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# Circular-Statistics-Based Estimators and Tests for the Index Parameter $\alpha$ of Distributions for High-Volatility Financial Markets

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**Abstract:** The distributions for highly volatile financial time-series data are playing an increasingly important role in current financial scenarios and signal analyses. An important characteristic of such a probability distribution is its tail behaviour, determined through its tail thickness. This can be achieved by estimating the index parameter of the corresponding distribution. The normal and Cauchy distributions, and, sometimes, a mixture of the normal and Cauchy distributions, are suitable for modelling such financial data. The family of stable distributions can provide better modelling for such financial data sets. Financial data in high-volatility markets may be better modelled, in many cases, by the Linnik distribution in comparison to the stable distribution. This highly flexible family of distributions is better capable of modelling the inflection points and tail behaviour compared to the other existing models. The estimation of the tail thickness of heavy-tailed financial data is important in the context of modelling. However, the new probability distributions do not admit any closed analytical form of representation. Thus, novel methods need to be developed, as only a few can be found in the literature. Here, we recall a recent novel method, developed by the authors, based on a trigonometric moment estimator using circular distributions. The linear data may be transformed to yield circular data. This transformation is solely for yielding a suitable estimator. Our aim in this paper is to provide a review of the few existing methods, discuss some of their drawbacks, and also provide a universal ($\forall \alpha \in (0, 2]$), efficient, and easily implementable estimator of $\alpha$ based on the transformation mentioned above. Novel, circular-statistics-based tests for the index parameter $\alpha$ of the stable and Linnik distributions are introduced and also exemplified with real-life financial data. Two real-life data sets are analysed to exemplify the methods recommended and enhanced by the authors.

**Keywords:** characteristic function-based estimator; estimation; fractional moment estimator; Hill estimator; index parameter; trigonometric method of moment estimator; wrapped Linnik; wrapped stable

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## 1. Introduction

In the modern era, there is an increasing need for modelling financial markets (and engineering sciences, e.g., signal detection) with high volatility. An important characteristic of such a probability distribution is its tail behaviour, determined through its tail thickness. There is a need for modelling such financial data. High variability has also been a common characteristic of modern circular data.

Corresponding circular distributions are characterised by heavy or long tails. The normal and Cauchy distributions, and, sometimes, a mixture of the normal and Cauchy distributions, are suitable for modelling such financial data. The family of stable distributions can provide better modelling for such financial data sets. Highly volatile financial time-series data may be better modelled, in many cases, by the Linnik distribution in comparison to the stable distribution, e.g., see Anderson and Arnold (1993). This highly flexible family of distributions is better capable of modelling the inflection points and tail behaviour compared to the existing popular flexible symmetric unimodal models.

There have been a lot of studies establishing the use of the Linnik family of distributions as a highly flexible, important, and useful family for modelling financial data. However, its implementation for real-life data seems to have been somewhat restricted, possibly because of the lack of a simple and efficient estimator of the parameter, particularly that of the index parameter $\alpha$. The estimation of the tail thickness of heavy-tailed financial data using the index parameter $\alpha$ is important in the context of modelling.

Our aim in this paper is to provide a universal (for all $\alpha \in (0, 2]$), efficient, and easily implementable estimator of $\alpha$ after presenting a review of the few existing methods. The issue behind the derivation is to study the advantages and also to point out the shortcomings of some of the estimators and hence to obtain better estimators that eliminate the effects of the shortcomings of the former ones.

We have observed that the circular-statistics-based estimators can be quite useful in this context, which is enhanced in this paper. Circular statistics are obtained for circular data. In many emerging real-life situations, we not only make observations on linear variables but also on circular ones, that is, on angular propagations, orientations, directional movements, and strictly periodic occurrences. Such data are referred to as directional data, which, in two dimensions, are known as circular data. Linear data may be transformed into circular data using the method of wrapping.

Here, we recall two highly flexible families of circular distributions, e.g., the wrapped stable family, in Section 2, and the wrapped Linnik family in Section 3, and the novel, universal, efficient, and easily implementable estimators of $\alpha$ derived from these are presented in Section 7. In Section 4, descriptions of the classical Hill estimator by Hill (1975), and its generalisation by Brilhante et al. (2013), are presented. In Section 5, the fractional moment estimator for the symmetric Linnik distribution proposed by Kozubowski (2001) is reviewed. The fractional moment estimator of the characteristic exponent used to measure the tail thickness for skewed stable distributions, proposed by Kuruoglu (2001), is particularised to obtain the same for the symmetric stable distribution in Section 5. In Section 6, the characteristic function-based estimator proposed by Anderson and Arnold (1993) is presented. In Section 7, the trigonometric method of moment estimators proposed by SenGupta (1996) and SenGupta and Roy (2023) is presented, which is further modified to obtain an improved estimator (as in SenGupta and Roy 2019, 2023) in Section 8. The trigonometric method of moment estimation is exploited here for symmetric circular distributions only. It can be used for asymmetric distributions as well. But the computations involved are complicated and time consuming and hence are not considered here. In Section 9, the performance of the estimators is discussed through extensive simulations, focusing on their estimated mean bias and estimated root-mean-square errors, as presented in Tables 1 and 2. In Section 10, the computed values of the estimators are obtained for two real-life financial data sets, which are presented in Table 3. In Section 11, novel tests for the index parameter $\alpha$ of the stable and Linnik distributions are introduced and also illustrated with real-life financial data. Some discussions and conclusions on the different estimators are provided in Section 12. In the Acknowledgement section, the authors express their acknowledgements.

## 2. The Symmetric Stable and Wrapped Stable Family of Distributions

The regular symmetric stable distribution is defined through its characteristic function given by

$$\psi_S(t) = \exp(it\mu - |\sigma t|^{\alpha}), \quad (1)$$

where $\mu$ is the location parameter, $\sigma$ is the scale parameter, and $\alpha$ is the index or shape parameter of the distribution.

Using Proposition 2.1 on page 31 of Jammalamadaka and SenGupta (2001), the following theorem is obtained (see SenGupta and Roy 2023).

**Theorem 1.** *(a) The trigonometric moment of order p for a wrapped stable distribution corresponds to the value of the characteristic function of the linear stable random variable at the integer value* $p = 1, 2, \ldots$ *(b) The characteristic function of the wrapped stable random variable* $\theta$ *at the integer p is*

$$\psi_{WS}(p) = E[\exp(ip(\theta - \mu))] = \exp(ip\mu - \rho^{p^\alpha}), \tag{2}$$

*where* $\rho = \exp(-\sigma^\alpha)$, $\mu$ *is the location parameter,* $\sigma$ *is the scale parameter,* $\alpha$ *is the index parameter and* $i = \sqrt{(-1)}$.

From the stable distribution, we can obtain the wrapped stable distribution (the process of wrapping is explained by Jammalamadaka and SenGupta (2001)). Suppose that $\theta_1, \theta_2, \ldots, \theta_m$ are a random sample of size $m$ drawn from the wrapped stable distribution (provided by Jammalamadaka and SenGupta (2001)), whose probability density function is given by

$$f(\theta, \rho, \alpha, \mu) = \frac{1}{2\pi}[1 + 2\sum_{p=1}^{\infty} \rho^{p^\alpha} \cos p(\theta - \mu)] \quad 0 < \rho \leq 1, 0 < \alpha \leq 2, 0 < \mu \leq 2\pi. \tag{3}$$

where $p = 1, 2, \ldots$ and the parameters explained as above.

## 3. The Symmetric Linnik and the Wrapped Linnik Family of Distributions

It was established by Pakes (1998) that the characteristic function of a symmetric ($\alpha$) Linnik (linear) distribution is given by

$$\psi_L(t) = \exp(it\mu)(1 + |t\sigma|^\alpha)^{-1}. \tag{4}$$

The density function cannot be written in an analytical form except for $\alpha$ = 2. The wrapping of this distribution yields the wrapped symmetric $\alpha$ Linnik family of distributions. However, this circular family differs from that of the symmetric wrapped stable family and none of these families is a sub-family of the other. In particular, taking $\alpha = 2$, for the wrapped symmetric stable family one gets the wrapped Cauchy, while for the wrapped symmetric Linnik family it gives the wrapped Laplace (double exponential) distribution.

Using Proposition 2.1 on page 31 of Jammalamadaka and SenGupta (2001), the following theorem is obtained (see SenGupta and Roy 2023).

**Theorem 2.** *(a) The trigonometric moment of order p for a wrapped Linnik distribution corresponds to the value of the characteristic function of the linear Linnik random variable at the integer value p. (b) The characteristic function of the wrapped Linnik random variable* $\theta$ *at the integer p is*

$$\psi_{WL}(p) = E[\exp(ip(\theta - \mu))] = \exp(ip\mu)(1 + (p\sigma)^\alpha)^{-1}.$$

The probability density function of wrapped Linnik distribution is defined as

$$f(\theta) = \frac{1}{2\pi}[1 + 2\sum_{p=1}^{\infty}((1 + (\sigma p)^\alpha)^{-1}) \cos p(\theta - \mu)], \tag{5}$$

where the parameter space is given by

$$\Omega = \Omega_1 \cup \Omega_2,$$
$$\Omega_1 = \{(\alpha, \sigma, \mu_0) : 1 \leq \alpha \leq 2, \sigma \geq 1, 0 \leq \mu_0 < 2\pi\} \quad \text{and}$$
$$\Omega_2 = \{(\alpha, \sigma, \mu_0) : 1 < \alpha \leq 2, \sigma < 1, 0 \leq \mu_0 < 2\pi\}.$$

We observe that these wrapped distributions preserve the parameter $\alpha$ for the corresponding linear distributions.

Without a loss of generality, we take $\mu = 0$ and $\sigma = 1$ in the following. The index parameter of the circular family of distributions plays an important role in determining the thickness and hence the tail behaviour of the distribution. There are, in fact, four possible names for the parameter $\alpha$. Some interpret it as the tail thickness parameter or the index parameter used to measure tail thickness mainly for heavy tailed distributions. Others interpret it as the characteristic exponent when it is present in an exponential form in a characteristic function. Sometimes, $\alpha$ is also defined as the shape parameter along with its three other companions viz. location parameter $\mu$, scale parameter $\sigma$, and skewness parameter $\beta$. For this paper, we assume the symmetric case that is $\beta = 0$. Several estimators of this parameter have been developed over time.

## 4. Hill Estimator and Its Generalisation

The classical Hill estimator (see Hill 1975; Dufour and Kurz-Kim 2010), is a simple non-parametric estimator based on order statistics. Given a sample of $n$ observations $X_1, X_2, \ldots X_n$ the Hill estimator is defined as

$$\hat{\alpha}_H = \Big[ \big( \frac{1}{k} \sum_{j=1}^{k} \ln X_{n+1-j:n} \big) - \ln X_{n-k:n} \Big]^{-1}$$

with standard error

$$SD(\hat{\alpha}_H) = \frac{k\hat{\alpha}_H}{(k-1)\sqrt{k-2}},$$

where $k$ is the number of observations which lie on the tails of the distribution of interest and is to be optimally chosen depending on the sample size, $n$, and tail thickness $\alpha$, as $k = k(n, \alpha)$ and $X_{j:n}$ denotes the $j$-order statistic of the sample of size $n$.

The asymptotic normality of the classical Hill estimator is provided by Goldie and Smith (1987) as

$$\sqrt{k}(\hat{\alpha_H^{-1}} - \alpha^{-1}) \xrightarrow{L} N(0, \alpha^{-2})$$

which leads to the following lemma

**Lemma 1.**

$$\hat{\alpha_H} - \alpha \xrightarrow{L} N\Big(0, \frac{1}{\alpha^2 k}\Big).$$

This estimator uses the linear function of the order statistics and can be used to estimate $\alpha \in [1, 2]$ only. Further, it is also "extremely sensitive" to the choice of the optimal number of tail observations $k$, which itself is a function of the unknown index parameter $\alpha$ being estimated.

The Hill estimator is scale invariant since it is defined in terms of the log of ratios but not location invariant. Therefore, centering needs to be performed in order to address the location invariance.

The classical Hill estimator is actually the logarithm of the geometric mean or the logarithm of the mean of order $p = 0$ of a set of statistics. This estimator has been generalized to a more general mean of order $p \geq 0$ of the same set of statistics by Brilhante et al. (2013) as follows:

$$\hat{\alpha}_{H_p} = \begin{cases} \frac{\left(1 - A_p^{-p}(k)\right)}{p}, & \text{if } p > 0 \\ \log_e A_0(k) \equiv \hat{\alpha}_H, & \text{if } p = 0, \end{cases}$$

where the class of statistics $A_p(k)$ is taken as the mean of order $p$ of the statistics $U_{ik}$ given by

$$U_{ik} = \frac{X_{n+1-i:n}}{X_{n-k:n}} = \frac{U(Y_{n+1-i:n})}{U(Y_{n-k:n})},$$

where $U(.)$ is the generalized inverse function of the cumulative distribution function $F$ of $X$ and using the distributional identity $X = U(Y)$ with $Y$ as a unit Pareto random variable and

$$A_p(k) = \begin{cases} \left(\frac{\sum_{i=1}^{k} U_{ik}^p}{k}\right)^{1/p}, & \text{if } p > 0 \\ \left(\prod_{i=1}^{k} U_{ik}\right)^{1/k}, & \text{if } p = 0 \end{cases} \quad (6)$$

Under the first order condition that the generalized inverse function $U(.)$ is of regular variation with index $\alpha$, the consistency of the generalized class of Hill estimators $\hat{\alpha}_{H_p}$ is established, provided $p < \frac{1}{\alpha}$. In addition, under the assumption of the second order condition, the asymptotic normality of $\hat{\alpha}_{H_p}$ can also be obtained (see Brilhante et al. 2013) as

$$\hat{\alpha}_{H_p} \equiv^d \alpha + \frac{\sigma_p(\alpha) Z_p(k)}{\sqrt(k)} + b_p(\alpha|\rho) A(n/k) + o_p\Big(A(n/k)\Big),$$

holds for all $p < \frac{1}{2\alpha}$ and $Z_p(k)$ is asymptotically standard normal and

$$\sigma_p(\alpha) = \frac{\alpha(1-p\alpha)}{\sqrt(1-2p\alpha)} \quad \text{and} \quad b_p(\alpha|\rho) = \frac{1-p\alpha}{1-p\alpha-\rho},$$

with $\rho$ being the second-order parameter, controlling the rate of convergence for the first order condition.

## 5. Fractional Moment Estimator

Another alternative estimator of the index parameter $\alpha$ is given by Kozubowski (2001) as the usual method of moment estimator with fractional order. If $x_1, x_2, \ldots, x_n$ are realizations from the symmetric Linnik distribution with index parameter $\alpha$ and scale parameter $\sigma$, then the $p$th absolute moment is

$$e(p) = E|Y|^p = \frac{p(1-p)\sigma^p \pi}{\alpha \Gamma(2-p) \sin(\pi p/\alpha) \cos(\pi p/2)},$$

where $0 < \alpha \leq 2$ and $0 < p < \alpha$. As suggested in Kozubowski (2001), using suitable choices of $p$ as $1/2$ and 1 and solving the respective equations, the fractional moment estimator of the the index parameter $\alpha$ can be obtained. This estimator is valid only for $\alpha > 1$. To overcome this restriction, a universal and efficient estimator for both stable and Linnik distributions will appear in our next works.

If $x_1, x_2, \ldots, x_n$ are realizations from the symmetric stable distribution with index parameter $\alpha$, scale parameter $\sigma$, and location parameter 0, then the $p$th absolute moment given by Kuruoglu (2001) is

$$E|Y|^p = \frac{\Gamma(\left(1-\frac{p}{\alpha}\right))}{\Gamma(1-p)} \frac{|\sigma|^{\frac{p}{\alpha}}}{\cos\left(\frac{p\pi}{2}\right)},$$

where $-1 < p < \alpha$, $p \neq 1$ and $\alpha \neq 1$. Using the method of moments with the corresponding sample moment,

$$A_p = \frac{1}{n} \sum_{i=1}^{n} |X_i|^p$$

and applying the following property of gamma function,

$$\frac{\Gamma(p)}{\Gamma(1-p)} = \frac{\pi}{\sin(p\pi)}, \quad p \neq 1$$

the fractional moment estimator of the index parameter $\alpha$ can be obtained.

## 6. Characteristic Function-Based Estimator

The characteristic function-based estimator of the index parameter of symmetric stable distribution (see Anderson and Arnold 1993) is obtained by the minimization of the objective function (where location parameter $\mu = 0$ and scale parameter $\sigma$ unknown) given by,

$$\hat{I'}_s(\alpha) = \sum_{i=1}^{n} w_i(\hat{\eta}(z_i) - \exp(-|\sigma z_i|^{\alpha}))^2, \quad (6)$$

where

$$\hat{\eta}(t) = \frac{1}{n} \sum_{j=1}^{n} \cos(t x_j), \qquad t \in R$$

and $x_1, x_2, \ldots, x_n$ are realizations from the symmetric stable($\alpha$) distribution with the theoretical characteristic function $\exp(-|\sigma z_i|^{\alpha})$, $z_i$ is the $i$th zero of the $m$th degree Hermite polynomial $H_m(z)$ and

$$w_i = \frac{2^{m-1} m! \sqrt{m}}{(m H_{m-1}(z))^2}.$$

Similarly, the characteristic function-based estimator for that of the symmetric Linnik distribution is obtained by the minimization of the objective function given by

$$I_l(\alpha, \sigma) = \sum_{i=1}^{n} w_i(\hat{\eta}(z_i) - (1 + |\sigma z_i|^{\alpha})^{-1})^2 \quad (7)$$

subject to the constraints, $1 < \alpha \leq 2$ and $\sigma > 0$, where $x_1, x_2, \ldots, x_n$ are realizations from the symmetric Linnik($\alpha$) distribution with the theoretical characteristic function $(1 + |\sigma z_i|^{\alpha})^{-1}$.

This estimator is consistent, as seen by Anderson and Arnold (1993). However, it cannot be obtained explicitly and needs to be obtained by solving the estimating equations in iterative methods such as the L-BFGS-B method used in R software (see Byrd et al. (1995)).

## 7. The Trigonometric Moment Estimator

It is known, in general, by Jammalamadaka and SenGupta (2001) that the characteristic function of $\theta$ at the integer $p$ is defined as,

$$\psi_\theta(p) = E[\exp(ip(\theta - \mu))] = \alpha_p + i\beta_p$$

$$\text{where} \quad \alpha_p = E\cos p(\theta - \mu) \quad \text{and} \quad \beta_p = E\sin p(\theta - \mu).$$

Further by, Jammalamadaka and SenGupta (2001) we know that, for the p.d.f given by (3),

$$\psi_\theta(p) = \rho^{p^\alpha}.$$

$$\text{Hence,} \quad E\cos p(\theta - \mu) = \rho^{p^\alpha} \quad \text{and} \quad E\sin p(\theta - \mu) = 0$$

Suppose $\theta_1, \theta_2, \ldots, \theta_m$ are a random sample of size $m$ drawn from the wrapped stable density given by (3). We define

$$\bar{C_1} = \frac{1}{m}\sum_{i=1}^{m}\cos\theta_i, \quad \bar{C_2} = \frac{1}{m}\sum_{i=1}^{m}\cos 2\theta_i, \quad \bar{S_1} = \frac{1}{m}\sum_{i=1}^{m}\sin\theta_i$$
$$\text{and} \quad \bar{S_2} = \frac{1}{m}\sum_{i=1}^{m}\sin 2\theta_i.$$

Then, we note that $\bar{R}_1 = \sqrt{\bar{C_1}^2 + \bar{S_1}^2}$ and $\bar{R}_2 = \sqrt{\bar{C_2}^2 + \bar{S_2}^2}$.

Using the method of trigonometric moments estimation, and equating $\bar{R}_1$ and $\bar{R}_2$ to the corresponding functions of the theoretical trigonometric moments, we get the estimator of the index parameter $\alpha$ of wrapped stable distribution (see SenGupta 1996):

$$\hat{\alpha_{WS}} = \frac{1}{\ln 2}\ln\frac{\ln \bar{R}_2}{\ln \bar{R}_1}.$$

Now, suppose $\theta_1, \theta_2, \ldots, \theta_m$ are a random sample of size $m$ drawn from the wrapped Linnik density given by (5). Using the method of trigonometric moments estimation, and equating the empirical trigonometric moments $\bar{R}_1$ and $\bar{R}_2$ to the corresponding theoretical moments, we get the estimator of index parameter $\alpha$ of wrapped Linnik distribution (as obtained for the wrapped stable distribution by SenGupta 1996),

$$\hat{\alpha_{WL}} = \frac{\ln[((1/\bar{R}_1 - 1)/(1/\bar{R}_2 - 1))]}{\ln(1/2)},$$

where $\bar{R}_j = \frac{1}{m}\sum_{i=1}^{m}\cos j(\theta_i - \bar{\theta})$, $j = 1,2$ and $\bar{\theta}$ is the mean direction given by $\bar{\theta} = \arctan\left(\frac{\bar{S_1}}{\bar{C_1}}\right)$. Note that $\bar{R}_1 \equiv \bar{R}$.

The asymptotic normality of the estimators $\hat{\alpha_{WS}}$ and $\hat{\alpha_{WL}}$ have been established in the following Theorems 3 and 4 respectively (see SenGupta and Roy 2019, 2023).

**Theorem 3.**
$$\sqrt{m}(\hat{\alpha_{WS}} - \alpha) \xrightarrow{L} N(0, \gamma'\Sigma\gamma),$$

*where*

$$\gamma = \frac{1}{\ln 2}\left(\frac{-\cos\mu_0}{\rho\ln\rho}, \frac{\cos 2\mu_0}{\rho^{2^\alpha}\ln\rho^{2^\alpha}}, \frac{-\sin\mu_0}{\rho\ln\rho}, \frac{\sin 2\mu_0}{\rho^{2^\alpha}\ln\rho^{2^\alpha}}\right)'$$

*and*

$$\underline{\gamma}'\Sigma\underline{\gamma} = \frac{1}{(\ln 2)^2}\left[\frac{1+\rho^{2^\alpha} - 2\rho^2}{2(\rho\ln\rho)^2} + \frac{1+\rho^{4^\alpha} - 2(\rho^{2^\alpha})^2}{2(\rho^{2^\alpha}\ln\rho^{2^\alpha})^2} + \frac{2\rho^{2^\alpha+1} - \rho - \rho^{3^\alpha}}{\rho\ln\rho\rho^{2^\alpha}\ln\rho^{2^\alpha}}\right].$$

**Theorem 4.** $\sqrt{m}(\hat{\alpha_{WL}} - \alpha) \xrightarrow{L} N(0, \gamma'\Sigma\gamma)$, *where*

$$\gamma = \frac{1}{\ln(1/2)}\begin{pmatrix} \frac{-\cos\mu_0(1+(\sigma)^\alpha)^2}{(\sigma)^\alpha} \\ \frac{\cos 2\mu_0(1+(2\sigma)^\alpha)^2}{(2\sigma)^\alpha} \\ \frac{-\sin\mu_0(1+(\sigma)^\alpha)^2}{(\sigma)^\alpha} \\ \frac{\sin 2\mu_0(1+(2\sigma)^\alpha)^2}{(2\sigma)^\alpha} \end{pmatrix} \text{ and}$$

$$\underline{\gamma}'\Sigma\underline{\gamma} = \frac{1}{(\ln(1/2))^2}\Bigg[-\frac{\cos^2 2\mu_0(1+\sigma^\alpha)(1+(2\sigma)^\alpha)^2}{2^\alpha\sigma^{2\alpha}} + \frac{\cos^2 2\mu_0(1+\sigma^\alpha)(1+(2\sigma)^\alpha)}{2^\alpha\sigma^{2\alpha}}$$
$$+ \frac{(1+(2\sigma)^\alpha)^4}{(2\sigma)^{2\alpha}} - \frac{(1+(2\sigma)^\alpha)^2}{(2\sigma)^{2\alpha}} - \frac{\sin^2 2\mu_0(1+\sigma^\alpha)(1+(2\sigma)^\alpha)^2}{2^\alpha\sigma^{2\alpha}}$$
$$+ \frac{\cos 3\mu_0 \sin\mu_0 \sin 2\mu_0(1+\sigma^\alpha)^2(1+(2\sigma)^\alpha)^2}{2^\alpha\sigma^{2\alpha}(1+(3\sigma)^\alpha)} - \frac{(1+(\sigma)^\alpha)^2}{(\sigma)^{2\alpha}} + \frac{(1+\sigma^\alpha)(1+(2\sigma)^\alpha)}{2^\alpha\sigma^{2\alpha}}$$
$$+ \frac{\sin^2\mu_0(1+\sigma^\alpha)(1+(2\sigma)^\alpha)}{2^\alpha\sigma^{2\alpha}} - \frac{\cos\mu_0\cos 2\mu_0\cos 3\mu_0(1+\sigma^\alpha)^2(1+(2\sigma)^\alpha)^2}{2^\alpha\sigma^{2\alpha}(1+(3\sigma)^\alpha)}$$
$$- \frac{3\cos\mu_0 \sin 2\mu_0 \sin 3\mu_0(1+\sigma^\alpha)^2(1+(2\sigma)^\alpha)^2}{2^{\alpha+1}\sigma^{2\alpha}(1+(3\sigma)^\alpha)} + \frac{\cos 2\mu_0(1+\sigma^\alpha)^4}{2\sigma^{2\alpha}} + \frac{\sin^2 2\mu_0(1+\sigma^\alpha)^4}{2\sigma^{2\alpha}(1+(2\sigma)^\alpha)}$$
$$+ \frac{\cos 2\mu_0(1+\sigma^\alpha)^4}{2\sigma^{2\alpha}(1+(2\sigma)^\alpha)} - \frac{\sin\mu_0\sin 3\mu_0\cos 2\mu_0(1+\sigma^\alpha)^2(1+(2\sigma)^\alpha)^2}{2^\alpha\sigma^{2\alpha}(1+(3\sigma)^\alpha)}\Bigg].$$

Where $m$ denotes the sample size and $\Sigma$ denotes the dispersion matrix of $\left(\bar{C_1}, \bar{S_1}, \bar{C_2}, \bar{S_2}\right)$ in both the above theorems.

Unlike for the previous estimators where at the most simulation results were given for the properties of the estimators, the asymptotic distributions obtained in the Theorems 3 and 4 establish rigorously the theoretical and the analytical properties of the trigonometric moment estimators. The estimators can be shown to be consistent and asymptotically normal(CAN) through the use of the theorems. Additionally, the usefulness of the theorems is to provide a methodology to rigorously test for the index parameter $\alpha$ which is illustrated in Section 11.

## 8. The Truncated Trigonometric Moment Estimator

The moment estimators $\alpha_{\hat{WS}}$ and $\alpha_{\hat{WL}}$ need not always remain in the support of the true parameter $\alpha$ (that is (0,2]). Hence, the moment estimators proposed above need not be proper estimators of $\alpha$. Hence, the modified estimators for wrapped stable and wrapped Linnik distribution free from this defect are, respectively, given by

$$\alpha_{WS}^{\hat{ttm}} = \begin{cases} \alpha_{\hat{WS}} & \text{if } 0 < \alpha_{\hat{WS}} < 2 \\ 2 & \text{if } \alpha_{\hat{WS}} \geq 2 \end{cases}$$

and

$$\alpha_{WL}^{\hat{ttm}} = \begin{cases} 1 & \text{if } \alpha_{\hat{WL}} \leq 1 \\ \hat{\alpha} & \text{if } 1 < \alpha_{\hat{WL}} < 2 \\ 2 & \text{if } \alpha_{\hat{WL}} \geq 2 \end{cases}$$

(since the support of $\alpha$ excludes non-positive values).

The asymptotic normality of the modified truncated estimators $\alpha_{WS}^{\hat{ttm}}$ and $\alpha_{WL}^{\hat{ttm}}$ are established, respectively, in the following theorems (see SenGupta and Roy 2019, 2023). We have

**Theorem 5.**

$$(\alpha_{WS}^{\hat{ttm}} - \alpha) \xrightarrow{L} N(0, V(\alpha_{WS}^{\hat{ttm}}))$$

*where* $V(\alpha_{WS}^{\hat{ttm}}) = E({\alpha_{WS}^{\hat{ttm}}}^2) - \alpha^2$
*where* $E({\alpha_{WS}^{\hat{ttm}}}^2) = \sigma^2\Big[\{a^*\phi(a^*) - b^*\phi(b^*) + \Phi(b^*) - \Phi(a^*)\}\Big] + \alpha^2\{\Phi(b^*) - \Phi(a^*)\} + 2\alpha\sigma\{\phi(a^*) - \phi(b^*)\}$
*where* $a^* = \frac{-\alpha}{\sqrt{\frac{\underline{\gamma}'\Sigma\underline{\gamma}}{m}}}$ *and* $b^* = \frac{2-\alpha}{\sqrt{\frac{\underline{\gamma}'\Sigma\underline{\gamma}}{m}}}$

**Theorem 6.**

$$(\alpha_{WL}^{\hat{ftm}} - \alpha) \xrightarrow{L} N(0, V(\alpha_{WL}^{\hat{ftm}}))$$

*where* $V(\alpha_{WL}^{\hat{ftm}}) = E({\alpha_{WL}^{\hat{ftm}}}^2) - \alpha^2$

*where* $E({\alpha_{WL}^{\hat{ftm}}}^2) = \Phi(a^*) + \sigma^2 \left[ \{a^*\phi(a^*) - b^*\phi(b^*) + \Phi(b^*) - \Phi(a^*)\} \right] + \alpha^2\{\Phi(b^*) - \Phi(a^*)\}$
$+2\alpha\sigma\{\phi(a^*) - \phi(b^*)\} + 4.[1 - \Phi(b^*)]$

*where* $a^* = \frac{1-\alpha}{\sqrt{\frac{\gamma'\Sigma\gamma}{m}}}$ *and* $b^* = \frac{2-\alpha}{\sqrt{\frac{\gamma'\Sigma\gamma}{m}}}$

$\sigma = \sqrt{\frac{\gamma'\Sigma\gamma}{m}}$

In both the above theorems, $\phi(.)$ and $\Phi(.)$ denote the p.d.f and c.d.f of a standard normal variable respectively.

## 9. Efficiency of the Estimators

It is naturally of interest to see how close these estimators are. Here, we briefly discuss this aspect with an empirical sample. The raw financial data can be transformed into circular data by using the method of wrapping (see, e.g., page 31 of Jammalamadaka and SenGupta (2001)). That is, for positive (linear) values, after dividing by $2\pi$, we take the remainder, while for negative (linear) values, we add $2\pi$ to the remainder to produce the corresponding circular values in $(0,2\pi]$. The fractional moment estimator, as suggested by Kozubowski (2001), for the Linnik distribution is valid when $\alpha > 1$ and that, for wrapped stable distribution, as suggested by Kuruoglu (2001), needs iterative techniques. The properties of this estimator also need to be studied. The efficiency of the estimators obtained using the four methods has been carried out, as suggested by the referees, by including the estimated bias (through the mean bias) and the standard errors (through the root mean square errors) of the estimators in Tables 1a,b and 2a,b. A comparison of the performance of the truncated trigonometric moment estimator $\alpha_{WS}^{\hat{ftm}}$ is made with that of the characteristic function-based estimator $\alpha_{WS}^{\hat{cf}}$ of $\alpha$ of wrapped stable distribution based on their mean bias and root mean square errors (RMSEs) for moderate sample sizes in Table 1a,b. In Table 1a,b, a simulation is performed for the values of $\alpha_{WS}^{\hat{ftm}}$ and $\alpha_{WS}^{\hat{cf}}$, each with sample size $n = 30, 50, 80$ and 100 when the skewness parameter $\beta = 0$. For each sample size $n$, 1000 replications are made. A similar simulation is performed in Table 2a,b for a comparison of the performance of the estimators of $\alpha$ of the wrapped Linnik distribution. It can be observed from Tables 1a,b and 2a,b that the mean bias and the root mean square error of the truncated trigonometric moment estimator of $\alpha$ is less than that of the characteristic function-based estimator for most sample sizes, indicating the efficiency of the former over the latter.

**Table 1.** (a) Data 1: Estimated bias (mean bias) and estimated standard error (RMSE) of the estimator of $\alpha$ of wrapped stable distribution. (b) Data 2: Estimated bias (mean bias) and estimated standard error (RMSE) of the estimator of $\alpha$ of wrapped stable distribution.

**(a) Data 1**

| Sample Size | Mean Bias ($\alpha_{WS}^{\hat{ftm}}$) | Mean Bias ($\alpha_{WS}^{\hat{cf}}$) | RMSE ($\alpha_{WS}^{\hat{ftm}}$) | RMSE ($\alpha_{WS}^{\hat{cf}}$) |
|---|---|---|---|---|
| 30 | 0.175 | 0.383 | 0.498 | 0.6697 |
| 50 | 0.1215 | 0.429 | 0.4286 | 0.667 |
| 80 | 0.014 | 0.457 | 0.363 | 0.656 |
| 100 | 0.029 | 0.478 | 0.3475 | 0.650 |

**Table 1.** *Cont.*

| (b) Data 2 | | | | |
|---|---|---|---|---|
| Sample Size | Mean Bias ($\hat{\alpha_{WS}^{ftm}}$) | Mean Bias ($\hat{\alpha_{WS}^{cf}}$) | RMSE ($\hat{\alpha_{WS}^{ftm}}$) | RMSE ($\hat{\alpha_{WS}^{cf}}$) |
| 30 | 0.009 | 1.087 | 0.267 | 1.341 |
| 50 | 0.179 | 1.138 | 0.438 | 1.353 |
| 80 | 0.128 | 1.225 | 0.552 | 1.384 |
| 100 | 0.042 | 1.236 | 0.141 | 1.389 |

**Table 2.** (a) Data 1: Estimated bias (mean bias) and estimated standard error (RMSE) of the estimator of $\alpha$ of wrapped Linnik distribution; (b) Data 2: Estimated bias (mean bias) and estimated standard error (RMSE) of the estimator of $\alpha$ of wrapped Linnik distribution.

| (a) Data 1 | | | | |
|---|---|---|---|---|
| Sample Size | Mean Bias ($\hat{\alpha_{WL}^{ftm}}$) | Mean Bias ($\hat{\alpha_{WL}^{cf}}$) | RMSE ($\hat{\alpha_{WL}^{ftm}}$) | RMSE ($\hat{\alpha_{WL}^{cf}}$) |
| 30 | 0.491 | 0.287 | 0.812 | 0.583 |
| 50 | 0.058 | 0.215 | 0.058 | 0.482 |
| 80 | 0.190 | 0.201 | 0.396 | 0.451 |
| 100 | 0.191 | 0.188 | 0.392 | 0.425 |
| **(b) Data 1** | | | | |
| Sample Size | Mean Bias ($\hat{\alpha_{WL}^{ftm}}$) | Mean Bias ($\hat{\alpha_{WL}^{cf}}$) | RMSE ($\hat{\alpha_{WL}^{ftm}}$) | RMSE ($\hat{\alpha_{WL}^{cf}}$) |
| 30 | 0.085 | 0.478 | 0.641 | 0.682 |
| 50 | 0.034 | 0.483 | 0.565 | 0.664 |
| 80 | 0.017 | 0.519 | 0.478 | 0.664 |
| 100 | 0.013 | 0.552 | 0.428 | 0.666 |

## 10. Examples

In this section, we consider the wrapped stable and the wrapped Linnik densities as possible underlying models of the financial data, on the Box–Jenkins common stock closing price data of IBM taken from Box et al. (1976), with the characteristic function estimate and the truncated trigonometric moment estimate, respectively. Further financial data considered in this section, as an example, are the gold price data which were collected per ounce in US dollars over the years 1980–2008. Gold is an important asset to mankind and is hence important in financial market. Aggarwal and Lucey (2007) have suggested some statistical procedures which provide the existence of psychological barriers in daily gold prices and also in change of gold prices from day to day. The prices, being in round numbers, present an obstacle with important effects on the conditional mean and variance of the gold price series around psychological barriers. Mills (2004) studied the properties of the daily gold price from 1971 to 2002 and found them to be characterised by the presence of autocorrelation, volatility and 15-day scaling. The distribution of daily returns of gold is highly leptokurtic and multi-period returns attain normality only after 235 days. Byström (2020) studied the link between happiness and gold price changes. He observed that there is no significant correlation between happiness and gold price changes. However, assuming the tails of the happiness distribution to be non-normal, the gold price change seems to increase particularly on a person's extremely unhappy days. However, the log returns (as in the analysis of stock data by Anderson and Arnold (1993)) data of the Indian gold market that we present here exhibit mild asymmetry, pronounced platykurtic and quite small first-order autocorrelation properties, which motivated us to study the symmetric Linnik distribution as an initial approximation of its distribution. The analysis of stock price data is generally carried out on a difference of order 1 in relation to the original series. So, denoting the original stock price data by $x_t$, they undergo transformation as

$z_t = 100(\ln(x_t) - \ln(x_{t-1}))$ which is then wrapped by the process as mentioned above. This transformation of log returns aims to achieve symmetry and reduce autocorrelation in the transformed series (for details, refer to SenGupta and Roy 2019, 2023). The Box–Jenkins data are denoted as data set 1, and the gold price data as data set 2, in the given tables. The computed estimates of $\alpha$ are shown in Table 3. Note that the values of the estimators $\hat{\alpha}$ by these two methods are quite different for each of the probability models. The values of the estimators are not comparable between the two families of distributions. However, within each family they determine a specific distribution. For example, an estimate of $\alpha$ close to 1 indicates a Cauchy (wrapped Cauchy) distribution in the family of stable (wrapped stable) distributions, while an estimate of $\alpha$ close to 2 indicates a Laplace (wrapped Laplace) in the family of Linnik (wrapped Linnik) distributions. With real life data sets, the use of these estimators can lead to quite different, possibly even contradictory, conclusions.

It can be observed from Figures 1 and 2 that the distribution of the log returns of the Box–Jenkins data is, while that of the gold price data is approximately symmetric with a certain amount of left skewness, whereas the gold price data are highly skewed in nature and the Box–Jenkins price data are bimodal. Still, we have used both the gold price and Box–Jenkins log return data sets as illustrations for our proposed estimators, as well as to explore their properties. We also note that both the methods of estimation based on trigonometric moments and characteristic function are not applicable to the two price data sets, since the underlying assumptions of the model are violated by the data sets.

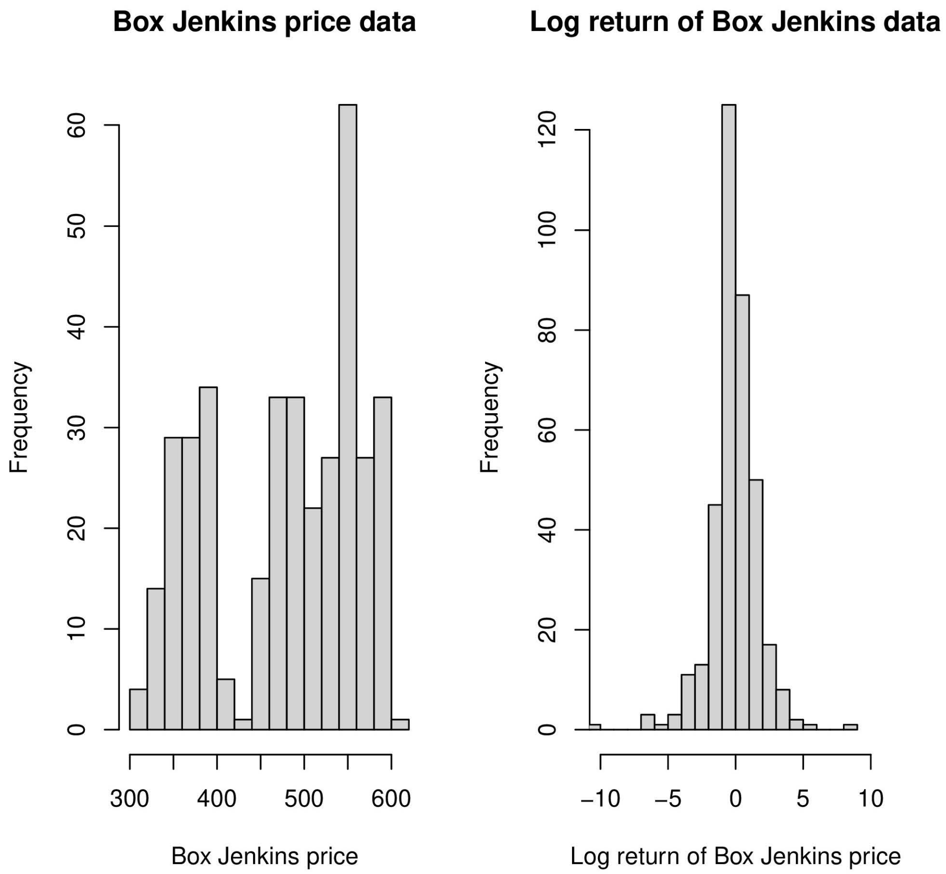


**Figure 1.** Histograms of Box–Jenkins price data and their logarithm return data.

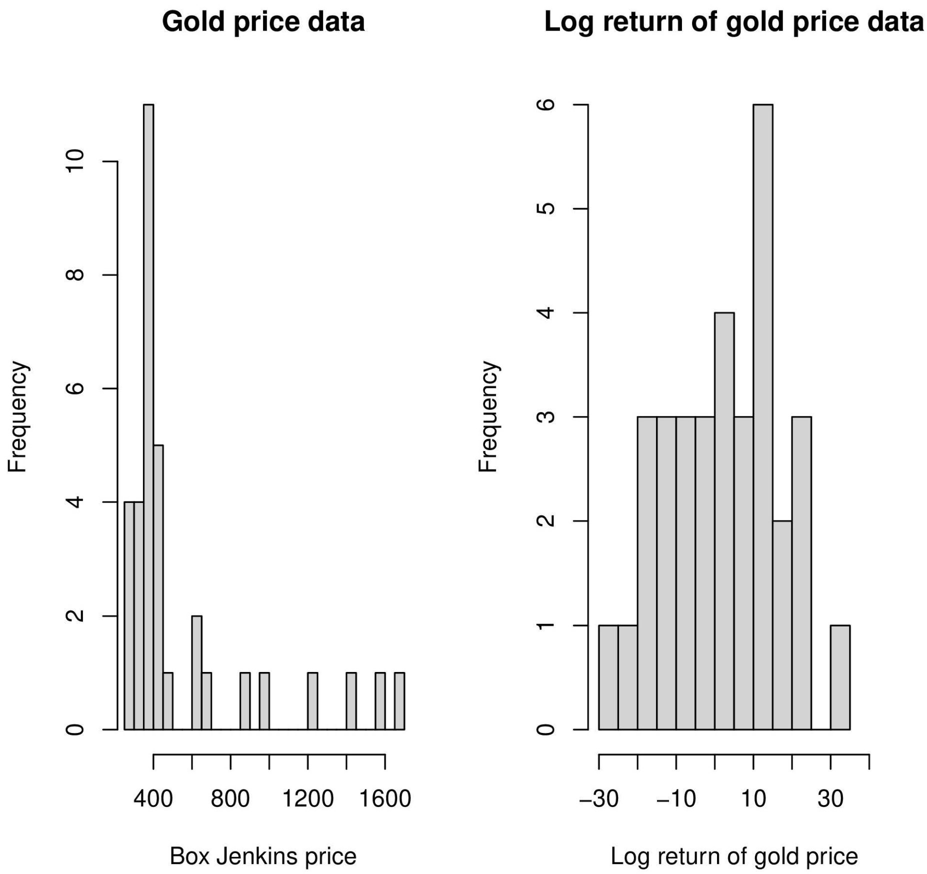


**Figure 2.** Histograms of gold price data and their logarithm return data.

**Table 3.** The estimates of $\alpha$.

| Data | $\alpha_{WS}^{\hat{ftm}}$ | $\alpha_{WL}^{\hat{ftm}}$ | $\alpha_{WS}^{\hat{cf}}$ | $\alpha_{WL}^{\hat{cf}}$ |
|---|---|---|---|---|
| 1 | 1.102854 | 1.941821 | 1.27487 | 2.0 |
| 2 | 0.3752206 | 1.263993 | 0.4149459 | 2.0 |

It can be observed from Table 3 that both the estimators are quite close to each other for the Box–Jenkins log return data, since they are symmetric in nature. The two estimators for the log return of the gold price data do not differ for wrapped stable distribution, but there seems to be an appreciable difference for wrapped Linnik distribution due to their differences in robustness against the asymmetric nature (e.g., the estimator of the location parameter by the mean and median give similar values for symmetric distribution but do not for asymmetric or skewed distribution, due to the difference in the robustness properties of the estimators). Thus, it is necessary that the assumptions of the symmetry of and independence in the data sets be verified in order to produce good estimates of the parameter by our proposed estimators as above.

## 11. Novel Tests for $\alpha$ Based on Circular Statistics

We are presenting here, to the best of our knowledge, the maiden attempt of testing for the index parameter of stable and Linnik distributions. Let $x_1, x_2, \ldots, x_n$ be realizations of symmetric stable ($\mu = 0, \sigma = 1, \alpha$) distribution. The choice of $\mu = 0$ and $\sigma = 1$ are justified, as given in Section 3. When the sample size $n$ is large, we can use the asymptotic distribution of $\alpha_{WS}^{\hat{ftm}}$, as stated in Theorem 5, to perform the test for the null hypothesis $H_0 : \alpha = \alpha_0$. Also, since the data have undergone logarithm ratio transformation, they are thus scale invariant and hence we can take the scale parameter $\sigma = 1$ in the expression of the estimator of the variance, $\widehat{V(\alpha_{WS}^{\hat{t}\hat{t}m})}$ to perform the test. Thus, the test statistics are given by

$$\frac{\alpha_{WS}^{\hat{t}\hat{t}m} - \alpha_0}{\sqrt{\widehat{V(\alpha_{WS}^{\hat{t}\hat{t}m})}}} \longrightarrow N(0,1)$$

where $\alpha_{WS}^{\hat{t}\hat{t}m}$ denotes the trigonometric truncated moment estimator of $\alpha$ for the data, assuming a stable distribution.

Let $x_1, x_2, \ldots, x_n$ be realizations of symmetric Linnik ($\mu = 0, \sigma = 1, \alpha$) distribution. When the sample size $n$ is large, we can use the asymptotic distribution of $\alpha_{WL}^{\hat{t}\hat{t}m}$, as stated in Theorem 6, to perform the test for the null hypothesis $H_0 : \alpha = \alpha_0$. Also, since the data have undergone logarithm ratio transformation, they are thu scale invariant and hence we can take the scale parameter $\sigma = 1$ in the expression of estimator of the variance, $\widehat{V(\alpha_{WL}^{\hat{t}\hat{t}m})}$ to perform the test. Thus, the test statistics are given by

$$\frac{\alpha_{WL}^{\hat{t}\hat{t}m} - \alpha_0}{\sqrt{\widehat{V(\alpha_{WL}^{\hat{t}\hat{t}m})}}} \longrightarrow N(0,1)$$

where $\alpha_{WL}^{\hat{t}\hat{t}m}$ denotes the trigonometric truncated moment estimator of $\alpha$ for the data assuming the Linnik distribution. Depending on the alternative hypothesis, the cut-off points of the tests can be determined from standard normal distribution tables.

A similar test can also be carried out based on a Hill estimator using Lemma 1, but it is not studied here because the determination of $k$ is complicated.

**Example:**

Anderson and Arnold (1993) have suggested the Linnik distribution for the financial data on Box–Jenkins based on their characteristic function-based method of estimation. We assume that the data come from a member of the Linnik family. In this family, the Linnik distribution is characterized by $\alpha = 2$. This has motivated us to rigorously verify their claim based on the corresponding test $H_0 : \alpha = 2$ against the alternative hypothesis $H_1 : \alpha < 2$. As per the suggestions of the referee, we perform a test for Laplace (a.k.a. double exponential) distribution corresponding to $\alpha = 2$ in the family of Linnik distributions. The test statistics as defined above are given by

$$\frac{\alpha_{WL}^{\hat{t}\hat{t}m} - 2}{\sqrt{\widehat{V(\alpha_{WL}^{\hat{t}\hat{t}m})}}}$$

The value of the test statistic is obtained as $-0.3456217$, implying that the null hypothesis of the claim of double exponential distribution is accepted both at the $5\%(1.645)$ and $1\%(2.326)$ levels of significance.

The Laplace distribution has been earlier used on an adhoc basis by Anderson and Arnold (1993) for the financial data on Box–Jenkins based on results of estimation. We have established it formally by providing rigorous proof through testing procedure which supports their findings.

## 12. Discussions and Conclusions

We have obtained a universal and efficient estimator of $\alpha$ which can be easily implemented in practice. We have studied the various properties of the estimators, pointed out their drawbacks and also obtained improved estimators eliminating these drawbacks. We have also compared the efficiency of some estimators, as observed in the above Tables 1 and 2. We have also introduced a novel method of testing for the index parameter of the stable and Linnik distributions. We thus hope that this maiden attempt will be useful for future analysis.

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