indicate a good fit when non-stationary time series data are involved. Granger and Newbold (1974) exhibit an example in which an equation has $R^2 = 0.997$ and the value of the Durbin–Watson (DW) statistic is 0.093, implying that when the residuals are strongly autocorrelated in time series regression, the interpretability of the coefficients could be questionable.

Since Granger and Newbold (1974) and others show that regression of independent and nonstationary time series could result in spurious outcomes, it is common to believe that regression of independent and nearly nonstationary time series could also get spurious outcomes. In this paper, we believe that there are some cases in which regression of independent and nearly nonstationary time series may not be spurious. Thus, we set up the following conjecture:

Conjecture 1. *Under some situations, the regression of two independent and nearly non-stationary series does not have any spurious problem at all.*

To examine whether the above conjecture could hold in some situations, we will discuss it in the next section.

3. Model Setup and Algorithm

In this section, we first state the model setting of generating two purely independent and nearly nonstationary time series, regressing one of them onto the other, and examining whether the corresponding regression is spurious. We then construct the algorithm for the simulation and discuss our simulation result in the next section.

3.1. Model Setup

We consider the simple linear regression in (1) between two unrelated nearly nonstationary AR(1) series X_t and Y_t such that

$$X_t = \alpha_1 X_{t-1} + \varepsilon_t \quad \text{and} \quad Y_t = \alpha_2 Y_{t-1} + e_t, \quad \text{with} \quad \varepsilon_t \overset{iid}{\sim} (0, \sigma_\varepsilon^2) \quad \text{and} \quad e_t \overset{iid}{\sim} (0, \sigma_e^2) \tag{7}$$

in which $0.9 < |\alpha_i| < 1$ $(i = 1, 2)$. For simplicity, we assume that both ε_t and e_t follow:

$$f(a; p) \propto \frac{1}{\sigma}\left\{1 + \frac{a^2}{k\sigma^2}\right\}^{-p} \qquad (-\infty < a < \infty)\,, \tag{8}$$

where $k = 2p - 1$ and $p \geq 2$. We note that $E(a) = 0$, $V(a) = \sigma^2$, and $t = \sqrt{(k/\nu)}\,(a/\sigma)$ follows a Student's t distribution with $\nu = 2p - 1$ degrees of freedom (df). For $1 \leq p < 2$, k is equated to 1 and in this case, σ in (8) is simply a scale parameter. When $\nu = 1$, it becomes a Cauchy distribution and when $\nu = \infty$, it becomes a normal distribution. Readers may refer to Pötzelberger (1990), Tiku and Wong (1998), Tiku et al. (1999, 2000), Wong and Bian (2005), Fu and Fu (2015), and others to know more properties of AR(1) series.

To simulate X_t and Y_t properly, without loss of generality, we will consider different factors that could affect the behavior of the time series. First, we consider the distribution of the error terms. We choose a time series that follows the following four different iid error distributions in our study:

Situation 1. *We assume that the distribution of the error terms ε_t and e_t defined in (7) follow the following situations:*

1. *a standard normal distribution: that is, both ε_t and $e_t \sim N(0,1)$;*
2. *a t-distribution with $df = 5$: that is, both ε_t and $e_t \sim t(5)$;*
3. *a t-distribution with $df = 2$: that is, both ε_t and $e_t \sim t(2)$; and*
4. *a t-distribution with $df = 1$: that is, both ε_t and e_t follow the standard Cauchy distribution.*

Second, we vary the lengths of the times series and simulate a time series with the following four different lengths in our study as stated in the following situations:

Situation 2. *We consider that the lengths of the times series Y_t and X_t defined in (7) to be: (i) T = 100; (ii) T = 200; (iii) T = 400; and (iv) T = 800.*

After deciding the error distribution and the lengths of the AR(1) processes, we now consider the different values of α_1 and α_2. In our model, since both X_t and Y_t are nearly nonstationary, we choose $0.9 < |\alpha_i| < 1$ $(i = 1, 2)$ and, in particular, we define $A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ and $A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$[1] and consider the following values for both α_1 and α_2 as stated in Situations 3 and 4:

Situation 3. *We consider that the values of both α_1 and α_2 such that $\alpha_1 \in A^+$ and $\alpha_2 \in A^-$.*

Situation 4. *We consider that the values of both α_1 and α_2 such that $\alpha_1 \in A^-$ and $\alpha_2 \in A^+$.*

We note that in this paper, we consider Situations 3 and 4 because when two autoregressive processes in which one is associated to the zero frequency; that is, the AR(1) with a positive coefficient in our paper, and the other is associated to the Nyquist frequency (π); that is, the AR(1) with a negative coefficient in our paper that has power at frequency π and completes a cycle every 2 observations, are independent or even asymptotically orthogonal. Readers may refer to Johansen and Schaumburg (1999), Ghysels and Osborn (2001), and del Barrio Castro et al. (2018, 2019) for more information. Readers may also refer to seasonal unit root tests, see, for example, del Barrio Castro et al. (2012) and Smith et al. (2009), and cointegration for processes integrated at different frequencies, see, for example, del Barrio Castro et al. (2020) with properties that are related to the series we are using in our paper.[2]

With four different error distributions, four different time series lengths, and the above 50 combinations of α_1 and α_2 values as stated in Assumptions 1, 2, 3, and 4, there are in total 800 cases of simulation in our study for the cases when autoregressive coefficients α_1 and α_2 have different signs.

Nevertheless, in this paper, we also study the cases when both autoregressive coefficients α_1 and α_2 are of the same signs, either positive or negative. Thus, we include the following situations in our study:

Situation 5. *We consider that the values of both α_1 and α_2 such that $\alpha_1 \in A^+$ and $\alpha_2 \in A^+$.*

Situation 6. *We consider that the values of both α_1 and α_2 such that $\alpha_1 \in A^-$ and $\alpha_2 \in A^-$.*

3.2. Algorithm

The two series X_t and Y_t are generated from independent error terms, and thus, they are expected not to be related. However, Granger and Newbold (1974) and others have shown that regression of independent nonstationary time series could result in spurious outcomes. In this paper, we believe that it is possible that when regressinng independent and nearly nonstationary Y_t and X_t as shown in Equation (1) may not be spurious under some situations as we stated in Conjecture 1. To check whether Conjecture 1 could hold under some situations, we set the following algorithm for each situation (different error distributions, different time series lengths, different combinations of α_1 and α_2) as described in Section 3.1:

Algorithm 1: For each situation (different error distributions, different time series lengths, different combinations of α_1 and α_2) as described in Section 3.1, we will conduct the following steps in our simulation:

1. Simulate 10,000 pairs of X_t and Y_t defined in (7) with coefficients described in Section 3.1.
2. For each pair of simulated X_t and Y_t, fit model in (1). Thus, in each subcase, we will obtain 10,000 $\hat{\beta}$'s and 10,000 corresponding *p*-values.
3. Plot the distribution of $\hat{\beta}$'s and record the standard errors and t-statistics of $\hat{\beta}$.
4. Use the T test defined in Equation (3) to test whether the null hypothesis H_0 in (2) hold. H_0 is rejected if *p*-value for the T test is less than 0.05. Calculate the proportion of significant $\hat{\beta}$'s or proportion of *p*-values that are less than 0.05 among the 10,000 fitted linear regression models in each subcase. This proportion is denoted as the rejection rate in this paper.

For each situation (different error distributions, different time series lengths, different combinations of α_1 and α_2) as described in Section 3.1, we will conduct simulation as described in Algorithm 1 and discuss the results in the next section.

4. Simulation

We follow Algorithm 1 to conduct simulation for each situation (different error distributions, different time series lengths, different combinations of α_1 and α_2) as described in Section 3.1. The simulation helps us to examine whether the T statistic as shown in Equation (3) for the model as shown in Equation (1) follow a Student t-distribution. If X_t and Y_t are unrelated, the true null hypothesis that all β coefficients are zero should be rejected around 5% of the time at the significance level of 5%. If the T test is good, that is, $\hat{\beta}$'s follow student t-distribution, the rejection rate should be close to 5%. If the rejection rate is significantly greater than 5%, then we conclude that there exists the spurious problem. In addition, we believe that it is possible that when regressing independent and nearly nonstationary Y_t and X_t as shown in (1) may not be spurious under some situations as we hypothesized in Conjecture 1. To check whether Conjecture 1 could hold under some situations, we discuss it in this section. We first discuss the results of the simulation for the cases when α_1 and α_2 are of different signs in the next subsection.

4.1. Simulation for the Cases When α_1 and α_2 Are of Different Signs

We first analyze cases as stated in Situation 3 and exhibit the results in Tables A1–A4 displaying in Appendix A that report the rejecting frequency of the T test when $(\alpha_1, \alpha_2) \in (A^+, A^-)$. From Tables A1–A4, one can observe that when choosing the values of both α_1 and α_2 as stated in Situation 3 are from $|0.9|$ to $|0.99|$, the rejection rate is about 0.0000 for any n and for any error distribution studied in our paper, except the situation when the error term follows a $t(1)$ in which the rejection rates are close to 0.0004.

We then analyze the cases as stated in Situation 4 and show the results in Tables A5–A8 displaying in Appendix B that report the rejecting frequency of the T test when $(\alpha_1, \alpha_2) \in (A^-, A^+)$. Similarly, from Tables A5–A8, one can observe that when choosing values of both α_1 and α_2 as stated in Situation 4 are between $|0.9|$ and $|0.99|$, the rejection rate is zero or close to zero for any n and any error distribution studied in our paper.

Our analysis shows that for all the cases when choosing values for (α_1, α_2) as stated in Situations 3 and 4 and when choosing values of both $|\alpha_1|$ and $|\alpha_2|$ are between 0.9 and 0.99, the rejection rates are much smaller than the 5% level of significance, implying that when (α_1, α_2) follow Situations 3 and 4 and when both $|\alpha_1|$ and $|\alpha_2|$ are between 0.9 and 0.99, all the corresponding regressions do not encounter any spurious problem for all the cases simulated in our paper, confirming that Conjecture 1 holds. In other words, our analysis shows that when independent Y_t and X_t follow nearly nonstationary AR(1) model and the autoregressive coefficients α_1 and α_2 have opposite signs, there is no spurious problem in the regression stated in Equation (1) and Conjecture 1 holds.

4.2. Simulation for the Cases When α_1 and α_2 Are of the Same Sign

We turn to examine whether the regression shown in Equation (1) is spurious for the cases when both α_1 and α_2 are of the same signs; that is, both α_1 and α_2 are positive or both are negative. To do so, we follow Algorithm 1 to conduct simulations for the cases when both α_1 and α_2 are positive and both are negative as displayed in Situations 5 and 6 and exhibit the results in Tables A9–A16 displaying in Appendices C and D, respectively.

We first discuss the cases when both α_1 and α_2 are positive as stated in Situation 5. Compared with the results in Tables A1–A8, all of the rejection rates in Tables A9–A12 are significantly higher than 5% and the rejecting frequency of the T test is higher than 49% for any n and any error distribution studied in our paper, except the situation when the error term follows $t(1)$ in which the rejection rates is higher than 32%. In addition, as n increases, or either α_1 or α_2 increases, or as the error distributions are further away from normal distribution, the rejecting rate increases even further.

We turn to discuss the cases when both α_1 and α_2 are negative as stated in Situation 6. Similar to the cases when both α_1 and α_2 are positive, when both α_1 and α_2 are negative, the rejecting frequency of the T test is higher than 50% for any n and for any error distribution studied in our paper, except the situation when the error term follows $t(1)$ in which the rejection rates is higher than 31%. In addition, Similar to the cases when both α_1 and α_2 are positive, as n increases, or either α_1 or α_2 increases, or as the error distributions are further away from normal distribution, the rejecting rate increases even further.

Our analysis shows that, different from all the cases when for α_1 and α_2 are of different signs, for all the cases when α_1 and α_2 are of the same signs, either positive or negative, as stated in Situations 5 and 6, respectively, and when both $|\alpha_1|$ and $|\alpha_2|$ are between 0.9 and 0.99, the rejection rate is much higher than the 5% level of significance for all the cases studied in our paper and it could be higher than 49%, implying that when (α_1, α_2) follow Situations 5 and 6 and when both $|\alpha_1|$ and $|\alpha_2|$ are between 0.9 and 0.99, the chance that the regressions being spurious is very high for all the cases simulated in our paper, which, in turn, rejects Conjecture 1 for all the cases in Situations 5 and 6.

5. Concluding Remarks

In this paper, we conjecture that under some situations, the regression of two independent and nearly non-stationary series does not have any spurious problem at all. To check whether our conjecture holds, we first generate two independent and nearly nonstationary AR(1) processes, $X_t = \alpha_1 X_{t-1} + \varepsilon_t$ and $Y_t = \alpha_2 Y_{t-1} + e_t$ in which $0.9 < |\alpha_1|, |\alpha_2| < 1$. We then regress Y_t on independent X_t to get $Y_t = \alpha + \beta X_t + u_t$ and check whether the proportion of rejecting the null hypothesis of the beta (β) to be zero. We first find that consistent with the literature that supports the hypothesis of regressing two independent and (nearly) nonstationary time series could be spurious, when both α_1 and α_2 are of the same signs, either positive or negative, and when the values of both $|\alpha_1|$ and $|\alpha_2|$ are between 0.9 and 0.99, the rejection rate is much bigger than the 5% level of significance in all the cases examined in our simulation and it could be higher than 49% in many cases, implying that the chance that the regressions being spurious is very high for all the cases when both α_1 and α_2 are of the same signs.

Nonetheless, for all the cases when for α_1 and α_2 are of different signs, then different from the literature, our results show that when both $|\alpha_1|$ and $|\alpha_2|$ are between 0.9 and 0.99, the rejection rates are much smaller than the 5% level of significance for all the cases studied in our paper, implying that when α_1 and α_2 are of different signs, regressing nearly nonstationary Y_t on independent and nearly nonstationary X_t will not get any spurious problem at all for all the cases being simulated in our paper.

We note that the literature shows that the regression of independent and (nearly) nonstationary time series could result in spurious outcomes. In this paper, we conjecture that under some situations, regression of two independent and nearly non-stationary series does not have any spurious problem at all, and in this paper, we aim to find some situations that our conjecture could hold. In this paper, we find that when $(\alpha_1, \alpha_2) \in (A^+, A^-)$ or (A^-, A^+), then our conjecture holds. We note that when $(\alpha_1, \alpha_2) \in (A^+, A^-)$ or (A^-, A^+), our conjecture holds which does not imply that these are only situations that our conjecture holds. There could have other situations that our conjecture could hold. We leave it to future studies to find other situations that our conjecture could hold. The purpose of our paper is to tell readers that when one finds regression of any two or more time series that do not have any spurious problem, this does not necessarily imply that the series are not independent. Thus, academics and practitioners should conduct some proper tests to show whether the series are independent.

Some academics may wonder whether there are some financial or economic time series that exhibit extreme negative autocorrelations. We believe there could have some financial or economic time series exhibit positive autocorrelations and some exhibit negative autocorrelations. We note that the time period used in our paper may not be daily or monthly, it should be set to fit the nature of the time series. It is well-known that stock returns could be overreacted or underreacted, this means that it could be positively auto-correlated or negatively auto-correlated and the true unobserved stock returns are positively auto-correlated or negatively auto-correlated. Whether they are extreme positively auto-correlated or negatively auto-correlated will depend on particular stocks. In addition, as we have mentioned before, when $(\alpha_1, \alpha_2) \in (A^+, A^-)$ or (A^-, A^+), our conjecture holds which does not imply that these are the only situations that our conjecture could hold. There may have other situations that our conjecture could hold. Some financial or economic time series could follow other situations that yet to be discovered, and thus, the conjecture could be important not only for statistics, but also for economics and finance. We also note that in our paper, we only consider (A^+, A^-) to cover nearly non-stationary series but do not cover the situations $A^+ = 1$ and $A^- = -1$. We do not cover the situation $A^+ = 1$ because this has been well-studied in the literature. On the other hand, we do not cover the situation $A^- = -1$ because this situation, we believe, is of no practice relevance.

We note that as far as we know, this paper is the first paper to discover that under some situations, the regression of two independent and nearly non-stationary series does not have any spurious problem at all. We follow Granger and Newbold (1974) and others to provide simulation results to show our discovery. Academics could follow Phillips (1986), Johansen and Schaumburg (1999), and others to provide formal proof of the finding in our paper to replace Brownian motions by using the OU processes with $exp(c/T)$ to approximate $(1 + c/T)$.[3] We will leave it to further research to develop the theoretical results to explain the phenomena discovered in this paper. We also note that in this paper, we get very good results by using $0.9 < |\alpha| < 1$. One may get good results by using the near-integrated approach. We will leave this to future studies.[4] Another problem in our study is that there is a serious problem with under-rejection. Further study could expose this problem and correct the test properly.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. $(\alpha_1, \alpha_2) \in (A^+, A^-)$

Table A1. Rejection rate, error $\sim$ N(0,1).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.9	0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.92	0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.95	0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.97	0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table A1. *Cont.*

α_2	α_1	*n* = 100	*n* = 200	*n* = 400	*n* = 800	Raverage	Saverage
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.99	0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A2. Rejection rate, error $\sim$ t(5).

α_2	α_1	*n* = 100	*n* = 200	*n* = 400	*n* = 800	Raverage	Saverage
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.9	0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.92	0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.95	0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
-0.97	0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.99	0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A3. Rejection rate, error ∼ t(2).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	0.9	0.0000	0.0001	0.0000	0.0001	0.0001	
	0.92	0.0001	0.0000	0.0000	0.0000	0.0001	
−0.9	0.95	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000
	0.97	0.0001	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.92	0.95	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0001	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0001	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9	0.0002	0.0000	0.0000	0.0000	0.0001	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.95	0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0001	0.0001	0.0000	0.0000	0.0001	
	0.99	0.0000	0.0001	0.0000	0.0000	0.0000	
Caverage		0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9	0.0001	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0001	0.0001	0.0000	0.0000	0.0001	
−0.97	0.95	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0003	0.0001	0.0001	0.0000	0.0001	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
−0.99	0.95	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A4. Rejection Rate, Error ∼ t(1).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	0.9	0.0020	0.0005	0.0007	0.0027	0.0015	
	0.92	0.0010	0.0006	0.0006	0.0013	0.0009	
−0.9	0.95	0.0009	0.0006	0.0002	0.0007	0.0006	0.0008
	0.97	0.0009	0.0007	0.0002	0.0001	0.0005	
	0.99	0.0009	0.0001	0.0000	0.0003	0.0003	
Caverage		0.0011	0.0005	0.0003	0.0010	0.0008	0.0008

Table A4. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
−0.92	0.9	0.0008	0.0005	0.0002	0.0006	0.0005	0.0004
	0.92	0.0012	0.0010	0.0002	0.0006	0.0008	
	0.95	0.0004	0.0004	0.0003	0.0003	0.0004	
	0.97	0.0005	0.0004	0.0000	0.0001	0.0003	
	0.99	0.0005	0.0002	0.0001	0.0000	0.0002	
Caverage		0.0007	0.0005	0.0002	0.0003	0.0004	0.0004
−0.95	0.9	0.0008	0.0007	0.0002	0.0005	0.0006	0.0004
	0.92	0.0010	0.0002	0.0001	0.0003	0.0004	
	0.95	0.0011	0.0003	0.0002	0.0001	0.0004	
	0.97	0.0007	0.0006	0.0000	0.0001	0.0004	
	0.99	0.0007	0.0001	0.0000	0.0001	0.0002	
Caverage		0.0009	0.0004	0.0001	0.0002	0.0004	0.0004
−0.97	0.9	0.0008	0.0006	0.0003	0.0003	0.0005	0.0004
	0.92	0.0009	0.0007	0.0001	0.0001	0.0005	
	0.95	0.0014	0.0001	0.0001	0.0002	0.0005	
	0.97	0.0002	0.0002	0.0001	0.0002	0.0002	
	0.99	0.0010	0.0002	0.0002	0.0000	0.0004	
Caverage		0.0009	0.0004	0.0002	0.0002	0.0004	0.0004
−0.99	0.9	0.0011	0.0003	0.0004	0.0000	0.0005	0.0003
	0.92	0.0005	0.0002	0.0001	0.0000	0.0002	
	0.95	0.0003	0.0003	0.0000	0.0001	0.0002	
	0.97	0.0009	0.0003	0.0001	0.0000	0.0003	
	0.99	0.0005	0.0000	0.0000	0.0000	0.0001	
Caverage		0.0007	0.0002	0.0001	0.0000	0.0003	0.0003
Oaverage		0.0008	0.0004	0.0002	0.0003	0.0004	0.0004

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Appendix B. $(\alpha_1, \alpha_2) \in (A^-, A^+)$

Table A5. Rejection rate, error ∼ N(0,1).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
0.9	−0.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.92	−0.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table A5. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
0.95	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
0.97	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
0.99	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Oaverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A6. Rejection rate, error $\sim$ t(5).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
0.9	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.92	0.0001	0.0000	0.0000	0.0000	0.0000	
0.92	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
0.95	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table A6. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
0.97	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.9	0.0001	0.0000	0.0000	0.0000	0.0000	
	−0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
0.99	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Oaverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A7. Rejection rate, error $\sim$ t(2).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.0000	0.0002	0.0000	0.0001	0.0001	
	−0.92	0.0000	0.0000	0.0000	0.0000	0.0000	
0.9	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0001	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0001	0.0000	0.0000	0.0000	0.0000
	−0.9	0.0001	0.0000	0.0000	0.0001	0.0001	
	−0.92	0.0000	0.0001	0.0000	0.0000	0.0000	
0.92	−0.95	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.9	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.92	0.0000	0.0001	0.0000	0.0000	0.0000	
0.95	−0.95	0.0002	0.0000	0.0000	0.0000	0.0001	0.0000
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0000	0.0001	0.0000	0.0001	0.0001	
Caverage		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.9	0.0001	0.0000	0.0000	0.0000	0.0000	
	−0.92	0.0002	0.0001	0.0000	0.0000	0.0001	
0.97	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	−0.97	0.0001	0.0001	0.0000	0.0000	0.0001	
	−0.99	0.0001	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

Table A7. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
0.99	−0.9	0.0002	0.0001	0.0000	0.0000	0.0001	0.0000
	−0.92	0.0001	0.0000	0.0000	0.0000	0.0000	
	−0.95	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.97	0.0000	0.0000	0.0000	0.0000	0.0000	
	−0.99	0.0001	0.0000	0.0000	0.0000	0.0000	
Caverage		0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
Oaverage		0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A8. Rejection rate, error $\sim$ t(1).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
0.9	−0.9	0.0014	0.0006	0.0009	0.0019	0.0012	0.0007
	−0.92	0.0010	0.0005	0.0004	0.0016	0.0009	
	−0.95	0.0011	0.0003	0.0004	0.0008	0.0007	
	−0.97	0.0008	0.0006	0.0000	0.0001	0.0004	
	−0.99	0.0008	0.0000	0.0001	0.0004	0.0003	
Caverage		0.0010	0.0004	0.0004	0.0010	0.0007	0.0007
0.92	−0.9	0.0009	0.0008	0.0002	0.0007	0.0007	0.0004
	−0.92	0.0012	0.0010	0.0002	0.0007	0.0008	
	−0.95	0.0004	0.0003	0.0003	0.0001	0.0003	
	−0.97	0.0004	0.0005	0.0000	0.0002	0.0003	
	−0.99	0.0005	0.0003	0.0001	0.0000	0.0002	
Caverage		0.0007	0.0006	0.0002	0.0003	0.0004	0.0004
0.95	−0.9	0.0009	0.0006	0.0002	0.0005	0.0006	0.0004
	−0.92	0.0017	0.0004	0.0002	0.0001	0.0006	
	−0.95	0.0009	0.0003	0.0003	0.0001	0.0004	
	−0.97	0.0008	0.0004	0.0001	0.0001	0.0004	
	−0.99	0.0004	0.0004	0.0000	0.0001	0.0002	
Caverage		0.0009	0.0004	0.0002	0.0002	0.0004	0.0004
0.97	−0.9	0.0008	0.0007	0.0005	0.0002	0.0006	0.0004
	−0.92	0.0009	0.0006	0.0000	0.0001	0.0004	
	−0.95	0.0014	0.0004	0.0002	0.0001	0.0005	
	−0.97	0.0004	0.0002	0.0000	0.0002	0.0002	
	−0.99	0.0012	0.0004	0.0001	0.0000	0.0004	
Caverage		0.0009	0.0005	0.0002	0.0001	0.0004	0.0004

Table A8. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
0.99	−0.9	0.0010	0.0007	0.0002	0.0001	0.0005	0.0003
	−0.92	0.0008	0.0006	0.0002	0.0000	0.0004	
	−0.95	0.0003	0.0002	0.0003	0.0001	0.0002	
	−0.97	0.0007	0.0004	0.0002	0.0000	0.0003	
	−0.99	0.0003	0.0001	0.0002	0.0001	0.0002	
Caverage		0.0006	0.0004	0.0002	0.0001	0.0003	0.0003
Oaverage		0.0008	0.0005	0.0002	0.0003	0.0005	0.0005

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Appendix C. $(\alpha_1, \alpha_2) \in (A^+, A^+)$

Table A9. Rejection rate, error ∼ N(0,1).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
0.9	0.9	0.5105	0.5173	0.5191	0.5202	0.5168	0.5669
	0.92	0.5128	0.5379	0.5446	0.5552	0.5376	
	0.95	0.5528	0.5704	0.5800	0.5764	0.5699	
	0.97	0.5640	0.5887	0.6054	0.6058	0.5910	
	0.99	0.5847	0.6203	0.6358	0.6370	0.6195	
Caverage		0.5450	0.5669	0.5770	0.5789	0.5669	0.5669
0.92	0.9	0.5193	0.5391	0.5422	0.5445	0.5363	0.5924
	0.92	0.5392	0.5596	0.5756	0.5744	0.5622	
	0.95	0.5633	0.5995	0.6120	0.6102	0.5963	
	0.97	0.5880	0.6239	0.6368	0.6458	0.6236	
	0.99	0.6028	0.6468	0.6584	0.6660	0.6435	
Caverage		0.5625	0.5938	0.6050	0.6082	0.5924	0.5924
0.95	0.9	0.5451	0.5805	0.5795	0.5877	0.5732	0.6354
	0.92	0.5660	0.5970	0.6131	0.6164	0.5981	
	0.95	0.6120	0.6391	0.6554	0.6504	0.6392	
	0.97	0.6252	0.6719	0.6853	0.6889	0.6678	
	0.99	0.6486	0.6981	0.7221	0.7248	0.6984	
Caverage		0.5994	0.6373	0.6511	0.6536	0.6354	0.6354
0.97	0.9	0.5591	0.5965	0.6113	0.6006	0.5919	0.6623
	0.92	0.5833	0.6107	0.6367	0.6350	0.6164	
	0.95	0.6282	0.6625	0.6897	0.6869	0.6668	
	0.97	0.6433	0.6984	0.7286	0.7300	0.7001	
	0.99	0.6767	0.7288	0.7614	0.7785	0.7364	
Caverage		0.6181	0.6594	0.6855	0.6862	0.6623	0.6623

Table A9. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
0.99	0.9	0.5775	0.6051	0.6274	0.6385	0.6121	0.6937
	0.92	0.6106	0.6462	0.6625	0.6677	0.6468	
	0.95	0.6454	0.6957	0.7225	0.7240	0.6969	
	0.97	0.6669	0.7251	0.7632	0.7687	0.7310	
	0.99	0.7103	0.7730	0.8071	0.8370	0.7819	
Caverage		0.6421	0.6890	0.7165	0.7272	0.6937	0.6937
Oaverage		0.5934	0.6293	0.6470	0.6508	0.6301	0.6301

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A10. Rejection rate, error $\sim$ t(5).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
0.9	0.9	0.5104	0.5152	0.5233	0.5151	0.5160	0.5667
	0.92	0.5228	0.5420	0.5447	0.5442	0.5384	
	0.95	0.5514	0.5588	0.5752	0.5877	0.5683	
	0.97	0.5608	0.5979	0.6092	0.6065	0.5936	
	0.99	0.5805	0.6134	0.6322	0.6435	0.6174	
Caverage		0.5452	0.5655	0.5769	0.5794	0.5667	0.5667
0.92	0.9	0.5207	0.5426	0.5490	0.5490	0.5403	0.5922
	0.92	0.5384	0.5598	0.5697	0.5690	0.5592	
	0.95	0.5757	0.5939	0.6042	0.6033	0.5943	
	0.97	0.5834	0.6336	0.6349	0.6319	0.6210	
	0.99	0.6102	0.6454	0.6607	0.6690	0.6463	
Caverage		0.5657	0.5951	0.6037	0.6044	0.5922	0.5922
0.95	0.9	0.5539	0.5724	0.5925	0.5902	0.5773	0.6438
	0.92	0.5729	0.5984	0.6114	0.6120	0.5987	
	0.95	0.6088	0.6413	0.6489	0.6484	0.6369	
	0.97	0.6198	0.6665	0.6827	0.6900	0.6648	
	0.99	0.6427	0.7039	0.7159	0.7235	0.6965	
Caverage		0.5996	0.6365	0.6503	0.6528	0.6348	0.6438
0.97	0.9	0.5723	0.5962	0.6093	0.6042	0.5955	0.6656
	0.92	0.5950	0.6304	0.6360	0.6392	0.6252	
	0.95	0.6291	0.6753	0.6931	0.6859	0.6709	
	0.97	0.6448	0.7061	0.7273	0.7295	0.7019	
	0.99	0.6669	0.7328	0.7639	0.7739	0.7344	
Caverage		0.6216	0.6682	0.6859	0.6865	0.6656	0.6656

Table A10. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
0.99	0.9	0.5752	0.6114	0.6252	0.6305	0.6106	0.6941
	0.92	0.6056	0.6394	0.6601	0.6695	0.6437	
	0.95	0.6478	0.7000	0.7093	0.7203	0.6944	
	0.97	0.6829	0.7336	0.7657	0.7784	0.7402	
	0.99	0.7092	0.7747	0.8099	0.8340	0.7820	
Caverage		0.6441	0.6918	0.7140	0.7265	0.6941	0.6941
Oaverage		0.5952	0.6314	0.6462	0.6499	0.6307	0.6307

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A11. Rejection rate, error $\sim$ t(2).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
0.9	0.9	0.5075	0.5134	0.5077	0.4965	0.5063	0.5631
	0.92	0.5198	0.5290	0.5310	0.5337	0.5284	
	0.95	0.5502	0.5810	0.5791	0.5798	0.5725	
	0.97	0.5705	0.5941	0.6000	0.5999	0.5911	
	0.99	0.5809	0.6237	0.6312	0.6338	0.6174	
Caverage		0.5458	0.5682	0.5698	0.5687	0.5631	0.5631
0.92	0.9	0.5160	0.5321	0.5458	0.5314	0.5313	0.5894
	0.92	0.5414	0.5557	0.5604	0.5591	0.5542	
	0.95	0.5730	0.5941	0.6058	0.6005	0.5934	
	0.97	0.5976	0.6195	0.6263	0.6381	0.6204	
	0.99	0.6202	0.6460	0.6618	0.6627	0.6477	
Caverage		0.5696	0.5895	0.6000	0.5984	0.5894	0.5894
0.95	0.9	0.5593	0.5797	0.5696	0.5736	0.5706	0.6341
	0.92	0.5831	0.5927	0.5959	0.5982	0.5925	
	0.95	0.6064	0.6430	0.6475	0.6449	0.6355	
	0.97	0.6354	0.6709	0.6858	0.6882	0.6701	
	0.99	0.6512	0.7045	0.7205	0.7321	0.7021	
Caverage		0.6071	0.6382	0.6439	0.6474	0.6341	0.6341
0.97	0.9	0.5736	0.5961	0.5993	0.6070	0.5940	0.6660
	0.92	0.5908	0.6260	0.6375	0.6308	0.6213	
	0.95	0.6407	0.6650	0.6868	0.6908	0.6708	
	0.97	0.6613	0.7048	0.7284	0.7340	0.7071	
	0.99	0.6761	0.7392	0.7588	0.7735	0.7369	
Caverage		0.6285	0.6662	0.6822	0.6872	0.6660	0.6660

Table A11. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
0.99	0.9	0.5866	0.6290	0.6315	0.6386	0.6214	0.6997
	0.92	0.6175	0.6508	0.6665	0.6676	0.6506	
	0.95	0.6494	0.6986	0.7236	0.7324	0.7010	
	0.97	0.6891	0.7337	0.7666	0.7798	0.7423	
	0.99	0.7113	0.7687	0.8146	0.8380	0.7832	
Caverage		0.6508	0.6962	0.7206	0.7313	0.6997	0.6997
Oaverage		0.6004	0.6317	0.6433	0.6466	0.6305	0.6305

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A12. Rejection rate, error $\sim$ t(1).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
0.9	0.9	0.4804	0.4400	0.3874	0.3251	0.4082	0.5000
	0.92	0.5024	0.4703	0.4126	0.3588	0.4360	
	0.95	0.5545	0.5287	0.4806	0.4259	0.4974	
	0.97	0.5883	0.5764	0.5489	0.4802	0.5485	
	0.99	0.6014	0.6250	0.6280	0.5858	0.6101	
Caverage		0.5454	0.5281	0.4915	0.4352	0.5000	0.5000
0.92	0.9	0.5148	0.4843	0.4174	0.3511	0.4419	0.5344
	0.92	0.5403	0.5129	0.4579	0.3836	0.4737	
	0.95	0.5703	0.5592	0.5233	0.4588	0.5279	
	0.97	0.6072	0.6253	0.5828	0.5254	0.5852	
	0.99	0.6256	0.6597	0.6602	0.6281	0.6434	
Caverage		0.5716	0.5683	0.5283	0.4694	0.5344	0.5344
0.95	0.9	0.5466	0.5294	0.4787	0.4128	0.4919	0.5906
	0.92	0.5742	0.5762	0.5168	0.4573	0.5311	
	0.95	0.6191	0.6213	0.5884	0.5311	0.5900	
	0.97	0.6406	0.6638	0.6519	0.5984	0.6387	
	0.99	0.6572	0.7063	0.7291	0.7128	0.7014	
Caverage		0.6075	0.6194	0.5930	0.5425	0.5906	0.5906
0.97	0.9	0.5802	0.5792	0.5415	0.4867	0.5469	0.6401
	0.92	0.6042	0.6231	0.5853	0.5240	0.5842	
	0.95	0.6435	0.6701	0.6516	0.6039	0.6423	
	0.97	0.6524	0.7112	0.7070	0.6761	0.6867	
	0.99	0.6907	0.7415	0.7681	0.7625	0.7407	
Caverage		0.6342	0.6650	0.6507	0.6106	0.6401	0.6401

Table A12. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	0.9	0.6017	0.6346	0.6252	0.6024	0.6160	
	0.92	0.6282	0.6511	0.6643	0.6414	0.6463	
0.99	0.95	0.6562	0.6983	0.7158	0.6990	0.6923	0.6951
	0.97	0.6832	0.7425	0.7651	0.7607	0.7379	
	0.99	0.7150	0.7673	0.8213	0.8295	0.7833	
Caverage		0.6569	0.6988	0.7183	0.7066	0.6951	0.6951
Oaverage		0.6031	0.6159	0.5964	0.5529	0.5921	0.5921

Note: ‘Raverage’ (stands for row average) is the average rejection rate for different values of n for the same case; ‘Caverage’ (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ for the same n for the same Situation; ‘Saverage’ (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^+ = \{0.99, 0.97, 0.95, 0.92, 0.9\}$ and different n for the same situation and ‘overall average’ in the 3 to 6 columns is the overall average for each n for all situations and all cases while ‘overall average’ in the last two column is the overall average for all the cases in the entire table.

Appendix D. $(\alpha_1, \alpha_2) \in (A^-, A^-)$

Table A13. Rejection rate, error $\sim$ N(0,1).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.5173	0.5196	0.5228	0.5219	0.5204	
	−0.92	0.5413	0.5469	0.5444	0.5419	0.5436	
−0.9	−0.95	0.5759	0.5852	0.5860	0.5783	0.5814	0.5761
	−0.97	0.6010	0.6006	0.5998	0.6084	0.6025	
	−0.99	0.6234	0.6360	0.6361	0.6348	0.6326	
Caverage		0.5718	0.5777	0.5778	0.5771	0.5761	0.5761
	−0.9	0.5407	0.5405	0.5435	0.5434	0.5420	
	−0.92	0.5637	0.5691	0.5687	0.5730	0.5686	
−0.92	−0.95	0.5945	0.6085	0.6101	0.6099	0.6058	0.6034
	−0.97	0.6255	0.6345	0.6330	0.6388	0.6330	
	−0.99	0.6600	0.6635	0.6746	0.6724	0.6676	
Caverage		0.5969	0.6032	0.6060	0.6075	0.6034	0.6034
	−0.9	0.5701	0.5755	0.5811	0.5755	0.5756	
	−0.92	0.5948	0.6122	0.6034	0.6077	0.6045	
−0.95	−0.95	0.6445	0.6582	0.6578	0.6568	0.6543	0.6478
	−0.97	0.6730	0.6868	0.6909	0.6867	0.6844	
	−0.99	0.7085	0.7217	0.7239	0.7267	0.7202	
Caverage		0.6382	0.6509	0.6514	0.6507	0.6478	0.6478
	−0.9	0.6038	0.6032	0.6149	0.6107	0.6082	
	−0.92	0.6284	0.6394	0.6403	0.6524	0.6401	
−0.97	−0.95	0.6850	0.6862	0.6896	0.6939	0.6887	0.6868
	−0.97	0.7172	0.7305	0.7268	0.7274	0.7255	
	−0.99	0.7546	0.7662	0.7781	0.7866	0.7714	
Caverage		0.6778	0.6851	0.6899	0.6942	0.6868	0.6868

Table A13. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.6254	0.6229	0.6409	0.6404	0.6324	
	−0.92	0.6588	0.6607	0.6595	0.6741	0.6633	
−0.99	−0.95	0.7059	0.7216	0.7328	0.7297	0.7225	0.7237
	−0.97	0.7521	0.7657	0.7790	0.7867	0.7709	
	−0.99	0.8010	0.8328	0.8365	0.8480	0.8296	
Caverage		0.7086	0.7207	0.7297	0.7358	0.7237	0.7237
Oaverage		0.6387	0.6475	0.6510	0.6530	0.6476	0.6476

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A14. Rejection rate, error $\sim$ t(5).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.5225	0.5248	0.5206	0.5201	0.5220	
	−0.92	0.5422	0.5476	0.5407	0.5536	0.5460	
−0.9	−0.95	0.5778	0.5881	0.5766	0.5753	0.5795	0.5783
	−0.97	0.6067	0.6046	0.6102	0.6062	0.6069	
	−0.99	0.6292	0.6359	0.6362	0.6473	0.6372	
Caverage		0.5757	0.5802	0.5769	0.5805	0.5783	0.5783
	−0.9	0.5404	0.5435	0.5434	0.5410	0.5421	
	−0.92	0.5674	0.5709	0.5717	0.5793	0.5723	
−0.92	−0.95	0.5939	0.5993	0.5988	0.6146	0.6017	0.6045
	−0.97	0.6354	0.6446	0.6352	0.6476	0.6407	
	−0.99	0.6550	0.6670	0.6704	0.6695	0.6655	
Caverage		0.5984	0.6051	0.6039	0.6104	0.6045	0.6045
	−0.9	0.5722	0.5734	0.5852	0.5819	0.5782	
	−0.92	0.6048	0.6094	0.6097	0.6123	0.6091	
−0.95	−0.95	0.6434	0.6499	0.6598	0.6439	0.6493	0.6501
	−0.97	0.6841	0.6888	0.6933	0.6898	0.6890	
	−0.99	0.7103	0.7263	0.7365	0.7265	0.7249	
Caverage		0.6430	0.6496	0.6569	0.6509	0.6501	0.6501
	−0.9	0.6033	0.6025	0.6084	0.6087	0.6057	
	−0.92	0.6363	0.6247	0.6339	0.6396	0.6336	
−0.97	−0.95	0.6801	0.6914	0.6917	0.6916	0.6887	0.6845
	−0.97	0.7161	0.7273	0.7294	0.7295	0.7256	
	−0.99	0.7502	0.7691	0.7751	0.7811	0.7689	
Caverage		0.6772	0.6830	0.6877	0.6901	0.6845	0.6845

Table A14. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.6334	0.6324	0.6417	0.6349	0.6356	
	−0.92	0.6624	0.6638	0.6688	0.6720	0.6668	
−0.99	−0.95	0.7057	0.7218	0.7296	0.7299	0.7218	0.7248
	−0.97	0.7579	0.7688	0.7798	0.7776	0.7710	
	−0.99	0.7967	0.8320	0.8402	0.8457	0.8287	
Caverage		0.7112	0.7238	0.7320	0.7320	0.7248	0.7248
Oaverage		0.6411	0.6483	0.6515	0.6528	0.6484	0.6484

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A15. Rejection rate, error $\sim$ t(2).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.5084	0.5231	0.5100	0.5033	0.5112	
	−0.92	0.5475	0.5319	0.5350	0.5316	0.5365	
−0.9	−0.95	0.5780	0.5737	0.5793	0.5691	0.5750	0.5721
	−0.97	0.6037	0.6119	0.5992	0.5984	0.6033	
	−0.99	0.6182	0.6434	0.6406	0.6354	0.6344	
Caverage		0.5712	0.5768	0.5728	0.5676	0.5721	0.5721
	−0.9	0.5371	0.5320	0.5341	0.5235	0.5317	
	−0.92	0.5705	0.5649	0.5668	0.5458	0.5620	
−0.92	−0.95	0.6131	0.5970	0.6031	0.5987	0.6030	0.6009
	−0.97	0.6299	0.6436	0.6354	0.6268	0.6339	
	−0.99	0.6733	0.6692	0.6711	0.6814	0.6738	
Caverage		0.6048	0.6013	0.6021	0.5952	0.6009	0.6009
	−0.9	0.5774	0.5781	0.5704	0.5729	0.5747	
	−0.92	0.5983	0.6062	0.5938	0.6024	0.6002	
−0.95	−0.95	0.6520	0.6551	0.6588	0.6521	0.6545	0.6495
	−0.97	0.6840	0.6888	0.6917	0.6902	0.6887	
	−0.99	0.7219	0.7321	0.7377	0.7270	0.7297	
Caverage		0.6467	0.6521	0.6505	0.6489	0.6495	0.6495
	−0.9	0.5938	0.6001	0.6034	0.5963	0.5984	
	−0.92	0.6337	0.6396	0.6328	0.6212	0.6318	
−0.97	−0.95	0.6930	0.6871	0.6899	0.6873	0.6893	0.6846
	−0.97	0.7265	0.7346	0.7349	0.7237	0.7299	
	−0.99	0.7635	0.7739	0.7808	0.7757	0.7735	
Caverage		0.6821	0.6871	0.6884	0.6808	0.6846	0.6846

Table A15. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.6359	0.6391	0.6361	0.6334	0.6361	
	−0.92	0.6581	0.6756	0.6675	0.6729	0.6685	
−0.99	−0.95	0.7182	0.7309	0.7299	0.7198	0.7247	0.7268
	−0.97	0.7606	0.7710	0.7822	0.7704	0.7711	
	−0.99	0.8147	0.8337	0.8424	0.8428	0.8334	
Caverage		0.7175	0.7301	0.7316	0.7279	0.7268	0.7268
Oaverage		0.6445	0.6495	0.6491	0.6441	0.6468	0.6468

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Table A16. Rejection rate, error ∼ t(1).

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.4695	0.4271	0.3674	0.3162	0.3951	
	−0.92	0.4981	0.4494	0.3963	0.3422	0.4215	
−0.9	−0.95	0.5477	0.5158	0.4538	0.4070	0.4811	0.4882
	−0.97	0.6006	0.5626	0.5176	0.4592	0.5350	
	−0.99	0.6476	0.6277	0.5966	0.5613	0.6083	
Caverage		0.5527	0.5165	0.4663	0.4172	0.4882	0.4882
	−0.9	0.4982	0.4573	0.4000	0.3320	0.4219	
	−0.92	0.5349	0.4850	0.4300	0.3641	0.4535	
−0.92	−0.95	0.5857	0.5357	0.4922	0.4330	0.5117	0.5174
	−0.97	0.6268	0.5963	0.5419	0.4870	0.5630	
	−0.99	0.6642	0.6678	0.6297	0.5857	0.6369	
Caverage		0.5820	0.5484	0.4988	0.4404	0.5174	0.5174
	−0.9	0.5549	0.5045	0.4531	0.3936	0.4765	
	−0.92	0.5894	0.5474	0.4904	0.4384	0.5164	
−0.95	−0.95	0.6371	0.6096	0.5487	0.5007	0.5740	0.5802
	−0.97	0.6833	0.6588	0.6224	0.5470	0.6279	
	−0.99	0.7314	0.7280	0.7055	0.6605	0.7064	
Caverage		0.6392	0.6097	0.5640	0.5080	0.5802	0.5802
	−0.9	0.5982	0.5616	0.5063	0.4546	0.5302	
	−0.92	0.6262	0.5916	0.5457	0.4965	0.5650	
−0.97	−0.95	0.6867	0.6579	0.6199	0.5563	0.6302	0.6343
	−0.97	0.7389	0.7106	0.6681	0.6232	0.6852	
	−0.99	0.7851	0.7794	0.7557	0.7235	0.7609	
Caverage		0.6870	0.6602	0.6191	0.5708	0.6343	0.6343

Table A16. *Cont.*

α_2	α_1	n = 100	n = 200	n = 400	n = 800	Raverage	Saverage
	−0.9	0.6437	0.6386	0.5923	0.5597	0.6086	
	−0.92	0.6721	0.6643	0.6393	0.6028	0.6446	
−0.99	−0.95	0.7295	0.7323	0.7047	0.6644	0.7077	0.7121
	−0.97	0.7847	0.7797	0.7563	0.7301	0.7627	
	−0.99	0.8333	0.8511	0.8384	0.8246	0.8369	
Caverage		0.7327	0.7332	0.7062	0.6763	0.7121	0.7121
Oaverage		0.6387	0.6136	0.5709	0.5225	0.5864	0.5864

Note: 'Raverage' (stands for row average) is the average rejection rate for different values of n for the same case; 'Caverage' (stands for column average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ for the same n for the same Situation; 'Saverage' (stands for Situation average) is the average rejection rate for different cases in which $\alpha_1 \in A^- = \{-0.99, -0.97, -0.95, -0.92, -0.9\}$ and different n for the same situation and 'overall average' in the 3 to 6 columns is the overall average for each n for all situations and all cases while 'overall average' in the last two column is the overall average for all the cases in the entire table.

Notes

1. We would like to thank the anonymous referee for introducing the notations of A^+ and A^-.
2. We thank the anonymous referee for providing us this information.
3. We would like to thank the anonymous referee for giving us this information.
4. We would like to thank the anonymous referee for giving us this information.

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