

Article

Bankruptcy Risk in Discounted Cash Flow Equity Valuation

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Abstract: We investigate the importance of bankruptcy risk in discounted cash flow (DCF) equity valuation. Our analyses first show how bankruptcy risk is incorporated in DCF valuation, where investment risk is captured by *cash flow certainty equivalents*. Within this general setting, we find that bankruptcy risk can be captured by discounting factors incorporating period-specific bankruptcy probabilities, allowing the numerators in a DCF valuation model to follow a binary *random walk*. Elaborating a model of this kind, we assess the value of the equity holders' *limited liability right* (the equity holders' right to hand over the firm to its creditors if bankruptcy occurs). Two valuation models commonly used in academic research and professional practice—the *Dividend Discount Model* (DDM) and the *Residual Income Valuation* (RIV) model—are addressed specifically. Our analyses show that bankruptcy probabilities are important for the estimation of the value drivers in both models. Even if bankruptcy probabilities are as low as 0.02, equity values might be severely exaggerated if bankruptcy risk is ignored in DDM or RIV. In particular, this holds for firms expected to have high future growth (conditioned on firm survival). For the RIV model to properly capture bankruptcy risk, we identify "*bankruptcy event accounting principles*" and an additional term that must be included in the model. We also show that bankruptcy risk under certain conditions can be handled through a specific calibration of the discounting rate/-s in all DCF models, allowing the value drivers—i.e., future dividends or residual income—to be forecasted *conditioned on firm survival*.

Keywords: bankruptcy risk; DCF; dividend discount model; discounting rate; equity valuation; financial distress; fundamental valuation; limited liability right; residual income valuation

JEL Classification: G33; M41



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1. Introduction

There is a long history in the accounting and finance literature of using financial statement information to predict financially distressed companies, with early attempts going back to [Smith and Winakor \(1935\)](#), [Beaver \(1966\)](#) and [Altman \(1968\)](#), and important methodological contributions in [Ohlson \(1980\)](#), [Shumway \(2001\)](#), and [Hensher and Jones \(2007\)](#). There is also an extensive literature contributing to the estimation of prediction models for different accounting regimes, industries or geographical areas (cf. [Skogsvik 1990](#); [Lennox 1999](#); [Dewaelheyns and Van Hulle 2006](#); [Altman and Sabato 2007](#); [Betz et al. 2014](#); [Ciampi 2015](#)). Estimated prediction models have typically generated robust prediction performance,¹ having the potential of being useful both for researchers and business professionals. However, prior literature is surprisingly silent on how to incorporate bankruptcy probabilities in valuation problems. Addressing this, we contribute by clarifying the role of bankruptcy risk in discounted cash flow (DCF) equity valuation.

Our analyses deal with DCF based valuation modelling, this being the dominant valuation paradigm in the literature as well as in professional practice for the valuation of equity securities. As a point of departure, we investigate how bankruptcy risk is

incorporated in the valuation theory proposed by Robichek and Myers (1965) and Rubinstein (1976), where the risk of future cash flows is captured by the discounting of *certainty equivalents* with the *risk-free interest rate*. Based on this theory, we focus on two common applications of DCF equity valuation: the *Dividend Discount Model* (DDM) and the *Residual Income Valuation* (RIV) model. For all our valuation models, we find that bankruptcy probabilities can be accommodated in the expected values of the *value drivers* or—given no bankruptcy recovery values to the equity holders—through a specific calibration of the discounting rates.

As indicated, the importance of bankruptcy risk in DCF valuation is far from resolved in prior literature (cf. Damodaran 2009). Casual observations of business managers and financial analysts indicate puzzlement regarding the importance of this risk. Even though there is a general understanding of the negative value effects of bankruptcies, the (ex ante) incorporation of bankruptcy risk in DCF valuation models is not well appreciated. Financial statements and analysts' forecasts of company earnings or dividends are in all but special cases conditioned on the "*going concern principle*", implying that the firm is expected to continue its operations forever. It is hence not surprising that equity valuation is commonly based on forecasts of earnings and/or dividends *conditioned on firm survival* (cf. Penman 2013, chp. 3). This "standard" approach will obviously not distort calculated equity values much when bankruptcy probabilities are negligible. However, if bankruptcy risk is more significant, ignoring this risk might lead to severely exaggerated valuations. For example, in a so-called "steady state" with a forecasted constant dividend conditioned on firm survival and a cost of equity capital of 10%, the equity value is exaggerated by more than +20% if a bankruptcy risk of 0.02 is ignored in the valuation (cf. Section 3.1).

Accounting based valuation models have since the late 1990:s become increasingly common in empirical research and professional practice. The RIV model (Preinreich 1938; Edwards and Bell 1961; Peasnell 1982; Ohlson 1995) has often been used to empirically assess the association between accounting numbers and stock market prices (cf. Harris et al. 1994; Hung and Subramanyam 2007; Anesten et al. 2019), or to estimate implied costs of capital (cf. Gebhardt et al. 2001; Claus and Thomas 2001). However, with the exception of Anesten et al. (2019), the effect of bankruptcy risk has not been considered in this literature.²

A competing paradigm to DCF valuation is the option based approach, as first explored in Black and Scholes (1973) and Merton (1973). This approach implicitly incorporates bankruptcy risk in the valuation of financial instruments. However, despite its theoretical elegance, the approach rests on assumptions that typically have low empirical validity in the context of equity valuation. For example, letting the bankruptcy event be triggered by insufficient financial debt payments can be too simplistic. Overdue operating liabilities (accounts payables, employee liabilities, tax liabilities, etc.) might also trigger bankruptcies. Furthermore, option based valuation hinges on exogenously determined market prices for the *underlying asset* and market values being *free of arbitrage*. In most real-world settings, the underlying asset in this context (i.e., the firm's total assets) cannot easily be traded, in turn making the assumption of arbitrage-free market pricing suspect (cf. Björk 2009, chp. 2 and 15). There are also a number of well-known implementation problems. For example, estimating the return volatility of the underlying asset is notoriously difficult, in particular when the time to maturity of financial debt is long and/or there are several layers of debt. Given these concerns, it is not surprising that option based equity valuation is rarely found in professional practice.

We contribute to the literature in several ways. First, we show how bankruptcy risk is incorporated in the general DCF valuation theory of Robichek and Myers (1965) and Rubinstein (1976). Within this framework, we incorporate the owners' *limited liability right*, allowing the equity holders to file for bankruptcy and not having to fulfil the firm's obligations. Second, we provide a solution for how to incorporate bankruptcy probabilities in the discounting rates, allowing the value drivers to be forecasted *conditioned on survival*.

This possibility has previously been addressed only for simplified specifications of DDM (Shaffer 2006; Saha and Malkiel 2012). We contribute by clarifying under what conditions such “calibrated” discounting rates can be used in DCF valuation.³ Third, we provide a linkage between accounting numbers and the firm’s operating cash flows in the event of bankruptcy, allowing accounting based valuation models to be consistent with DCF equity valuation.⁴

Our analyses should be helpful for the financial management of private and public firms. For example, in the pursuit of merger and acquisition (M&A) strategies, managers need to know the value effects of bankruptcy risk in order not to bid excessively high prices for target firms. Another example is the management of financially distressed firms, where managers need to understand the importance of bankruptcy risk when approaching investment banks or when planning to make new issues of equity capital. Managers who do not comprehend how bankruptcy risk affects the value of owners’ equity of distressed firms, might simply fail to secure the needed financing for their firms. Bankruptcy risk is also important for the application of various financial statement accounting principles. For example, equity valuation is directly related to impairment tests of acquired goodwill in the consolidated financial statements of parent companies, where the ignorance of this risk might lead to erroneous overvaluations of recognised goodwill and net income numbers.

The paper is organised as follows. In the next section, we analyse how bankruptcy risk is incorporated in the generic DCF valuation approach, where the riskiness of a firm’s operating cash flows is captured through *certainty equivalents* (Section 2.1). Within this setting, we determine the importance of the owners’ *limited liability right* when a firm goes bankrupt (Section 2.2). In Section 3, we show how bankruptcy risk is incorporated in the *Dividend Discount Model (DDM)* (Section 3.1) and the *Residual Income Valuation (RIV)* model (Section 3.2). Concluding remarks and suggestions for future research follow in the last section of the paper.

2. Bankruptcy Risk and Equity Valuation

2.1. The Valuation Framework

In DCF equity valuation, the value of owners’ equity is determined through the discounting of future cash flows to the equity holders. As shown in prior literature (Robichek and Myers 1965; Rubinstein 1976; Fama 1977; Myers and Turnbull 1977; Stark 1986; Christensen and Feltham 2009), this can be achieved in two ways: either through the discounting of *risk-adjusted* cash flows (“*certainty equivalents*”) with the *risk-free rate*, or the discounting of *expected values* of cash flows with a *risk-adjusted* discounting rate. As shown in Robichek and Myers (1965) and Fama (1977), the latter alternative is more restrictive in the sense that a risk-adjusted discounting rate implies that the riskiness of future cash flows increases geometrically in a specific manner over time. In order to address the importance of bankruptcy risk in the most general setting, we start our modelling with risk-adjusted cash flows. We use the following notation, where \tilde{X} denotes the random variable X and the valuation date is $t = 0$:

\widetilde{FTE}_t = cash flow to equity holders at the end of period t ,

\widetilde{OCF}_t = operating cash flow (after company taxes) at the end of period t ,

$r_{f,t}$ = risk-free interest rate in period t ,

$CE_0(\tilde{X}_t)$ = certainty equivalent of cash flow \tilde{X}_t , to be received at the end of period t ,

$E_0(\tilde{X}_t)$ = expected value of a cash flow \tilde{X}_t , to be received at the end of period t ,

$p_{fail,t}$ = probability of bankruptcy at the end of period t , conditioned on the firm being a going concern at the beginning of the period, and

$E_0(\widetilde{RecV}_t^\dagger)$ = expected recovery value of the firm’s assets, to be received at the beginning of period $t + 1$ if the firm goes bankrupt in period t .

To make our modelling less complex, we make the following assumptions in this section:

- The firm has *no financial debt* at date $t = 0$ and will not issue any debt in the future.
- The firm's *operating cash flows* after company taxes are received by the equity holders at the end of future periods. If an operating cash flow is *negative* in some period t , the equity holders have to contribute this amount at the end of the period.
- The firm can file for bankruptcy at the end of future periods, after the operating cash flow of the period is received or settled by the equity holders. If the firm files for bankruptcy at the end of period t , there will be no cash flows to the equity holders in any following period $t + 1, t + 2, \dots \infty$.

Given the above assumptions, the value of owners' equity, $V_0(EQ)$, in DCF valuation is the present value of the firm's operating cash flows. In line with [Robichek and Myers \(1965\)](#), we hence write:

$$V_0(EQ) = \sum_{t=1}^{\infty} \frac{CE_0(\widetilde{FTE}_t)}{\prod_{\tau=1}^t (1 + r_{f,\tau})} \tag{1}$$

Introducing *certainty equivalent ratios* (cf. [Robichek and Myers 1965](#)),⁵ (1) can equivalently be written:

$$V_0(EQ) = \sum_{t=1}^{\infty} \frac{cer_0(\widetilde{FTE}_t) \cdot E_0(\widetilde{FTE}_t)}{\prod_{\tau=1}^t (1 + r_{f,\tau})} \tag{2}$$

where:

$cer_0(\widetilde{X}_t) = CE_0(\widetilde{X}_t) / E_0(\widetilde{X}_t)$ = certainty equivalent ratio assessed at $t = 0$ for the cash flow \widetilde{X}_t .

Since the firm has no financial debt, $E_0(\widetilde{FTE}_t)$ in (2) is a function of $p_{fail,t}$, \widetilde{OCF}_t and $RecV_t^{\dagger}$:

$$E_0(\widetilde{FTE}_t) = \left(\prod_{\tau=1}^t P_{\tau}\right) \cdot E_0(\widetilde{OCF}_t^*) + \left(\prod_{\tau=1}^{t-1} P_{\tau}\right) \cdot p_{fail,t} \cdot E_0(\widetilde{OCF}_t^{\dagger} + \widetilde{RecV}_t^{\dagger}) + \left[1 - \prod_{\tau=1}^{t-1} P_{\tau}\right] \cdot 0 \tag{3}$$

where:

$P_t = (1 - p_{fail,t})$ = probability of *survival* at the end of period t , conditioned on the firm being a going concern at the beginning of period t ,

\widetilde{OCF}_t^* = operating cash flow after company taxes at the end of period t , conditioned on the firm being a going concern at the end of period t , and

$\widetilde{OCF}_t^{\dagger}$ = operating cash flow after company taxes at the end of period t , conditioned on the firm going bankrupt at the end of period t .

The probability $p_{fail,t}$ in (3) is the one-period probability of the firm filing for bankruptcy in period t , provided that the firm is a going concern at the end of the prior period. Such probabilities have in prior research been estimated in theoretical models of business failure (cf. [Wilcox 1971](#); [Vins0 1979](#)) or statistical models, ranging from simple relative frequencies of bankruptcies (provided by for example Moody's Investor Service; www.moodys.com) to sophisticated applications of probit/logit analysis (cf. [Ohlson 1980](#); [Zavgren 1985](#); [Skogsvik 1990](#); [Shumway 2001](#); [Hensher and Jones 2007](#); [Ciampi 2015](#)). Throughout our analyses, bankruptcy probabilities are assumed to be *unbiased*.⁶ Furthermore, in this section, we assume that the equity holders have to settle any negative operating cash flows at the end of period t . We here thus ignore the "*limited liability right*" that the equity holders of a limited liability firm have (this assumption will be relaxed in Section 2.2 below).

Writing the discounting factors $(1 + r_{f,t}) = R_{f,t}$, $V_0(EQ)$ in (2) can be restated:

$$V_0(EQ) = \sum_{t=1}^{\infty} \left[\frac{\left(\prod_{\tau=1}^{t-1} P_{\tau} \right) \cdot \left[(1 - p_{fail,t}) \cdot E_0(\widetilde{OCF}_t^*) + p_{fail,t} \cdot E_0(\widetilde{OCF}_t^{\dagger} + \widetilde{RecV}_t^{\dagger}) \right]}{\left(\prod_{\tau=1}^t R_{f,\tau} \right) / cer_0(\widetilde{FTE}_t)} \right] \quad (4)$$

In line with DCF valuation theory, the value of owners' equity in (4) is calculated as the sum of discounted expected cash flows to the equity holders, using period-specific discount factors, $\left[\left(\prod_{\tau=1}^t R_{f,\tau} \right) / cer_0(\widetilde{FTE}_t) \right]$. Note that the expected values of cash flows— $E_0(\widetilde{OCF}_t^*)$ conditioned on *firm survival* and $E_0(\widetilde{OCF}_t^{\dagger} + \widetilde{RecV}_t^{\dagger})$ conditioned on *bankruptcy*—are multiplied by period-specific probabilities of survival and bankruptcy, respectively, assessed at the valuation date $t = 0$.

$V_0(EQ)$ in (4) shows that the inclusion of bankruptcy risk in DCF equity valuation typically *cannot* be performed through some “weighing” of equity values calculated conditioned on *firm survival* or *bankruptcy*. This has been suggested in prior literature (cf. Damodaran 2009), claiming that firm values can be calculated as a weighted average of values conditioned on survival or bankruptcy, where the weights are the cumulative survival and bankruptcy probabilities, respectively, over the forecasting period. (4) shows that such an approach would only work in a *one-period setting*, i.e., when only $(1 - p_{fail,1})$ and $p_{fail,1}$ are present in the valuation formula. This would be a very simplistic business venture, not being representative of the multiperiod forecasting of cash flows that normally takes place in DCF valuation.

Setting $R_{f,t} / P_{\tau} = R_{f,\tau}^*$ and rewriting expression (4), the value of owners' equity is found to constitute a *random walk* of expected cash flows *conditioned on firm survival or bankruptcy* in future periods, discounted to a present value with the bankruptcy calibrated factors $\prod_{\tau=1}^t R_{f,\tau}^*$ multiplied by period-specific values of $(1 - p_{fail,t}) / cer_0(\widetilde{FTE}_t)$:

$$V_0(EQ) = \sum_{t=1}^{\infty} \left[\frac{(1 - p_{fail,t}) \cdot E_0(\widetilde{OCF}_t^*) + p_{fail,t} \cdot E_0(\widetilde{OCF}_t^{\dagger} + \widetilde{RecV}_t^{\dagger})}{\left(\prod_{\tau=1}^t R_{f,\tau}^* \right) \cdot \left(\frac{1 - p_{fail,t}}{cer_0(\widetilde{FTE}_t)} \right)} \right] \quad (4a)$$

As an example of how the valuation model in (4a) works, assume that:

- (1) Bankruptcy probabilities are the same over time, $p_{fail,t} = \bar{p}_{fail} = 0.02$, and the risk-free rate is constant, $r_{f,t} = \bar{r}_f = 4.0\%$. This means that $R_{f,\tau}^*$ in (4a) is equal to $(1 + 0.04) / (1 - 0.02) = 1.0612$ for all future periods.
- (2) The expected operating cash flows *conditioned on firm survival* are constant over time; $E_0(\widetilde{OCF}_t^*) = 200$ MUSD. If the firm goes bankrupt in some period, the cash flow to the equity investors is $E_0(\widetilde{OCF}_t^{\dagger} + \widetilde{RecV}_t^{\dagger}) = -500$ MUSD.
- (3) The equity holders' certainty equivalent ratios $cer_0(\widetilde{FTE}_t) = cer_0(\widetilde{FTE}_1) = 0.95$ all future periods.

Knowing that $\prod_{\tau=1}^t R_{f,\tau}^* = \left[(1 + \bar{r}_f) / (1 - \bar{p}_{fail}) \right]^t = (1.0612)^t$ when $r_{f,t}$ and $p_{fail,t}$ are constants, $V_0(EQ)$ in (4a) is:

$$V_0(EQ) = \sum_{t=1}^{\infty} \left[\frac{(1 - 0.02) \cdot 200 + 0.02 \cdot (-500)}{(1.0612)^t \cdot \left(\frac{1 - 0.02}{0.95} \right)} \right] = \frac{196 + (-10)}{0.0612} \cdot \frac{0.95}{0.98} = 2946.2 \text{ MUSD} \quad (4b)$$

Applying the model in (4a), we find that the value of owners' equity is 2946.2 MUSD. Note that if bankruptcy risk had been ignored, the value of owners' equity would have been much higher:

$$V_0(EQ) = \sum_{t=1}^{\infty} \frac{200}{(1.04)^t \cdot \left(\frac{1}{0.95}\right)} = \frac{200}{0.04} \cdot 0.95 = 4750.0 \text{ MUSD} \tag{4c}$$

Ignoring a bankruptcy risk of only $\bar{p}_{fail} = 0.02$ in the numerical example thus leads to an overvaluation of +61.2% ($= 4750.0/2946.2 - 1$) of owners' equity. Obviously, this is not a trivial error, illustrating the importance of properly incorporating bankruptcy risk in DCF equity valuation.

The valuation model in (4a) constitutes a general formula for the valuation of owners' equity in an *unlevered proprietorship* (where the equity holders have no limited liability right). Note that if the equity holders are *risk-neutral* (meaning that $cer_0(\widetilde{FTE}_t) = 1.0$ for $t = 1, 2, \dots, \infty$), the random walk numerators divided by the survival probabilities $(1 - p_{fail,t})$ would only be divided by the calibrated discounting factors $R_{f,\tau}^*$. This shows that bankruptcy probabilities have a *compounding effect* on the discounting factors, in the same way as a risk premium being added to the risk-free rate would have. Also, note that the denominators in (4a) are a function of the risk-free rates ($R_{f,\tau} = 1 + r_{f,\tau}$), bankruptcy probabilities ($p_{fail,\tau}$), and the risk aversion metrics $cer_0(\widetilde{FTE}_t)$. Since $E_0(\widetilde{OCF}_t^*)$ presumably is larger than $E_0(\widetilde{OCF}_t^+ + \widetilde{RecV}_t^+)$, additional algebra shows that the numerators *decrease* and the denominators *increase* in (4a) if the bankruptcy probabilities increase. Clearly, this is in line with economic intuition—if the risk of bankruptcy increases, the value of owners' equity $V_0(EQ)$ should go down.

2.2. Incorporating the Equity Holders' Limited Liability Right

In most legal jurisdictions, the ownership of a *limited liability firm* allows the equity holders to declare bankruptcy and hand over the firm's assets to its creditors. A *limited liability right* of this kind makes it possible for equity holders to avoid having to settle the negative cash flows of a distressed firm if the firm files for bankruptcy. In the previous section, this possibility was not taken into consideration in the valuation of owners' equity, as we assumed that the equity holders' cash flow always would be equal to $\widetilde{OCF}_t^+ + \widetilde{RecV}_t^+$ if the firm files for bankruptcy in a period.

If the equity holders have a limited liability right, they can avoid negative cash flows if the firm fails. Hence, their final cash flow if the firm declares bankruptcy in a period is:

$$\widetilde{FTE}_t^+ = \text{Max} \left[\left(\widetilde{OCF}_t^+ + \widetilde{RecV}_t^+ \right), 0 \right] \tag{5}$$

where:

\widetilde{FTE}_t^+ = cash flow to the equity holders of a limited liability firm at the end of period t , if the firm goes bankrupt in period t .

(5) means that if bankruptcy occurs in period t , \widetilde{FTE}_t^+ is equal to $\left(\widetilde{OCF}_t^+ + \widetilde{RecV}_t^+ \right)$ if this value is positive, but equal to 0 if $\left(\widetilde{OCF}_t^+ + \widetilde{RecV}_t^+ \right) \leq 0$.

Replacing $E_0(\widetilde{OCF}_t^\dagger + \widetilde{RecV}_t^\dagger)$ in formula (4a) above with $E_0[\text{Max}[(\widetilde{OCF}_t^\dagger + \widetilde{RecV}_t^\dagger), 0]]$, we obtain a new equity value, $V_0(EQ_{limliab})$:

$$V_0(EQ_{limliab}) = \sum_{t=1}^{\infty} \left[\frac{(1 - p_{fail,t}) \cdot E_0(\widetilde{OCF}_t^*) + p_{fail,t} \cdot E_0[\text{Max}[(\widetilde{OCF}_t^\dagger + \widetilde{RecV}_t^\dagger), 0]]}{\left(\prod_{\tau=1}^t R_{f,\tau}^*\right) \cdot \left(\frac{1 - p_{fail,t}}{cer_0(\widetilde{FTE}_t)}\right)} \right] \quad (6)$$

If $E_0[\text{Max}[(\widetilde{OCF}_t^\dagger + \widetilde{RecV}_t^\dagger), 0]] = 0$ for $t = 1, 2, \dots, \infty$ in (6), the value of owners' equity can be simplified:

$$V_0(EQ_{limliab}) = \sum_{t=1}^{\infty} \left[\frac{E_0(\widetilde{OCF}_t^*)}{\left(\prod_{\tau=1}^t R_{f,\tau}^*\right) / \left(cer_0(\widetilde{FTE}_t)\right)} \right] \quad (6a)$$

In (6a) the value driver \widetilde{OCF}_t^* in the numerator is *conditioned on firm survival*, and the importance of bankruptcy risk is handled in the denominator. The bankruptcy probabilities are incorporated in the discounting factors $\prod_{\tau=1}^t R_{f,\tau}^*$ and the pricing of bankruptcy risk is captured by the certainty equivalent ratios, $cer_0(\widetilde{FTE}_t)$. If the risk associated with the operating cash flows \widetilde{OCF}_t^* increase over time, the certainty equivalent ratios decrease over time. Assume for example that $cer_0(\widetilde{FTE}_t) = cer_0(\widetilde{FTE}_1) \cdot \varphi^{t-1}$ where $0 < \varphi \leq 1.0$, and (6a) can be rewritten:

$$V_0(EQ_{limliab}) = cer_0(\widetilde{FTE}_1) \cdot \sum_{t=1}^{\infty} \left[\frac{E_0(\widetilde{OCF}_t^*)}{\left(\prod_{\tau=1}^t R_{f,\tau}^*\right) / \varphi^{t-1}} \right] \quad (6b)$$

When $\varphi < 1.0$, (6b) shows that the value of owners' equity is *negatively* affected by this parameter, in line with the intuition that the riskiness of cash flows is *negatively* associated with equity values. If the risk of future cash flows is *constant* over time, $\varphi = 1.0$ and investor risk aversion will be captured solely by $cer_0(\widetilde{FTE}_1)$ for all cash flows.

Also note that if $\varphi = 1.0$ in (6b), risk aversion *cannot* be transformed into a constant risk premium being added to the (bankruptcy calibrated) risk-free rates, since future values of $E_0(\widetilde{OCF}_t^*)$ should be multiplied by the parameter $cer_0(\widetilde{FTE}_1)$ *only once* and discounted to a present value with the bankruptcy calibrated risk-free rates. This insight is important for the valuation of owners' equity when the riskiness of future equity cash flows is the *same over time*. If a constant risk premium (erroneously) would be added to the risk-free rates in such cases (perhaps in line with some "standard professional policy"), calculated equity values will become *negatively* affected.

Returning to the numerical example in Section 2.1, but now assuming that $E_0(\widetilde{FTE}_t^\dagger) = 0$, $cer_0(\widetilde{FTE}_1) = 0.95$ and $\varphi = 1.00$, the value of owners' equity is:

$$V_0(EQ_{limliab}) = 0.95 \cdot \sum_{t=1}^{\infty} \frac{200}{\frac{(1.0612)^t}{1.00}} = \frac{0.95 \cdot 200}{0.0612} = 3104.58 \text{ MUSD} \quad (6c)$$

$V_0(EQ_{limliab})$ in (6c) is higher than $V_0(EQ)$ in (4b) above (2946.2 MUSD) due to the value of the equity holders' limited liability right. Furthermore, note that *if* the implied discounting rate of the first period's operating cash flow, $(200 / \frac{0.95 \cdot 200}{1.0612} - 1) = 11.71\%$ would (erroneously) be used in the present value calculation of the operating cash flows, the value of owners' equity would be $200 / 0.1171 = 1708.63$ MUSD. Adding a risk premium

of $11.71\% - 6.12\% = 5.59\%$ to the (bankruptcy calibrated) risk-free rate would hence lead to an *undervaluation* of owners' equity of about -45% ($1708.63/3104.58 - 1$). Clearly, compounding the implied risk premium of the first operating cash flow $E_0(\widetilde{OCF}_1^*) = 200$, would cause the value of owners' equity to be *strongly undervalued* when $\varphi = 1.00$.

If the equity holders' limited liability right does not affect values of $p_{fail,t}, \widetilde{OCF}_t^*, \widetilde{OCF}_t^+$ or \widetilde{RecV}_t^+ , the equity value $V_0(EQ_{limliab})$ in (6) cannot be lower than the equity value in (4a) above. If we simplify the models such that $p_{fail,t} = \bar{p}_{fail}, r_{f,t} = \bar{r}_f$ and $cer_0(\widetilde{FTE}_t) = 1, 0$ (i.e., investors are *risk neutral*), the value of the limited liability right is:

$$V_0 - V_0(EQ_{limliab}) = \left(\frac{\bar{p}_{fail}}{1 - \bar{p}_{fail}} \right) \cdot \sum_{t=1}^{\infty} \left[\frac{E_0 \left[\text{Max} \left[\left(\widetilde{OCF}_t^+ + \widetilde{RecV}_t^+ \right), 0 \right] - \left(\widetilde{OCF}_t^+ + \widetilde{RecV}_t^+ \right) \right]}{\left(\bar{R}_f^* \right)^t} \right] \tag{7}$$

Since $\bar{R}_f^* = (1 + \bar{r}_f) / (1 - \bar{p}_{fail}) = 1 + (\bar{r}_f + \bar{p}_{fail}) / (1 - \bar{p}_{fail})$, setting $E_0(\widetilde{OCF}_t^+ + \widetilde{RecV}_t^+) = Y < 0$, we find:

$$V_0(EQ_{limliab}) - V_0(EQ) = \left(\frac{\bar{p}_{fail}}{1 - \bar{p}_{fail}} \right) \cdot \frac{[0 - Y]}{\bar{r}_f} = \left(\frac{\bar{p}_{fail}}{\bar{r}_f + \bar{p}_{fail}} \right) \cdot (-Y) > 0 \tag{8}$$

Since Y is negative, (8) shows that the value of the limited liability right is *positive* when \bar{r}_f and \bar{p}_{fail} are positive. Additional algebra shows that the value is *positively* related to \bar{p}_{fail} and *higher* the more negative $Y = E_0(\widetilde{OCF}_t^+ + \widetilde{RecV}_t^+)$ is. The limited liability right should hence be particularly valuable for the equity holders of firms that are more likely to become distressed, have a larger downside risk of future operating cash flows, and/or have low recovery values of their assets. Equity holders should prefer the limited liability status of a firm when running high risk business operations. This result allows for straightforward empirical hypotheses, implying for example that firms in high risk industry sectors predominantly will be limited liability firms, or that limited liability firms will pursue business models with higher operating risks than firms where the equity holders have full responsibility for the firm's liabilities.

3. Bankruptcy Risk in DDM and RIV with Risk-Adjusted Discounting Rates

In this section, we develop our analyses in Section 2 for DCF equity valuation models which are commonly used in academic research and professional practice. In Section 3.1 we investigate DDM and in Section 3.2 the RIV model. In order to simplify our modelling we henceforth assume:

Assumption 1. The risk-free rate is constant, equal to \bar{r}_f and $R_{f,t} = 1 + \bar{r}_f = \bar{R}_f$.

Assumption 2. Equity investors' certainty equivalent ratios are geometrically decreasing over time, such that $cer_0(\widetilde{FTE}_t) = [cer_0(\widetilde{FTE}_1)]^t$.

The above assumptions mean that the denominators in the DCF model in (4) above can be rewritten as:

$$\prod_{\tau=1}^t R_{f,\tau} / cer_0(\widetilde{FTE}_t) = \left[\bar{R}_f / cer_0(\widetilde{FTE}_1) \right]^t \tag{9}$$

Writing $\bar{R}_f / cer_0(\widetilde{FTE}_1)$ in (9) as a function of a risk-adjusted discounting rate ρ_{EQ} , we have:

$$\left[\bar{R}_f / cer_0(\widetilde{FTE}_1) \right]^t = (1 + \rho_{EQ})^t \tag{10}$$

(10) means that the risk-adjusted discounting rate is:

$$\rho_{EQ} = \frac{1 + \bar{r}_f}{\text{cer}_0(\widetilde{FTE}_1)} - 1 \tag{10a}$$

Assumption 2 implies that future equity cash flows in a specific geometric fashion become riskier over time, an assumption that allows for a *constant risk-adjusted expected return* ρ_{EQ} (cf. Robichek and Myers 1965; Rubinstein 1976; Fama 1977; Christensen and Feltham 2009). The difference $(\rho_{EQ} - \bar{r}_f)$ is then the *expected return premium* that equity investors require to accept the risk associated with future cash flows.⁷

Whether *expected* market returns include a risk premium to compensate for bankruptcy risk is at present unsettled. Some prior studies have found a *positive* association between bankruptcy risk and stock returns (cf. Vassalou and Xing 2004; Penman et al. 2007), but other studies have documented a *negative* relation (cf. Dichev 1998; Piotroski 2000; Griffin and Lemmon 2002; Fama and French 2006; Campbell et al. 2008; Chava and Purnanandam 2010).⁸ However, it should be stressed that all our results hold *disregarding* whether bankruptcy risk is a priced risk factor or not.⁹ We *only* require that values of $\rho_{EQ,t}$ can be assessed.

3.1. DCF Equity Valuation (DDM)

In order for our analyses to be valid for publicly traded limited liability firms, we henceforth assume that the firm is a *limited liability* corporation. Furthermore, we assume:

Assumption 3. *The firm is financed by owners' equity and financial debt. Financial debt has an annual coupon after company taxes equal to $r_{coup}^x \cdot \text{Face}_{t-1}$ (r_{coup}^x = interest coupon rate after company taxes and Face_{t-1} = face value of financial debt at time $t - 1$) to be paid at the end of future periods. As long as the firm is a going concern, cash flows to the equity holders consist of dividend payments equal to $\widetilde{FTE}_t^* = (\widetilde{OCF}_t^* - r_{coup}^x \cdot \text{Face}_{t-1}) \cdot pr_t$, where pr_t = dividends paid at the end of period t divided by the net income of the period.*

Assumption 4. *If $E_{t-1}(\widetilde{OCF}_t) - r_{coup}^x \cdot \text{Face}_{t-1}$ is negative in some future period t , and the value of owners' equity conditioned on firm survival at the end of the period is less than $r_{coup}^x \cdot \text{Face}_{t-1} - E_{t-1}(\widetilde{OCF}_t)$, the firm files for bankruptcy at the end of the period.*

Assumption 5. *If the firm goes bankrupt in some period t , the creditors receive $\theta \cdot \tilde{A}_{t-1}^* \leq (1 + r_{coup}) \cdot \text{Face}_{t-1}$, where r_{coup} = interest coupon rate before company taxes, $\tilde{A}_{t-1}^* \geq 0$ is the book value of the firm's (net) assets at the end of period $t - 1$, and the equity holders receive nothing.*

Assumption 4 means that if the firm's expected operating cash flow is smaller than $r_{coup}^x \cdot \text{Face}_{t-1}$, and the value of owners' equity conditioned on the firm being a going concern is too small to compensate for the need of equity funding, the firm declares bankruptcy. As the firm is assumed to be a limited liability corporation, the equity holders then hand over the firm's assets to the creditors. In this way the equity holders "cut their losses", even though it implies that their shares become worthless.

Since the equity holders' certainty equivalent ratios are decreasing as postulated in Assumption 2 above, the expected cost of equity capital will be constant and equal to ρ_{EQ} as shown in (10a). Given this, we obtain an expression for the equity value where only expected dividends conditioned on firm survival multiplied by period specific survival probabilities, remain in the numerators:

$$V_0(EQ_{limliab}) = \sum_{t=1}^{\infty} \frac{[\prod_{\tau=1}^t P_{\tau}] \cdot E_0(\widetilde{FTE}_t^*)}{(1 + \rho_{EQ})^t} \tag{11}$$

Since both $\prod_{\tau=1}^t P_{\tau}$ and $(1 - \rho_{EQ})^t$ include t factors, (11) can equivalently be written with bankruptcy calibrated discounting rates, denoted $\rho_{EQ,t}^*$:

$$V_0(EQ_{limliab}) = \sum_{t=1}^{\infty} \frac{E_0(\widetilde{FTE}_t^*)}{\prod_{\tau=1}^t (1 + \rho_{EQ,t}^*)} \tag{12}$$

where:

$$\rho_{EQ,t}^* = (\rho_{EQ} + p_{fail,t}) / (1 - p_{fail,t}) = \text{bankruptcy calibrated expected equity return period } t.$$

(12) shows that the expected dividends *conditioned on firm survival* directly can be discounted with the bankruptcy calibrated discounting rates, $\rho_{EQ,t}^*$. Provided that analysts' forecasts of $E_0(\widetilde{FTE}_t^*)$ presume that the firm is a going concern, discounting such forecasts with $\rho_{EQ,t}^*$ will handle bankruptcy risk in the DDM model. In this sense, (12) is a simple but valid DCF model when the forecasts of the value drivers are conditioned on survival.

A common elaboration of DDM is the "Gordon model" (Gordon 1962). This model can easily be derived from (11) when $p_{fail,t} = \bar{p}_{fail}$ and the (unconditioned) growth rate of future dividends, g , is constant and less than ρ_{EQ} :

$$V_0(EQ)^{GORDON} = \frac{E_0(\widetilde{FTE}_1)}{\rho_{EQ} - g} = \frac{(1 - \bar{p}_{fail}) \cdot E_0(FTE_1^*)}{\rho_{EQ} - g} \tag{13}$$

Knowing the relationship between ρ_{EQ}^* and ρ_{EQ} , and since g must be a function of the dividend growth rate conditioned on survival (g^*) and bankruptcy ($g^\dagger = -100\%$),¹⁰ we can rewrite (13) as in (13a):

$$V_0(EQ)^{GORDON} = \frac{E_0(\widetilde{FTE}_1^*)}{\rho_{EQ}^* - g^*} \tag{13a}$$

where: $g^* = (g + \bar{p}_{fail}) / (1 - \bar{p}_{fail})$.

The valuation models in (13) and (13a) are equivalent, highlighting the importance of being consistent in the forecasting of dividends, the choice of the equity discounting rate and the dividend growth parameter.

A numerical example illustrating the impact of bankruptcy risk in the Gordon model is provided in Table 1. Equity values in the table are calculated in line with (13), with an expected dividend payment at $t = 1$ conditioned on firm survival equal to 10 and an (unconditioned) cost of equity capital $\rho_{EQ} = 10\%$. The growth rate g^* and the bankruptcy probability \bar{p}_{fail} vary between 0 and 8%, and 0 and 0.05, respectively.

Table 1 shows that if $g^* = 4\%$ and $\bar{p}_{fail} = 0$, the value of owners' equity is 166.7. Given the same growth rate but a bankruptcy probability of 0.02, the equity value drops by about 27% to 121.3. This is obviously not a trivial decrease in the equity value, illustrating the importance of not ignoring bankruptcy risk in DCF equity valuation. The table also illustrates that the sensitivity of values to variations in \bar{p}_{fail} increases when the growth rate g^* is higher. For example, if $g^* = 7\%$ and \bar{p}_{fail} goes from 0 to 0.02, the equity value is reduced by about 43% ($= 190.7/333.3 - 1$), i.e., much more than when $g^* = 4\%$. The importance of recognizing bankruptcy risk in DDM should thus be particularly important for firms which—conditioned on being going concerns—are forecasted to have strong growth prospects.

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if $g^* = 7\%$ and \bar{p}_{fail} goes from 0 to 0.02, the equity value is reduced by about -43% ($= 190.7/333.3 - 1$), i.e., much more than when $g^* = 4\%$. The importance of recognizing bankruptcy risk in DDM is thus particularly important for firms which—conditioned on being going concerns—are forecasted to have strong growth prospects.

Table 1. Value of owners' equity in DDM—Gordon model. $V_0(EQ)^{GORDON} = \frac{(1-\bar{p}_{fail}) \cdot E_0(\widetilde{FTE}_1^*)}{(\rho_{EQ}-g)}$; where $E_0(\widetilde{FTE}_1^*) = 10$, $\rho_{EQ} = 10\%$, $g = g^*(1-\bar{p}_{fail}) - g^+ \cdot \bar{p}_{fail}$ and $g^+ = -1.0$. Notation: \bar{p}_{fail} = bankruptcy probability at the beginning of future periods, \widetilde{FTE}_1^* = dividend payment to equity holders at the end of period $t = 1$ conditioned on firm survival, ρ_{EQ} = expected equity return, g^* = growth rate of dividends conditioned on firm survival, and g^+ = growth rate of dividend if the firm goes bankrupt.

Probability of Bankruptcy (\bar{p}_{fail}):						
Growth (g^*)	0	0.01	0.02	0.03	0.04	0.05
0	100.0	90.0	81.7	74.6	68.6	63.3
1%	111.1	98.9	88.9	80.6	73.6	67.6
2%	125.0	109.8	97.6	87.7	79.5	72.5
3%	142.9	123.3	108.2	96.1	86.3	78.2
4%	166.7	140.6	121.3	106.4	94.5	84.4
5%	200.0	163.6	138.0	119.0	104.3	92.7
6%	250.0	195.7	160.1	135.1	116.5	102.2
7%	333.3	243.2	190.7	156.2	131.9	113.8
8%	500.0	321.4	235.6	185.1	151.9	128.4

3.2. Residual Income Valuation (RIV)

Provided that the *clean surplus relation* (i.e., that net income, dividends and new issues of equity capital explain periodic changes in equity book values) holds in expectation in the financial statements and “mark-to-market” accounting is used in the firm’s transactions with its owners, a well-known reformulation of DDM is the *Residual Income Valuation (RIV)* model (cf. [Preinreich 1938](#); [Edwards and Bell 1961](#); [Ohlson 1995](#); [Skogsvik 2002](#)). Given the same assumptions as in Section 3.1, the RIV model is:

$$V_0(EQ_{limliab}) = BV_0 + \sum_{t=1}^{\infty} \frac{E_0(\tilde{I}_t - \rho_{EQ} \cdot \tilde{BV}_{t-1})}{(1 + \rho_{EQ})^t} \tag{14}$$

where:

\tilde{I}_t = accounting net income in period t , and

\tilde{BV}_t = equity book value (ex dividend, cum equity issue) at the end of period t .

The expected value of residual income, $E_0(\tilde{I}_t - \rho_{EQ} \cdot \tilde{BV}_{t-1})$ in (14), constitutes a probability weighted average of three possible outcomes:

- The firm is a going concern at the end of the period t , with an expected outcome equal to $(\prod_{\tau=1}^t P_t) \cdot E_0(\tilde{I}_t^* - \rho_{EQ} \cdot \tilde{BV}_{t-1}^*)$.
- The firm files for bankruptcy in period t , with an expected outcome equal to $(\prod_{\tau=1}^{t-1} P_t) \cdot p_{fail,t} \cdot E_0(\tilde{I}_t^+ - \rho_{EQ} \cdot \tilde{BV}_{t-1}^*)$.
- The firm has filed for bankruptcy in some prior period, with an expected outcome equal to $[1 - [(\prod_{\tau=1}^t P_t) + \sum_{\tau=1}^t (\prod_{\tau=1}^{\tau-1} P_t) \cdot p_{fail,\tau}]] \cdot 0 = 0$.

Since the clean surplus relation must hold when the firm is a going concern as well as if it goes bankrupt in period t , it is required:

$$\widetilde{BV}_t^* = \widetilde{BV}_{t-1}^* + \widetilde{I}_t^* - \widetilde{FTE}_t^* \tag{15a}$$

$$\widetilde{BV}_t^\dagger = \widetilde{BV}_{t-1}^\dagger + \widetilde{I}_t^\dagger - \widetilde{FTE}_t^\dagger \tag{15b}$$

In line with Assumption 5 above, $\widetilde{FTE}_t^\dagger = 0$, which implies:

$$\widetilde{I}_t^\dagger = \widetilde{BV}_t^\dagger - \widetilde{BV}_{t-1}^\dagger \tag{16}$$

The firm’s net income in the bankruptcy period must hence be equal to $(\widetilde{BV}_t^\dagger - \widetilde{BV}_{t-1}^\dagger)$ for RIV to be consistent with DDM. Unfolding the numerators in the summation terms of (14) above, the RIV model thus incorporates bankruptcy risk as follows:

$$\begin{aligned} V_0(EQ_{limliab}) = & BV_0 + \sum_{t=1}^{\infty} \frac{(\prod_{\tau=1}^t P_t) \cdot E_0(\widetilde{I}_t^* - \rho_{EQ} \cdot \widetilde{BV}_{t-1}^*)}{(1+\rho_{EQ})^t} + \\ & + \sum_{t=1}^{\infty} \frac{(\prod_{\tau=1}^{t-1} P_t) \cdot p_{fail,t} \cdot E_0(\widetilde{BV}_t^\dagger - (1+\rho_{EQ}) \cdot \widetilde{BV}_{t-1}^\dagger)}{(1+\rho_{EQ})^t} \end{aligned} \tag{17}$$

The equity value in (17) is the sum of three main terms. As in (14), the first term is the book value (ex dividend, cum equity issue) of owners’ equity at $t = 0$. The first summation term is the present value of future expected residual income *conditioned on firm survival*, and the second summation term is the present value of expected residual income if *bankruptcy* occurs in some future period.

Since we have assumed that the equity holders receive nothing if the firm files for bankruptcy, a useful simplification of the model in (17) is to set $\widetilde{BV}_t^\dagger = 0$. From this, it follows that the firm’s net income in the bankruptcy period includes:

- The equity holders’ *limited liability gain or loss* of not having to settle the firm’s operating cash flow $\widetilde{OCF}_t^\dagger$.
- The *realization gain or loss* when the (net) assets of the firm are liquidated. If the book value of these assets at the end of the bankruptcy period is $\widetilde{A}_{t-1}^* \cdot (1 + \Delta)$, this gain or loss is $[\theta \cdot \widetilde{A}_{t-1}^* - \widetilde{A}_{t-1}^* \cdot (1 + \Delta)]$.
- The equity holders’ *limited liability gain* when the firm’s financial debt obligation is settled, i.e., $(1 + r_{coup}^x) \cdot Face_{t-1} - \theta \cdot \widetilde{A}_{t-1}^*$.

The firm’s net income in the bankruptcy period, \widetilde{I}_t^\dagger , is thus:

$$\begin{aligned} \widetilde{I}_t^\dagger &= \widetilde{OCF}_t^\dagger + \widetilde{A}_{t-1}^* \cdot \Delta - r_{coup}^x \cdot Face_{t-1} + [-\widetilde{OCF}_t^\dagger] + \\ &+ [\theta \cdot \widetilde{A}_{t-1}^* - \widetilde{A}_{t-1}^* \cdot (1 + \Delta)] + [(1 + r_{coup}^x) \cdot Face_{t-1} - \theta \cdot \widetilde{A}_{t-1}^*] = \\ &= Face_{t-1} - \widetilde{A}_{t-1}^* = -\widetilde{BV}_{t-1}^* \end{aligned} \tag{18}$$

Assumption 5 together with the clean surplus relation in (15b), confirms that $\widetilde{I}_t^\dagger = -\widetilde{BV}_{t-1}^*$ when $\widetilde{BV}_t^\dagger = 0$. If the firm goes bankrupt in some period, the “bottom-line” net income is thus equal to the loss of the opening book value of owners’ equity. Note that $\widetilde{I}_t^\dagger = -\widetilde{BV}_{t-1}^*$ is a *necessary and sufficient* condition for the RIV model to be equivalent to DDM when the equity holders’ bankruptcy cash flow $\widetilde{FTE}_t^\dagger = 0$.

In line with DDM in Section 3.1, the RIV model can be rewritten to only include forecasts of residual income *conditioned on firm survival* when $\widetilde{FTE}_t^+ = 0$ and (hence) $\widetilde{I}_t^+ = -\widetilde{BV}_{t-1}^*$. Since $\rho_{EQ,t}^* = (\rho_{EQ} + p_{fail,t}) / (1 - p_{fail,t})$, we can rewrite (17) as follows:

$$\begin{aligned}
 V_0(EQ_{limliab}) &= BV_0 + \sum_{t=1}^{\infty} \frac{E_0(\widetilde{I}_t^* - \rho_{EQ} \cdot \widetilde{BV}_{t-1}^*)}{\prod_{\tau=1}^t (1 + \rho_{EQ,t}^*)} + \\
 &+ \sum_{t=1}^{\infty} \frac{\left(\frac{p_{fail,t}}{1 - p_{fail,t}}\right) \cdot E_0[(-1 - \rho_{EQ}) \cdot \widetilde{BV}_{t-1}^*]}{\prod_{\tau=1}^t (1 + \rho_{EQ,t}^*)} = \\
 &= BV_0 + \sum_{t=1}^{\infty} \frac{E_0\left[\widetilde{I}_t^* - \rho_{EQ} \cdot \widetilde{BV}_{t-1}^* - \left(\frac{p_{fail,t} \cdot (1 + \rho_{EQ})}{1 - p_{fail,t}}\right) \cdot \widetilde{BV}_{t-1}^*\right]}{\prod_{\tau=1}^t (1 + \rho_{EQ,t}^*)} = \\
 &= BV_0 + \sum_{t=1}^{\infty} \frac{E_0[\widetilde{I}_t^* - \rho_{EQ,t}^* \cdot \widetilde{BV}_{t-1}^*]}{\prod_{\tau=1}^t (1 + \rho_{EQ,t}^*)}
 \end{aligned} \tag{19}$$

(19) shows that *bankruptcy calibrated discounting rates*, $\rho_{EQ,t}^*$, can substitute for the second summation term in (17) when net income in the bankruptcy period equals $-\widetilde{BV}_{t-1}^*$. The RIV model in (19) constitutes a versatile rewrite of the RIV model in (17),¹¹ allowing for forecasts of *residual income conditioned on survival* in the numerators.

A numerical example illustrating the application of the RIV model in (19) is shown in Table 2. The table includes forecasts of net income ($E_0(\widetilde{I}_t^*)$), dividend payments ($E_0(\widetilde{FTE}_t^*)$), opening book values ($E_0(\widetilde{BV}_{t-1}^*)$) and equity book returns ($E_0(\widetilde{ROE}_t^*) = E_0(\widetilde{I}_t^*) / E_0(\widetilde{BV}_{t-1}^*)$), conditioned on firm survival. From period $t = 5$ onwards, net income and dividends are expected to grow by 5% per period conditioned on firm survival. Assuming that the expected cost of equity capital $\rho_{EQ} = 10\%$, $\bar{p}_{fail} = 0.02$, and that the equity holders receive nothing if bankruptcy occurs, the bankruptcy calibrated cost of capital is $\rho_{EQ}^* = (0.10 + 0.02) / (1 - 0.02) = 12.24\%$. The value of owners' equity according to (19) is then:

$$\begin{aligned}
 V_0(EQ_{limliab}) &= 1000 + \left(\frac{200 - 0.1224 \cdot 1000}{1.1224} + \frac{250 - 0.1224 \cdot 1100}{1.1224^2} + \right. \\
 &+ \frac{240 - 0.1224 \cdot 1200}{1.1224^3} + \left. \frac{264 - 0.1224 \cdot 1320}{1.1224^4} \right) + \\
 &+ \frac{264 \cdot 1.05 - 0.1224 \cdot 1386}{(0.1224 - 0.05) \cdot 1.1224^4} = \\
 &= 1000 + 291.11 + 936.04 = 2227.15 \text{ MUSD}
 \end{aligned} \tag{19a}$$

Table 2. Numerical example—explicit forecast periods $t = 1$ to 4 and steady state from $t = 5$. Equity return = 10%, probability of bankruptcy = 0.02, equity book value = 1000 MUSD at the beginning of period $t = 1$, and dividend payout ratio = 0.75 from period $t = 5$ onwards.

	Period:				
	1	2	3	4	5
Net income:	200	250	240	264	(+5%)
Dividend:	100	150	120	198	(+5%)
Opening book value:	1100	1200	1320	1386	(+5%)
Equity book return:	20.0%	22.7%	20.0%	20.0%	20.0%

The equity value in (19a) consists of:

- (1) The equity book value at $t = 0$; $BV_0 = 1000$ MUSD.
- (2) The present value of expected residual income in periods $t = 1$ to $t = 4$; 291.11 MUSD.
- (3) The horizon value at the end of period $t = 4$, representing the capitalized value of expected residual income in the steady state periods $t = 5, 6, \dots, \infty$; 936.04 MUSD.

$V_0(EQ_{limliab})$ in (19a) is 2227.15 MUSD, implying a market-to-book ratio of about 2.23 (=2227.15/1000). Notably, the equity value is about 120% higher than the book value, caused by the expected equity book returns $E_0(\widetilde{ROE}_t^*)$ being higher than the bankruptcy calibrated cost of equity capital $\rho_{EQ}^* = 12.24\%$ in future periods (cf. the last row in Table 2).¹²

As a further elaboration of (19), set $p_{fail,t} = \bar{p}_{fail}$ (implying that $\rho_{EQ,t}^* = \rho_{EQ}^*$, since we have previously assumed that $\rho_{EQ,t} = \rho_{EQ}$), a steady state setting with constant growth conditioned on survival equal to g^* , $g^* < \rho_{EQ}^*$ and we obtain:

$$V_0(EQ_{limliab}) = BV_0 + \frac{E_0(\widetilde{I}_1^*) - \rho_{EQ}^* \cdot BV_0}{\rho_{EQ}^* - g^*} \tag{20}$$

Assume furthermore that $E_0(\widetilde{I}_1^*)$ is predicted as a weighted average of I_{-1} (net income in the prior period $t = -1$) and a “sustainable” net income equal to $\rho_{EQ}^* \cdot BV_0$:

$$E_0(\widetilde{I}_1^*) = \delta \cdot I_{-1} + (1 - \delta) \cdot \rho_{EQ}^* \cdot BV_0 \tag{21}$$

where:

$\delta =$ profitability persistence parameter, $0 \leq \delta \leq 1$.

The parameter δ is a measure of the persistence of I_{-1} and $\rho_{EQ}^* \cdot BV_0$ depicts the expected equilibrium net income conditioned on firm survival.¹³ Replacing $E_0(\widetilde{I}_1^*)$ in (20) with (21), we find:

$$\begin{aligned} V_0(EQ_{limliab}) &= \left[1 - \frac{\delta \cdot \bar{p}_{EQ}^*}{\rho_{EQ}^* - \bar{g}_{ss}^*} \right] \cdot BV_0 + \left[\frac{\delta}{\rho_{EQ}^* - \bar{g}_{ss}^*} \right] \cdot I_{-1} = \\ &= [C_1] \cdot BV_0 + [C_2] \cdot I_{-1} \end{aligned} \tag{22}$$

(20a) shows that the value of owners’ equity is explained by the equity book value at $t = 0$ (BV_0) multiplied by the factor C_1 and the net income in period $t = -1$ multiplied by the factor C_2 . Straightforward algebra shows that the derivative of C_1 with respect to \bar{p}_{fail} is *positive*, while the derivative of C_2 with respect to \bar{p}_{fail} is *negative*. This means that the weight on the equity book value *increases*, and the weight on net income *decreases* if \bar{p}_{fail} goes up. This provides guidance for the interpretation of estimated coefficients of BV_0 and I_{-1} in empirical research, where stock prices are regressed on equity book values and net income. Since $\rho_{EQ}^* = (\rho_{EQ} + \bar{p}_{fail}) / (1 - \bar{p}_{fail})$, (22) implies that such coefficients will be affected by the magnitude of bankruptcy risk among the sample firms. In line with observations for US firms in for example Barth et al. (1998), our analysis predicts that the coefficient of BV_0 will *increase*, and the coefficient of I_{-1} will decrease when the bankruptcy risk of the firms increases.

4. Concluding Remarks

Bankruptcy prediction models have been around for decades, providing business managers and academics with important information about the financial health of firms. Prior literature has however been surprisingly silent on the use of such models, in particular with regard to DCF equity valuation modelling. Even though there is a general understanding of the severe negative value effects of bankruptcies, the importance of bankruptcy risk in equity valuation models is not well developed. A common idea appears to be that “some adjustment” of the discounting rate(s) can be done to handle the impact of bankruptcy

risk. However, at the same time it appears to be less clear how to precisely articulate such an adjustment.

We investigate the role of bankruptcy risk in DCF equity valuation models. In a general setting, where investment risk is captured by certainty equivalent ratios, we clarify how bankruptcy risk affects the value drivers in DCF valuation. We also show how bankruptcy risk can be incorporated in the discounting factors, allowing the numerators in DCF valuation to be forecasted as random walks of expected cash flows conditioned on firm survival or bankruptcy, or to be conditioned solely on the firm being a going concern. Regarding the importance of corporate and bankruptcy laws, we find that the equity holders' *limited liability right* (the right to hand over a firm's assets to its creditors, without having to make any capital contributions when the firm files for bankruptcy) is value enhancing and that its value is positively related to bankruptcy risk.

In an effort to make our modelling more applied, we show how bankruptcy risk is included in commonly used DDM equity valuation and accounting based residual income valuation (RIV). We show that bankruptcy probabilities are instrumental for the assessment of expected values of the value drivers in such models. As a special case, requiring that the equity holders' recovery values are 0, bankruptcy probabilities can be incorporated through a specific calibration of the discounting rate(-s). Interestingly, this allows for predictions of the value drivers conditioned solely on *firm survival* in both DDM and RIV. Regarding the role of accounting principles in RIV, a *bankruptcy recognition principle* is proposed to incorporate bankruptcy risk in RIV when the equity holders' recovery values are 0. The principle means that net income should be equal to the loss of the opening book value of owners' equity if the firm goes bankrupt in some period. When the equity holders' recovery values are negligible and the clean surplus relation holds, residual income numbers based on this principle are necessary and sufficient for RIV to be consistent with DDM.

Our results highlight potential weaknesses in professional DCF equity valuation. In professional practice, present value calculations are often performed through the discounting of expected equity cash flows *conditioned on* the firm being a *going concern*, with a constant, risk-adjusted cost of equity capital. Obviously, discounting cash flows *conditioned on firm survival* simplifies the estimation of the numerators in DCF valuation, but it makes the denominators significantly more complex. Necessary conditions for a valuation model of this kind to be valid are that the riskiness of future cash flows increases in a specific fashion over time, that bankruptcy probabilities are the same in future periods, and that the equity holders' recovery values are negligible. We also show that ignoring bankruptcy risk in DDM and RIV modelling leads to more or less exaggerated equity valuations.

Regarding the applicability of our analyses of bankruptcy risk in DCF equity valuation, there are—as in most model based research—issues that need further attention. For example, we have assumed that expected costs of equity capital, $\rho_{EQ,t}$, are available. We believe this to be a reasonable first step when investigating the role of bankruptcy risk in DCF equity valuation, but recognize that it indeed would be interesting to know more about the *market pricing of bankruptcy risk*. In addition to a return premium due to covariance risk (as in CAPM), it is possible that the Fama and French risk factors "Size" and "Book-to-market" capture at least parts of a return premium caused by bankruptcy risk. There is a need for more theoretical and empirical research on this issue. As a more practical limitation of our analyses, we have assumed that the equity holders can make assessments of periodic operating cash flows and bankruptcy recovery values, and determine whether the firm should file for bankruptcy in some period. However, such values can be difficult to estimate, meaning that the timing of the bankruptcy event becomes uncertain. In a US legal setting, financially distressed firms can obtain Chapter 11 protection and continue their operations for extended periods of time. We encourage future research to provide more knowledge about this. Furthermore, knowing more about the taxation of financially distressed firms would be helpful for assessing the timing and magnitude of the equity holders' cash flows. More descriptive research, which also can be case based, looking more closely into the tax consequences of the bankruptcy event would thus be worthwhile.

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Notes

- 1 Cf. [Bellovary et al. \(2007\)](#), where 165 bankruptcy prediction studies from 1965 to 2004 are reviewed.
- 2 A comprehensive survey of the valuation of defaultable financial securities in prior literature can be found in [Uhrig-Homburg \(2002\)](#).
- 3 [Shaffer \(2006\)](#) shows how to include bankruptcy probabilities in the denominator of the *Gordon model*, but leaves out the requirement to express dividend growth *conditioned on firm survival* when calibrated discounting rates are used.
- 4 [Realdon \(2013\)](#) investigates how bankruptcy probabilities can be incorporated in RIV modelling, but only for settings limited by restrictive assumptions (for example, presuming that “risk neutral” probabilities are available and ignoring longer-term bankruptcy probabilities). Our modelling is based on less restrictive assumptions and model specifications.
- 5 A certainty equivalent ratio is the *risk-free* amount that investors are willing to take instead of a *risky* cash flow, divided by the *expected value* of the cash flow.
- 6 Methodological issues concerning statistically estimated probabilities are discussed in [Zmijewski \(1984\)](#), and a Bayesian approach for the estimation of unbiased probabilities is derived in [Skogsvik and Skogsvik \(2013\)](#).
- 7 If expected equity returns are in line with CAPM, the return premium $(\rho_{EQ} - \bar{r}_f)$ reflects “systematic risk” as captured by equity betas.
- 8 The latter studies are in line with the CAPM based numerical analyses of bankruptcy risk in [Jennergren \(2013\)](#). If equity investors’ certainty equivalent ratios are *geometrically decreasing* over time such that $\rho_{EQ} = (1 + \bar{r}_f) / cer_0 (\overline{FTE}_1) - 1$, the relation between ρ_{EQ} and p_{fail} depends on the derivative $\delta(cer_0 (\overline{FTE}_1) / \delta(p_{fail})) = \Delta(cer)$. One might intuitively expect that $\Delta(cer) < 0$ (i.e., that certainty equivalent ratios *decrease* when p_{fail} *increases*), implying a positive association between ρ_{EQ} and p_{fail} . However, since one cannot rule out that $\Delta(cer) \geq 0$, in particular if the equity holders’ limited liability right is taken into consideration, a *negligible* or even *negative* association between ρ_{EQ} and p_{fail} remain possible.
- 9 See note 8.
- 10 The derivation of g^* follows from the definition of *unconditioned* growth, i.e., $g = g^*(1 - \bar{p}_{fail}) + g^\dagger \cdot \bar{p}_{fail} = g^*(1 - \bar{p}_{fail}) - 1, 0 \cdot \bar{p}_{fail}$.
- 11 If both $\rho_{EQ,t}$ and $p_{fail,t}$ are constants, equal to ρ_{EQ} and \bar{p}_{fail} , respectively, one gets a RIV model that closely resembles the model specification in (14), i.e., $V_0(EQ_{limliab}) = BV_0 + \sum_{t=1}^{\infty} \frac{E_0(\tilde{I}_t - \rho_{EQ}^* \cdot \tilde{B}V_{t-1}^*)}{(1 + \rho_{EQ}^*)^t}$.
- 12 Also note that $V_0(EQ_{limliab}) = 2227.15$ MUSD is consistent with DDM valuation. Applying this valuation model as expressed in (12) in Section 3.1, one gets: $V_0(EQ_{limliab}) = \left(\frac{100}{1.1224} + \frac{150}{1.1224^2} + \frac{120}{1.1224^3} + \frac{198}{1.1224^4} \right) + \frac{198 \cdot 1.05}{(0.1224 - 0.05) / (1.1224)^4} = 417.79 + 1809.36 = 2227.15$ MUSD.
- 13 In order to keep our modelling simple, we sidestep the impact of (unconditional) accounting conservatism here. However, accounting conservatism would not change our conclusion about the effects of bankruptcy risk on statistically estimated coefficients of book value and net income in the main text.

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