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Stability and Bifurcations in Banks and Small Enterprises—A Three-Dimensional Continuous-Time Dynamical System

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Abstract: Here, we discuss a three-dimensional continuous-time Lotka–Volterra dynamical system, which describes the role of government in interactions with banks and small enterprises. In Italy, during the COVID-19 emergency, the main objective of government economic intervention was to maintain the proper operation of the bank–enterprise system. We also review the effectiveness of measures introduced in response to the COVID-19 pandemic lockdowns to avoid a further credit crunch. By applying bifurcation theory to the system, we were able to produce evidence of the existence of Hopf and zero-Hopf bifurcating periodic solutions from a saddle focus in a special region of the parameter space, and we performed a numerical analysis.

Keywords: credit crunch; simulation; credit big data; nonlinear analysis; periodic solutions; stability; dynamical system; zero-Hopf bifurcation

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1. Introduction

In this paper, we consider a three-dimensional continuous-time Lotka–Volterra dynamical system (see Bischi and Tramontana (2010) for a discrete case), which describes the role of government in interactions with banks and small enterprises. This work follows up on other contributions in the literature that discuss the credit crunch and its effects on the bank–enterprise dynamical system (Ditzen 2018; Liu and Fan 2017; Marasco et al. 2016; Tsai 2017; Wang et al. 2018; Wei et al. 2018; Desogus and Venturi 2019).

Interactions between banks and enterprises are highly complex and nonlinear. Due to regulations, financial institutions, especially commercial banks, can only engage in financial intermediation. Their operations therefore depend on the caliber and financial standing of their clientele. Where lending strategies are based on credit risk and profit criteria, banks conduct thorough assessments of each counterparty before disbursing loans. Each calculated weighting of the aggregation of segmented units is recorded in the bank's risk portfolios. Likewise, businesses require bank credit to support investments and keen management to account for the inherent disparities of their working capital. Concurrently, businesses generate positive flows for the banking system, either through income creation, a proportion of which is deposited in banks and supplies banks with funds, or intermediation charges that provide bank revenue.

When bank leverage falls below a certain threshold (Desogus and Venturi 2019), its power to intervene is reduced as its role changes. The resulting reduction in supervisory provisions, greater availability of liquidity, and increased containment of portfolio risk indicators is often perceived to be a positive, short-term phenomenon. Increasingly acute over time, however, excessive credit restrictions tend to harm the positive environmental factors that help maintain a productive enterprise system (Rozendaal et al. 2016).

In this work, we improve the analysis of the bank–enterprise two-dynamical system. (see Desogus and Venturi 2019; Desogus and Casu 2020b). Indeed, we have noted the significance of positive effects generated by an efficient banking sector that provides liquidity to the business sector so that the banks themselves are kept healthy and performing (Iyer

Rajkamal et al. 2014). We have also found that negative effects are produced by the failure of this situation, and we scrutinized the effectiveness of measures introduced in response to the COVID-19 pandemic lockdowns to avoid a further credit crunch. Such critical events could have occurred because of the generalized impairment of creditworthiness following the nationwide stoppages (Caggiano et al. 2017; Petrosky-Nadeau 2013); therefore, we examined the further development of critical default trends in the populations of enterprises and banks in Italy.

Most of the research and studies have been investigated how government interventions aimed at supporting bank lending to the productive sector have indeed developed analyses and considerations on measures that affect bank capitalization, that is, direct injections of liquidity into the system aimed at stimulating credit operations for companies (cf. Laeven and Valencia 2013; cf. also Tan et al. 2020). In contrast, this paper instead questions the effects of government intervention strategies based on guarantee measures, while at the same time, it considers certain scenarios characterized by financial crises and measures aimed at counteracting a credit crunch. Such measures are implemented through nominal ceiling allocations, which do not imply an immediate monetary transfer from the government to the banks, bringing with it (also) advantages in public accounting. In particular, we analyze a form of state intervention in business that is accompanied by a government guarantee, up to full coverage even in borderline cases, and zero weighting of bank provisions on the portion guaranteed by the fund, which when structured as they have been—with their immediate enforceability ensured—entails a significant mitigation of the banks' credit risks.

This work currently represents an innovation in the recent scientific literature, both because of its approach, which is due to its conceptual choice of constructing and using mathematical models emanating from Lotka–Volterra dynamical systems in continuous time for the analysis of the topic, and because of its relevant ability to provide results capable of describing the complex solutions of the system. At the same time, these models succeed in intercepting the instantaneous variations caused by the persistence of the interconnections between banks, enterprises and government in the broader economic activity aimed at wealth production. The structure of the system is supported by a large database and records on business demographics and bank credit flows for Italy and the United Kingdom, which were collected, systematized and broken down into total disbursement volumes and NPLs, which confirm the general outcomes of our system of equations.

This paper goes so far as to establish that government support of bank lending through the provision of public loan guarantees may represent a best practice, particularly in contrast to the side effects of financial crises on the deleveraging of firms by banks. By replicating the examples presented in this paper, the mathematical model proposed can also be used as a tool for measuring the proper implementation of newly established guarantee funds. From a mathematical point of view, we use bifurcation theory (Nishimura and Shigoka 2019; Zhao and Zhao 2016; Neri and Venturi 2007) to confirm the behavior of the dynamic solutions generated by these governments acts against credit restrictions, as empirically observed in the data. Some numerical simulation is presented.

2. The Dataset

Data from Italy relating to time intervals included in the second decade of the 2000s (the precise period is indicated in the caption of each table and figure; see Tables 1–4) illustrate the correlation between the contraction of credit—especially what was made available to micro and small enterprises—and an increase in the mortality rate of those enterprises affected. As a consequence, this relationship also brought about increased levels of impaired credit.

Table 1 depicts the progressive reduction in loan disbursements, which we have deemed an independent variable. Adverse consequences to stakeholders in the production sector can be tied to this downward trend. Crucially, however, Tables 2 and 3 show the disparate impacts of the credit crunch on enterprises of different sizes: whilst established micro enterprises exited the market more readily than new companies of a similar size, macro enterprises remained relatively unaffected (Bassetto et al. 2015). Indeed, based on the data in Tables 1 and 2, the correlation between the reduction in credit granted to the productive sector and the active population of SMEs is +0.73. The aggregate number of viable small and medium-sized enterprises (SMEs) in the market is only marginally corrected by the incidence of macro-enterprises.

Considered alongside each other, Table 1 and the negative relationship just outlined further correlate with the rise in net non-performing loans (NPLs) prior to the first quarter of 2017, whilst the data in Tables 1 and 4 generate a Bravais–Pearson coefficient of −0.57 for March 2017. Where data in Table 5 on the number of impaired loans from June 2017 to September 2019 may suggest a pivot to a downward trend, the shift was in fact caused by the European Central Bank guidelines, announced in March 2017, to incentivize the sale of NPLs and strengthen monitoring processes (European Central Bank 2017). The responses by banks to these measures are reflected in the provisions and losses recorded in their financial statements. According to 2018 and 2019 reports by the ABI (Italian Banking Association) (ABI-Cerved 2018a, 2018b, 2019b, 2019a), disposals ranged from EUR 50 to 70 billion per period. In fact, the estimated outlook for the 2020 to 2021 period, prior to the COVID-19 emergency, forecast renewed increases in NPLs (ABI-Cerved 2019a).

Table 1. Gross lending to enterprises in Italy, in billions of euros (January 2012–January 2020).

Month	Loans	Month	Loans	Month	Loans
01-2012	1876.24	07-2014	1737.30	01-2017	1627.13
02-2012	1867.07	08-2014	1714.68	02-2017	1626.02
03-2012	1843.10	09-2014	1723.49	03-2017	1622.25
04-2012	1855.38	10-2014	1714.31	04-2017	1610.56
05-2012	1848.45	11-2014	1709.79	05-2017	1614.94
06-2012	1839.29	12-2014	1690.08	06-2017	1591.44
07-2012	1842.29	01-2015	1694.07	07-2017	1554.87
08-2012	1824.73	02-2015	1686.08	08-2017	1534.32
09-2012	1814.34	03-2015	1694.93	09-2017	1524.81
10-2012	1816.01	04-2015	1688.14	10-2017	1527.56
11-2012	1822.06	05-2015	1679.99	11-2017	1528.31
12-2012	1803.78	06-2015	1694.54	12-2017	1531.91
01-2013	1807.28	07-2015	1693.20	01-2018	1531.41
02-2013	1804.57	08-2015	1675.37	02-2018	1538.13
03-2013	1784.33	09-2015	1679.76	03-2018	1528.03
04-2013	1779.04	10-2015	1659.50	04-2018	1528.99
05-2013	1771.14	11-2015	1680.46	05-2018	1531.15
06-2013	1757.39	12-2015	1659.19	06-2018	1468.33
07-2013	1761.55	01-2016	1656.96	07-2018	1471.66
08-2013	1737.36	02-2016	1653.73	08-2018	1453.09
09-2013	1737.09	03-2016	1648.84	09-2018	1449.99
10-2013	1725.21	04-2016	1638.35	10-2018	1445.76
11-2013	1712.06	05-2016	1650.04	11-2018	1452.22
12-2013	1706.80	06-2016	1652.76	12-2018	1412.34
01-2014	1755.44	07-2016	1644.17	01-2019	1412.80
02-2014	1748.48	08-2016	1637.58	02-2019	1407.97
03-2014	1741.68	09-2016	1635.69	03-2019	1383.85
04-2014	1734.24	10-2016	1634.59	04-2019	1389.70
05-2014	1719.44	11-2016	1639.76	05-2019	1385.88
06-2014	1730.64	12-2016	1618.88	06-2019	1370.11
				07-2019	1374.30
				08-2019	1349.54
				09-2019	1346.08
				10-2019	1337.47
				11-2019	1334.66
				12-2019	1312.60
				01-2020	1325.77

Own processing based on data from the Bank of Italy, ISTAT and Chambers of Commerce.

Table 2. Total number of enterprises in Italy (2012–2018).

Year	2012	2013	2014	2015	2016	2017	2018
B	2451	2336	2257	2186	2250	2318	2332
C	417,306	407,344	396,422	389,317	399,458	404,528	406,508
D	8926	10,169	10,459	10,775	10,015	10,042	10,056
E	8967	9121	9146	9231	9060	9230	9301
F	572,412	549,846	529,103	511,405	534,824	537,348	537,853
G	1,163,413	1,153,640	1,123,134	1,105,227	1,128,117	1,129,703	1,130,596
H	131,755	129,865	125,688	123,625	127,651	127,817	128,172
I	307,878	313,207	312,013	315,464	312,000	311,478	312,009
J	97,280	95,989	96,997	98,381	96,933	96,916	97,080
K	91,434	93,031	95,209	96,173	93,199	93,340	93,393
L	235,434	243,564	239,134	238,273	237,137	237,095	237,067
M	710,017	691,700	705,895	714,934	700,468	700,308	700,406
N	143,770	139,362	139,898	139,595	139,959	140,415	140,724
P	26,890	27,677	29,088	29,566	28,360	28,257	28,304
Q	259,400	261,056	277,295	285,231	269,170	269,050	269,191
R	63,054	62,704	64,169	65,022	63,165	63,351	63,404
S	202,065	199,902	203,180	203,680	200,831	200,794	200,857
Total	4,442,452	4,390,513	4,359,087	4,338,085	4,352,597	4,361,988	4,367,254

Own processing based on data from the Bank of Italy, ISTAT and Chambers of Commerce. Business categories: B: extraction of minerals from quarries and mines, C: manufacturing, D: supply of electricity, gas, steam and air conditioning, E: supply of water, sewerage, waste management and environmental remediation services, F: construction, G: wholesale and retail trade, repair of motor vehicles and motorcycles, H: transport and storage, I: accommodation and food service businesses, J: information and communications services, K: financial and insurance service businesses, L: real estate businesses, M: professional, scientific and technical businesses, N: rental and travel agencies, business support services, P: education, Q: healthcare and social services, R: arts, sports, entertainment and amusement businesses, S: other service businesses.

Table 3. Number of micro enterprises in Italy (2012–2018).

	2012	2013	2014	2015	2016	2017	2018
B	1907	1850	1775	1712	1796	1795	1795
C	345,293	338,015	328,486	321,837	330,613	330,526	330,459
D	8380	9610	9916	10,205	9448	9445	9443
E	6485	6688	6748	6816	6628	6626	6625
F	548,709	528,592	509,648	492,388	515,477	515,341	515,237
G	1,124,546	1,116,087	1,086,631	1,068,659	1,089,768	1,089,481	1,089,262
H	119,126	117,430	113,241	110,756	114,173	114,143	114,120
I	288,119	294,007	292,996	295,706	290,253	290,177	290,119
J	91,274	89,895	91,020	92,279	90,353	90,329	90,311
K	88,998	90,637	92,831	93,799	90,799	90,775	90,757
L	234,738	242,874	238,492	237,637	236,437	236,374	236,327
M	702,053	683,778	698,154	707,020	691,902	691,720	691,581
N	132,452	128,082	128,721	128,394	128,327	128,294	128,268
P	25,239	25,957	27,351	27,781	26,359	26,352	26,347
Q	253,160	254,655	270,894	278,646	262,123	262,054	262,001
R	60,658	60,382	62,001	63,011	60,997	60,981	60,969
S	198,593	196,542	199,755	200,185	197,103	197,051	197,011
Total	4,229,730	4,185,081	4,158,660	4,136,831	4,142,556	4,141,465	4,140,633

Own processing based on data from the Bank of Italy, ISTAT and Chambers of Commerce.

Table 4. Net non-performing loans in Italy, in billions of euros (March 2012–March 2017).

Month	NPLs	Month	NPLs
03-2012	80.37	03-2015	140.10
06-2012	85.17	06-2015	145.66
09-2012	88.63	09-2015	149.29
12-2012	93.42	12-2015	151.42
03-2013	97.33	03-2016	147.87
06-2013	103.64	06-2016	149.68
09-2013	108.90	09-2016	151.24
12-2013	117.51	12-2016	154.03
03-2014	125.35	03-2017	150.49
06-2014	130.28		
09-2014	133.52		
12-2014	136.32		

Own processing based on data from the Bank of Italy, ISTAT and Chambers of Commerce.

Table 5. Net nonperforming loans in Italy, in billions of euros (June 2017–September 2019).

Month	Net NPLs
06-2017	150.25
09-2017	133.97
12-2017	128.59
03-2018	125.78
06-2018	99.45
09-2018	92.28
12-2018	73.55
03-2019	67.46
06-2019	66.08
09-2019	62.21

Own processing based on data from the Bank of Italy, ISTAT and Chambers of Commerce.

Framed as businesses with strategic plans for profit maximization, banks thus continue to covet corporate savings, whilst at the same time they implement risk reduction methods while absorbing risk capital. This entails reducing credit availability, which diminishes short-term guarantee assets and administrative costs. Weakening the resilience of the system, this strategic framework tends to undermine the growth potential of the enterprise population and its ability to maintain stable mortality and birth rates.

The trickle-down effects, however, result in losses for banks caused by reduced funds from deposits and increased costs from impaired assets and net losses (Bernanke et al. 1994; Wehinger 2014). Where companies are also the source of salary payments and income for employees—and, by extension, of the entire economic system—these adverse consequences can pro-cyclically affect macroeconomic conditions on a national scale (Buera et al. 2015).

Moving away from the dynamics of periodic stocks, the graph in Figure 1, which is based on data from Tables 1–5, tracks the percentage change in recorded monthly flows in terms of credit disbursed, demographic rate of enterprises (or the ratio of enterprises entering to those leaving the market), and the number of NPLs in Italy from 2012 to 2018. The data for NPLs only cover the first quarter of 2017, in light of our earlier discussion of regulatory changes for NPL management. Alongside the natural time lag caused by the macro-complexity of the objects analyzed (Kurkina 2017), Figure 1 confirms that the contraction of credit granted by banks to maximize their profit margins can be correlated with a progressive decline in (performing) firms in the market and with a general increase in NPLs. Recovering this disbursed credit will also catalyze a reversal of recent trends for NPLs and enterprises.

This reasoning, of course, has greater validity and application for banks operating primarily in the credit market, as a simultaneous diversification of assets would mitigate the cause-and-effect mechanism of the considerations discussed (Baldini and Causi 2020). To confirm our empirical conclusions, a comparative survey of similar data from the United

Kingdom (UK) was conducted. Tables 6–8 detail the value of loans disbursed, business demographics and the conditions of NPLs in the UK. In addition, in this case, the datasets relate to time intervals included in the second decade of the 2000s (and, similarly to what was noted for the Italian data, the precise period is indicated in the caption of each table and figure). Because it is situated outside the euro area, the UK was taken as a comparative reference. In this case, we considered micro, small, and medium-sized enterprises (mSME), according to the provisions of the European taxonomic framework. As was observed in Italy, the UK study showed an inverse correlation between the amount of loans disbursed and the number of companies performing in the UK market. This, in turn, correlated with the level of NPLs for the period, accounting for the adjustment delay.

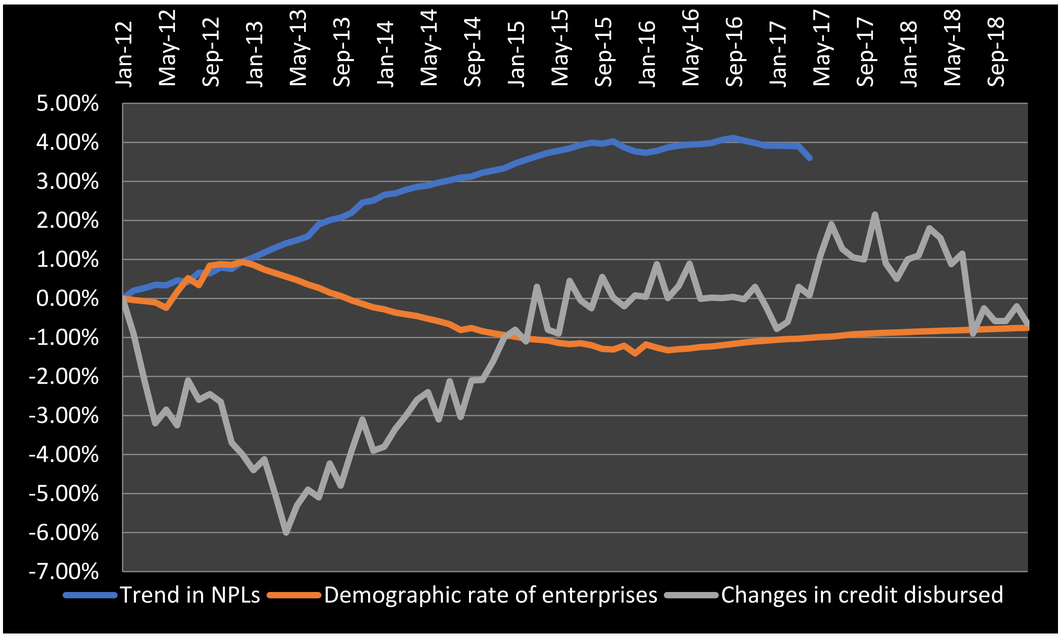

Figure 1. Changes in credit disbursed, population of enterprises in the market and number of NPLs in Italy from January 2012 to September 2018. The graph shows a general upward trend of 4.00% in NPLs from January 2012 to May 2017. There is a rise of 1.00% in enterprise population from January 2012 to January 2013, followed by a decline of 2.00% and stagnation until September 2018. Credit disbursement sees a sharp decline of 6.00% from January 2012 to May 2013, recovering gradually to reach original levels in 2015 and increasing a further 2.00% in 2017, before dropping back in 2018. Based on data in Tables 1–5.

Table 6. Gross lending to enterprises in the United Kingdom, in billions of pounds (September 2010–June 2019).

Month	Loans	Month	Loans	Month	Loans
09-2010	1170.00	09-2013	1014.60	09-2016	1421.70
12-2010	1167.40	12-2013	1245.50	12-2016	1485.90
03-2011	1154.50	03-2014	1165.20	03-2017	1444.10
06-2011	1134.10	06-2014	1312.70	06-2017	1409.30
09-2011	1177.70	09-2014	1316.50	09-2017	1454.30
12-2011	1103.40	12-2014	1549.00	12-2017	1419.20
03-2012	1011.70	03-2015	1427.90	03-2018	1408.80
06-2012	950.60	06-2015	1442.40	06-2018	1501.50
09-2012	910.20	09-2015	1414.40	09-2018	1407.90
12-2012	935.20	12-2015	1502.70	12-2018	1451.90
03-2013	1012.90	03-2016	1542.50	03-2019	1410.80
06-2013	1024.70	06-2016	1470.30	06-2019	1372.20

Own processing of data from the Bank of England.

Table 7. Total number of enterprises in the United Kingdom (2009–2018).

	2009	2010	2011	2012	2013
A	135,049.21	132,628.21	132,044.91	133,995.51	134,798.33
B, D, E	24,862.45	24,416.75	24,309.36	24,668.47	24,816.26
C	233,580.46	229,393.11	228,384.24	231,757.99	233,146.54
F	856,693.17	841,335.40	837,635.21	850,008.96	855,101.69
G	478,605.48	470,025.61	467,958.44	474,871.24	477,716.37
H	275,882.23	270,936.54	269,744.97	273,729.70	275,369.72
I	163,062.87	160,139.67	159,435.38	161,790.60	162,759.95
J	296,969.39	291,645.67	290,363.02	294,652.33	296,417.71
K	78,419.78	77,013.97	76,675.26	77,807.93	78,274.10
L	92,910.06	91,244.48	90,843.19	92,185.15	92,737.46
M	723,907.38	710,930.03	707,803.36	718,259.21	722,562.58
N	404,215.90	396,969.60	395,223.73	401,062.07	403,464.99
P	274,440.67	269,520.83	268,335.48	272,299.39	273,930.85
Q	305,627.51	300,148.59	298,828.54	303,242.90	305,059.75
R	221,942.53	217,963.81	217,005.21	220,210.86	221,530.23
S	266,270.39	261,497.01	260,346.95	264,192.86	265,775.74
Total	4,832,439.48	4,745,809.28	4,724,937.25	4,794,735.17	4,823,462.27
	2014	**2015**	**2016**	**2017**	**2018**
A	141,754.01	147,097.77	153,438.79	152,991.04	152,487.44
B, D, E	26,096.80	27,080.58	28,247.96	28,165.53	28,072.82
C	245,177.04	254,419.59	265,386.98	264,612.56	263,741.54
F	899,225.45	933,123.95	973,348.59	970,508.29	967,313.68
G	502,366.82	521,304.77	543,776.91	542,190.13	540,405.41
H	289,578.97	300,495.35	313,448.95	312,534.29	311,505.52
I	171,158.46	177,610.70	185,267.05	184,726.43	184,118.36
J	311,713.04	323,463.82	337,407.54	336,422.96	335,315.56
K	82,313.09	85,416.09	89,098.16	88,838.17	88,545.74
L	97,522.77	101,199.13	105,561.57	105,253.53	104,907.07
M	759,847.24	788,491.54	822,481.43	820,081.37	817,381.92
N	424,284.02	440,278.45	459,257.75	457,917.60	456,410.28
P	288,065.84	298,925.19	311,811.10	310,901.22	309,877.83
Q	320,801.01	332,894.39	347,244.64	346,231.35	345,091.67
R	232,961.32	241,743.37	252,164.32	251,428.49	250,600.86
S	279,489.93	290,025.98	302,528.28	301,645.48	300,652.55
Total	5,072,355.81	5,263,570.67	5,490,470.02	5,474,448.44	5,456,428.25

Own processing of data from the UK Department for Business, Energy and Industrial Strategy. Business categories: B, D, E: mining and quarrying; supply of electricity, gas and air conditioning; supply of water, sewerage, waste management and environmental remediation services, C: manufacturing, F: construction, G: wholesale and retail trade, repair of motor vehicles and motorcycles, H: transport and storage, I: accommodation and food service businesses, J: information and communications services, K: financial and insurance service businesses, L: real estate businesses, M: professional, scientific and technical businesses, N: administrative and support services, P: education, Q: healthcare and social services, R: arts, entertainment and recreation businesses, S: other service businesses.

In particular, we examined the effects of the bank rescue package of GBP 50 billion, which was issued by the British government in response to the 2009 to 2010 financial crisis and recession (Wong 2009). Where the package was designed to increase the amount of money available for banks to lend, Figure 2 highlights simultaneous regrowth of loan disbursements, recovery of the enterprise population and a decline in NPLs in the first months of 2012.

Table 8. Net NPLs in the United Kingdom, in billions of pounds (September 2010–June 2019).

Month	NPLs	Month	NPLs	Month	NPLs
09-2010	110.71	09-2013	113.14	09-2016	32.02
12-2010	124.78	12-2013	98.19	12-2016	29.66
03-2011	124.78	03-2014	98.19	03-2017	29.66
06-2011	124.78	06-2014	98.19	06-2017	29.66
09-2011	124.78	09-2014	98.19	09-2017	29.66
12-2011	125.00	12-2014	52.19	12-2017	23.19
03-2012	125.00	03-2015	52.19	03-2018	23.19
06-2012	125.00	06-2015	52.19	06-2018	23.19
09-2012	125.00	09-2015	52.19	09-2018	23.19
12-2012	113.14	12-2015	32.02	12-2018	33.85
03-2013	113.14	03-2016	32.02	03-2019	33.85
06-2013	113.14	06-2016	32.02	06-2019	33.85

Own processing of data from the World Bank and International Monetary Fund.

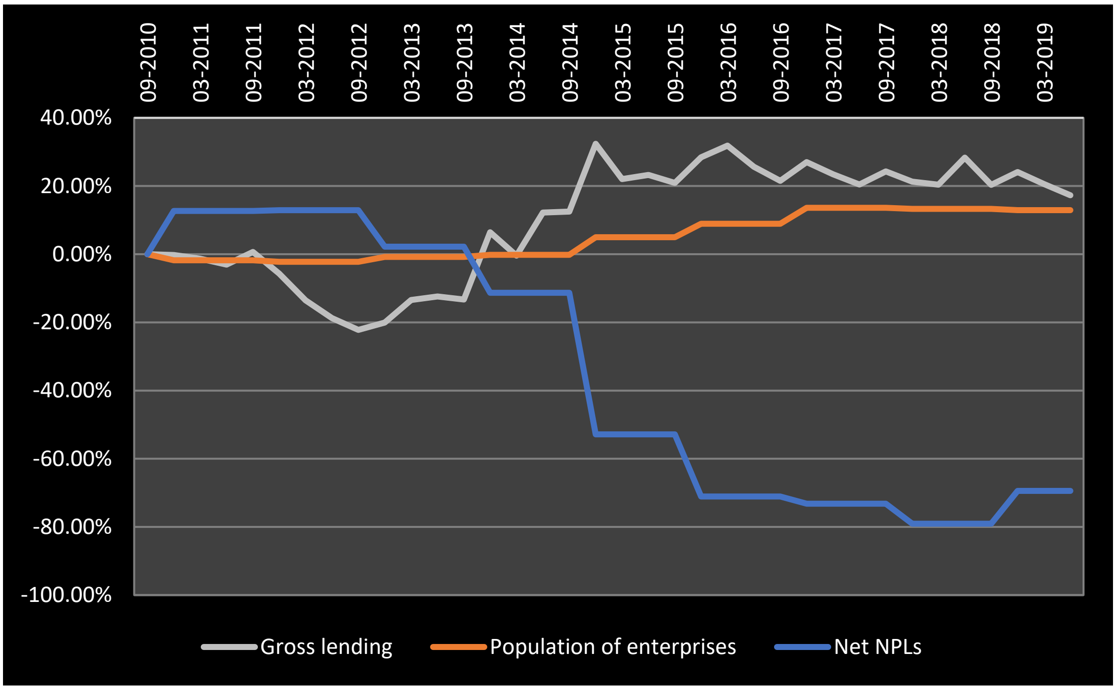

Figure 2. Percentage changes in credit disbursed, population of enterprises in the market and number of NPLs in the United Kingdom from September 2010 to March 2019. The graph shows a general decline in net NPLs in the UK from September 2010 to March 2019, dropping sharply in 2014 and falling by 70.00% by March 2019. The population of enterprises remains relatively unchanged until September 2014, after which there is a gradual increase of about 15.00% by March 2019. Gross lending declines 20.00% by September 2012, then rises 50.00% by September 2014, before fluctuating regularly to reach 20.00% above September 2010 levels in March 2019. Based on data in Tables 6–8.

These trends in the UK economy confirm that ad hoc government intervention that supports the provision of credit to SMEs has a positive impact on the productive fabric. By mitigating, and even negating, the effects of a credit crunch, the intervention helped to stabilize the banking system and prevented the stratification of impaired positions and NPLs in loan portfolios.

3. The General Model

We would now like to consider a purely dynamical nonlinear system:

$$\dot{z}_1 = f(z_1, z_2, z_3) \tag{1}$$

$$\dot{z}_2 = g(z_1, z_2, z_3) \quad (2)$$

$$\dot{z}_3 = h(z_1, z_2, z_3) \quad (3)$$

where the independent state variables are z_1, z_2 and z_3; z_1 represents the population of banks, z_2 represents the enterprises, and z_3 the government intervention. Equation (1) thus describes the traditional imbalance caused by dynamic adjustments in the production market (Calcagnini et al. 2019). Equation (2) refers to the corresponding imbalance in the credit market and repercussions for business demographics. Equation (3) represents government initiatives to help support enterprises to stay in the market. The intrinsic relationship between the populations of banks and enterprises is encompassed in this system of differential equations, in which the number of performing loans of one population is dependent on that of the other population. The populations of banks and enterprises, especially portfolio SMEs, are more sensitive to varying levels of financial support; respectively:

$$\dot{z}_1 = z_1 f_1(z_1,\ z_2)$$
$$\dot{z}_2 = z_2 f_2(z_1,\ z_2)$$

(Desogus and Venturi 2019).

As such, banks find that restricting credit volumes increases their short-term performance, forcing enterprises out of the market and increasing the number of NPLs. These new circumstances should prompt banks to expand their lending business, which would then reactivate these cycles. Therefore, to preserve the bank–enterprise relationship, it is necessary to maintain the macro-system (see inter al. Desogus and Casu 2020a; Degryse and Van Cayseele 2000). This means that the stabilization of levels of leverage in the productive and business sectors should be consistent with the dynamic models profiled above. The recursive phases discussed earlier should be guaranteed, even when standardizing regulations are introduced. This is even more significant for fragile and/or partially impaired economic scenarios or under persistently unfavorable economic conditions (De Angelo and Roll 2015).

Although the exogenous events that led to the unfavorable economic situation were—more or less—temporary, the repercussions appeared to have been immediately absorbed within the business cycle. Indeed, there was an exacerbation of the phase trajectories, which are normally pseudo-elliptical, toward an intensification of the credit contraction period and a manifestation of a progressive delay in the spontaneous rebound reaction of the bank–small-enterprise system (Ganong and Liebman 2018).

In Italy, the main objective of government economic intervention during the COVID-19 emergency was to maintain proper operations of the bank–enterprise system. Here, it is crucial to note that the 2020 economic crisis was caused by exogenous elements—and not by pathologies internal to the system, as for example happened in the 2009–2011 crisis. Management of the current situation therefore focuses on ensuring temporary compensation for the sudden halt in productive activities, placing less emphasis on restoring economic and financial aspects that have themselves been damaged.

As delineated in Decree Laws No. 18 and No. 23 of 2020, the Italian government shifted its resource flows to direct public guarantees—in the model that we are considering, this effect is mathematically expressed by F—making them also immediately enforceable and with zero weighting on allocations (D'Ignazio Alessio and Menon 2020). Other measures included reinsurance for the credit guarantee consortia, offering free access and coverage of between 80% and 90% for most lending operations. Instead of injecting liquidity into the productive system (and households) with 'helicopter money' or other direct forms, the Italian government focused on strengthening public guarantee funds. In this way, the government allowed banks to retain their role of financial intermediation unchanged, whilst also encouraging a quantitative expansion of credit provisions.

In light of this, it is apparent that a strong productive sector fosters a healthy banking sector, which in turn cultivates favorable circumstances for enterprises. It is therefore necessary for us to delve further into the dynamics of the bank–enterprise system as we take into consideration the limits imposed by the macroeconomic and idiosyncratic components peculiar to each enterprise in the population z_2.

Hence, we would like to now consider the following three-dimensional continuous-time Lotka–Volterra model involving the population of banks being z_1, the population of enterprises being z_2, and the government intervention as z_3. Adapting Equation (3) for a purely three-dimensional economic nonlinear model, the system of equations as independent state variables is then:

$$\begin{cases} \dot{z}_1 = \delta(\alpha z_1 z_2 - \tau z_2 + F - k z_1) \\ \dot{z}_2 = \mu(h z_2 - \beta z_1 z_2 + \eta z_1 z_3 + (\beta - F) z_3) \\ \dot{z}_3 = -\tau z_2 + F \end{cases} \tag{S}$$

This arrangement represents the interpretation of the purely dynamical nonlinear system formed by Equations (1)–(3), which has been constructed taking the contribution of government intervention on systemic effects through guarantees provided by the central fund into account. In this sense, government action is always aimed at MSMEs, which end up being the beneficiaries of support through the government guarantee. Instead, the population of banks receives these effects 'reflexively'. Therefore, the mere algebraic sum of the two components τz_2 and F has been correctly imputed in the re-elaboration of the third equations (and replaced in the first), since there is no direct relationship between banks and the government intervention in support of the provision of credit. This reflexivity can be seen in the second equation of (S), in the $\eta z_1 z_3$ contribution. Even the bank financing operations, provided by the Central Bank, (TLTRO—targeted longer-term refinancing operations) do not determine any interaction between z_3 and z_1, being, for all intents and purposes, loans, albeit dependent on the subsequent granting of credit to the enterprises by the borrowing banks (Castellacci and Choi 2014; Ledenyov and Ledenyov 2012; Hori and Futagami 2019).

The first equation is characterized from the following parameters: $\delta,\ \alpha, \tau,\ F$, where δ is an adjustment parameter in the traditional imbalance in the production market equation, α is an interaction parameter between the population of the bank and enterprises in the first equation, β is an interaction parameter between the population of the bank and enterprises in the second equation, τ is the decrease rate of the population of the enterprises, and k is the decrease rate of the population of the banks, with F as the activation of the government guarantee fund. In a situation in which adequate leverage is available, the second equation is characterized by the following parameters: $\mu,\ h, \beta,\ \eta$, where μ is an adjustment parameter in the credit market equation, h is the growth rate of the population of enterprises, and η is an interaction parameter between the population of the bank and the government intervention.

The vector of parameters $\omega \equiv (\alpha, \beta, \delta,\ \eta, \mu, \tau, k, h, F)$ abides inside the parameter space $\mathbf{\Omega} \equiv \Re^6_+ \times \Re \times \Re \times (0, \beta), \forall\, \omega \in \mathbf{\Omega}$.

3.1. Steady States and Local Stability Properties

Let $P(z_1, z_2, z_3)$ be a generic point. Recall that a stationary (equilibrium) point P^* (z_1^*, z_2^*, z_3^*) of our system (S) is any solution such that:

$$\dot{z}_1 = 0; \quad \dot{z}_2 = 0; \quad \dot{z}_3 = 0 \tag{4}$$

The differential equations in (S), solved for $\dot{z}_1 = 0$, $\dot{z}_2 = 0$, and $\dot{z}_3 = 0$ imply the following steady state value: P_1^* (z_1^*, z_2^*, z_3^*), with:

$$z_1^* = 0; \;\; z_2^* = \frac{F}{\tau}; \;\; z_3^* = -\frac{hF}{\tau(F - \beta)} \tag{5}$$

As mentioned in Section 3, intervention through fund F ensures indirect support for the disbursement of credit through collateral coverage payable; at the time of its activation, the fund will necessarily be $0 < F < \beta$. That is, β represents an interaction parameter between the populations of banks and companies and also signals a (reciprocal) influence on the effects of a credit contraction. The incidence of F will therefore tend asymptotically toward the β parameter, with the effectiveness of F being reduced as it approaches β.

In Italy, this was acutely apparent as new operational provisions to reform the guarantee fund came into force on 15 March 2019 (pursuant to a 6 March 2017 inter-ministerial decree) (MISE—Ministero dello Sviluppo Economico 2019). The main changes included the redefinition of intervention methods as direct guarantees, of reinsurance, and counter-guarantees. In addition, there was also the application of a valuation model based on the probability of default by beneficiary companies over all of the fund's operations, the reorganization of measures covered, maximum guaranteed amounts, and the introduction of operations focusing on tripartite risk (Hassan et al. 2022). In other words, the FCG (Central Guarantee Fund) is now equipped with a rating system for incoming applications, which makes guarantee percentages inversely proportional to the credit risk posed by the beneficiary company; stable companies, which ipso facto have interaction parameters (α and β), would be compatible with regular funding requirements and would receive moderate F assistance, whilst companies that are more at risk would receive greater F support. By considering feasible parameter values, we are presenting, for notational convenience, the following subset:

$$\mathbf{\Omega}_1 \equiv \{\omega \in \mathbf{\Omega} :\ \alpha \in \Re_+, \beta \in \Re_+, \delta \in \Re_+, \eta \in \Re_+, \mu \in \Re_+,\ k \in \Re,\ \tau \in \Re_+,\ \ h \in \Re,\ 0 < F < \beta\}$$

where the system's steady state solution is called P_1^* (z_1^*, z_2^*, z_3^*).

3.2. Local Analysis

As is well-known, in a hyperbolic equilibrium point P^*, the local dynamical properties of a nonlinear system are described, for brevity, in terms of the Jacobian matrix (Refaai et al. 2022). Hence, let J denote the Jacobian matrix of system (S). So, simple algebra leads to the following (3×3) matrix:

$$J = \begin{vmatrix} \delta(\alpha z_2 - k) & \delta(\alpha z_1 - \tau) & 0 \\ \mu(-\beta z_2 + \eta z_3) & \mu(h - \beta z_1) & \mu(\eta z_1 + \beta - F) \\ 0 & -\tau & 0 \end{vmatrix} \tag{6}$$

Let J be evaluated at the equilibrium point P_i^*: $J\left(P_i^*\right) = J_i^*$.

Then, the Jacobian matrix J can be evaluated at the steady-state value P_1^*, $J\left(\ P_1^*\right) = J_1^*$ for brevity, which is given by:

$$J(\ P_1^*) = \begin{vmatrix} \delta\left(\frac{\alpha F}{\tau} - k\right) & -\delta\tau & 0 \\ \mu\left(-\frac{\beta F}{\tau} + \frac{\eta h F}{\tau(-\beta+F)}\right) & \mu h & \mu(\beta - F) \\ 0 & -\tau & 0 \end{vmatrix} \tag{7}$$

We therefore obtain:

$$Tr(J_1^*) = \delta\left(\frac{\alpha F}{\tau} - k\right) + \mu h \tag{8}$$

$$Det(J_1^*) = -\delta(\alpha F - k\tau)\mu(-\beta + F) \tag{9}$$

$$B(J_1^*) = \delta\left(\frac{\alpha F}{\tau} - k\right)\mu h + \mu(\beta - F)\tau \tag{10}$$

where $Tr\left(J_1^*\right)$ is the trace of $J\left(\ P_1^*\right)$, and $Det\left(J_1^*\right)$is the determinant of $J\left(\ P_1^*\right)$, and $B\left(J_1^*\right)$ is the sum of the principal minor of $J\left(\ P_1^*\right)$. Note that the eigenvalues of $J_1{}^*$ are the solutions of the characteristic equation:

$$det(\lambda\mathbf{I} - J_1^*) = \lambda^3 - Tr(J_1^*)\lambda^2 + B(J_1^*)\lambda - Det(J_1^*) \tag{11}$$

where $\mathbf{I}$ is the identity matrix.

We focus on local analysis in the set $\mathbf{\Omega}_1$

Proposition 1. *Let $\omega \in \Omega_1$ then:*

(*a*) *If $k < \frac{F\alpha}{\tau}$, $h^* < h < 0$, there exist two subsets $\mathbf{\Omega}_1^A$ and $\mathbf{\Omega}_1^B$ such that when $\omega \in \mathbf{\Omega}_1^A$, J_1^* has one eigenvalue with a positive real part and two eigenvalues with negative real parts, and when $\omega \in \mathbf{\Omega}_2^A$, J_1^* has three eigenvalues with positive real parts. This means that if $\omega \in \mathbf{\Omega}_2^A$, we will have instability.*

(*b*) *If $k > \frac{F\alpha}{\tau}$, $h^* < h < 0$, there exist two subsets $\mathbf{\Omega}_1^A$ and $\mathbf{\Omega}_1^B$such that when $\omega \in \mathbf{\Omega}_1^A$, J_1^* has three eigenvalues with negative real parts, and when $\omega \in \mathbf{\Omega}_2^A$, J_1^* has one eigenvalue with a negative real part and two eigenvalues with positive real parts. This means that the equilibrium P_1^* will be locally unique.*

Proof. These results were obtained by applying the Routh–Hurwitz stability criterion to the system (S), according to which the number of the positive eigenvalues of the Jacobian matrix $J\left(P_1^*\right)$, evidently evaluated at the steady states P_1^*, will be equal to the number of variations of the sign in the scheme:

$$-1; \quad Tr(J_1^*); \quad -B({J_1}^*) + \frac{Det\left(J_1^*\right)}{Tr\left(J_1^*\right)}; \quad Det(J_1^*)$$

We define:

$$G(J_1^*) = -B(J_1^*) + \frac{Det\left(J_1^*\right)}{Tr\left(J_1^*\right)}$$

Case 1a. Let $\omega \in \mathbf{\Omega}_1^A$ be and $k < \frac{F\alpha}{\tau}$, then:

When $Tr\left(J_1^*\right)$, $Det\left(J_1^*\right)$ and $B\left(J_1^*\right)$ are positive, the sign of $G\left(J_1^*\right)$ can be positive. In this case, we have one eigenvalue with a positive real part and two eigenvalues with negative parts, so P_1^* will be an unstable saddle.

Case 1b. Let $\omega \in \mathbf{\Omega}_2^A$ be and $k < \frac{F\alpha}{\tau}$, then:

When $Tr\left(J_1^*\right)$, $Det\left(J_1^*\right)$ and $B\left(J_1^*\right)$ are positive, the sign of $G\left(J_1^*\right)$ can be negative. In this case, we have three eigenvalues with positive real parts, so P_1^* will be a completely unstable saddle.

Case 2a. Let $\omega \in \mathbf{\Omega}_1^A$ be and $k > \frac{F\alpha}{\tau}$, then:

Both $Tr\left(J_1^*\right)$ and $Det\left(J_1^*\right)$ are always negative, and the sign of $G\left(J_1^*\right)$ is negative. In this case, we have three eigenvalues with negative real parts, so P_1^* will be a stable saddle.

Case 2b. Let $\omega \in \mathbf{\Omega}_2^A$ be and $k > \frac{F\alpha}{\tau}$, then:

Both $Tr\left(J_1^*\right)$ and $Det\left(J_1^*\right)$ are always negative, and the sign of $G\left(J_1^*\right)$ is positive. In this case, we have one eigenvalue with a negative real part and two eigenvalues with positive real parts, so P_1^* is a saddle focus. □

4. Global Analysis

Here, we need to go beyond the conventional stability analysis and use bifurcation theory. We have chosen h as the bifurcation parameter to examine the existence of Hopf bifurcating closed orbits from the steady state: P_1^* (z_1^*, z_2^*, z_3^*).

Lemma 1. *If $\omega \in \mathbf{\Omega}_1$, then there exists at least one value $h = h^*$ such that $J^*\left(P_1^*\right)$ has a pair of purely imaginary roots.*

Proof. Since $G(h) = -B\left(J_1^*\right)\, Tr\left(J_1^*\right) + Det\left(J_1^*\right)$ changes sign in $\mathbf{\Omega}_1$, by the Routh–Hurvitz criterion, we state that J^* has one positive (real) eigenvalue and two complex conjugate roots whose real parts can be either positive or negative. It means that the two complex conjugate roots of J_1^* can be either positive or negative. Furthermore, since the real parts of the complex conjugate roots vary continuously with respect to h, there must exist at least one value $h = h^*$ such that $G(h) = 0$. When this occurs, by Vieta's theorem, J_1^* has a simple pair of purely imaginary eigenvalues. The sign of $Det\left(J_1^*\right)$ is independent of h; Vieta's theorem has been used properly. (Q.E.D.) □

Lemma 2. *If $\omega \in \mathbf{\Omega}_1$, then the derivative of the real part of the complex conjugate eigenvalues with respect to h, evaluated at h = h*, will always be different from zero.*

Proof. To prove that $\frac{dRe\ \lambda(h)}{dh}$ cannot be zero at the bifurcation point h^*, by following the strategy developed by Benhabib and Miyao (1981), we show that:

$$\text{Sign } \frac{\text{dRe } \lambda(h)}{dh}\bigg|_{h^*} = \text{Sign } \left(\frac{\text{dTr}}{dh}\ B - Tr\frac{dB}{dh} + \frac{dDet}{dh}\right) = \text{Sign } \frac{\text{d}G(h)}{dh}\bigg|_{h^*}$$

Whereas $G(h)$ is a second-degree polynomial in h changing sign at $h = h^*$ (see Lemma 1 proof), the bifurcation points cannot coincide with the minimum or maximum of the function. Therefore, there must be a neighborhood of h^* where the derivative of $G(h)$ with respect to h is different from zero. (Q.E.D.) □

Theorem 1. *Assuming the hypotheses of Lemmas 1 and 2, then, there will be a continuous function $h(\rho)$ with $h(0) = h*$, and for all that are small enough $\rho = 0$, there will be a continuous family of non-constant positive periodic solutions P_1^* $(z_1^*(\rho), z_2^*(\rho), z_3^*(\rho))$ for the dynamical system (S), which will then collapse to the stationary point P_1^* (z_1^*, z_2^*, z_3^*) as $\rho \to 0$.*

Proof. (It follows from the Hopf bifurcation theorem; see Appendix A.) □

Example 1. *Let $\omega \in \mathbf{\Omega}_1^A$, $k < \frac{F\alpha}{\tau}$ and $h\hat{} \cong -0.001050519739$;*

Set $\omega \equiv (\alpha, \beta, \delta,\ \eta, k, \mu, \tau, F) = (0.00478,\ 0.02787,\ 1,\ 0.042, -0.0250,\ 1,\ 0.03453,\ 0.025)$.

According to Proposition 1, the equilibrium point $P_1{}^*$ will be a saddle focus with three eigenvalues with a real positive part:

$$\lambda_{\text{R}} = 0.01450660897;\ \lambda_c, \overline{\lambda_c} = 0.006451815026 \pm 0.0123613182\ \mathbf{I}.$$

Example 2. *Let $\omega \in \mathbf{\Omega}_2^A$, $k < \frac{F\alpha}{\tau}$ and $\hat{h} \cong -0.002750519739$;*

Set $\omega \equiv (\alpha, \beta, \delta,\ \eta, k, \mu, \tau, F) = (0.00478,\ 0.02787,\ 1,\ 0.042, -0.025,\ 1,\ 0.03453,\ 0.025)$

According to Proposition 1, the equilibrium point $P_1{}^*$ will be a saddle focus with one real eigenvalue with a real positive part and two complex eigenvalues with a real negative part.

$$\lambda_{\text{R}} = 0.03593045781;\ \lambda_c, \overline{\lambda_c} = -0.00510109393 \pm 0.007237778242\ \mathbf{I}.$$

Example 3. *Let $\omega \in \mathbf{\Omega}_1$ and $k < \frac{F\alpha}{\tau}$ and $h^* \cong -0.001750529$;*

Set $\omega \equiv (\alpha, \beta, \delta,\ \eta, k, \mu, \tau, F) = (0.00478,\ 0.02787,\ 1,\ 0.042, -0.025,\ 1,\ 0.03453,\ 0.025)$.

Then a Hopf bifurcation will result with eigenvalues

$$\lambda_{\text{R}} = 0.02671023902;\ \lambda_c, \overline{\lambda_c} = \pm 1.170195840\ \mathbf{I}.$$

Then, we know that there is a continuous family of non-constant positive periodic solutions $P^*(z^*{}_1(\rho), z^*{}_2(\rho), z^*{}_3(\rho))$ for the dynamical system (S), which collapses at the equilibrium point $P^*{}_1$ $(z^*{}_1, z^*{}_2, z^*{}_3)$ as $\rho \to 0$.

5. The Zero-Hopf Bifurcation

We will use F as a bifurcation parameter to show that the linearization matrix of the righthand side of the system (S), evaluated at the steady state, will have a zero eigenvalue. More specifically:

$$Tr(J_h^*) = \delta\left(-k + \frac{\alpha F}{\tau}\right) + h^*$$

When we consider that $Tr\left(J_h^*\right) = -k\delta + \delta\frac{\alpha F}{\tau} + h^* = 0$, since we have assumed that $\delta = 1$, then we should remember that:

- ➢ Where δ is an adjustment parameter in the traditional imbalance in the production market equation;
- ➢ $-k$ is the rate of decrease in the bank loans' performance correlated to periods of negative demographics of SMEs;
- ➢ α is the interaction parameter between banks and enterprises;
- ➢ τ describes the damping (or absorption effect)—on enterprises—of the oscillatory motion of the system dynamics and, in particular, of the compensatory intervention of the fund.

Therefore, if $Tr\left(J_h^*\right) = 0$ then $k = \frac{h^*}{\delta} + \frac{F\alpha}{\tau}$, i.e., when the rate of decrease in the positive performance of the banking portfolios is equal to the product of the fund. However, there will be a difference among the damping and bank–enterprise interactions, normalized by the relationship with the absorption parameter.

We can also write the previous relation with respect to F:

$$\hat{F} = -\frac{\tau(-k\delta + h^*)}{\delta\alpha}$$

In this situation, the intervention of the fund manages to maintain the condition of the system, yet without improving the performance indicators of the companies or the banks.

Theorem 2 (Gavrilov–Guckenheimer bifurcation). *Let $(h.F)\epsilon\ \mathbf{\Omega}_1$. Furthermore, let $u = u(h^*\hat{F})$ and $v = v(h^*\hat{F})$ such that at the same time $B\left(J_1^*\right) < 0$, and $Tr\left(J_1^*\right) = Det\left(J_1^*\right) = 0$.*

Then, $J_h{}^*$ has one real zero eigenvalue $\lambda_R = Tr(J_1^*) = 0$ and two purely imaginary eigenvalues given by $\lambda_c, \overline{\lambda_c} \pm i\xi$ where $\xi = \sqrt{B\left(J_1^*\right)}$.

Proof. We consider the matrix J_h^* that represents the Jacobian matrix J_1^* put into a normal form:

$$J_h^* = T^{-1}J_1^* \quad T = \begin{bmatrix} 0 & -\xi & 0 \\ \xi & 0 & 0 \\ 0 & 0 & TrJ_1^* \end{bmatrix} \tag{12}$$

where $Tr\left(J_h^*\right) = Tr\left(J_1^*\right)$ and $\xi = \sqrt{B\left(J_1^*\right)}$ are evaluated at the bifurcation point $h = h^*$.

Let the parameters $h = h^*$and $F = \hat{F}$ choose, such that:

$$\hat{F} = \frac{\tau(k\delta - h^*)}{\delta\alpha} \tag{13}$$

Then: $Tr\left(\ J_h^*\right) = \mathrm{Tr}\left(J_1^*\right) = 0$.
So, we can rewrite $J_h^* = T^{-1}J_1^*\ T$ as:

$$J_{h,F}{}^* = \begin{bmatrix} 0 & -\xi & 0 \\ \xi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

$\mathrm{Det}(J_{h,F}{}^*)$ in (14) vanishes and $J_{h,F}{}^*$ has at least one eigenvalue equal to zero.
Let $B(J_{h,F}{}^*) = \xi^2$. □

We can now show analytically and numerically that there is a Gavrilov–Guckenheimer two-bifurcation codimension. This phenomenon, which has recently been closely studied, takes the form of a pitchfork–Hopf interaction (Bella and Mattana 2018; Bosi and Desmarchelier 2018; Bella et al. 2022), which is a linear degeneracy that can be associated with the onset of a 2-torus trapping region in the three-dimensional space enclosed by a two-dimensional surface (see Figure 3).

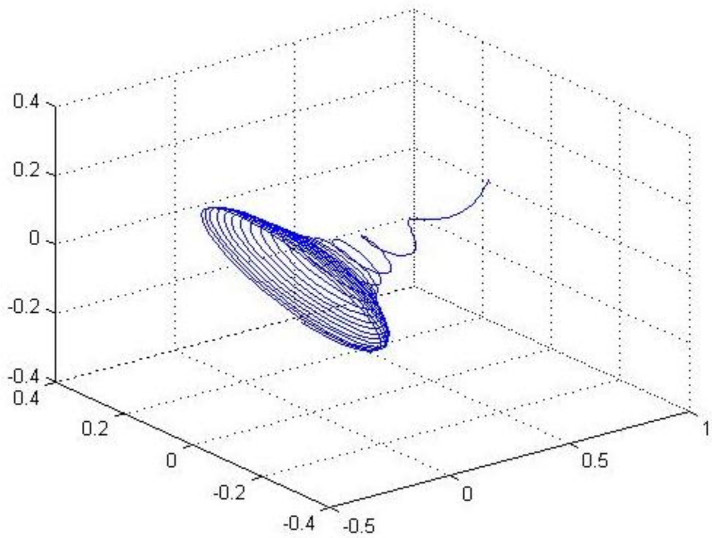

Figure 3. The Hopf cycle with $h^* = -0.001750529$.

Considering the case of the initial conditions $(w_1(0), w_2(0), w_3(0)) = (0.1, 0.1, -0.2)$, then the attractor will have the form represented in Figure 4.

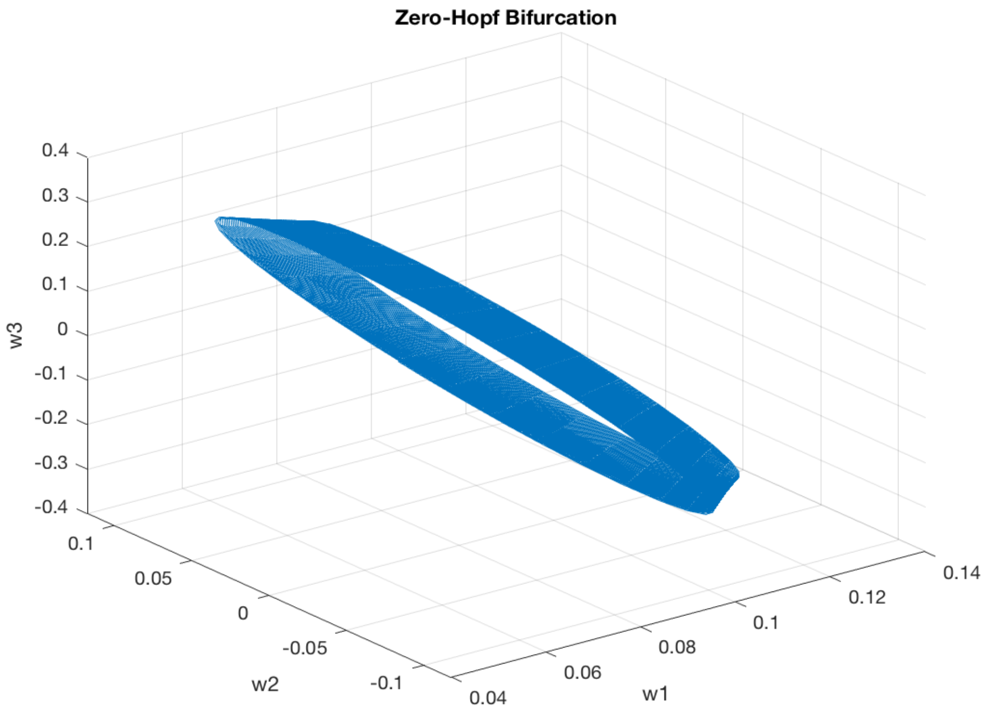

Figure 4. The zero-Hopf cycle.

6. Discussion

As the nonlinear differential equations defining this system are continuous and derivable complex functions, this system is also conditioned by a set of interaction and adjustment parameters that make each population dependent on the other. Conducting a dynamical analysis of the unique steady-state model, we applied a Jacobian matrix $J(P^*)$ to describe the local dynamical properties of the hyperbolic equilibrium points $P_1{}^*$. Considering the derived eigenvalues J^* in the parameter space $\mathbf{\Omega}_1$, we found that the equilibrium path is locally unique. Furthermore, by applying the Routh–Hurwitz criterion, we ascertained the fundamental stability of the system such that, for each instance that the fund was relied upon, $0 < F < \beta$. In practice, this means that the fund was structured to provide a guarantee that was inversely proportional to the credit risk posed by the company under review.

Where the economic crisis resulting from the COVID-19 pandemic and subsequent lockdowns in 2020 was caused by factors external to the financial economic system, we found that measures introduced by the Italian government were not focused on correcting the existing bank–enterprise system or aspects of it that may have suffered because of the crisis. Rather, those interventions sought to maintain the fabric of the bank–enterprise system healthy by attending to the unexpected lapses in productive activities. In particular, the Italian government made adjustments to the system by providing collateral coverage, payable through its public guarantee fund F, and by encouraging the continuation and increase in credit transactions.

Exploring the impacts of this intervention on the economic financial system and considering the growth rate of the small enterprises as bifurcation parameters, we were able to prove the existence of a stable Hopf cycle. Following from the collapse of all sufficiently small growth rates of the population of enterprises $\rho(h)$ in a continuous family of non-constant positive periodic solutions to stationary point $P_1{}^*$, we produced evidence of the existence of Hopf and zero-Hopf periodic solutions and that these tended to bifurcate from a saddle focus in a particular region of the parameter space. In addition, we observed the simultaneous occurrence of a zero eigenvalue and two purely imaginary eigenvalues (Hopf bifurcation), which gave rise to a Gavrilov–Guckenheimer bifurcation. In treating government intervention and the growth rate of the population of enterprises as two bifurcation parameters, we were able to deduce the existence of a two-bifurcation-related codimension with the persistence of a pre-existing Hopf limit cycle.

From this, we noted that whilst the intervention of fund F allowed for the condition of the system to be maintained, there was no indication that it improved the performance of companies or banks. In other words, when the relationship between the performance of banks, the absorption parameter and the interaction parameter between banks and enterprises is equal to the effect of the fund, the system achieves stability with no instrumental positive or negative change.

As we continue to operate with the uncertainties and instability introduced by the COVID-19 pandemic, understanding the underlying mechanisms of the options available to governments for ensuring continued stability and function of our financial economic systems is crucial (Xu et al. 2022). Of potentially critical importance is identifying at what point, if at all, the Gavrilov–Guckenheimer bifurcation may bring the system into chaos. Further investigation may identify how the measures implemented by the Italian government in 2020 might be able to move beyond maintaining pre-existing performance levels. Perhaps these measures could also be applied to improving the performance of banks and companies, or in combination with other instruments to achieve similar or enhanced results. Additional research could also consider the feasibility and effectiveness of these measures in economies with structures dissimilar to Italy's.

Since the COVID-19 pandemic was an exogenous force, this paper does not address impairments of creditworthiness or strategies for avoiding or mitigating credit crunches caused by poor internal structures. Likewise, it offers only a partial contribution to discussions on how to respond if an economic financial system were to suffer from contemporaneous exogenous and endogenous shocks (for instance, if the conditions of the COVID-19 pandemic and the 2009–2011 crisis were to occur at the same time).

7. Conclusions

This paper is attempting to provide some valuable insights on the interconnections and dependencies between banks, enterprises and the government in the interest of preventing a credit crunch enjoined by external factors. With the help of dynamical systems analysis and bifurcation theory, we have analyzed how the intervention of the guarantee fund in Italy has maintained the stability of the financial and economic system.

Drawing on Bischi and Tramontana (2010) for the discrete application of similar dynamical systems and on earlier two-dimensional modeling of the bank–enterprise system by Desogus and Casu (2020b), we have developed a three-dimensional continuous-time Lotka–Volterra dynamical model that demonstrates the interactions between populations of banks and enterprises and the government in a given financial system. Specifically, we focused on those interactions that are facilitated by credit transactions and the role of the guarantee fund in supporting the continuation of credit exchanges during periods of economic crisis.

This modeling has been informed by the cyclical trends that characterize economic financial models and that cause these systems to oscillate between states of stability and instability. This trajectory was confirmed through a comparative analysis of data from Italy and the UK that outlines how ad hoc government intervention in support of credit disbursement has helped to alleviate pressures within the system, to decrease the risk of impaired credit ratings and non-performing loans and to prevent a looming credit crunch.

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Appendix A

To apply Hopf's bifurcation theorem, we need first to put the system (S) into a normal form. Several steps are required.

a First, we need to translate the fixed point $P_1{}^*$ to the origin $t_1 = z_1 - z_1^*$, $t_2 = z_2 - z_2^*$, $t_3 = z_3 - z_3^*$; $\check{h} = h - h^*$; $t,\ z,\ z^* \in \Re^3$ under which system (S) becomes:

$$\begin{cases} \dot{t_1} = \delta\left(-kt_1 + \alpha t_1 t_2 + \frac{\alpha t_1 F}{\tau} + F - \tau\left(t_2 + \frac{F}{\tau}\right)\right) \\ \dot{t_2} = \left(\check{h} + h^*\right)\left(t_2 + \frac{F}{\tau}\right) - \beta t_1 t_2 + \frac{\beta t_1 F}{\tau} + \eta t_1\left(t_3 + \frac{F\left(\check{h}+h^*\right)}{\tau(\beta - F)}\right) + (\beta - F)\left(t_3 + \frac{F\left(\check{h}+h^*\right)}{\tau(\beta - F)}\right) \\ \dot{t_3} = F - \tau\left(t_2 + \frac{F}{\tau}\right) \end{cases} \quad (H)$$

b As a second step, we need to separate the linear part of the vector field from the rest. Formally, this means that our system becomes:

$$\dot{t} = f(t) = J_1^*\, t + F(t)$$

where $J^*(P_1^*)$ corresponds to the Jacobian of system (S) and where $F(t)$ is computed by the usual Taylor expansion, and only terms of order 2 and higher are included:

$$F_1 = \delta\alpha t_1 t_2; \quad F_2 = -\beta t_1 t_2 + \eta t_1 t_3; \quad F_3 = 0$$

$$\begin{vmatrix} F_1 \\ F_2 \\ F_3 \end{vmatrix} = \begin{vmatrix} \delta\alpha t_1 t_2 \\ -\beta t_1 t_2 + \eta t_1 t_3 \\ 0 \end{vmatrix}$$

c Finally, let T be the matrix that transforms J_1^* into a Jordan canonical form: $t = Tw$. Hence, we can write our system as:

$$T\dot{w} = J_1^* \, Tw + \check{F}(Tw)$$

or

$$\dot{w} = T^{-1} J_1^* \, Tw + T^{-1}\check{F}(Tw)$$

which is a form that simplifies the linear part of system (H) as much as possible. The calculation of T—the matrix of coordinate change for system (S)—requires the determination of the basis vectors e_1, e_2, e_3 associated with the eigenvalues of J_1^* at the bifurcation point. By substituting $BTr = Det$ in the characteristic polynomial at the bifurcation point, the real eigenvalue, λ^R, is positive and equal to Tr, whereas the complex conjugate eigenvalues, λ_c and $\overline{\lambda_c}$, are purely imaginary, with $\lambda_c = i\sqrt{B(J_1^*)}$ and $\overline{\lambda_c} = -i\sqrt{B(J_1^*)}$. Now, we can define $\xi = \sqrt{B(J_1^*)}$.

Since we know the form of the eigenvalues of J_1^* at the bifurcation point $h = h^*$:

$$\lambda_r = Tr(J_1^*); \; \lambda_c, \overline{\lambda_c} = \pm\xi i$$

The calculation of the basis vectors is not complicated. An eigenvector of J_1^* with the eigenvalue $Tr(J_1^*)$ is e_1, the eigenspace of J_1^* corresponding to the complex eigenvalues $\lambda_c, \overline{\lambda_c} = \pm\xi i$ where the orthogonal complement of the transpose of J_1^* corresponds to the real eigenvalue $\lambda_R = Tr(J_1{}^*)$. So, we choose e_1, e_2, e_3. Now we have the basis, and we can compute T.

$$e_1 = \begin{vmatrix} \frac{j_{11}^* v_1 + j_{12}^*}{\xi} \\ \frac{j_{21}^* v_1 + j_{22}^*}{\xi} \\ \frac{j_{32}^* v_2}{\xi} \end{vmatrix}; \quad e_2 = \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}; \quad e_3 = \begin{vmatrix} \frac{(j_{22}^* - \lambda_r)\lambda_r - j_{23}^* j_{32}^*}{j_{21}^* j_{32}^*} \\ -\frac{\lambda_r}{j_{32}^*} \\ 1 \end{vmatrix}$$

$$T = \begin{vmatrix} e_{11} & 1 & e_{31} \\ e_{22} & 1 & e_{32} \\ e_{33} & 0 & 1 \end{vmatrix}$$

$$\boldsymbol{T} = \begin{vmatrix} \frac{j_{11}^* v_1 + j_{12}^*}{\xi} & 1 & \frac{(j_{22}^* - \lambda_r)\lambda_r - j_{23}^* j_{32}^*}{j_{21}^* j_{32}^*} \\ \frac{j_{21}^* v_1 + j_{22}^*}{\xi} & 1 & -\frac{\lambda_r}{j_{32}^*} \\ \frac{j_{32}^*}{\xi} & 0 & 1 \end{vmatrix} = \begin{vmatrix} T_{11} & 1 & T_{13} \\ T_{21} & 1 & -\frac{\lambda_r}{j_{32}^*} \\ \frac{j_{32}^*}{\xi} & 0 & 1 \end{vmatrix} \quad \text{(A1)}$$

Setting $D = -\frac{1}{\beta}(\lambda_3 - \xi T_{11} + \xi T_{21} + T_{13} j_{32}^*)$, we can write:

$$T^{-1} = \frac{1}{D} \begin{vmatrix} 1 & -1 & T_{13} \\ -\frac{1}{\xi}(\lambda_3 + \xi T_{21}) & 1 & -\frac{\lambda_3}{j_{32}^*} \\ -\frac{1}{\xi} j_{32}^* & \frac{1}{\xi} j_{32}^* & \frac{1}{\xi}(\xi T_{11} - \xi T_{21}) \end{vmatrix} \quad \text{(A2)}$$

Finally, we get the system put into a normal form:

$$J_h^* = T^{-1} J_1^* \quad T = \begin{bmatrix} 0 & -\xi & 0 \\ \xi & 0 & 0 \\ 0 & 0 & Tr J_1^* \end{bmatrix}$$

We call $J_h{}^*$ the Jacobian matrix J_1^* put into a normal form.

Where $Tr\left(J_h^*\right) = Tr\left(J_1^*\right)$ and $B\left(J_1^*\right) = \xi^2$ is evaluated at the bifurcation point $h = h^*$, such that:

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{bmatrix} 0 & -\xi & 0 \\ \xi & 0 & 0 \\ 0 & 0 & Tr J \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} \check{F}_{1a} w_1 w_2 + \check{F}_{1b} w_1 w_3 + \check{F}_{1c} w_2 w_3 + \check{F}_{1d} w_1^2 + \check{F}_{1e} w_2^2 + \check{F}_{1f}\, w_3^2 \\ \check{F}_{2a} w_1 w_2 + \check{F}_{2b} w_1 w_3 + \check{F}_{2c} w_2 w_3 + \check{F}_{2d} w_1^2 + \check{F}_{2e} w_2^2 + \check{F}_{2f}\, w_3^2 \\ \check{F}_{3a} w_1 w_2 + \check{F}_{3b} w_1 w_3 + \check{F}_{3c} w_2 w_3 + \check{F}_{3d} w_1^2 + \check{F}_{3e} w_2^2 + \check{F}_{3f}\, w_3^2 \end{pmatrix} \quad \text{(A3)}$$

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