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Article

Properties of VaR and CVaR Risk Measures in High-Frequency Domain: Long–Short Asymmetry and Significance of the Power-Law Tail

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Abstract: This study investigates the properties of risk measure, value at risk (VaR) and conditional VaR (CVaR), using high-frequency Bitcoin data. These data allow us to conduct a high statistical analysis. Our findings reveal a disparity in VaR and CVaR values between the left and right tails of the return probability distributions. We refer to this disparity as "long–short asymmetry". In the high-frequency domain, the tail distribution can be accurately described by a power-law function. Moreover, the ratio of CVaR to VaR is expected to be determined solely by the power-law exponent. Through empirical analysis, we confirm that this ratio property holds true for high confidence levels. Furthermore, we investigate the relationship between risk measures (VaR and CVaR) and realized volatility. We observe that they trace a trajectory in a two-dimensional plane. This trajectory changes gradually, indicating periods of both high and low risk.

Keywords: risk measure; value at risk; conditional value at risk; expected shortfall; power-law function; realized volatility; Bitcoin; Rachev ratio

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1. Introduction

Risk management plays a central role in various financial sectors. Its purpose is to prevent unexpected substantial losses in trading and operations by closely monitoring these risks. Although there are various sources of financial risk, such as credit and liquidity risks, we will focus specifically on market risk, which refers to the risk associated with market price changes. The widely accepted risk measure is known as the value at risk (VaR). VaR provides a single numerical value that summarizes the overall risk of a portfolio (see, for example, Abad et al. (2014); Duffie and Pan (1997); Gourieroux and Jasiak (2010); Linsmeier and Pearson (2000)). For our analysis, we will consider a simple portfolio consisting of a single asset, such as a stock. In this context, changes in asset prices are described as returns r, and historical return data form a return probability distribution $P(r)$. VaR is defined as the maximum loss at a given confidence level, denoted as X%, over a given time horizon, denoted as T. Figure 1 shows a schematic drawing for the VaR approach. "$VaR(X)_L$" stands for the VaR for the long position at the confidence level of X% and is defined so that the probability of the left tail ($-\infty < r \leq VaR(X)_L$) becomes (100-$X$)%. Similarly, "$VaR(X)_S$" for the short position at the confidence level of X% is defined as the right tail ($VaR(X)_S \leq r < \infty$) of the return probability distribution.

One drawback of VaR is its inability to provide information on potential losses beyond the VaR threshold. This limitation becomes particularly significant when dealing with the tail of the probability distribution. In scenarios where the tail is heavier than that of a normal distribution, the potential loss can exceed what would be expected under normal distribution assumptions. Empirical evidence shows that asset return distributions often exhibit fat tails, which are recognized as stylized facts (Cont 2001). To address this issue and incorporate tail information, an improved risk measure known as conditional VaR (CVaR), or expected shortfall with coherent properties, has been introduced (Acerbi and

Tasche 2002; Artzner et al. 1999). CVaR is defined as the average value of VaR that exceeds the VaR at a given confidence level, denoted as $X\%$.

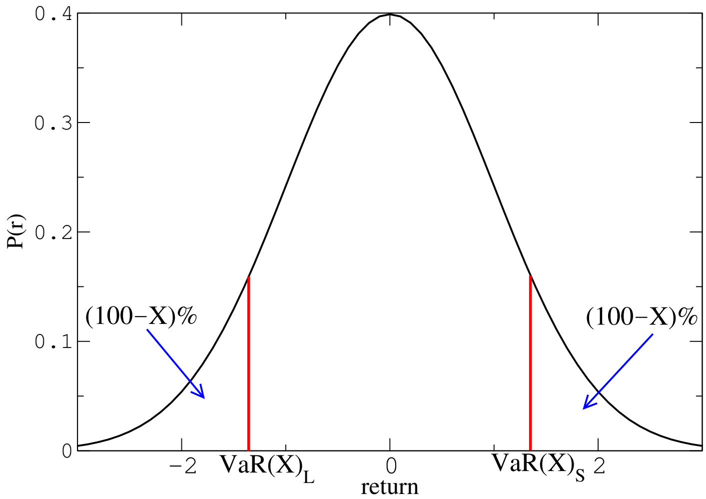

Figure 1. Schematic drawing of the VaR approach.

Usually, VaR and CVaR calculations in risk management focus on the left tail of the return probability distribution. This region corresponds to potential losses in a long position. Conversely, the right tail represents losses in a short position. When assuming a symmetrical return probability distribution, there would be no distinction between VaR and CVaR values in the left and right tails. However, it cannot be guaranteed that empirical distributions will exhibit symmetry. We refer to the difference between VaR and CVaR associated with asymmetric distributions as "long–short asymmetry". One of our objectives was to empirically investigate the presence of such long–short asymmetry. Measuring long–short asymmetry poses various challenges, primarily due to the limited availability of accurate probability distributions. This limitation becomes more pronounced when using one-day or longer time returns, as the shorter sampling period results in fewer statistics. For instance, if we collect daily returns for one year, we would have only 365 (or approximately 250) working days' worth of data. For risk measures in the cryptocurrency market, based on daily returns, see, e.g., Almeida et al. (2022).

To address this issue, we analyze the data in the high-frequency domain, which involves collecting a large amount of statistical data. In this study, we specifically utilize 1 min Bitcoin data. In cryptocurrency markets, Bitcoin is traded continuously for 24 h, allowing us to gather 52,560 1 min return data points over a 1-year period. This extensive dataset enables us to conduct a thorough statistical analysis.

Long–short asymmetry is closely associated with the reward–risk ratios that are defined as ratios between a reward measure and a risk measure (Cheridito and Kromer 2013). We calculate one of the reward–risk ratios, the Rachev ratio (Biglova et al. 2004), which is given by a ratio of the CVaR for the short position to the CVaR for the long position. Since the Rachev ratio deviates from one for asymmetrical distribution, it also quantifies the long–short asymmetry.

Our second objective is to explore the relationship between the tail exponent and risk measures. In the high-frequency domain, the tails of the return probability distributions are fat-tailed and exhibit power-law behavior (Gabaix 2009; Gopikrishnan et al. 1998, 1999; Pan and Sinha 2007; Plerou et al. 1999). Let α be the power-law exponent of the cumulative return distribution. As we will explain later, under the assumption of a power-law probability distribution, the ratio of CVaR to VaR is $\alpha/(\alpha-1)$; we empirically verify the presence of this $\alpha/(\alpha-1)$ relationship at high confidence levels.

We also investigate the relationship between realized volatility (RV) (Andersen and Bollerslev 1998; Andersen et al. 2003; McAleer and Medeiros 2008) and the risk measures.

Our findings reveal that these variables form a trajectory that exhibits periods of both high and low risk.

The remainder of this paper is organized as follows: Section 2 describes the methodology and data used in this study. Section 3 presents the empirical results. Finally, we discuss and conclude our findings in Section 4.

2. Methodology and Data

First, we calculate the VaR and CVaR for the return probability distribution $P(r)$. Here, for simplicity, we assume that the mean of the returns is zero. The VaR at a confidence level $X\%$ is denoted as $VaR(X)_L$ and is defined for the long position (left tail) as

$$p_X = \int_{-\infty}^{VaR(X)_L} P(r)dr, \tag{1}$$

where $p_X = 1 - X/100$. Similarly, the VaR for the short position, $VaR(X)_S$, is defined in the right tail as

$$p_X = \int_{VaR(X)_S}^{\infty} P(r)dr. \tag{2}$$

CVaR is defined as the average VaR that exceeds the VaR at confidence level $X\%$. The CVaR for the long position at confidence level $X\%$ is given by

$$CVaR(X)_L = \int_{-\infty}^{VaR(X)_L} rP(r)dr/p_X. \tag{3}$$

Similarly, the CVaR for the short position is given by

$$CVaR(X)_S = \int_{VaR(X)_S}^{\infty} rP(r)dr/p_X. \tag{4}$$

Next, we calculate the VaR and CVaR for specific forms of $P(r)$. Note that although we calculate VaR and CVaR for the long position, we obtain the same expression for both of the short positions, except with regard to the sign. First, we assume that $P(r)$ is a normal distribution with a standard deviation σ, that is, $P(r) = \exp(-\frac{r^2}{2\sigma^2})/\sqrt{2\pi\sigma^2}$. Then, Equation (1) is as follows (Hull 2018):

$$VaR(X)_L = -\sigma N^{-1}(X), \tag{5}$$

where $N^{-1}(X)$ denotes the inverse cumulative normal function. Similarly, we obtain Equation (3).

$$CVaR(X)_L = -\sigma\frac{\exp(-K^2/2)}{\sqrt{2\pi}p_X}, \tag{6}$$

where $K \equiv N^{-1}(X)$. Using Equations (5) and (6), the ratio of CVaR to VaR, denoted by R_{norm}, is

$$R_{norm} = \frac{\exp(-K^2/2)}{\sqrt{2\pi}Kp_X}. \tag{7}$$

Second, we assume that the total probability distribution $P(r)_t$, defined as $P(r)_t = P(r)_{tail} + P_o(r)$, consists of two parts: the tail $P(r)_{tail}$ and the other $P_o(r)$. $P(r)_{tail}$ is described by a power-law function as

$$P(r)_{tail} = c|r|^{-(\alpha+1)}, \tag{8}$$

where the constant c is the normalization factor determined such that $\int_{-\infty}^{\infty} P(r)_t = 1$. The actual value of c is not significant. At the tail, we assume that only $P(r)_{tail}$ contributes to the calculations of VaR and CVaR. Using Equation (8), Equation (1) is calculated to be[1]

$$p_X = \frac{c}{\alpha} |VaR(X)_L|^{-\alpha}. \tag{9}$$

Similarly, Equation (3) leads to the following:

$$CVaR(X)_L = \frac{c}{p_X(\alpha - 1)} |VaR(X)_L|^{-(\alpha-1)}. \tag{10}$$

Using Equations (9) and (10), the ratio of $CVaR(X)_L$ to $VaR(X)_L$ becomes:

$$R_{power} = \frac{\alpha}{\alpha - 1}. \tag{11}$$

Interestingly, at any confidence level for the tail distributions described by a power-law function, R_{power} is determined by α only.

To quantify the differences in VaR and CVaR between the left and right tails, we compute

$$D_{VaR}(X) = VaR_S(X) - |VaR_L(X)|, \tag{12}$$

and

$$D_{CVaR}(X) = CVaR_S(X) - |CVaR_L(X)|, \tag{13}$$

at a given confidence level $X\%$. $D_{VaR}(X)$ and $D_{CVaR}(X)$ take zero values for symmetrical distributions. We also calculate the Rachev ratio (R-ratio) (Biglova et al. 2004) defined as

$$R\text{-}ratio = \frac{CVaR_S(X)}{|CVaR_L(X)|}. \tag{14}$$

Similarly, we can also define a ratio by VaR (V-ratio) as

$$V\text{-}ratio = \frac{VaR_S(X)}{|VaR_L(X)|}. \tag{15}$$

Both the R-ratio and V-ratio will take the value of 1 for symmetric distributions. The R-ratio and V-ratio are related to D_{VaR} and D_{CVaR} as follows:

$$R\text{-}ratio - 1 = D_{CVaR}/|CVaR_L|, \tag{16}$$

and

$$V\text{-}ratio - 1 = D_{VaR}/|VaR_L|. \tag{17}$$

For the tail distributions described by a power-law function, the ratio of R-ratio to V-ratio, denoted as R_{RV}, is given by

$$R_{RV} \equiv \frac{R\text{-}ratio}{V\text{-}ratio} = \frac{\alpha_S}{\alpha_S - 1} \frac{\alpha_L - 1}{\alpha_L}, \tag{18}$$

where α_S (α_L) is the power-law exponent at the right (left) tail of the return probability distribution.

In this study, we used Bitcoin data traded on Bitstamp exchanges[2] from 1 January 2015 to 21 May 2022. In the early stages of the Bitcoin market, characterized by low liquidity, we observed market properties that differed from those of liquid markets, such as developed-country stock markets (Di Matteo et al. 2005). For example, the Hurst exponent of the return time series in the early stages of the Bitcoin market was found to be less than 0.5, indicating the anti-persistence of the series (Urquhart 2016). It is argued that the anti-persistence seen in the cryptocurrency market can be attributed to the low liquidity of the market (Wei 2018).

The power-law exponents α of the tail return distributions were significantly lower than the expected values of 3 based on return distributions in developed countries (Begušić et al. 2018; Drożdż et al. 2018; Easwaran et al. 2015; Takaishi 2021a). Due to the low liquidity on the Bitstamp exchange before 2013 (Takaishi and Adachi 2020), we selected a period after 2015 when liquidity was sufficiently high.

From the 1 min price data p_t, we construct 1 min return data using $r_t = \ln p_t - \ln p_{t-1}$. Table 1 describes the descriptive statistics of the whole 1 min return data. The kurtosis was found to be about 98, which is considerably high, implying that the return distribution is fat-tailed. It is known that as the time scale of returns increases, the return distributions approach the Gaussian distribution. On the Bitcoin market, the kurtosis reaches the value of 3 (the kurtosis of the Gaussian distribution) at the time scale of two weeks (Takaishi 2018 2021b).

Table 1. Descriptive statistics for the whole sample of 1 min returns. The values in parentheses indicate one-sigma errors estimated by the Jackknife method.

Mean	Standard Deviation	Kurtosis	Skewness	Nobs
$1.2(8) \times 10^{-6}$	0.00135(8)	98(44)	−0.3(2)	3.87M

We analyze the data within a 1-year window containing 52,560 return data points. The window is then shifted by one day to capture time-varying properties and enable further investigation. To calculate VaR and CVaR, we first sort the 52,560 return data points in ascending (descending) order for long (short) positions. Then, $VaR(X)$ at the confidence level X is obtained from the 52,560 $\times$ $(X/100)$th value in the sorted data[3]. Similarly, $CVaR(X)$ is obtained from the average of the sorted data from the first to the 52,560 $\times$ $(X/100)$th data point.

3. Empirical Results

Figures 2 and 3 show the time evolution of VaR and CVaR, respectively. In the figures, positive (negative) values correspond to the VaR and CVaR regarding the short (long) position or the right (left) tail of the return probability distribution. The magnitude of the VaR and CVaR is found to be relatively small, i.e., an order of about 0.005∼0.01 since the variation in 1 min returns that we use here is also small. As described in Table 1, the standard deviation of 1 min returns is small, ∼0.00135. Here, note that the standard deviation of 1-day returns is calculated to be ∼4.63 (Takaishi 2021b), which is bigger than that of 1 min returns.

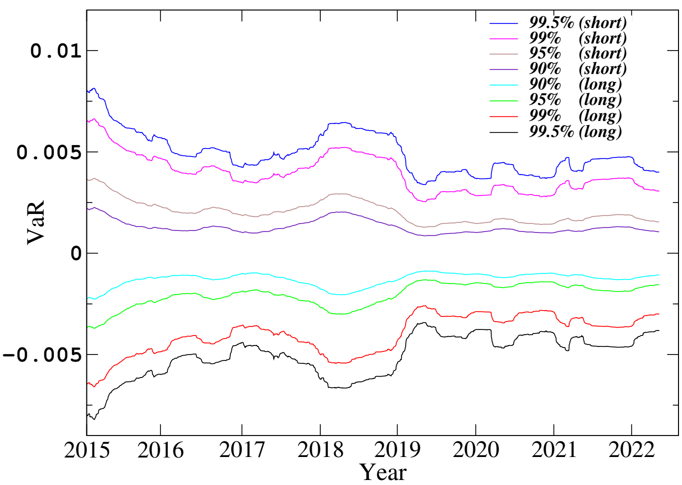

Figure 2. Time evolution of VaR at confidence levels $X = 99.5\%, 99\%, 95\%$, and 90%.

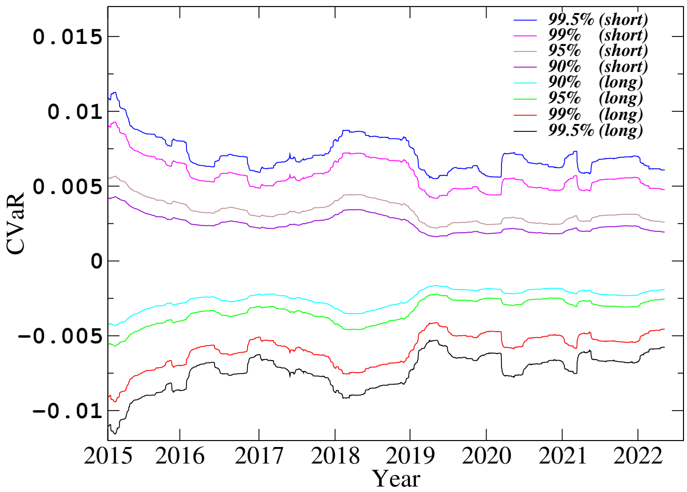

Figure 3. Time evolution of CVaR at confidence levels $X = 99.5\%, 99\%, 95\%$, and 90%.

It is evident from the figure that VaR and CVaR are not constant and vary over time on the Bitcoin market. The VaR and CVaR show similar time variation patterns. Namely, we observed that the magnitude of VaR and CVaR increases around 2015 and 2018, implying that the market risk was higher around 2015 and 2018 compared to other periods. This high and low risk pattern will be more clear when we analyze both risk measures (VaR and CVaR) and the RV simultaneously (we will return to this point later).

Figures 4 and 5 display the time evolutions of D_{VaR} and D_{CVaR}, respectively. D_{VaR} and D_{CVaR} quantify long–short asymmetry in VaR and CVaR. It is evident from the figures that D_{VaR} and D_{CVaR} predominantly take non-zero values, indicating the presence of long–short asymmetry in VaR and CVaR.

This asymmetry is more pronounced at high confidence levels, highlighting that risks can differ between the left and right tails at the same confidence level. At high confidence levels, D_{VaR} and D_{CVaR} take mostly negative values except for in some periods, which means that in the period we studied here, the long position is riskier than the short position.

The significance of long–short asymmetry should be compared to the magnitude of VaR or CVaR. For example, at the confidence level $X = 99.5\%$, the absolute value of CVaR (D_{CVaR}) around 2016 is about 0.01 (0.0005), which results in $|D_{CVaR}/CVaR| \simeq 0.05$. Thus, in this case, the significance of long–short asymmetry is about 5%. The significance of long–short asymmetry is also measured by directly comparing long and short positions. Such measurements are the V-ratio and R-ratio.

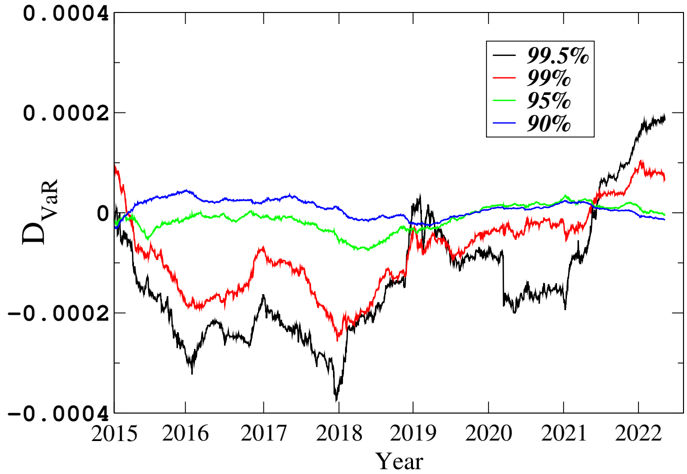

Figure 4. Time evolution of D_{VaR} at confidence levels of $X = 99.5\%, 99\%, 95\%$, and 90%.

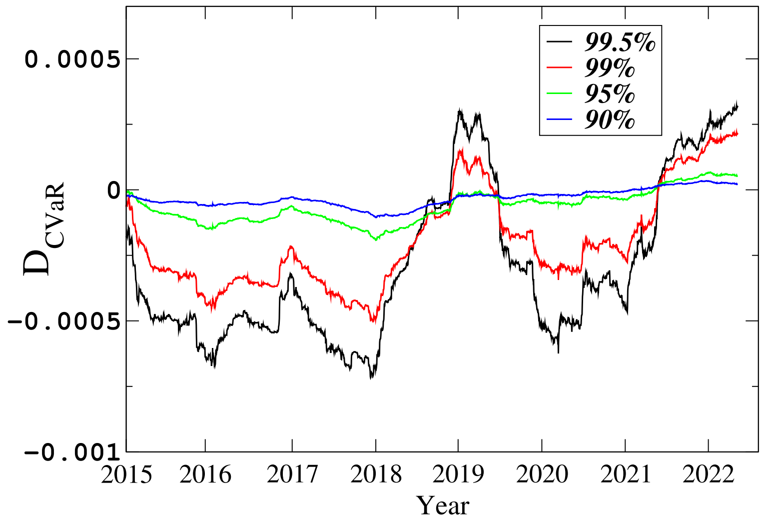

Figure 5. Time evolution of D_{CVaR} at confidence levels of $X = 99.5\%, 99\%, 95\%,$ and 90%.

Figures 6 and 7 show the time evolutions of the V-ratio and R-ratio, respectively. These ratios exhibit similar time variations with D_{VaR} and D_{CVaR}, and predominantly deviate from 1, indicating the presence of long–short asymmetry. Moreover, since the V-ratio and R-ratio are defined by Equations (14) and (15), they could represent the significance of long–short asymmetry. For example, at the confidence level $X = 99.5\%$, the V-ratio from 2016 to 2018 is about 0.95, implying that there is about a 5% difference in the VaR between long and short positions. Although the magnitude of D_{VaR} is small at the confidence level $X = 90\%$, the V-ratio from 2016 to 2018 is around 1.03 (3% difference in VaR), which means that the long–short asymmetry could be significant at lower confidence levels.

"D_{VaR}, D_{CVaR}" and "V-ratio, R-ratio" can also identify the period in which the return probability distributions are approximately symmetrical. For symmetrical distributions, D_{VaR} and D_{CVaR} take values of zero at any confidence level. Similarly, the V-ratio and R-ratio take one at any confidence level. Therefore, the criterion that D_{VaR} and D_{CVaR} are zero and that the V-ratio and R-ratio are one at any confidence level can help to provide information on symmetrical distributions. For example, from Figures 4–7 we recognize that the middle of 2021 matches the criterion approximately, and thus, the return probability distribution is expected to be symmetrical approximately in the middle of 2021.

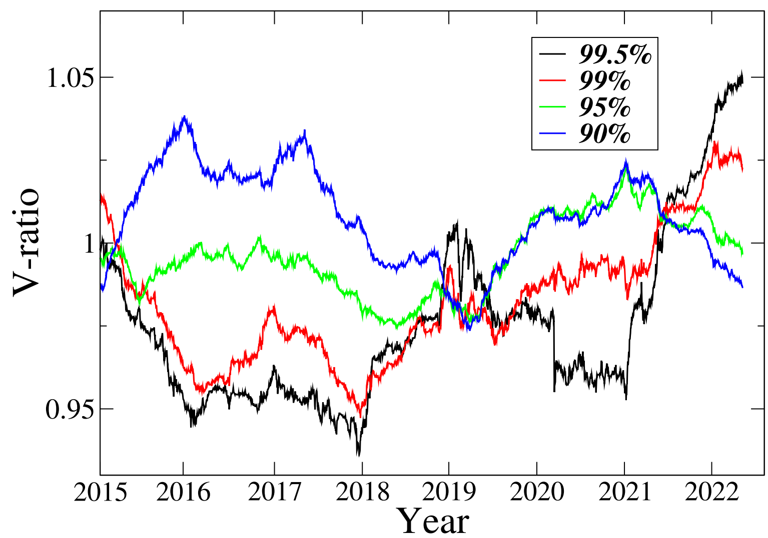

Figure 6. Time evolution of V-ratio at confidence levels of $X = 99.5\%, 99\%, 95\%,$ and 90%.

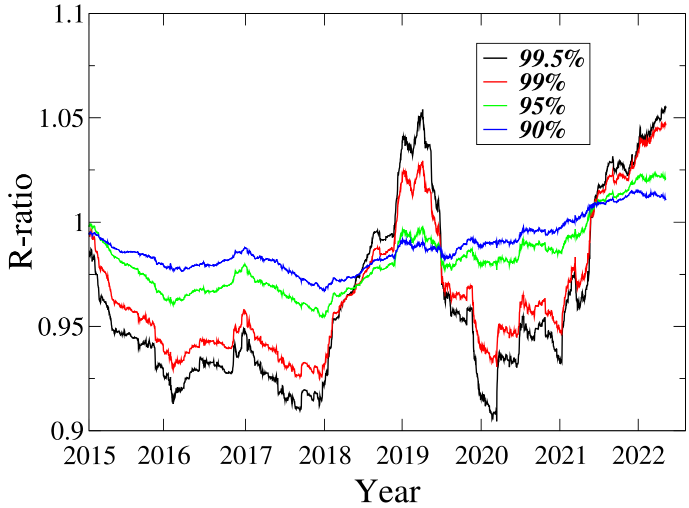

Figure 7. Time evolution of R-ratio at confidence levels of $X = 99.5\%, 99\%, 95\%$, and 90%.

To examine the significance of the power-law distribution with regard to the VaR and CVaR, we determine the power-law exponent α by fitting the tail data to $\sim |r|^{\alpha+1}$ using the Hill estimator (Hill 1975). The Hill estimator estimates the power-law exponent α with the equation

$$\frac{1}{\alpha+1} = \frac{1}{k}\sum_{i=1}^{k}(\ln r^{(i)} - \ln r^{(k)}), \tag{19}$$

where $r^{(1)} \geq r^{(2)} \geq \cdots \geq r^{(k)}$ is the order statistics for the tail data.

Figure 8 shows the time evolution of α obtained from the left and right tails of the return probability distribution. The power-law exponent α varies considerably over time around $\alpha = 3$. The values of α are higher than the value of $\alpha = 2$ observed in the early stage of the Bitcoin market (Easwaran et al. 2015). The low value of α could be related to the liquidity in the early stage of the Bitcoin market. The liquidity is also considered to be the origin of the low Hurst exponents observed in the cryptocurrency markets (Wei 2018). In the period we studied here, the liquidity of the Bitcoin market is expected to be high (Takaishi and Adachi 2020) and the power-law exponent α comes close to the value of $\alpha = 3$ observed in the stock market (Gopikrishnan et al. 1998, 1999).

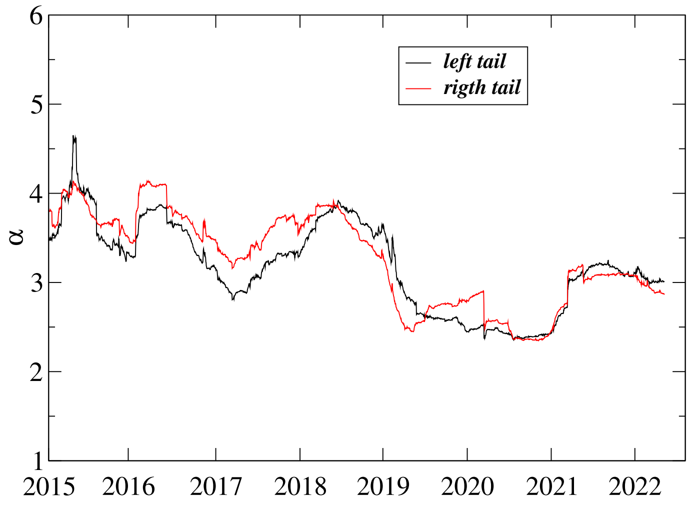

Figure 8. Time evolution of the power-law exponent α.

As suggested by Equation (11), the ratio CVaR/VaR can be expressed as $\alpha/(\alpha-1)$ for the power-law tail. In Figures 9 and 10, we show the time evolution of the ratio alongside the corresponding values of $\alpha/(\alpha-1)$. The dashed straight lines in the figures represent the theoretical results under a normal distribution assumption obtained with Equation (7). The empirical ratios consistently exceed those derived from the normal assumption, suggesting that the empirical return distributions exhibit fatter tails compared to the normal distribution. The results at high confidence levels (99.5% and 99%) closely align with the $\alpha/(\alpha-1)$ law. This indicates that the probability distributions are well described by the power-law function in the region corresponding to high confidence levels. As the confidence level decreases, the results deviate from $\alpha/(\alpha-1)$, which implies that in the region with lower confidence levels, the probability distributions are not well approximated by the power-law function.

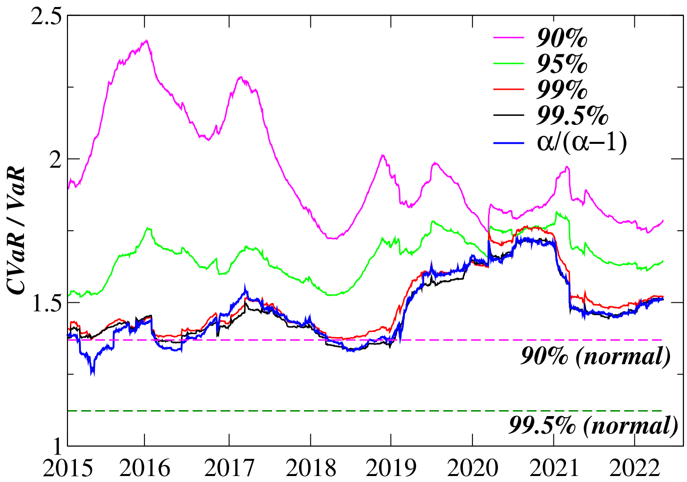

Figure 9. Time evolution of CVaR/VaR for the long position at confidence levels of $X = 99.5\%$, 99%, 95%, and 90%. The dashed lines show the theoretical values under the normal distributional assumption, obtained with Equation (7).

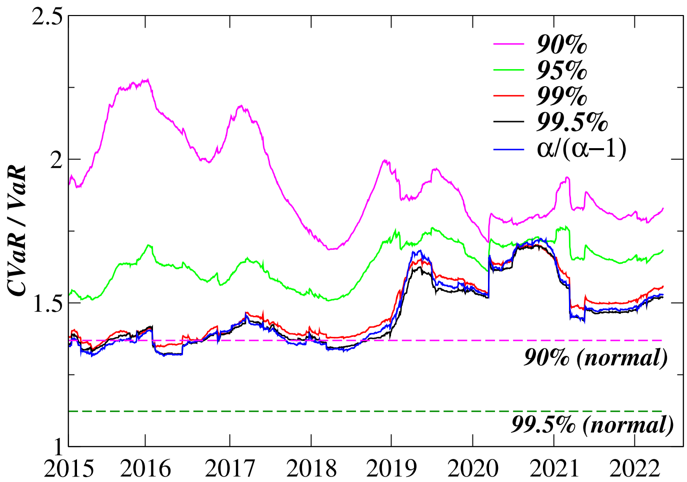

Figure 10. Time evolution of CVaR/VaR for the short position at confidence levels of $X = 99.5\%$, 99%, 95%, and 90%. The dashed lines show the theoretical values under the normal distributional assumption, obtained with Equation (7).

To visualize the $\alpha/(\alpha-1)$ law more clearly, we plot CVaR/VaR as a function of α in Figures 11 and 12. At high confidence levels ($X = 99.5\%$ and 99%), the ratio aligns well with the line $\alpha/(\alpha-1)$, indicating that the return probability distribution at these high confidence levels is consistent with the power-law distribution. For lower confidence levels, the ratio deviates from the $\alpha/(\alpha-1)$ law and moves above the curve of the $\alpha/(\alpha-1)$ law. The deviation from the curve of the $\alpha/(\alpha-1)$ law implies that the return probability distribution departs from the power-law distribution. Thus, it is concluded that the tail distributions corresponding to lower confidence levels are not well described by the power-law function.

In the absence of a specific distributional assumption, such as the normal distribution, the variance alone does not provide accurate values for VaR and CVaR. However, the magnitudes of VaR and CVaR are correlated with the variance. To investigate the correlation between risk measures and variance, we use RV as a proxy for variance. The daily RV is constructed as a sum of squared intraday returns,

$$RV = \sum_{i=1}^{n} r_{i,\Delta}^2, \tag{20}$$

where $r_{i,\Delta}$ is the number of returns sampled at a Δ-minute sampling frequency and n is the number of returns sampled in a day. We calculate the daily RV from the 5 min returns (Liu et al. 2015). Since we consider the VaR and CVaR calculated over a 1-year window, we use an RV averaged over the same 1-year window.

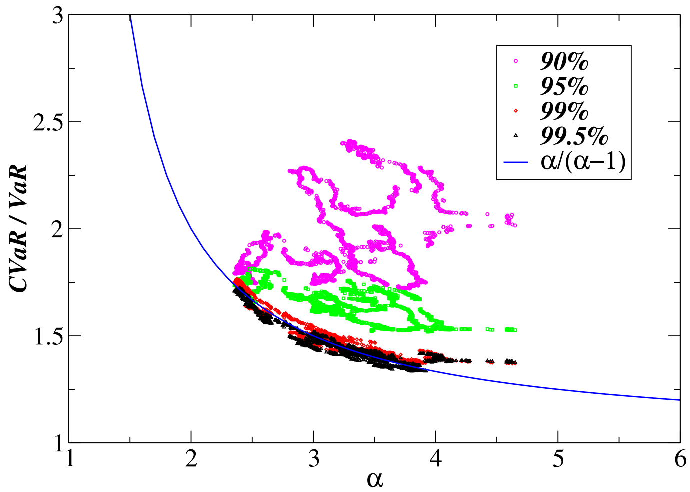

Figure 11. Time evolution of CVaR/VaR for long position at confidence levels of $X = 99.5\%, 99\%, 95\%$, and 90% as a function of α.

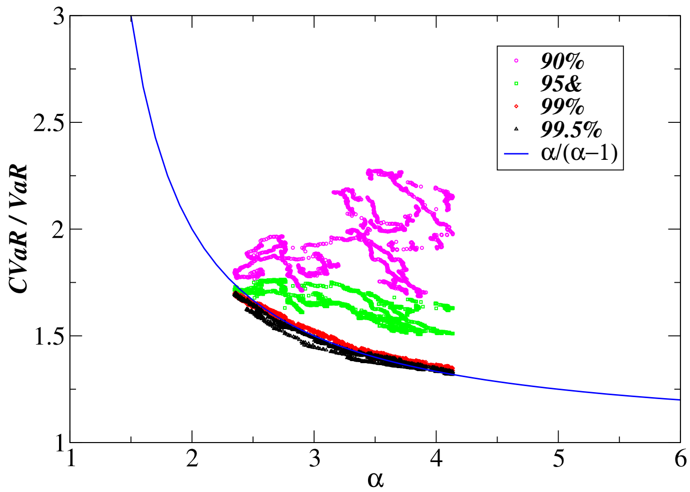

Figure 12. Time evolution of CVaR/VaR for short position at confidence levels of $X = 99.5\%$, $99\%, 95\%$, and 90% as a function of α.

Figure 13 shows the RV averaged over a 1-year window of data. The RV also varies over time and its time variation pattern is very similar to those of the VaR and CVaR, which indicates that the RV is correlated with the VaR and CVaR. Using the RV, we make two-dimensional plots of the risk measures and RV at confidence levels of $X = 99.5\%$ and

90%, as shown in Figures 14 and 15. These plots reveal a strong correlation between the VaR (CVaR) and RV. In our analysis of the 1-year window, we observed that the magnitude of the risk appears to change, resulting in trajectories representing changes in the VaR(CVaR)–RV plane. The trajectories found in the VaR–RV and CVaR–RV planes are very similar, and we also observed the similar trajectories for the short position (not shown here). These trajectories enable us to identify periods of high or low risk. The periods around 2015 and 2018 are classified as high risk, while the periods around 2017 and 2019 correspond to low-risk periods.

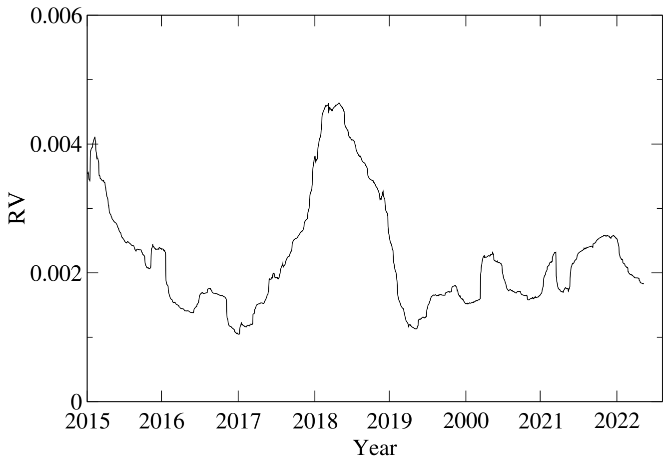

Figure 13. Time evolution of the daily RV averaged over a one-year window.

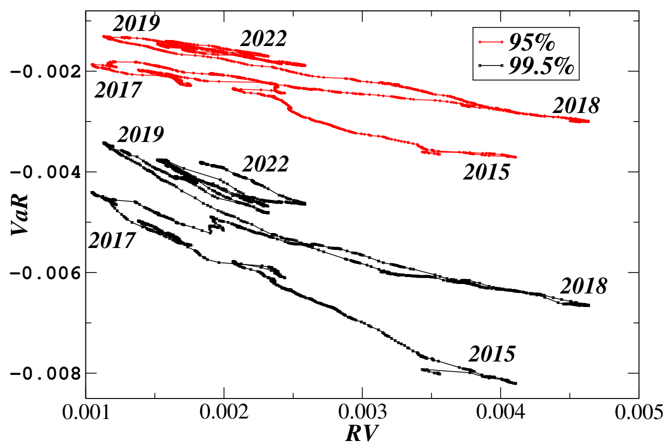

Figure 14. VaR for the long position versus RV at confidence levels of $X = 99.5\%$ and 95%.

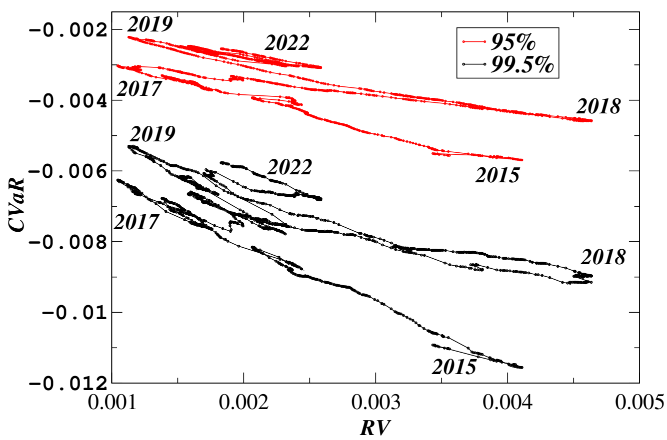

Figure 15. CVaR for the long position versus RV at confidence levels of $X = 99.5\%$ and 95%.

4. Discussion and Conclusions

Using high-frequency Bitcoin data, we conducted an investigation into the properties of the risk measures VaR and CVaR. By analyzing the risk measures in the left and right tails of the return probability distribution, we discovered "long–short asymmetry" for VaR and CVaR. This finding implies that the risk differs between long and short positions, particularly at high confidence levels. Such divergence in risks between long and short trades can offer valuable insights for trading strategies, particularly in the realm of high-frequency trading.

Furthermore, we observed that the ratios, CVaR/VaR, at high confidence levels align well with the $\alpha/(\alpha-1)$ law derived from the power-law distributional assumption. The presence of the $\alpha/(\alpha-1)$ law suggests that VaR and CVaR are no longer independent when the power-law distribution assumption holds.

Moreover, we observed a strong correlation between the risk measures and the RV, which resulted in the formation of trajectories in a two-dimensional plane. These trajectories unveiled periods of high and low risk. It would be intriguing to explore whether the high- and low-risk periods are associated with other measures, such as market efficiency (the Hurst exponent) (Bariviera 2017; Urquhart 2016), multifractality (Takaishi 2018), and inverted volatility asymmetry (Bouri et al. 2017; Katsiampa 2017; Stavroyiannis and Babalos 2017; Takaishi 2021b).

It is worth noting that our findings are based solely on Bitcoin data. Further investigations should be conducted using other assets to ascertain the universality of the "long–short asymmetry" and the $\alpha/(\alpha-1)$ law. Since previous studies (Gopikrishnan et al. 1998 1999) have already revealed that stock price returns exhibit the power-law probability distributions, we should expect the $\alpha/(\alpha-1)$ law for $CVaR/VaR$ on the stock markets.

In this study, we employed a 1-year window to investigate the time-varying properties. Consequently, the observed properties were averaged over the course of a year. To capture more dynamic changes within a year, smaller windows would be necessary. Although analyzing smaller windows poses some challenges due to reduced statistical data, it could yield interesting insights into more dynamic fluctuations.

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Notes

[1] In the context of the Pareto distribution, the similar expression can be found in Abad et al. (2014); Gourieroux and Jasiak (2010).

[2] http://www.bitstamp.net, accessed on 22 May 2022.

[3] When "52,560 × $(X/100)$" is not a multiple of an integer, we interpolate two neighbor returns.

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