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Daily and Weekly Geometric Brownian Motion Stock Index Forecasts

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Abstract: In this manuscript, daily and weekly geometric Brownian motion forecasts are obtained and tested for reliability for three indexes, DJIA, NASDAQ and S&P 500. A twenty-year rolling window is used to estimate the drift and diffusion components, and applied to obtain one-period-ahead geometric Brownian motion index values and associated probabilities. Expected values are estimated by totaling up the product of the index value and its associated probabilities, and test for reliability. The results indicate that geometric Brownian-simulated expected index values estimated using one thousand simulations can be reliable forecasts of the actual index values. Expected values estimated using one or ten simulations are not as reliable, while those obtained using at least one hundred simulations could be useful.

Keywords: stock index; DJIA; S&P 500; NASDAQ; geometric Brownian motion; daily and weekly forecasts; simulation; rolling window; probability; expected value

JEL Classification: C6; G1

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1. Introduction

Continuing research in Finance (Cowles 1933; Fama 1965a, 1965b, 1970; Lim and Luo 2012; Malkiel and Fama 1970) provides the theoretical foundation to obtaining accurate forecasts of future stock index values, especially if they are considered to follow a martingale process (Campbell et al. 1997; Fama 1965b). A number of different innovative and sophisticated techniques have been used to model and forecast future stock indexes. A few of these are (a) complementary ensemble empirical mode decomposition method (Chen and Pan 2016); (b) price-reversion model (Chiang-Lin et al. 2022), machine learning (Demirel et al. 2021; Pabuçcu 2019; Tang et al. 2023); (c) neural networks (Fatima and Uddin 2022; Kamalov 2020; Qiu and Song 2016); (d) locally stationary wavelets (Fryzlewicz 2005; Nason et al. 2000); (e) fuzzy time-series modeling (Tsai et al. 2019); (f) parameterized non-linear ordinary differential equations and dynamic system (Li et al. 2018); (g) genetic algorithms (Abraham et al. 2022; Pimenta et al. 2018); (h) times-series forecasting (Chávez et al. 2023; Idrees et al. 2019); and (i) geometric Brownian motion (GBM) (Alhagyan 2024; Farida et al. 2018; Karimov 2017; Kundu 2021; Maruddani and Trimono 2018; Reddy and Clinton 2016; Sinha 2024c), to name a few. This manuscript adds to the literature by applying the GBM approach to obtain and test the accuracy of daily and weekly forecasts of the Down Jones Industrial Average (DJIA), the NASDAQ Composite (NASDAQ), and the S&P 500 index values. The results indicate that accurate and reliable forecasts can be obtained using the GBM-based methodology that requires estimating expected values from numerous simulations (Sinha 2021, 2024a, 2024b, 2024d).

$$\frac{dP}{P} = \mu dt + \sigma dW_t \tag{1}$$

Equation (1) presents the stochastic differential equation commonly used to represent GBM (Benninga 2014; Musiela and Rutkowski 2005; Navin 2007; Sengupta 2004). In this equation, μ represents the drift term; σ is the diffusion term; and the standard normal random Wiener variable is represented by dW_t, with the Wiener term expected to have a standard deviation of one, and an average of zero. The variable P represents the stock price, and dt is the change in time.

An implementable closed-form solution of GBM used in numerous manuscripts (Benninga 2014; Benninga and Mofkadi 2021; Hull 2018; Reddy and Clinton 2016; Sinha 2021, 2024a, 2024b, 2024d) and also in this manuscript is presented in Equation (2). In this equation, $P_{0,i}$ is the observed index values at time 0 for an index, while $P_{sim,t,i}$ is the GBM-simulated value after time period Δt. The drift and diffusion terms represented by μ and σ are usually estimated as the historical mean and standard deviation of the returns (Alhagyan 2024; Benninga 2014; Benninga and Mofkadi 2021; Estember and Maraña 2016; Hersugondo et al. 2022; Sengupta 2004; Urama and Ezepue 2018) of the index. ε is a standard normal random variable with a mean of zero and a standard deviation of one.

$$P_{sim,t,i} = P_{0,i} e^{((\mu_{t,i} - \frac{1}{2}\sigma_{t,i}^2)\Delta t + (\sigma_{t,i}\varepsilon\sqrt{\Delta t}))} \tag{2}$$

The biologist Robert Brown discovered Brownian motion in 1827 (Maruddani and Trimono 2018), and mathematical solutions were provided by Norbert Weiner and Einstein (1905), while Bachelier (1900) has been widely credited to be the first to use GBM to price assets. The GBM literature has grown to be a popular methodological procedure, resulting in its own extremely vast and exhaustive literature, with many authors making significant contributions to the literature, a few of which are briefly discussed here. Kundu (2021) applied the GBM technique to the Nifty 50, an index from the National Stock Market Index of India. He applied the technique to daily closings for the period between 13 July 2020 and 9 June 2021 for a total of two hundred and twenty-six days. Maruddani and Trimono (2018) forecasted prices of two stocks from Indonesia in a portfolio using a multi-dimensional GBM approach for the period between 4 January 2016 and 21 April 2017. Farida et al. (2018) applied their GBM model to daily closing prices of stocks that are part of the Jakarta Composite Index for the period between January 2014 and December 2014. Hersugondo et al. (2022) applied the GBM and Value at Risk to predict the Jakarta Islamic Index, and index comprised of Sharia stocks from 16 August 2021 to 23 August 2021. Prasad et al. (2022) accurately simulated stock prices from the Bombay Stock Exchange. Nordin et al. (2024) applied the GBM to the five ASEAN countries for the period between 2017 and 2022. Reddy and Clinton (2016) compared GBM-simulated daily prices to actual prices of Australian stocks listed on the S&P/ASX 50 Index for the period between 1 January 2013 and 31 December 2014. They estimated the drift term as the expected return by applying stock beta by applying the Capital Asset Pricing Model framework, as well as using historical standard deviation. They found simulated prices moving in the same directions at least half of the times in their sample period. Brătian et al. (2022) applied the GBM approach to the GBM framework to crisis and non-crisis financial situation and found the application to make suitable stock index forecasts.

Besides for stocks and stock indexes, GBM has been used to model prices of a number of different assets. Hamdan et al. (2020), Roslan and Halim (2024), and Sinha (2024a) estimated applied GBM to gold prices, while Germansah et al. (2023) tested GBM's forecastability of gold derivatives. Al-Harthy (2007), Alhagyan (2024), and Croghan et al. (2017) applied GBM to energy prices. The prices of crude palm oil using GBM were researched by Ibrahim et al. (2021). Ramos et al. (2019) attempted to forecast the prices of iron ore using GBM. Abbas and Alhagyan (2023) and Alhagyan (2022) applied GBM to exchange-rate forecasting. Summarizing all the literature on GBM is beyond the scope of this manuscript, but the works of Abidin and Jaffar (2014), Hoyyi et al. (2019), Kayal and Maheswaran (2018), Kayal and Mondal (2020), Ladde and Wu (2009), Liu et al. (2020), Paluszek and Thomas (2020), Shafii et al. (2019), and Tie et al. (2018) should be mentioned, among numerous other manuscripts.

One of the major shortcomings of the GBM modeling is the use of insufficient iterations (Kumar et al. 2024; Sinha 2021). Each independently generated value of ε in Equation (2) results in a simulated value of $P_{sim,t,i}$. However, as ε is a random variable, numerous GBM-simulated values can be obtained, as its value changes idiosyncratically within the requirement of its value having an average of zero and a standard deviation of one. Using this logic, Sinha (2021, 2024d) simulated values of S&P 500 for a number of simulations, ranging from one to one hundred thousand simulations, and estimated an expected value from those at the monthly, quarterly, and yearly frequencies. Sinha (2021) found that, as the number of simulations increases, the average sum of the simulated normal distribution probabilities increases to over 99% as the number of simulations increases to one thousand simulations. Sinha (2024d) explore this further and statistically tested the reliability of the GBM-based expected values for DJIA, NASDAQ, and S&P 500 index values, and found that reliable GBM values can be obtained by using one thousand simulations. Sinha (2024d) differs from Sinha (2021) in regard to how the drift and diffusion terms are estimated, and like Sinha's work in (2024d), this manuscript also differs from (2021) as in regard to how the drift and diffusion are calculated. Given Sinha's (2021, 2024d) findings, expected value are estimated using one thousand simulations in this manuscript. As part of robustness checks, one, ten, fifty, and one hundred simulations are also considered. This manuscript differs from Sinha (2021) and (Sinha 2024d) as the GBM process is applied to make daily and weekly forecasts as opposed to monthly, quarterly, and annual predictions. Sinha (2021) also only tested the S&P 500 index values. This manuscript differs from Sinha (2021) in how the drift and diffusion terms are calculated as well. In Sinha (2021), the drift and diffusion terms were estimated using the regression model proposed by Marquering and Verbeek (2004). Like Sinha (2024d) and Sinha (2024b), the historical mean and standard deviation of log returns are used as proxies for the drift and diffusion terms.

Summarizing, this manuscript contributes to the literature in Finance, as it shows how the GBM methodological approach can be used to obtain reliable forecasts of stock indices at the daily and weekly frequencies. The methodological innovation in this manuscript lies in the fact that it uses the GBM simulation to obtain numerous forecasts, and from these forecasts and their associated probabilities, we can estimate an expected value, and applies it to the daily and weekly frequencies. The expected values are reliable and accurate forecasts of next-day or next-week actual values, as the case maybe. The results of this manuscript also confirm that if expected values are based on a single or a few simulations, then those forecasts are not very reliable or accurate. The results in this manuscript indicate that GBM-based expected-value estimates based on at least one hundred simulations may be useful, but those based on one thousand forecasts may be accurate and reliable.

In this manuscript, Section 2 provides an overview of the data used. Section 3 is the methodology, while Section 4 discusses the results. Section 5 is the conclusion.

2. Data

The data for the manuscript primarily consists of daily and weekly index values for three stock indexes: Dow Jones Industrial Average (DJIA), NASDAQ Composite (NASDAQ), and S&P 500 from 30 December 1983 to 29 December 2023. The data consists of index values for only the actual trading days between these two dates. The data extracted from DataStream was facilitated by Bradley University's subscription to Refinitiv Workspace[1]. The data obtained through this subscription are not publicly available; however, daily index values for DJIA, NASDAQ, and S&P 500 are available for free through websites like Yahoo Finance, Google Finance, and MSN Finance, to name a few such sites. The daily returns were calculated by taking the natural logarithms of adjacent index values using the formula shown in Equation (3). In this equation, r is the log-return; i refers to the three indexes, DJIA, NASDAQ, and S&P 500; and P_t and P_{t-1} are adjacent prices, with P_{t-1} occurring earlier than P_t.

$$r_i = ln\left(\frac{P_{i,t}}{P_{i,t-1}}\right) \tag{3}$$

Table 1 provides descriptive statistics. This table contains the mean return (μ) and standard deviation (σ) of the daily returns estimated using Equation (3) at both the daily and weekly frequency. The formulae used for them are shown in Equations (4) and (5), respectively. These are standard formulae that are normally used for estimating averages and standard deviations. The table also provides the largest daily and weekly increases and decreases.

$$\mu_i = \frac{\sum r_i}{n} \tag{4}$$

$$\sigma_i = \sqrt{\frac{\sum (r_i - \overline{r_i})^2}{n-1}} \tag{5}$$

Table 1. Descriptive statistics.

	DJIA	**NASDAQ**	**S&P 500**
		Daily Frequency	
Mean return	0.03%	0.04%	0.03%
Min return	−25.63%	−13.15%	−22.90%
Max return	10.76%	13.25%	10.96%
Standard deviation	1.13%	1.39%	1.15%
		Weekly Returns	
Mean return	0.16%	0.19%	0.16%
Min return	−20.03%	−29.18%	−20.08%
Max return	12.08%	17.38%	11.42%
Standard deviation	2.33%	2.94%	2.33%

In this table are the basic descriptive statistics, like the mean and standard deviation of the DJIA, NASDAQ composite, and S&P 500, as well as the maximum and minimum returns at both the daily and weekly frequency for the period between 2 January 1984 and 29 December 2023.

The daily mean return for the three indexes over the whole sample period was 0.03%, for the DJIA and S&P 500 and 0.04% for NASDAQ. During this period, the largest one-day decreases of 25.63% and 22.90% for the DJIA and S&P 500, respectively, were observed on 19 October 1987. The largest one-day decrease for the NASDAQ was observed as 13.15% on 16 March 2020. The largest one-day increase of 10.76% for the DJIA was observed on 24 March 2020, while the largest one-day increase for the NASDAQ was observed on 3 January 2001, and it was 13.25%. The largest increase of 10.96% for the S&P 500 was on 13 October 2008. The daily standard deviation for the DJIA, NASDAQ, and S&P 500 over the entire sample period was 1.13%, 1.39%, and 1.15%, respectively.

3. Methodology

The primary methodological approach in this manuscript is to obtain forecasts of stock index values and then compare those to actual values Sinha (2021). This approach is similar to that of Sinha (2024d), and also used in Sinha (2024a, 2024b). The expected index values are obtained through GBM simulations, which requires two important components: drift and diffusion. These were calculated using Equations (4) and (5), respectively. Using those components, one thousand values were simulated using the GBM framework shown in Equation (2). Probabilities for these simulated values were also obtained, and, using the probabilities, the expected index values were estimated using Equation (7). Equation (8) was used to test the reliability of the forecasted index values. The differences in the means and standard deviations of the actual and expected values were also tested. The process was also repeated for different numbers of simulations: 1, 10, 50, and 100 simulations. The methodological approach is similar to that of Sinha (2021), expected for how the drift and diffusion components were estimated and for the number of number of simulations. The details of the methodology are described in the following sections.

3.1. Estimating Drift and Diffusion

The drift and diffusion terms used in the GBM framework for the simulation were proxied by the average returns and standard deviation of the returns estimated using Equation (3). The drift and diffusion terms were estimated using the standard formulae for averages and standard deviations, using formulae similar to Equations (4) and (5), respectively, for all the three indexes, DJIA, NASDAQ, and S&P 500, separately. A rolling-window approach was used similar to the approaches used in Sinha (2021, 2024a, 2024b, and 2024d), where the window was based on a twenty-year period. In this manuscript, the window is based on a twenty-year period as well, but at the daily and weekly frequency. This works out to five thousand and thirty-eight days at the daily frequency, and one thousand and forty-three weeks at the weekly frequency. For each window, a day or a week is dropped at the beginning of the window, and a day or a week is added at the end, as the windows move along.

3.2. Geometric Brownian Motion and Expected Index Value Estimation

The GBM framework, as represented in Equation (2), has been used in a number of manuscripts and books (Benninga and Mofkadi 2021; Hull 2018; Maruddani and Trimono 2018; Musiela and Rutkowski 2005; Navin 2007; Reddy and Clinton 2016; Sinha 2021, 2024a, 2024b, 2024d). In this manuscript, the GBM was simulated one thousand times, unlike in some other works (Sinha 2021, 2024a, 2024b, 2024d), where one hundred thousand simulations were carried out. In Sinha (2021), the expected price based on only one simulation was also estimated. In Sinha (2024d), a different number of GBM simulations were carried out, namely 1, 10, 50, 100, 200, 300, 400, 500, and 1000 simulations. In these simulations, it was found that the expected price based on one thousand simulations produced sufficiently reliable forecasted values. Accordingly, in this study, one thousand simulations were initially carried out. As part of the robustness check, the expected index values based on one, ten, fifty, and one hundred simulations were also estimated.

$$r_{sim,t,i} = ln\left(\frac{P_{sim,t,i}}{P_{0,i}}\right) \tag{6}$$

Using Equation (6), the returns for the simulated prices were estimated. The simulated returns were sorted from the lowest to the highest, and cumulative normally distributed functions for each of these returns were calculated around the means and standard deviations estimated using the rolling-window process described in Section 3.1. The probability of each simulated return was estimated by taking the difference between the cumulative distribution function of two adjacent simulated and sorted returns. As this probability is the same as the probability of the simulated index value, the expected index value was estimated by totaling up the multiplication of the simulated index value with its associated probability, as shown in Equation (7). This process is similar to that used in Sinha's works (Sinha 2021, 2024a, 2024b, 2024d). This calculation was carried out for each index at the daily and weekly frequency between 2 January 2004 and 29 December 2023. As part of robustness check, expected values were also calculated for the one, ten, fifty, and one hundred simulation runs.

$$Exp\_P_{t,i} = \sum_{1}^{1000} P_{sim,t,i} * Probability_{P_{sim,t,i}} \tag{7}$$

3.3. Testing Expected Index Values

The efficacy of the expected value as forecasts of the actual value was tested using for three different ways. First, a plot of the actual value against the GBM-based expected index value was created for the three indexes, DJIA, NASDAQ, and S&P 500, at the daily and weekly frequencies. These plots were based on one thousand simulation runs. Figure 1 is the plot of the natural log of the actual DJIA index value against the natural log the expected DJIA index value estimated using the geometric Brownian motion simulation for the period between 2 January 2024 and 29 December 2023, at the daily frequency. Figure 2 is the plot for NASDAQ, and Figure 3 for S&P 500 at the daily frequency. Figures 4–6 are the plots for the three indexes at the weekly frequency.

$$ln(P_{t,i}) = \alpha_i + \beta_i ln\left(Exp\_P_{t,i}\right) + \varepsilon_i \quad (8)$$

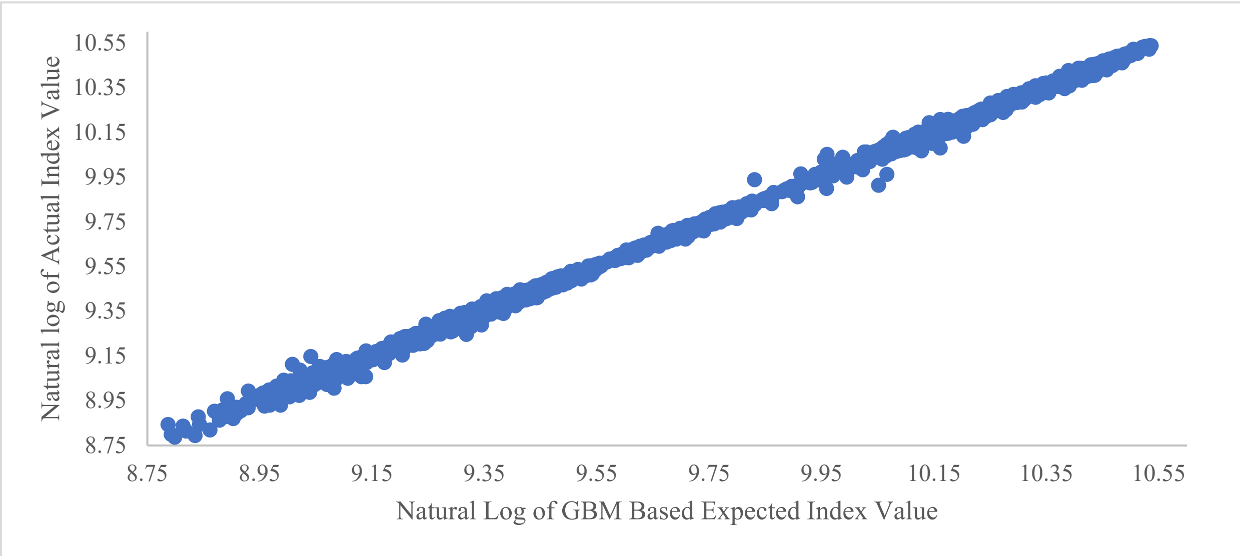

Figure 1. Actual vs. expected DJIA index value (daily frequency).

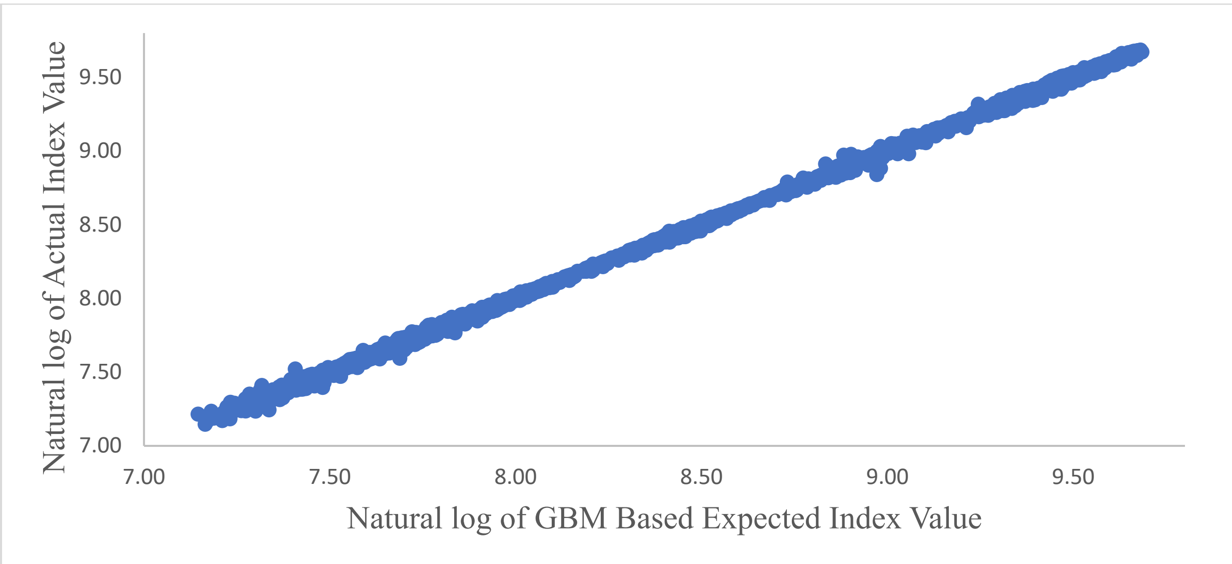

Figure 2. Actual vs. expected NASDAQ index value (daily frequency).

Second, the accuracy of the GBM-based expected index value to serve as a proxy for the actual index value was tested by using the regression equation depicted in Equation (8). If the expected index value serves as accurate forecasts, the plots shown in Figures 1–6 will exhibit a linear relationship between the actual and expected index value, and this will result in β_i, the coefficient of the expected index value, in Equation (8) having a value of 1. Also, if the GBM-based expected values were accurate forecasts of the actual index value, the value of the intercept coefficient, α_i, would be equal to zero. This logic is similar to that mentioned in the works of Sinha (2021, 2024a, 2024b, 2024d), as well as in the work of Grinblatt and Titman (1992). The values of the slope coefficient different from one and the value of the intercept different from zero would not necessarily mean that the expected index values are not useful, especially if the equation has a very high R-square and adjusted R-square, and if the expected and actual index values exhibit a high positive correlation coefficient. As these regressions are simple regressions, the correlation coefficients were obtained by taking the square root of R-square. The results of the regressions at the daily frequency are presented in Table 2, while Table 3 has the results for the weekly frequency. For Tables 2 and 3, the GBM-based expected index values were based on one thousand simulations. Tables 6 and 7 also provide coefficients for regression tests, but in these results, they are for one, ten, fifty, and one hundred simulations. The tables have results for all three indexes: DJIA, NASDAQ, and S&P 500.

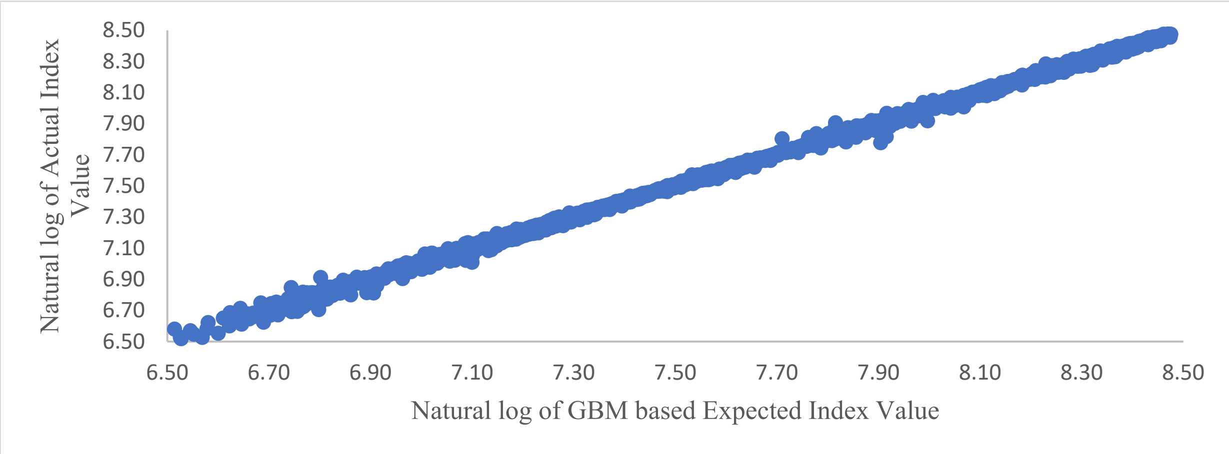

Figure 3. Actual vs. expected S&P 500 index value (daily frequency).

Table 2. Daily forecast regression results.

	Dependent Variable: Actual Index Value		
Regression Parameters	**DJIA**	**NASDAQ**	**SP500**
Intercept	0.00	0.00	0.00
t-stat ($H_0 = 0$)	0.27	0.45	0.49
GBM-based expected value	1.00	1.00	1.00
t-stat ($H_0 = 0$)	**2704.85**	**3535.37**	**2802.96**
t-stat ($H_0 = 1$)	0.03	0.07	0.14
N	5033	5033	5033
F-stat	**7,316,212**	**12,500,000**	**7,856,604**
R-square	99.93%	99.96%	99.94%
Adj R-Square	99.93%	99.96%	99.94%
ρ	1.00	1.00	1.00

This table presents results of the regression with the natural log of the actual index value as the dependent variable, while the natural log of the GBM-based expected index value is the independent variable. The expected value is based on one thousand GBM simulations. The regressions are based on daily frequency data for the period between 2 January 2004 and 29 December 2023. The highlighted t-stat and F-stat values are significant at 1% significance. ρ is the correlation coefficient between the natural log of the actual and expected index values.

Third, the efficacy of the GBM-based expected index values as proxies for the actual index values was also investigated by testing the differences in standard deviation and means between the two sets of index values. The logic here is that, if the expected index values are good proxies, there would be no difference between their means and standard deviation and those of the actual index values. The differences in means and standard deviations were tested using the procedure discussed in the work of Berenson et al. (2015), and also used in the works of Sinha (2021, 2024a, 2024d).

$$\mathrm{F-stat_i} = \frac{(\ln(\text{GBM Expected Index Value}))^2}{(\ln(\text{Actual Index Value}))^2} \tag{9}$$

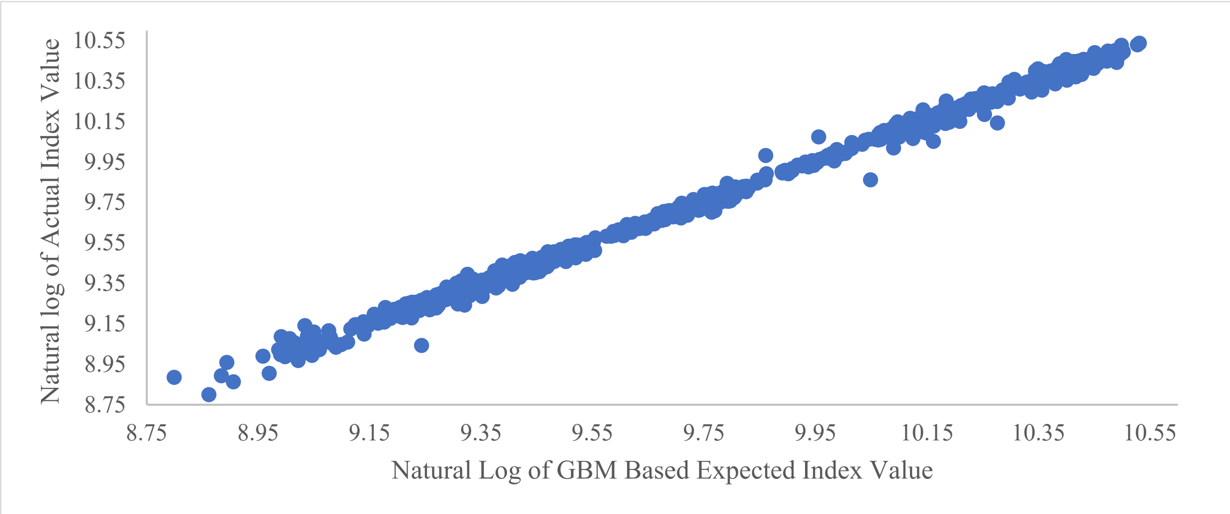

Figure 4. Actual vs. expected DJIA index value (weekly frequency).

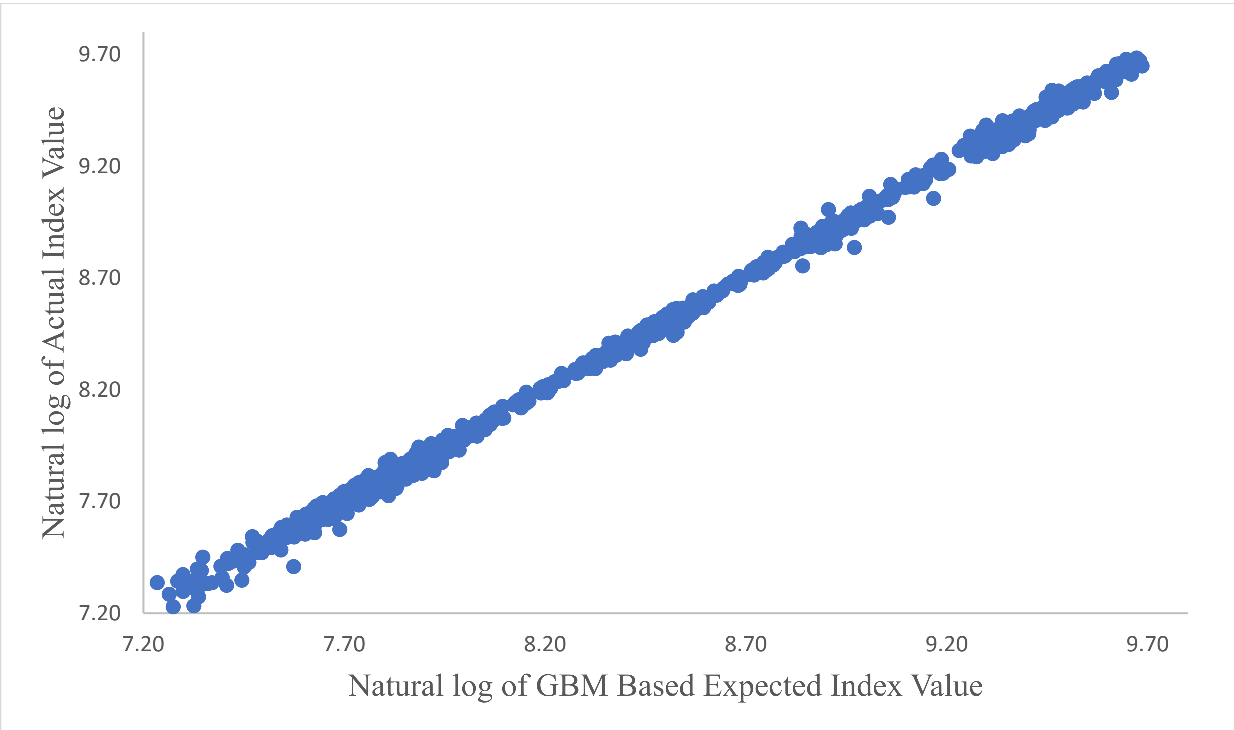

Figure 5. Actual vs. expected NASDAQ index value (weekly frequency).

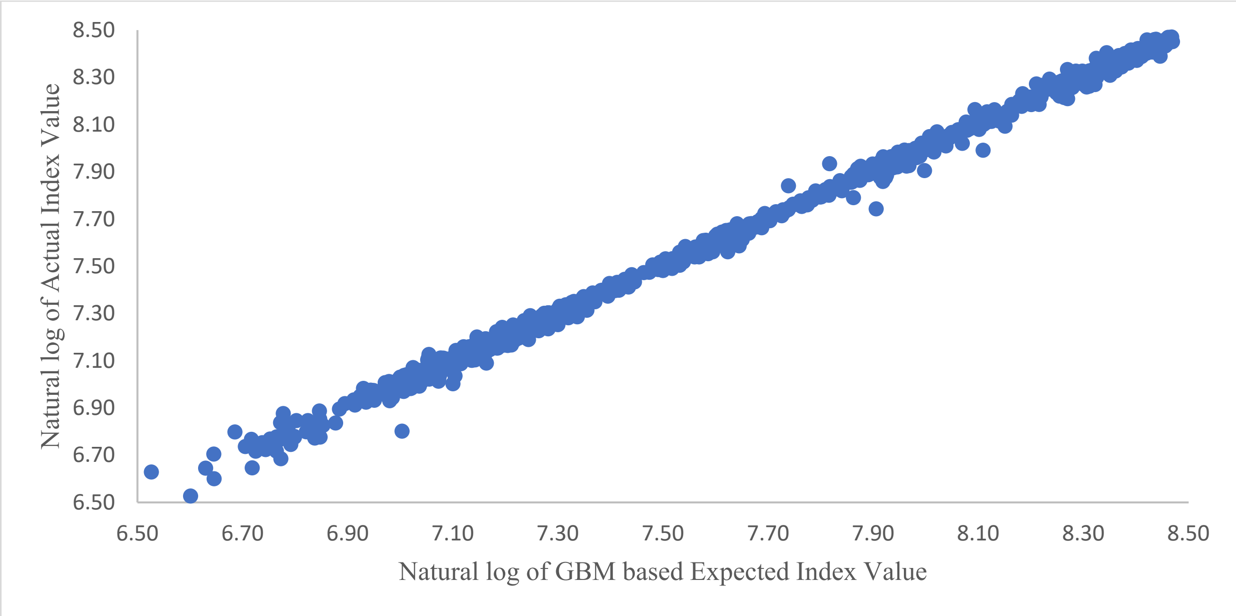

Figure 6. Actual vs. expected S&P 500 index value (weekly frequency).

Table 3. Weekly forecast regression results.

	Dependent Variable: Actual Index Value		
Regression Parameters	**DJIA**	**NASDAQ**	**S&P 500**
Intercept	0.00	0.00	0.00
t-stat (H_0 = 0)	0.05	−0.10	0.05
GBM-based expected value	1.00	1.00	1.00
t-stat (H_0 = 0)	**589.28**	**802.67**	**632.97**
t-stat (H_0 = 1)	0.01	−0.18	−0.02
N	1044	1044	1044
F-stat	347,255	644,286	400,652
R-square	99.70%	99.84%	99.74%
Adj R-square	99.70%	99.84%	99.74%
ρ	1.00	1.00	1.00

This table presents results of the regression with the natural log of the actual index value as the dependent variable, while the natural log of the GBM-based expected index value is the independent variable. The expected value is based on one thousand GBM simulations. The regressions are based on weekly frequency data for the period between 2 January 2004 and 29 December 2023. The highlighted t-stat and F-stat values are significant at 1% significance. ρ is the correlation coefficient between the natural log of the actual and expected index values.

The differences in standard deviation were tested using the F-stat calculated by dividing the variance of the GBM-based expected index value by the variance of the actual index value, as shown above in Equation (9). The critical value for the F-stat is obtained from a distribution with $n_{1,i} - 1$ and $n_{2,i} - 1$; however, in this manuscript, n_1 is equal to n_2.

$$\sigma_{pooled,i} = \frac{(n_1 - 1)\sigma^2_{AIV,i} + (n_2 - 1)\sigma^2_{EIV,i}}{(n_{1,i} - 1) + (n_{2,i} - 1)} \quad (10)$$

The differences in means were tested using a pooled variance approach calculated using the formula shown in Equation (10). A t-stat calculated using Equation (11) was used to test the hypothesis of no difference between the means of the actual index value, and the GBM-based expected index value. In Equation (11), $\mu_{1,i} - \mu_{2,i}$ was hypothesized to have a value equal to zero. In these equations, *AIV* represents the actual index value, while *EIV* represents the GBM expected index values, with $n_{1,i} = n_{2,i}$.

$$t_{stat,i} = \frac{(\overline{\mu}_{EIV,i} - \overline{\mu}_{AIV,i}) - (\mu_{1,i} - \mu_{2,i})}{\sqrt{\sigma^2_{pooled,i}\left(\frac{1}{n_{1,i}} + \frac{1}{n_{2,i}}\right)}} \tag{11}$$

The results of the tests for differences in standard deviation and means are presented in Tables 4 and 5 for all three indexes with one thousand GBM simulations, respectively. The results for one, ten, fifty, and one hundred simulations are presented in Tables 8 and 9, respectively.

Table 4. Daily forecast differences in standard deviations and means.

Index	DJIA	NASDAQ	S&P 500
Standard Deviations			
Actual	0.4373	0.6799	0.4792
Expected	0.4371	0.6798	0.4791
Pooled	0.4372	0.6798	0.4791
F-stat	1.00	1.00	1.00
p-value	0.98	0.99	0.98
Means			
Actual	9.7265	8.3744	7.5495
Expected	9.7255	8.3735	7.5486
Difference	0.0009	0.0009	0.0009
t-stat	0.11	0.07	0.1
p-value	0.92	0.95	0.92

The results of tests for differences between the standard deviation and means of the natural log of the actual index value, and the natural log of the GBM-based expected index are presented in this table. The expected GBM index value is based on one thousand simulations. The tests are at the daily frequency for the period between 2 January 2004 and 29 December 2023. The standard deviations are hypothesized to be the same, while the differences in means are hypothesized to be equal to zero.

Table 5. Weekly forecast differences in standard deviations and means.

Index	DJIA	NASDAQ	S&P 500
Standard Deviations			
Actual	0.4378	0.6803	0.4797
Expected	0.4372	0.6796	0.479
Pooled	0.4375	0.6799	0.4793
F-stat	1.00	1.00	1.00
p-value	0.96	0.97	0.97
Means			
Actual	9.7267	8.3746	7.5497
Expected	9.7261	8.3738	7.5489
Difference	0.0006	0.0008	0.0008
t-stat	0.03	0.03	0.04
p-value	0.97	0.98	0.97

The results of tests for differences between the standard deviation and means of the natural log of the actual index value, and the natural log of the GBM based expected index are presented in this table. The expected GBM index value is based on one thousand simulations. The tests are at the weekly frequency for the period between 2 January 2004 and 29 December 2023. The standard deviations are hypothesized to be the same, while the differences in means are hypothesized to be equal to zero.

4. Results

The plots for actual vs. expected index values are presented in Figures 1–3 at the daily frequency for the for the period between 2 January 2004 and 29 December 2023, and are based on one thousand GBM simulations. The plots are scatter plots and are obtained using the natural logs of the actual index values against the natural log of the GBM-based expected index values. Figure 1 is the plot for DJIA. The plot shows a linear relationship and a strong positive correlation between the actual DJIA and the expected DJIA.

Figure 2 is the scatter plot of the natural log of actual NASDAQ index values against the natural log of the GBM-based expected index value of NASDAQ at the daily frequency. Just like the plot for DJIA in Figure 1, Figure 2 also suggests a strong correlation and positive linear relationship between the actual and expected index values for NASDAQ.

Figure 3 plots the natural log of S&P 500 against the natural log of the expected S&P 500 index values calculated using GBM simulations. Just like the plots for DJIA and NASDAQ, Figure 3 also suggests a positive correlation and a strong linear relationship between the actual and expected index values for S&P 500.

Figures 4–6 are the plots at the weekly frequency for the DJIA, NASDAQ, and S&P 500, respectively. These plots, just like those in Figures 1–3, have the natural log of the actual index value plotted against the natural log of the GBM-based expected index values for the period between 2 January 2024 and 29 December 2023. Just like the plots in Figures 1–3, these plots also suggest strong correlations and positive linear relationships between the actual and expected values for all three indexes.

The results of regression Equation (8) for the daily frequency are presented in Table 2. As it was hypothesized, for the GBM-based expected index values to be reliable predictors of the actual index value, the intercept term, α, in regression Equation (12) should have a value of zero, while the coefficient of the expected value, β, should have a value of 1. The results presented in Table 2 are consistent with this hypothesis. The expected index value in Table 2 is based on one thousand GBM simulations.

$$ln(P_{t,i}) = \hat{\alpha}_i + \hat{\beta}_i ln\left(Exp\_P_{t,i}\right) \quad (12)$$

The values of the intercept coefficient, α, are numerically equal to zero, and statistically, the hypothesis that they are equal to zero cannot be rejected for all three indexes. The t-sat for the hypothesis for that the intercept equal to zero was found to be 0.27, 0.45, and 0.49 for the DJIA, NASDAQ, and S&P 500 respectively. The coefficients of the expected value, β, have a numerical value of 1.00 at two decimal places, and the hypothesis that they are not equal to one is rejected, with t-stats equal to 0.03, 0.07, and 0.14, respectively, for the three indexes. In addition, the hypothesis that the intercept is equal to zero is rejected at the 1% confidence level as well. The R-square and the adjusted R-squares for all three indexes are very high, and so are the F-stats. The correlation coefficients between the natural log of the actual index value and the GBM-based expected index value are also found to be equal to one. These results are consistent with the plots observed in Figures 1–3.

The results for the weekly frequency, as presented in Table 3, are similar to the those at the daily frequency, and they are consistent with what is observed in the plots in Figures 4–6. Also, just like Table 2, the expected index value is estimated using one thousand simulations. The coefficient of the intercept, α, is numerically equal to zero for all three indexes: DJIA, NASDAQ, and S&P 500. With the t-stat for the hypothesis that the intercept is not equal to zero being 0.05, -0.1, and 0.05, respectively, the hypothesis can be easily rejected at the 1% significance level. The coefficient for the slope coefficient, β, for the GBM-based expected index value is numerically equal to 1 for all three indexes. With t-stat for the value of the coefficient equal to one being 0.01, -0.18, and -0.02, respectively, the hypothesis that they are different from one can be rejected for all three indexes. The hypothesis that the coefficients are different from zero can be rejected at the 1% confident level as well. The regression equations show very high R-squares and adjusted R-squares, as well correlation coefficients equal to one.

The results for the tests for the differences in standard deviation and means are presented in Table 4 for the daily frequency. For DJIA, the difference in standard deviation between the natural log of the actual index and the natural log of the GBM-based expected index value is 0.0002. For NASDAQ and S&P 500, it is 0.0001. The F-stat and its *p*-value would not reject the hypothesis of no difference between the standard deviations of actual and expected index value for all three indexes. The differences in means of the actual and expected index values are found to be 0.0009 for all three indexes. The t-stat estimated using the pooled variance using Equation (11) and the associated *p*-value do not reject the hypothesis of no-difference between the averages of the natural logs of the actual and GBM-based expected index values.

The differences in standard deviation and means between the natural log of the actual index value and the natural of GBM-based expected index values at the weekly frequency are presented in Table 5. The GBM-based expected index value was calculated using one thousand simulations. Differences in standard deviation are found to be 0.0006 for DJIA, and 0.0007 for NASDAQ and S&P 500. The F-stat and its *p*-value do not reject the hypothesis of no differences in standard deviation. The differences in the averages of the natural logs of the actual and expected index values are found to be 0.0006 for the DJIA, and 0.0008 for both the NASDAQ and S&P 500. The t-stat and associate *p*-values do not reject the hypothesis of no differences in means.

As part of robustness check, GBM-based expected index values were estimated at one, ten, fifty, and one hundred simulations and tested for their reliability as forecasts. They were tested with simple regressions, and tests for differences in standard deviation and means. The tests were applied to all three indexes, DJIA, NASDAQ, and S&P 500. The simple regression test results are presented in Tables 6 and 7 for the daily and weekly frequency, respectively. The test results for differences in standard deviation and means are presented in Tables 8 and 9 for the daily and weekly frequency, respectively.

For the daily frequency results presented in Table 6, one can see that the intercept of the simple regression is very high when only one simulation was used to estimate the GBM-based expected index value. It is 8.38 for DJIA, 6.11 for NASDAQ, and 6.32 for S&P 500. Not only are these numbers not numerically equal to zero, but t-stats for coefficients reject the hypothesis that they are equal to zero for all three indexes. It is also observed for all three indexes that, as the number of simulations increases, the numerical value of the intercept decreases to 0.01 for all three indexes at one hundred simulations. Statistically, the hypothesis that they are equal to zero can be rejected, which is different from what is observed for one thousand simulations in Table 2. The coefficient of the slope term for the GBM-based expected index value is observed to be 0.15, 0.31, and 0.19 for DJIA, NASDAQ, and S&P 500, respectively, for the expected index value based on one simulation. The t-stat test statistics also reject the hypotheses that they are equal to zero or one. As the number of simulations increases to one hundred, the value of the slope coefficient starts increasing and is numerically equal to one at fifty and one hundred simulations for all three indexes. Statistically, at least, at fifty simulations, the value of the slope coefficient is neither equal to zero nor one for all three indexes. At the one hundred simulations, just like at one thousand simulations, the slope coefficient is statistically different from zero, but not from one. The correlation coefficient increases to one, as the number of simulations increases to one hundred. The R-square, adjusted R-square, and the F-stats, all increase as the number of simulations increases to one hundred.

Table 6. Daily forecast regression results for different simulations.

	Dependent Variable: Actual DJIA Value			
No. of simulations	**1**	**10**	**50**	**100**
Intercept	8.38	0.60	0.03	0.01
t-stat (H_0 = 0)	**189.62**	**19.74**	**4.69**	2.52 [a]
GBM-based expected value	0.15	0.95	1.00	1.00
t-stat (H_0 = 0)	**30.81**	**300.58**	**1313.19**	**1992.11**
t-stat (H_0 = 1)	**168.38**	**16.58**	2.06 [a]	0.50

Table 6. *Cont.*

N	5033	5033	5033	5033
F-stat	**949.46**	**90,350**	**1,724,457**	**3,968,518**
R-square	15.88%	94.73%	99.71%	99.87%
Adj R-square	15.86%	94.72%	99.71%	99.87%
ρ	0.40	0.97	1.00	1.00
	Dependent Variable: Actual NASDAQ Value			
No. of simulations	**1**	**10**	**50**	**100**
Intercept	6.11	0.28	0.03	0.01
t-stat (H_0 = 0)	**126.74**	**16.52**	**7.16**	**3.63**
GBM-based expected value	0.31	0.98	1.00	1.00
t-stat (H_0 = 0)	**47.51**	**479.04**	**1902.72**	**2800.65**
t-stat (H_0 = 1)	**107.07**	**10.88**	**2.59**	0.39
N	5033	5033	5033	5033
F-stat	**2257**	**229,478**	**3,620,359**	**7,843,636**
R-square	30.97%	97.85%	99.86%	99.94%
Adj R-square	30.95%	97.85%	99.86%	99.94%
ρ	0.56	0.99	1.00	1.00
	Dependent Variable: Actual S&P 500 Value			
No. of Simulations	**1**	**10**	**50**	**100**
Intercept	6.32	0.43	0.04	0.01
t-stat (H_0 = 0)	**175.92**	**19.44**	**7.29**	**3.46**
GBM-based expected value	0.19	0.96	1.00	1.00
t-stat (H_0 = 0)	**34.60**	**320.81**	**1447.83**	**2108.72**
t-stat (H_0 = 1)	**150.21**	**15.00**	**3.57**	0.70
N	5033	5033	5033	5033
F-stat	**1197.27**	**102,921**	**2,096,202**	**4,446,701**
R-square	19.22%	95.34%	99.76%	99.89%
Adj R-square	19.21%	95.34%	99.76%	99.89%
ρ	0.44	0.98	1.00	1.00

This table presents results of the regression, with the natural log of the actual index value as the dependent variable, while the natural log of the GBM-based expected index value is the independent variable. The expected value is based on four different GBM simulations, namely 1, 10, 50, and 100. The regressions are based on daily frequency data for the period between 2 January 2004 and 29 December 2023. The highlighted t-stat and F-stat values are significant at 1% significance. Super-script [a] indicates significance at 5% level. ρ is the correlation coefficient between the natural log of the actual and expected index values.

The regression results for the weekly forecast when the GBM-based expected index value was estimated based on one, ten, fifty, and one hundred simulations are presented in Table 7. In this table, just like in Table 2, Table 3, and Table 6, the simple regression has the natural log of the actual log of the actual index as the dependent variable, and it has the natural log of the GBM-based expected index value as the independent variable. The values of the intercept term for this equation are numerically equal to 8.30, 6.20, and 6.48 for DJIA, NASDAQ, and S&P 500, respectively, when the expected index is calculated using only one simulation. The hypothesis that the value is also equal to zero is rejected at the 1% confidence level for all three indexes. As the number of simulations increases, the value of the intercept decreases and is equal to 0.01 for one hundred simulations for all three indexes. The hypotheses that they are equal to zero cannot be rejected, just like in Table 3, when one thousand simulations were used. The slope coefficient has a value of 0.16, 0.30, and 0.16 with one simulation for the three indexes, respectively, and increases to 1.00 at one hundred simulations for all three. The hypothesis that the slope coefficient is equal to zero or one is rejected when the expected value is estimated using one and ten simulations. At the fifty and one hundred simulations, the hypothesis that the slope coefficient is equal to zero is rejected, but the hypothesis that it is equal to one cannot be rejected for all three indexes. The correlation coefficients at one simulation level are 0.41, 0.54, and 0.40 for DJIA, NASDAQ, and S&P 500, respectively, and they increase to one at the fifty-simulation level for all three indexes. The R-square, adjusted R-square, and the F-stat also increase as the number of simulations increases.

Table 7. Weekly forecast regression results at different simulations.

	Dependent Variable: Actual DJIA Value			
No. of simulations	**1**	**10**	**50**	**100**
Intercept	8.30	0.61	0.05	0.01
t-stat (H_0 = 0)	**84.76**	**8.41**	2.31 [a]	0.54
GBM-based expected value	0.16	0.95	1.00	1.00
t-stat (H_0 = 0)	**14.63**	**126.06**	**448.22**	**538.55**
t-stat (H_0 = 1)	**75.29**	**7.08**	1.46	0.03
N	1044	1044	1044	1044
F-stat	**213.99**	**15,891**	**200,897**	**290,038**
R-square	17.04%	93.85%	99.48%	99.64%
Adj R-square	16.96%	93.84%	99.48%	99.64%
ρ	0.41	0.97	1.00	1.00
	Dependent Variable: Actual NASDAQ Value			
No. of simulations	**1**	**10**	**50**	**100**
Intercept	6.20	0.30	0.02	0.01
t-stat (H_0 = 0)	**57.64**	**7.43**	1.63	0.78
GBM-based expected value	0.30	0.98	1.00	1.00
t-stat (H_0 = 0)	**20.55**	**200.75**	**620.84**	**735.23**
t-stat (H_0 = 1)	**48.79**	**5.07**	0.20	−0.03
N	1044	1044	1044	1044
F-stat	**422.35**	**40,301**	**385,448**	**540,565**
R-square	28.84%	97.48%	99.73%	99.81%
Adj R-square	28.77%	97.48%	99.73%	99.81%
P	0.54	0.99	1.00	1.00
	Dependent Variable: Actual S&P 500 Value			
No. of simulations	**1**	**10**	**50**	**100**
Intercept	6.48	0.39	0.03	0.01
t-stat (H_0 = 0)	**84.25**	**7.41**	1.99 [a]	1.07
GBM-based expected value	0.16	0.96	1.00	1.00
t-stat (H_0 = 0)	**14.08**	**137.49**	**484.99**	**564.98**
t-stat (H_0 = 1)	**71.97**	**5.50**	0.81	0.33
N	1044	1044	1044	1044
F-stat	198.31	18,902	235,218	319,200

Table 7. *Cont.*

R-square	15.99%	94.78%	99.56%	99.67%
Adj R-square	15.91%	94.77%	99.56%	99.67%
P	0.40	0.97	1.00	1.00

This table presents results of the regression with the natural log of the actual index value as the dependent variable, while the natural log of the GBM-based expected index value is the independent variable. The expected value is based on four different GBM simulations, namely 1, 10, 50, and 100. The regressions are based on weekly frequency data for the period between 2 January 2004 and 29 December 2023. The highlighted t-stat and F-stat values are significant at 1% significance. Super-script [a] indicates significance at 5% level. ρ is the correlation coefficient between the natural log of the actual and expected index values.

The results for the differences in standard deviation and means between the natural log of the actual index value and the natural log of the GBM-based expected index values at one, ten, fifty, and one hundred simulations are presented in Table 8 for the daily frequency. When the expected index value was calculated using only one simulation, the hypothesis of no difference in standard deviation could be rejected for DJIA, NASDAQ, and S&P 500 at the 1% significance level. As the number of simulations increases from one to ten to fifty and one hundred, the hypothesis of no difference in standard deviation cannot be rejected for NASDAQ and S&P 500. For the DJIA, the hypothesis cannot be rejected when its expected value is calculated using fifty and one hundred simulations, and not when it is calculated using one and ten simulations. It is rejected with 1% confidence for expected value based on one simulation, and at the 10% confidence level, in the case of ten simulations. Just like the standard deviation, the hypothesis for no differences between the means of the natural logs of the actual and expected index values can be rejected at the 1% significance levels for all three indexes when the expected value is estimated using only one and ten simulation runs. The hypothesis of no difference cannot be rejected when the expected value is calculated using fifty and one hundred simulations for all three indexes.

Table 8. Daily forecast differences in standard deviations and means for different simulations.

		Index: DJIA		
No. of simulations	**1**	**10**	**50**	**100**
Standard Deviations				
Actual	0.4373	0.4373	0.4373	0.4373
Expected	1.1262	0.449	0.4373	0.4371
Pooled	0.8543	0.4432	0.4373	0.4372
F-stat	**6.63**	1.05 [b]	1.00	1.00
p-value	0.00	0.06	0.99	0.98
Means				
Actual	9.7265	9.7265	9.7265	9.7265
Expected	8.7256	9.6292	9.7071	9.7166
Difference	1.0008	0.0972	0.0194	0.00984
t-stat	**58.77**	**11.00**	2.23 [a]	1.13
p-value	0.00	0.00	0.03	0.26
		Index: NASDAQ		
No. of simulations	**1**	**10**	**50**	**100**
Standard Deviations				
Actual	0.6799	0.6799	0.6799	0.6799
Expected	1.2315	0.6878	0.6804	0.6798
Pooled	0.9947	0.6839	0.6801	0.6798
F-stat	**3.28**	1.02	1.00	1.00
p-value	0.00	0.41	0.96	0.99

Table 8. *Cont.*

		Means		
Actual	8.3744	8.3744	8.3744	8.3744
Expected	7.3571	8.2781	8.3543	8.3647
Difference	1.0173	0.0963	0.0201	0.0097
t-stat	**51.3**	**7.07**	1.49	0.72
p-value	0.00	0.00	0.14	0.47
		Index: S&P 500		
No. of simulations	**1**	**10**	**50**	**100**
		Standard Deviations		
Actual	0.4792	0.4792	0.4792	0.4792
Expected	1.1213	0.4898	0.4798	0.4791
Pooled	0.8623	0.4845	0.4795	0.4791
F-stat	**5.48**	1.04	1.00	1.00
p-value	0.00	0.12	0.93	0.99
		Means		
Actual	7.5495	7.5495	7.5495	7.5495
Expected	6.5428	7.4502	7.5302	7.5396
Difference	1.0067	0.0993	0.0193	0.00993
t-stat	**58.57**	**10.28**	2.02 [a]	1.04
p-value	0.00	0.00	0.04	0.30

The results of tests for differences between the standard deviation and means of the natural log of the actual index value and the natural log of the GBM-based expected index are presented in this table. The expected value is based on four different GBM simulations, namely 1, 10, 50, and 100. The tests are at the daily frequency for the period between 2 January 2004 and 29 December 2023. The standard deviations are hypothesized to be the same, while the differences in means are hypothesized to be equal to zero. Highlighted values indicate significance at 1% level, while superscripts a and b indicate significance at the 5% and 10% significance levels.

Table 9 presents the results of the tests for the differences in standard deviation and means between the natural log of the actual index value and the natural log of the GBM-based expected index value for the weekly frequency. The patterns in the results are similar to those observed at the daily frequency. When the expected index value is estimated using only one simulation, the hypothesis of no difference in standard deviation and means can be rejected. The hypothesis cannot be rejected when fifty and one hundred simulations are used to estimate the expected index value for all three indexes. When the expected value is estimated using ten simulations, the means are found to be different but not the standard deviations.

Table 9. Weekly forecast differences in standard deviations and means for different simulations.

		Index: DJIA		
No. of Simulations	**1**	**10**	**50**	**100**
		Standard Deviations		
Actual	0.4378	0.4378	0.4378	0.4378
Expected	1.1105	0.4479	0.4381	0.4371
Pooled	0.8441	0.4429	0.438	0.4374
F-stat	**6.43**	1.05	1.00	1.00
p-value	0.00	0.46	0.98	0.96
		Means		
Actual	9.7267	9.7267	9.7267	9.7267
Expected	8.7371	9.6293	9.7084	9.7174
Difference	0.9895	0.0974	0.0183	0.00926
t-stat	**26.79**	**5.02**	0.96	0.48
p-value	0.00	0.00	0.34	0.63

Table 9. *Cont.*

	Index: NASDAQ			
No. of Simulations	**1**	**10**	**50**	**100**
	Standard Deviations			
Actual	0.6803	0.6803	0.6803	0.6803
Expected	1.2324	0.6886	0.6796	0.6796
Pooled	0.9954	0.6844	0.6799	0.6799
F-stat	**3.28**	1.02	1.00	1.00
p-value	0.00	0.69	0.97	0.97
	Means			
Actual	8.3746	8.3746	8.3746	8.3746
Expected	7.3496	8.2786	8.3553	8.3654
Difference	1.0250	0.0960	0.0193	0.00917
t-stat	**23.53**	**3.2**	0.65	0.31
p-value	0.00	0.00	0.52	0.76
	Index: S&P 500			
No. of Simulations	**1**	**10**	**50**	**100**
	Standard Deviations			
Actual	0.4797	0.4797	0.4797	0.4797
Expected	1.1719	0.4856	0.4794	0.4792
Pooled	0.8954	0.4827	0.4795	0.4794
F-stat	**5.97**	1.03	1.00	1.00
p-value	0.00	0.69	0.99	0.97
	Means			
Actual	7.5497	7.5497	7.5497	7.5497
Expected	6.5167	7.4491	7.5315	7.5398
Difference	1.0330	0.1006	0.0182	0.00993
t-stat	**26.36**	**4.76**	0.87	0.47
p-value	0.00	0.00	0.39	0.64

The results of tests for differences between the standard deviation and means of the natural log of the actual index value and the natural log of the GBM-based expected index are presented in this table. The expected value is based on four different GBM simulations, namely 1, 10, 50, and 100. The tests are at the weekly frequency for the period between 2 January 2004 and 29 December 2023. The standard deviations are hypothesized to be the same, while the differences in means are hypothesized to be equal to zero. Highlighted values indicate significance at 1% level.

The results of this manuscripts indicate that, despite their randomness and the inherent uncertainty, accurate stock index forecasts can be obtained by using the GBM approach. In fact, Brătian et al. (2022) considers GBM to be one of the primary tools for modern quantitative finance. The inherent strength of the GBM modeling process lies in the fact that it allows for simulating purely random forecasts given a particular set of values for the drift and diffusion terms, and this is consistent with the assumption that stock index values follow a true martingale or random walk process (Campbell et al. 1997; Fama 1965b). The GBM process also allows forecasts to be independent and normally distributed (Dash 2019), thereby allowing for associated probability of forecasts to be used for the estimation of expected values using Equation (7) in this manuscript. The expected values based on one thousand simulations at the daily and weekly frequency for DJIA, NASDAQ, and the S&P 500 are found to be very reliable forecasts when they are compared to the actual observed index values.

This manuscript ignores any economic policy influences on the estimation of the drift and diffusion terms, although stocks market returns are also supposed to be influenced by economic policies (Ginn 2023; Khojah et al. 2023; Luo and Zhang 2020). An approach to incorporate economic policy and financial factors influence on the GBM modeling process would be to incorporate such factors while estimating the drift and diffusion terms.

While there are numerous approach to incorporate such variables to incorporate while estimating the drift and diffusion terms, Sinha (2021) used the Marquering and Verbeek (2004) approach to estimate expected drift and diffusion to use in GBM simulations. The results were found to be useful, but Sinha (2024d) found the expected index values based on historical drift and diffusion values were just as good as, if not better than, those values when using the Marquering and Verbeek (2004) approach. The results in this manuscript indicates that reliable one-day, and one-week, ahead forecasts can be obtained using GBM simulations, when the drift and diffusion terms are based on the previous twenty years.

5. Conclusions

Ever since Brownian motion was discovered by the biologist Robert Brown in 1927, and mathematical solutions provided by mathematicians and physicists (Einstein 1905, 1956; Wiener 1923), the GBM approach has been used to price financial assets (Bachelier 1900; Black and Scholes 1973; Hull and While 1987; Samuelson 1965, 1973), among numerous others mentioned earlier in this manuscript. At the core, the application of GBM comes down to forecasting future values that are dependent on two terms, drift and diffusion, usually proxied by average returns and standard deviation, along with a standard normal random variable that has a mean of zero and a standard deviation of one, depicted in Equation (2) in this manuscript. Each time the value of the random variable, ε, in Equation (2) changes, a unique value for a future asset price can be simulated. The accuracy and reliability of the simulated price can be tested by comparing the simulated value to the actual price. The possibility of simulating numerous asset prices values exists, as ε is a random variable that can assume values randomly. In fact, one major shortcoming of the literature in GBM is that most studies use insufficient iterations (Kumar et al. 2024; Sinha 2021). Addressing this issue, Sinha (2021) uses one hundred thousand simulations to forecast monthly, quarterly, and annual values of the S&P 500 stock index. Having obtained numerous simulations, the issue arises as to which simulated value ought to be compared to the actual value. Sinha (2021) addresses this issue by simulating probabilities along with the simulated prices and calculates an expected value by totaling up the multiplication of the simulated value and its associated probability. Comparing the GBM-based expected index value to the actual value, Sinha (2021) found that those values could be used as proxies for the actual value.

In Sinha (2021), the drift and diffusion terms in the GBM model were calculated using a regression model developed by Marquering and Verbeek (2004) for estimating future returns and standard deviation. The GBM-based expected index values were found to be more reliable than using a single simulation. Sinha (2024d) applies estimated GBM-based expected values and compares them to actual values, but uses historical average returns and standard deviations as the drift and diffusion terms while simulating future expected values. The results of statistical tests in Sinha (2024d) are at least as good, if not better than those of Sinha (2021). Sinha (2024a) uses this approach to gold prices and finds the GBM-based expected values to be reliable forecasts. This manuscript differs from Sinha (2021, 2024d), as it applies and tests the approach using daily and weekly frequencies of DJIA, NASDAQ, and S&P 500. The results indicate that GBM-based expected index values are highly correlated with the actual values when expected values are estimated using one thousand simulations. Regression results also indicate the GBM-based expected values to be good predictors of the actual values. Tests of differences in means and standard deviations do not indicate any differences between those of the actual and the simulated values based on one thousand simulations. The results also indicate that expected values based on low levels of simulations may not be as accurate and useful as those at a higher number of simulations.

Although the methodological approach in this manuscript is similar to that of Sinha (2021, 2024d), the results show that the GBM-based expected forecasts at the daily and weekly frequencies are more accurate and reliable than those at the monthly, quarterly, and yearly frequencies. The differences in the results perhaps depend on the values of the drift and diffusions used in the GBM simulations. In Sinha (2021), the estimates of drift and diffusion are based on the empirical methodology described in Marquering and Verbeek (2004), an approach that uses economic and financial variables that are not statistically significant. The drift and diffusion terms estimation procedure in this manuscript is the same as that in Sinha (2024d), but the reliability in this manuscript seems to be better than that in Sinha (2024d). Perhaps, this is due to the frequency of the data used in this manuscript. In Sinha (2024d), the frequency is annual, quarterly, and monthly, while in this manuscript, it is daily and weekly. As the daily and weekly returns and standard deviations are fractions of the yearly, quarterly, and monthly values, it is to be expected that the dispersion of the forecasts at the daily and weekly forecasts would be less than those at the other frequencies, resulting in better expected values. Farida et al. (2018) and Kundu (2021) also find that the GBM forecasts are useful for making daily stock index forecasts, but this manuscript differs from them in the length of the data sample, the stock indexes researched, and the methodological approach. Farida et al. (2018) and Kundu (2021) also find that the GBM forecasts are useful to make daily stock index forecasts, but this manuscript differs from them in the length of the data sample, the stock indexes researched, and the methodological approach.

While this manuscript contributes to the literature by showing that accurate daily and weekly forecasts of stock indexes, especially DJIA, NASDAQ composite, and S&P 500, can be obtained estimated expected stock index values from values obtained from numerous GBM simulations, this procedure could be further developed by incorporating advancements and developments occurring in the GBM approach, especially skew Brownian motion (Corns and Satchell 2007; Hussain et al. 2023; Pasricha and He 2022; Zhu and He 2018), fractional Brownian motion (Alhagyan 2024; Alhagyan et al. 2016; Araneda 2024; Dhesi et al. 2016; Misiran et al. 2012), and irrational fractional Brownian motion (Dhesi and Ausloos 2016; Dhesi et al. 2016, 2021). Additionally, comparing and contrasting the results obtained from the methodology applied in this manuscript to those obtained from skew Brownian motion, fractional Brownian motion, and irrational fractional Brownian motion could also be the focus of further research. In this manuscript, the rolling window is based on a twenty-year period. As there may not be a theoretical justification for a twenty-year period, a further area of research could be to identify an optimal window by estimating drift and diffusion terms based on windows ranging from one, two, three, five, or ten years.

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Note

[1] Refinitiv DataStream is a data analytics product offered by the London Stock Exchange Group (LSEG), a provider of global economics and financial market data. The data service strives to provide data of the highest quality and claims to have a value accuracy rate of 99.85%.

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