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Optimal Market Completion through Financial Derivatives with Applications to Volatility Risk

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Abstract: This paper investigates the optimal choices of financial derivatives to complete a financial market in the framework of stochastic volatility (SV) models. We first introduce an efficient and accurate simulation-based method applicable to generalized diffusion models to approximate the optimal derivatives-based portfolio strategy. We build upon a double optimization approach, i.e., expected utility maximization and risk exposure minimization, already proposed in the literature, demonstrating that strangle options are the best choices for market completion within equity options. They lead to lower investors' risk exposure for a wide range of strikes compared to the lesser flexibility of calls, puts, and strangles. Furthermore, we explore the benefit of using volatility index derivatives and conclude that they could be more convenient substitutes when short-term maturity equity options are not available.

Keywords: expected utility theory; constant relative risk aversion (CRRA) utility; optimal derivative choice; volatility risk; volatility index (VIX) options

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1. Introduction

Financial markets are often modeled as a system of contingents on states mirroring the real-world economy. This generates a concept widely used in economic and finance literature, namely the complete market, which is described as 'a market for every good'. Earlier studies assumed that the number of securities equals the number of states of nature and investigated the optimal allocation, placing all the capital at once (see Arrow 1964; Arrow and Debreu 1954). Recognizing that investors benefit from adjusting allocation with a change in market status, more recent researchers have focused on the idea of a dynamically complete market, which is defined as a market wherein a self-financing strategy can replicate any contingent claim.

The study of portfolio choice in a dynamically complete market under a continuous-time framework can be traced back to the seminal work of Merton (1969), who computed the optimal allocation and consumption policy with a dynamic programming technique, assuming that the stock price follows a geometric Brownian motion (GBM). In this framework, the uncertainty is reflected in the Brownian motion, which captures the randomness of a stock's return; hence, investors can achieve the best portfolio performance with investments only in the stock and a cash account.

However, the financial market is ever-evolving and becoming increasingly complex; for instance, substantial evidence suggests that a single Brownian motion or source of randomness is insufficient to explain the movements of a single stock or index. Researchers have had to incorporate so-called stylized facts such as stochastic volatility (SV) or stochastic interest rates in their modeling to mimic this new reality. These stylized facts are captured via adding new 'state variables' (e.g., new random processes for SV). These state variables have been recognized as essential factors in the portfolio allocation process.

The importance of adding financial derivatives into a portfolio for market completion was demonstrated in Liu and Pan (2003), confirming that investors can improve portfolio performance when adding as many linearly independent equity options as new state variables in the portfolio composition. They do this to hedge the risk of the new state variables, therefore achieving significant improvement in portfolio performance compared to incomplete market investment (e.g., investing solely on stock and cash account). This work was extended in many directions. For example, Escobar et al. (2017) constructed optimal portfolios with the addition of options to hedge new state variables accounting for stochastic correlation. Moreover, Li et al. (2018) solved derivative-based strategies under an asset–liability management (ALM) framework with the mean-variance criterion. In a similar setting, the optimal complete and incomplete strategy for the 4/2 SV model was derived in Cheng and Escobar-Anel (2021), which demonstrated the superiority of the complete market portfolio.

Although the literature cited above strongly supports the addition of derivatives to complete the market, investors may complete the market in many ways due to the variety of derivatives in the market. Therefore, investors effectively have a non-unique solution to the problem (i.e., an infinite number of strategies, each linked to a derivative choice, producing the same maximum expected utility). The issue of infinitely many solutions and the optimal choice of derivatives was studied in the recent paper (Escobar-Anel et al. 2022) in the context of the Black—Scholes—Merton model. The paper proposed an optimization criterion (i.e., additional to the maximization of the utility, namely risk exposure minimization) to produce a unique, meaningful solution, thus deriving a practical derivative selection methodology for investors. The inner portfolio optimization solution is derived extending (Zhu et al. 2023). The risk exposure minimization criterion can be motivated from many angles, especially regarding regulatory constraints intended to control investors' exposure to risky assets and protect investors' capital in the event of a market crash.

In this paper, we study the optimal financial derivatives for market completion in the famous setting of SV models, with emphasis on the celebrated Heston model (see Heston 1993). Our findings allow investors to improve the performance of their portfolios while reducing the overall risk exposure and accounting for the most important stylized fact of stock prices, stochastic volatility. Details of the contributions will be provided later as bullet points.

There are two major hurdles for our derivatives-based portfolio allocation problem. First, given that the complexity of advanced models with many state variables jeopardizes the solvability of the utility maximization allocation problem, closed-form solutions are often unavailable. This hurdle can be overcome using approximation methods for dynamic portfolio choice problems. Brandt et al. (2005), inspired by the least-squares Monte Carlo method (see Longstaff and Schwartz 2001), recursively estimated the value function and optimal allocation following a dynamic programming principle. This method was later named the BGSS, and Cong and Oosterlee (2017) utilized the stochastic grid bundling method for conditional expectation estimation, introduced in Jain and Oosterlee (2015), further enhancing the accuracy of BGSS. Additionally, Zhu and Escobar-Anel (2022) targeted unsolvable continuous-time models, proposing an efficient and accurate simulation-based method, namely the polynomial affine method for constant relative risk aversion utility (PAMC). The second hurdle appears in the complexity of derivatives' price dynamics, which could lead to highly non-linear stochastic differential equations in contrast to traditional asset classes. In this paper, we overcome the two hurdles simultaneously by unifying the PAMC and using an options Greek approximation technique. Notably, the broad applicability of this methodology laid the foundation for the derivatives selection study within a generalized model family.

As mentioned above, we focus on investors concerned about volatility risk and seek the best derivatives to attain market completion. The seminal paper by Heston (1993) recognized the mean-reverting pattern of volatilities and introduced the well-known Heston (GBM 1/2) model. Later, extensions, such as the GBM 3/2 (see Heston 1997) and GBM

4/2 (see Grasselli 2017), were developed to capture the volatility surface better. These led to notable successes in the valuation of European equity options and semi-closed-form solutions for the option price, and Greeks are generally accessible using Fourier transformation. Popular equity options, such as call, put, straddle, and strangle options, are ideal for investors to manage the volatility risk. Furthermore, the volatility index (VIX), a measure of the stock market's volatility based on S&P 500 index options provided by the Chicago Board Options Exchange (CBOE), affords investors an alternative way to assess the volatility risk. The effectiveness of VIX products in portfolio performance enhancement has been confirmed in the literature: see Doran (2020), Chen et al. (2011), and Warren (2012). Hence, in this paper, we compare two categories of derivatives, namely equity options and VIX options, in terms of optimal dynamic completion.

The contributions of the paper are as follows:

1. The multitude of financial derivatives available in the market offers investors non-unique optimal choices regarding expected utility theory (EUT) maximization. We are interested in an optimal choice of derivatives. In this paper, we extend the extra optimization criterion proposed in Escobar-Anel et al. (2022), namely risk exposure minimization, from the family of GBM to SV models. This aids investors with practical derivative selection in a popular stock market modeling setting.
2. The PAMC-indirect numerical method is proposed to approximate the optimal allocation for a constant relative risk aversion (CRRA) investor investing in the derivatives market. The superior accuracy and efficiency of the methodology are verified using the Heston model.
3. Targeting equity and volatility risk, we first consider the optimal choice among equity options (e.g., calls, puts, straddles, and strangles). We demonstrate that strangles are the best options for minimizing risk exposure. They perform better even for a larger range of strike prices than the other options.
4. We also investigate the usage of financial derivatives on the VIX to complete the market, and we conclude that investors would prefer VIX options to equity strangles when only medium to long-term maturity options are available.

The remainder of this paper is organized as follows: Section 2 presents the investor's problem (i.e., the two criteria for optimal allocation [utility maximization] and optimal market completion [risk exposure minimization]). Section 3 details an efficient approximation method for derivatives-based portfolio allocation. The optimal market completion targeting volatility risk within an equity option and a VIX option is studied in Section 4, followed by the conclusion in Section 6. Appendix A presents the mathematical proofs, while Appendix B provides an alternative approximation method and a numerical examination of accuracy and efficiency for the two methods.

2. Investor's Problem

In this section, we introduce a market completion framework using financial derivatives. We define a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a right-continuous filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. The market is frictionless (i.e., no transaction cost and market impact), and a risk-free cash account M_t, a stock S_t, and an investor with constant relative risk aversion (CRRA) utility, $U(W) = \frac{W^{1-\gamma}}{1-\gamma}$ exist. The market dynamics are summarized as follows:

$$\begin{cases} \frac{dM_t}{M_t} = rdt \\ \frac{dS_t}{S_t} = (r + \lambda^S \sigma^S)dt + \sigma^S dB_t^S \\ dH_t = \mu^H dt + \sigma^H dB_t^H \\ < dB_t^S, dB_t^H >= \rho_{SH} dt. \end{cases} \tag{1}$$

where B_t^H and B_t^S are Brownian motions with correlation $\rho_{SH} \in (-1, 1)$, and the interest rate r is constant. We impose a market viability assumption, namely the absence of arbitrage of the first kind. This is that the discounted asset-prices must be semimartingales; see Kardaras and Platen (2011) and the seminal paper of Delbaen and Schachermayer (1994). Our main result can also be regarded as reminiscent of the Fundamental Theorem of Asset Pricing. State variable H_t follows a generalized diffusion process, where $\mu^H = \mu^H(t, H_t)$ denotes the drift and $\sigma^H = \sigma^H(t, H_t)$ denotes volatility. The market price of risk and the volatility of stock could be functions of both the stock price and the state variable, respectively; that is, $\lambda^S = \lambda^S(t, H_t, \ln S_t)$ and $\sigma^S = \sigma^S(t, H_t, \ln S_t)$. This framework is quite flexible as it embeds popular models among practitioners, solvable within portfolio optimization; for instance, the Heston stochastic volatility (SV) model Heston (1993) solved in Kraft (2005), the 3/2 SV model of Heston (1997) and the 4/2 SV model Grasselli (2017) solved within Cheng and Escobar-Anel (2021), various constant elasticity of volatility (CEV)-related models, see Anel and Fan (2024) for an overview, and recently the stochastic elasticity of volatility with stochastic volatility (SEV-SV) model in Escobar-Anel and Fan (2023).

In this market, the number of investable risky assets is less than the number of risk drivers, hence market incompleteness. To eliminate the welfare loss resulting from the unhedgeable risk drivers, we introduce a set of financial derivatives:

$$\Omega_O^{(n)} = \left\{ \bar{O}_t = [O_t^{(1)}, O_t^{(2)}, ..., O_t^{(n)}]^T \mid O_t^{(i)} \neq 0, i = 1, ..., n \text{ and } \mathrm{rank}(\Sigma_t) = 2, \, t \in [0, T] \right\}.$$

We can think of these derivatives as having payoff $O_t^{(i)} = \xi_t^{-i}\mathbb{E}[\xi_{t+T} G_i(H_{t+T}, S_{t+T}) \mid \mathcal{F}_t]$ with ξ being the exogenously given pricing kernel in the economy with dynamics $\frac{d\xi_t}{\xi} = -rdt - \Lambda_t dB_t$, we will consider specific payoff in the applications. We assume that an investor allocates in an element of Ω_O; that is, a specific $\bar{O}_t = [O_t^{(1)}, O_t^{(2)}, ..., O_t^{(n)}]^T$ ($n \geq 2$). Please note that by arbitrage arguments, the dynamics of the extended market are as follows:

$$\begin{cases} \frac{dM_t}{M_t} = rdt \\ d\bar{O}_t = diag(\bar{O}_t)[(r \cdot \mathbb{1} + \Sigma_t \Lambda)dt + \Sigma_t dB_t] \\ dH_t = \mu^H dt + \sigma^H dB_t^H \\ < dB_t^S, dB_t^H >= \rho_{SH} dt, \end{cases} \tag{2}$$

where $B_t = [B_t^S, B_t^H]^T$ and Σ_t represents the $n \times 2$ variance matrix of $\bar{O}_t$; the first column $(i, 1)$ represents the sensitivity of $O_t^{(i)}$ to the underlying asset S_t (i.e., $\frac{\partial O_t^{(i)}}{\partial S_t} S_t \frac{1}{O_t^{(i)}} \sigma^S$); and the second column $(i, 2)$ represents the sensitivity of $O_t^{(i)}$ to the state variable H_t (i.e., $\frac{\partial O_t^{(i)}}{\partial H_t} \frac{1}{O_t^{(i)}} \sigma^H$). $\Lambda = [\lambda^S, \lambda^H]^T$, where $\lambda^H = \lambda^H(t, H_t, \ln S_t)$ denotes the market price of volatility risk. The setting above is also very flexible, as it permits not only any derivatives but also a variety of models as identified from Equation (1). Rank 2 variance matrix Σ_t guarantees the completeness of the market. As observed above, and for simplicity, we assume that the derivatives in $\Omega_O^{(n)}$ will be rolled over; this means they always maintain the same time to maturity and a non-zero value, see Liu and Pan (2003) for pioneering work with this common assumption. Please note that the investor is not prohibited from trading on the stock, which is included in $\Omega_O^{(n)}$ as a special derivative.

Let $\Omega_\pi^{(O)}$ denote the space of admissible strategies satisfying the standard conditions, where the element $\pi_t = [\pi_t^{(1)}, \pi_t^{(2)}, ..., \pi_t^{(n)}]^T$ represents the proportions of the investor's wealth in the derivatives W_t satisfies

$$\begin{aligned} \frac{dW_t}{W_t} &= \pi_t^T diag^{-1}(\bar{O}_t) d\bar{O}_t + (1 - \pi_t^T \mathbb{1}) \frac{dM_t}{M_t} \\ &= (r + \pi_t^T \Sigma_t \Lambda)dt + \pi_t^T \Sigma_t dB_t. \end{aligned} \tag{3}$$

The investor's objective is to maximize the expected utility of their wealth at terminal time T; this places the problem in the expected utility framework popularized in the seminal work of Merton (1975); hence, their problem at time $t \in [0, T]$ can be written as

$$V(t, W, H, \ln S) = \max_{\pi_{s \geq t} \in \Omega_\pi^{(O)}} \mathbb{E}(U(W_T) \mid \mathcal{F}_t), \tag{4}$$

where $U(x)$ is a flexible utility function, chosen of the type CRRA (constant relative risk aversion, $U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \gamma > 0, \gamma \neq 1$) as an example [1]. Please note that this function could depend explicitly on the current level of wealth, time, stock price, and variance level; these are all the Hamilton-Jacobi-Bellman (HJB) equation variables. The associated HJB equation for the value function V follows the principles of stochastic control and is given by

$$\begin{aligned}\sup_{\pi_t} \Big\{ V_t + W_t V_W (r + \pi_t^T \Sigma_t \Lambda) + \frac{1}{2} W_t^2 V_{WW} (\pi_t^T \Sigma_t \Phi \Phi^T \Sigma_t^T \pi_t) + W_t V_{WH} \sigma^H (\pi_t^T \Sigma_t A) + W_t V_{W \ln S} \sigma^S (\pi_t^T \Sigma_t B) \Big\} \\ + V_H \mu^H + \frac{1}{2} V_{HH} (\sigma^H)^2 + V_{\ln S} (r + \lambda^S \sigma^S - (\sigma^S)^2/2) + \frac{1}{2} V_{\ln S \ln S} (\sigma^S)^2 + V_{H \ln S} \sigma^H \sigma^S \rho_{SH} = 0,\end{aligned} \tag{5}$$

where $\Phi = \begin{bmatrix} 1 & 0 \\ \rho_{SH} & \sqrt{1 - \rho_{SH}^2} \end{bmatrix}$, $A = [\rho_{SH}, 1]^T$ and $B = [1, \rho_{SH}]^T$.

Next, we change variables to simplify mathematical calculations, similarly to Liu and Pan (2003), creating a new artificial market. This market consists of three assets: a risk-free money account M_t and two pure factor assets $S_t^{(S)}$ and $S_t^{(H)}$:

$$\begin{cases} \frac{dM_t}{M_t} = r dt \\ \frac{dS_t^{(S)}}{S_t^{(S)}} = (r + \lambda^S) dt + dB_t^S \\ \frac{dS_t^{(H)}}{S_t^{(H)}} = (r + \lambda^H) dt + dB_t^H \\ dH_t = \mu^H dt + \sigma^H dB_t^H \\ < dB_t^S, dB_t^H >= \rho_{SH} dt. \end{cases} \tag{6}$$

Compared to the original market, the market state variable is still H_t; nonetheless, here, the investor can put their money in the hypothetical pure factor assets $S_t^{(S)}$ and $S_t^{(H)}$, which have a unit exposure on B_t^S and B_t^H, respectively. Let $\eta_t = [\eta_t^{(1)}, \eta_t^{(2)}]^T$ be the allocation on the pure factors (also known as exposures in the literature: see Liu and Pan (2003)); $\hat{W}_t$ denotes the investor's wealth process, and $\hat{V}(t, \hat{W}, H, \ln S)$ represents the value function in the artificial market. Similarly, the associated HJB equation becomes

$$\begin{aligned}\sup_{\eta_t} \Big\{ \hat{V}_t + \hat{W}_t \hat{V}_{\hat{W}} (r + \eta_t^T \Lambda) + \frac{1}{2} \hat{W}_t^2 \hat{V}_{\hat{W}\hat{W}} (\eta_t^T \Phi \Phi^T \eta_t) + \hat{W}_t \hat{V}_{\hat{W}H} \sigma^H (\eta_t^T A) + \hat{W}_t \hat{V}_{\hat{W} \ln S} \sigma^S (\eta_t^T B) \Big\} \\ + \hat{V}_H \mu^H + \frac{1}{2} \hat{V}_{HH} (\sigma^H)^2 + V_{\ln S} (r + \lambda^S \sigma^S - (\sigma^S)^2/2) + \frac{1}{2} \hat{V}_{\ln S \ln S} (\sigma^S)^2 + \hat{V}_{H \ln S} \sigma^H \sigma^S \rho_{SH} = 0.\end{aligned} \tag{7}$$

If the solution of the associated HJB PDEs exists, then it is easy to verify that

$$\hat{V}(t, \hat{W}, H, \ln S) = V(t, W, H, \ln S) \tag{8}$$

$$\hat{W}_t = W_t \tag{9}$$

$$\Sigma_t^T \pi_t^* = \eta_t^*. \tag{10}$$

where we could also write $\pi_t^* = \Sigma_t (\Sigma_t^T \Sigma_t)^{-1} \eta_t^*$. Furthermore, if the number of derivatives in O_t is greater than 2 (i.e., $n \geq 2$), there are infinitely many optimal strategies, all producing the same maximum value function.

Aside from the expected utility maximization, the investor is also concerned with the size of their risky allocations. For instance, an institutional investor may have to keep their gross allocation exposure under a certain level due to regulatory constraints. In other words, even a tiny exposure could be significant for capital safety regarding unmodelable risk, such as a financial crisis. Hence, we consider an additional derivative selection criterion, namely risk exposure minimization, introduced in Escobar-Anel et al. (2022):

$$\min_{\bar{O}_t \in \Omega_O^{(n)}} \left\| \underset{\pi_t \in \Omega_\pi^{(O)}}{\arg\max} \mathbb{E}(U(W_T) \mid \mathcal{F}_t) \right\|_1 , \tag{11}$$

where $\|\pi_t\|_1 = \sum_{i=1}^{n} \left|\pi_t^{(i)}\right|$ represents the ℓ_1 norm of allocations at time t. Please note that this objective is equivalent to maximizing the cash position while shorting less. Escobar-Anel et al. (2022) demonstrated that the redundancy offers no additional help with either the investor's expected utility or their risky asset exposure in the case of two one-factor assets[2].

In the following proposition, we demonstrate a generalized conclusion that applies to any diffusion model.

Proposition 1. *Assume that an optimal solution for Problem* (11) *exists for* $n \geq 2$*; then,* (11) *leads to the same minimal* ℓ_1 *norm for any* $n \geq 2$*. In addition, an optimal strategy exists for Problem* (11) *such that the number of non-zero allocations is less than or equal to* 2[3].

Proof. See Appendix A.1. □

Proposition 1 demonstrates that investors do not need to consider a portfolio with size $n > 2$. Working with $n = 2$ is sufficient for both Problems (4) and (11). Hence, we only study the most straightforward case when given a complete market setting (i.e., $n = 2$).

3. Polynomial Affine Method for CRRA Utilities in Financial Derivatives Market

In this section, we introduce a methodology to compute derivatives-based portfolio strategies. This method is required to find the optimal candidate composition $\bar{O}_t \in \Omega_O^{(2)}$ for risk exposure minimization.

Complexity in assets' dynamic models often jeopardizes the analytic solvability of HJB PDE; this means that closed-form solutions are only sometimes available. Motivated by this fact, Zhu and Escobar-Anel (2022) proposed a simulation-based method to approximate the optimal strategy for continuous-time portfolios within EUT (i.e., the PAMC). The original PAMC method only applies to asset classes, such as equity, fixed income, and currency, where asset dynamics are known explicitly. However, the PAMC can easily be extended to financial derivatives markets with proper modifications. The new method, namely the PAMC-indirect, is introduced in Section 3.1. Furthermore, an alternative method is described in Appendix B. The performances of both methodologies are demonstrated in the case of the Heston model, and the comparison to the theoretical solution confirms the excellent accuracy and efficiency of the PAMC-indirect method.

It should be noted that after introducing a sufficient number of derivatives to complete the market, i.e., fixing the pricing kernel ξ, the CRRA portfolio optimization problem could be solved via martingale duality. This is, the optimal wealth is proportional to $\xi_t^{-1/\gamma} f(t, H_t, S_t)$, with optimal exposure obtained numerically via a simulation of $f(t, H_t, S_t)\mathbb{E}[(\xi_T/\xi_t)^{1-1/\gamma} \mid \mathcal{F}_t]$. Nonetheless, this approach requires a complete market while our method still works with fewer derivatives (partially complete, i.e., incomplete markets).

3.1. The PAMC-Indirect

Inspired by the quadratic affine model family (see Liu 2006), the PAMC approach assumes that the value function has the following representation:

$$V(t, W, H, \ln S) = \frac{W^{1-\gamma}}{1-\gamma} f(t, H, \ln S), \tag{12}$$

where $f(t, H, \ln S)$ is approximated by an exponential polynomial function of order k; that is, $\exp\{P_k\}$. The PAMC method utilizes the Bellman equation and the fact that the value function at re-balancing time is the conditional expectation of the value function at $t + \Delta t$; that is,

$$V(t, W_t, H_t, \ln S_t) = \max_{\pi_t} \mathbb{E}(V(t + \Delta t, W_{t+\Delta t}, H_{t+\Delta t}, \ln S_{t+\Delta t}) \mid \mathcal{F}_t).$$

The PAMC expands the value function at $t + \Delta t$ with respect to wealth W, state variable H and log stock price $\ln S$, and it considers a sufficiently small re-balancing interval Δt such that the infinitesimal $o(\Delta t)$ terms are omitted. Then, the value function $V(t, W_t, H_t, \ln S_t)$ is rewritten as a quadratic function of the portfolio strategy, and the optimal strategy is immediately solved with the first order condition given the information at $t + \Delta t$. Proposition 2 displays the optimal strategy η_t^* estimation in the artificial pure factor market (6).

Proposition 2. *Given the approximation of the value function at the next re-balancing time $t + \Delta t$ (i.e., $\frac{W^{1-\gamma}}{1-\gamma} \exp\{P_k\}(t + \Delta t, H, \ln S)$), the optimal strategy at time t is given by*

$$\eta_t^* = \frac{1}{\gamma}(\Phi\Phi^T)^{-1}(\Lambda + \frac{\partial P_k}{\partial H}\sigma^H A + \frac{\partial P_k}{\partial \ln S}\sigma^S B). \tag{13}$$

Proof. See Appendix A.2. □

The PAMC-indirect inherits the recursive approximation structure of the PAMC. After the generation of paths of asset price and state variables, the optimal pure factor strategies at last re-balancing time $T - \Delta t$ can be directly computed with (13) because $P_k(T, H, \ln S) = 0$; the path-wise expected utilities are obtained through simulation. Furthermore, the expected utilities are regressed over stock price $S_{T-\Delta t}$ and state variable $H_{T-\Delta t}$, and the regression function approximates the $V(T - \Delta t, W, H, \ln S)$. Then, the method moves backward, and similar procedures are conducted at each re-balancing time until the optimal initial strategy of the pure factor portfolio (i.e., η_0^*) is obtained.

Finally, the PAMC-indirect calculates the portfolio variance matrix Σ_t, which depends on the option price O_t, Delta $\frac{\partial O_t}{\partial S_t}$ and the sensitivity to the state variable $\frac{\partial O_t}{\partial H_t}$. The optimal derivatives strategy π_0^* is solved with (10). Only in some special cases (e.g., the Black-Scholes model) are option prices solved analytically. Various approximation methods for option price and Greeks are available in the existing literature. The option style and underlying assets model should determine the choice of such methods. For example, an accurate Fourier transform (FT) approximation is an ideal choice when the semi-closed-form solution of an option is available (e.g., the Heston model, the Ornstein–Uhlenbeck 4/2 model), while a simple Monte Carlo simulation is universal for options with a deterministic exercise date; and a least-squares Monte Carlo method is applicable when considering American style options.

We clarify the notation in Table 1 and detail the PAMC-indirect in Algorithm 1.

Algorithm 1: PAMC-indirect

Input: S_0,W_0,H_0

Output: Optimal trading strategy π_0^*

Initialization;

Generating n_r paths of $B_t^{m,S}$, $B_t^{m,H}$,S_t^m, H_t^m,$S_t^{m,S}$,$S_t^{m,H}$ $\quad for \quad m = 1...n_r$;

while $t = T - \Delta t$ **do**

  for $m = 1...n_r$ **do**

    Directly compute optimal allocation $\eta_{T-\Delta t}^m$ with Equation (13) where $P_k = 0$ at time T;

    for $n = 1...N$ **do**

      Generate $\hat{S}_T^{m,n,S}$ and $\hat{S}_T^{m,n,H}$ given $S_{T-\Delta}^{m,S}$ and $S_{T-\Delta}^{m,H}$;

    Compute wealth $\hat{W}_T^{m,n}(\eta_{T-\Delta t}^m)$ at the terminal time given the wealth at $W_{T-\Delta t} = W_0$, the transformed value function is estimated by $\hat{v}^m = \ln\left[(1-\gamma)\frac{1}{N}\sum_{n=1}^{N} U(\hat{W}_T^{m,n}(\pi_{T-\Delta t}^m))\right] - (1-\gamma)\ln W_0$;

  Regress $\hat{v}^m$ over the polynomial of $H_{T-\Delta t}^m$ and $\ln S_{T-\Delta t}^m$, and obtain the function $L_{T-\Delta t}(H, \ln S)$;

for $t = T - 2\Delta t$ *to* Δt **do**

  for $m = 1...n_r$ **do**

    Directly compute optimal allocation η_t^m with Equation (13) where $P_k = L_{t+\Delta t}(H, \ln S)$;

    for $n = 1...N$ **do**

      Generate $\hat{S}_{t+\Delta t}^{m,n}$, $\hat{H}_{t+\Delta t}^{m,n}$, $\hat{S}_{t+\Delta t}^{m,n,S}$ and $\hat{S}_{t+\Delta t}^{m,n,H}$ given S_t^m, H_t^m, $S_t^{m,S}$ and $S_t^{m,H}$;

    Compute wealth $\hat{W}_{t+\Delta t}^{m,n}(\eta_t^m)$ at the terminal given the wealth at $W_t = W_0$, the transformed value function is estimated by $\hat{v}^m = \ln\left[\frac{1}{N}\sum_{n=1}^{N}(W_{t+\Delta t}^{m,n}(\pi_t^m))^{1-\gamma} exp(L_{t+\Delta t}(\hat{H}_{t+\Delta t}^{m,n}, \ln \hat{S}_{t+\Delta t}^{m,n}))\right] - (1-\gamma)\ln W_0$;

  Regress $\hat{v}^m$ over the polynomial of H_t^m and $\ln S_t^m$, and obtain the function $L_t(H, \ln S)$;

while $t = 0$ **do**

  η_0^* is obtained with Equation (13) and where the $P_k = L_{\Delta t}(H, \ln S)$;

  Apply approximation methods and obtain the price of $O_0(H_0, \ln S_0)$ as well as its sensitivity $\frac{\partial O_0}{\partial S_0}(H_0, \ln S_0)$ and $\frac{\partial O_0}{\partial H_0}(H_0, \ln S_0)$;

  Compute the variance matrix Σ_0, and the optimal allocation $\pi_0^* = \Sigma_0(\Sigma_0^T\Sigma_0)^{-1}\eta_0^*$;

return π_0^*

Table 1. Notation and definitions.

Notation	**Meaning**
$B_t^{m,S}$,$B_t^{m,H}$	Brownian motion at time t in mth simulated path
S_t^m	Stock price at time t in mth simulated path
$S_t^{m,S}$	Pure factor asset S_t^S at time t in mth simulated path
$S_t^{m,H}$	Pure factor asset S_t^H at time t in mth simulated path
H_t^m	State variable Stock price at time t in mth simulated path
O_t^m	Derivatives price at time t in mth simulated path
n_r	Number of simulated paths
Σ_t^m	Variance matrix of portfolio composition at time t in mth simulated path
N	Number of simulations to compute expected utility for a given set (W_0, S_t^m, H_t^m)
$\hat{W}_{t+\Delta t}^{m,n}(\pi^m)$	The simulated wealth level at $t+\Delta t$ given the wealth, the allocation and other state variables at t are W_0, π^m, S_t^m, and H_t^m
$\hat{S}_{t+\Delta t}^{m,n}$	A simulated stock price at $t+\Delta t$ given S_t^m
$\hat{H}_{t+\Delta t}^{m,n}$	A simulated state variable at $t+\Delta t$ given H_t^m
$\hat{O}_{t+\Delta t}^{m,n}$	A simulated option price at $t+\Delta t$
$V(t, W, \ln S, H)$	Value function at time t given wealth W, stock price S and state variable H
$\hat{v}^m$	Estimation of $P_k(t, \ln S_t^m, H_t^m) = \log(f(t, \ln S_t^m, H_t^m))$ in (12). Regress and in regression; superscript m indicates the corresponding regressor $(\ln S_t^m, H_t^m)$
$L_t(H, \ln S)$	The regression function to be used to approximate $P_k(t, \ln S, H)$
η_t^m	Optimal strategy at time t in mth simulated path

4. Derivatives Selection

In this section, we study derivative selection for market completion-that is, (11)—for $n = 2$ within subsets of the derivative set $\Omega_O^{(2)}$. The derivative selection problem is rewritten as

$$\min_{\bar{O}_t \in \Omega_O^{(2,C)}} \left\| \arg\max_{\pi_t \in \Omega_\pi^{(O)}} \mathbb{E}(U(W_T) \mid \mathcal{F}_t) \right\|_1, \tag{14}$$

where $\Omega_O^{(2,C)}$ is a derivative set defined by

$$\Omega_O^{(2,C)} = \left\{ \bar{O}_t = [S_t, O_t^{(C)}]^T | O_t^{(C)} \in C, t \in [0, T] \right\}.$$

The portfolio composition $\bar{O}_t \in \Omega_O^{(2,C)}$ consists of a stock S_t and a derivative security $O_t^{(C)}$; superscript C represents the candidate set of derivative type; and T_{op} denotes the time to maturity of $O_t^{(C)}$. This setting coincides with a popular practical strategic investment implementation (i.e., eliminating unhedgeable risk factors of a pure-stock portfolio with financial derivatives). We use the Heston SV model given in (15) as the proxy of the market dynamics.

$$\begin{cases} \frac{dM_t}{M_t} = rdt \\ \frac{dS_t}{S_t} = (r + \lambda X_t)dt + \sqrt{X_t} dB_t^S \\ dX_t = \kappa^X(\theta^X - X_t)dt + \sigma^X \sqrt{X_t} dB_t^X \\ dO_t^{(C)} = (rO_t^{(C)} + \lambda \frac{\partial O_t^{(C)}}{\partial S_t} S_t X_t + \lambda^X \frac{\partial O_t^{(C)}}{\partial X_t} \sigma^X X_t)dt + \frac{\partial O_t^{(C)}}{\partial S_t} S_t \sqrt{X_t} dB_t^S + \frac{\partial O_t^{(C)}}{\partial X_t} \sigma^X \sqrt{X_t} dB_t^X \\ < B^S, B^X >_t = \rho_{SX} \end{cases} \tag{15}$$

The Heston model is a specific case of the generalized diffusion model (1) with $\lambda^S = \lambda\sqrt{X_t}$, $\lambda^H = \lambda^X\sqrt{X_t}$, $\sigma^S = \sqrt{X_t}$, $\mu^X = \kappa^X(\theta^X - X_t)$ and $\sigma^H = \sigma^X\sqrt{X_t}$. We employed a representative market-calibrated set of parameters (see Table 2), given in Liu and Pan (2003); see also Escobar and Gschnaidtner (2016) for a review of parameter values. The optimal allocation for the model (15) can be written explicitly with Equations (13) and (10) as follows

$$
\begin{aligned}
\pi_t^S &= \frac{1}{\gamma(1-\rho_{SX}^2)}(\lambda - \rho_{SX}\lambda^X) - \pi_t^O \frac{S_t}{O_t^{(C)}} \frac{\partial O_t^{(C)}}{\partial S_t} \\
\pi_t^O &= \left(\frac{O_t^{(C)}}{\gamma\sigma^X(1-\rho_{SX}^2)}(\lambda^X - \rho_{SX}\lambda) + \frac{O_t^{(C)}}{\gamma}\frac{\partial P_k}{\partial X_t} \right) \frac{1}{\frac{\partial O_t^{(C)}}{\partial X_t}}.
\end{aligned} \tag{16}
$$

The representation indicates that the optimal allocation on option π_t^O solely depends on the choice of option $O_t^{(C)}$ (i.e., π_t^O is a function of the option's sensitivity to the instantaneous variance and option price). In contrast, the optimal allocation on the stock π_t^S is determined by the ratio of the option's sensitivity to the instantaneous variance and the sensitivity to the stock.

Table 2. Parameter value for the Heston model.

Parameter	**Value**	**Parameter**	**Value**
T	1 year	ρ_{SX}	-0.4
θ^X	0.0169	σ^X	0.25
κ^X	5.0	λ	4.0
λ^X	-7.1	T_{op}	0.1 year
Δt	$\frac{1}{60}$	*period*	60
r	0.05	X_0	θ^X
S_0	1.0	M_0	1.0
W_0	1	γ	4
N	2000	n_r	100

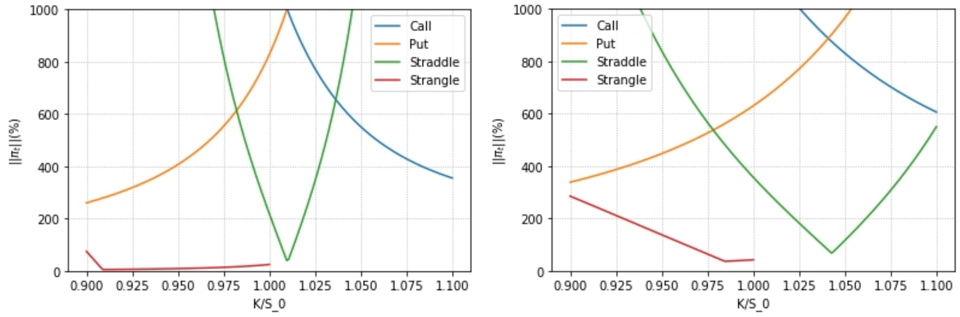
4.1. Derivatives Selection within Options on Stock

We start the selection among four popular equity options. Specifically, the candidate set is given by[4]

$$C = \{\textit{Call option, Put option, Straddle, Strangle}\}.$$

For simplicity, we only consider European-style derivatives. Call (i.e., payoff $(S-K)^+$) and put (i.e., payoff $(K-S)^+$) options are the most common products traded in the market. Additionally, a straddle (i.e., payoff $(S-K)^+ + (K-S)^+$) is a commonly used product when investors expect the underlying asset to deviate from the spot price; hence, the long position of a straddle is approximately a long volatility position. Compared with a straddle synthesized by purchasing a call and a put with the same strike price and maturity, a strangle (i.e., payoff $(S-K_1)^+ + (K_2-S)^+$) has a more flexible structure, as it takes long positions on out-of-the-money (OTM) put and call, which is a cheaper way to acquire exposure to volatility[5].

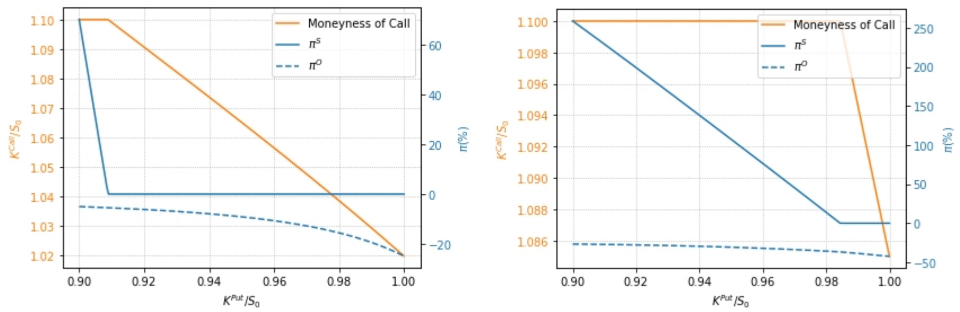
Figure 1 displays the risk exposure $\|\pi_t\|_1$ of portfolios as a function of derivative moneyness K/S_0, where K is the strike price of the options. Figure 1a exhibits risk exposure given options with maturity $T_{op} = 0.1$, and Figure 1b displays results when the option maturity is $T_{op} = 0.5$. In both cases, investors reduce their risk exposure with OTM put and call options. Puts and calls could lead to illiquid choices, whereas a straddle achieves minimum $\|\pi_t\|_1$ when near at-the-money (ATM). The optimal moneyness of a straddle option shifts to the right as maturity T_{op} increases. The risk exposure with a strangle decreases as its component put option moves deeper OTM. Furthermore, even the strangle consisting of a near-ATM put and call outperforms other options. We consequently conclude that the strangle minimizes the risk exposure.



(**a**) Maturity = 0.1 (**b**) Maturity = 0.5

Figure 1. The $\|\pi_t\|_1$ versus moneyness: The Y-axis is the risk exposure of a portfolio containing different derivatives. The X-axis indicates the moneyness K/S_0 of calls, puts, and straddles. The strangle is synthesized with an OTM put and an OTM call. Given the moneyness of the OTM put indicated by the X-axis, the strike price of the OTM call is the one achieving minimum $\|\pi_t\|_1$ within the range $[S_0, 110\%S_0]$.

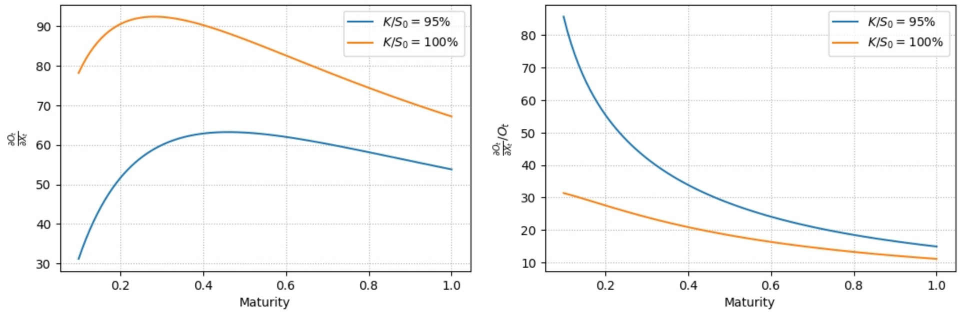
The turning point on the left tail of the strangle's risk exposure in Figure 1 is further studied in Figure 2, where we illustrate how the optimal moneyness of an OTM call, an allocation on stock π_t^S and an allocation on strangle π_t^O vary with the moneyness of an OTM put. Note the practical range selected for the moneyness of an OTM call; that is, $K^{Call}/S_0 \in [S_0, 110\%S_0]$. It is shown that if the put option's strike price, starting at the spot price, moves in the direction of OTM, the corresponding optimal moneyness of the call option also becomes deeply OTM. The OTM call reaches the boundary earlier than the put, which leads to the turning point. Before the turning point, allocation on the stock π_t^S continues to be minor, and π_t^O gradually approaches 0; hence, the total risk exposure $\|\pi_t\|_1$ assumes a decreasing trend. However, π_t^S increases rapidly after the turning point, and $\|\pi_t\|_1$ consequently rises as π_t^O continues to drop. Moreover, Figure 2a,b compare strangles with maturity $T_{op} = 0.1$ and $T_{op} = 0.5$, respectively. The turning point for a longer maturity strangle is more easily reached, which makes it less preferable.



(**a**) Maturity = 0.1 (**b**) Maturity = 0.5

Figure 2. Impact of the OTM put's moneyness on the strangle. The left vertical axis indicates the optimal moneyness of the OTM call within $[100\%S_0, 110\%S_0]$. The right vertical axis indicates the allocation of the stock and the strangle.

Equation (16) demonstrates that the allocation on the option is determined by the ratio of the Vega to the option price. Therefore, in Figure 3, we investigate the relationship between the Vega of the strangle and the time to maturity to provide further insight for the comparison of maturity in Figures 1 and 2. Figure 3a illustrates the Vega versus the maturity of an ATM strangle (the moneyness of component put option $K/S_0 = 100\%$), and an OTM strangle (the moneyness of component put option $K/S_0 = 95\%$). For an especially short-term maturity strangle, the terminal payoff does not have sufficient time to react to the change in the volatility state. Therefore, the Vega is small. For the long-term maturity strangle, a change in the instantaneous variance also has a negligible impact on the option price because of its mean-reverting nature. Hence, the Vegas of both strangles are concave in time to maturity, which peaks at around 0.3 years. The impact from time to maturity on the ratio of Vega to price is illustrated in Figure 3b, where $\frac{\partial O_t^{(C)}}{\partial X_t} / O_t^{(C)}$ is always positive and monotonically decreases with maturity, which leads to an increasing $|\pi_t^O|$. In Figure 2, π_t^S is close to 0 before the boundary, and $|\pi_t^O|$ increases with maturity; hence, we conclude that a short-term maturity strangle is preferable.



(**a**)Vega $\frac{\partial O_t^{(C)}}{\partial X_t}$ (**b**)Vega to Price $\frac{\partial O_t^{(C)}}{\partial X_t} / O_t^{(C)}$

Figure 3. Sensitivity of strangle option price $O_t^{(C)}$ to instantaneous variance X_t versus time to maturity T_{op}. The legend indicates the moneyness of the component put, and the call option achieves minimum risk exposure. Please note that there is no boundary for the strike price of the component call.

4.2. Derivatives Selection within VIX Products

Next, we study an investor accessing the VIX of the stock at hand, such as the VIX for the S&P 500. In this case, the investor can directly access the volatility risk by investing in products based on the VIX. The VIX has drawn investors' attention since its origin in 1993; not only is it a real-time indicator of the market sentiment, but also products such as VIX futures and VIX options are popular for hedging volatility risk. In this section, we explore products on the VIX. We consider a candidate set

$$C = \{\textit{VIX call, VIX put, VIX straddle, Strangle}\}.$$

Please note that a strangle is the best option for minimizing risk exposure in Section 4.1. VIX calls and VIX puts are call and put options, respectively, based on the value of the VIX. A VIX straddle is an instrument synthesized by the long position of a VIX call and a VIX put with the same strike price.

Given the definition of VIX as specified in the CBOE white paper (CBOE 2003), Lin (2007) solved the VIX^2 in closed form as a function of instantaneous variance X_t. Under the Heston model, we have

$$VIX_t^2 = \frac{1}{\tau}(a_\tau X_t + b_\tau)$$
$$a_\tau = \frac{1 - \exp -\kappa_v^* \tau}{\kappa_v^*}, \quad b_\tau = \theta_v^*(\tau - a_\tau) \quad \kappa_v^* = \kappa_v + \lambda^X \sigma_v, \quad \theta_v^* = \frac{\kappa_v \theta_v}{\kappa_v^*}, \quad \tau = \frac{30}{365}, \tag{17}$$

where VIX_t^2 is linear with the instantaneous variance X_t. Computing a VIX option's price and Greeks is easy via Monte Carlo simulation, enabling us to find elements in the variance matrix Σ_t.

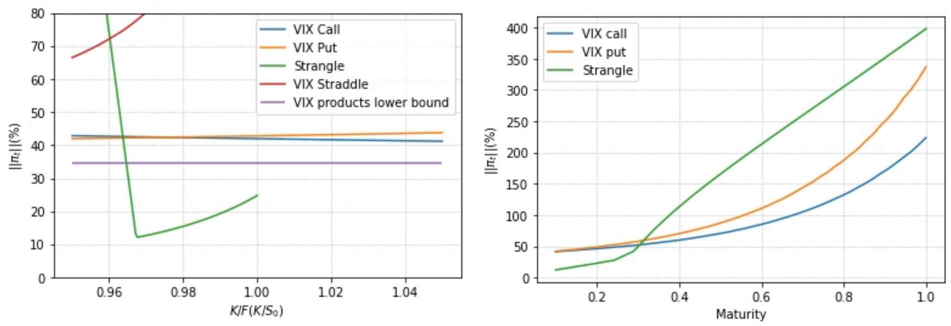
Unlike options on the stock, by investing in VIX products, the investor acquires exposure only on the volatility risk; hence, the variance matrix Σ_t is diagonal. Moreover, the equity-neutral position of VIX products leads to a specific case of (16):

$$
\begin{aligned}
\pi_t^S &= \frac{1}{\gamma(1-\rho_{SX}^2)}(\lambda - \rho_{SX}\lambda^X) \\
\pi_t^O &= \left(\frac{O_t^{(C)}}{\gamma\sigma^X(1-\rho_{SX}^2)}(\lambda^X - \rho_{SX}\lambda) + \frac{O_t^{(C)}}{\gamma}\frac{\partial P_k}{\partial X_t} \right) \frac{1}{\frac{\partial O_t^{(C)}}{\partial X_t}}.
\end{aligned} \tag{18}
$$

In this case, the allocation on the stock is invariant to the choice of VIX products, which thus becomes a natural lower bound for risk exposure (i.e., $\|\pi_t\|_1 \geq |\pi_t^S|$).

The risk exposure when investors hedge the volatility risk with VIX calls and puts is displayed in Figure 4a. On the one hand, calls and puts on the VIX have properties similar to those on the stock: OTM options tend to achieve smaller risk exposure. On the other hand, a VIX straddle is less efficient in hedging the volatility risk because it is relatively insensitive to the volatility, and a more significant risk exposure $\|\pi_t\|_1$ is needed for investors compared to the cases of VIX calls and puts. The risk exposure with the equity strangle is displayed for comparison purposes; here, the turning point resulting from the boundary of moneyness on the OTM call is still evident. Moreover, the strangle achieves a much smaller risk exposure than the VIX products. We therefore conclude that equity strangle is superior when the time to maturity T_{op} for candidate products is minor ($T_{op} = 0.1$).

Figure 4b illustrates how the option maturity T_{op} affects the risk exposure $\|\pi_t\|_1$. It indicates that an OTM VIX call and an OTM VIX put are preferable in (a), and a similar conclusion is verified numerically for any $T_{op} \in (0, 1]$. Therefore, risk exposure for the best VIX call ($K = 105\% S_0$) and VIX put ($K = 95\% S_0$) are plotted in Figure 4b. In addition, the minimum risk exposure within a pre-specified region of moneyness is also displayed. As the volatility time series exhibits a mean-reverting property, the VIX options with long-term maturity are insensitive to the instantaneous variance; hence, it has little effect in hedging the volatility risk. The figure also suggests that a large allocation on the long-term maturity VIX option is needed, such that the risk exposure increases rapidly with maturity. A strangle achieves smaller risk exposure when short-term maturity products are available in the market, aligning with the result in Figure 4a.



(**a**) (**b**)

Figure 4. The $\|\pi_t\|_1$ for VIX products. (**a**) The $\|\pi_t\|_1$ versus moneyness: the time to maturity of both VIX options and the strangle is 0.1 year. The X-axis indicates the moneyness of VIX options and the OTM put in the strangle; the strike price of the OTM call is the one achieving minimum $\|\pi_t\|_1$ within the range $[S_0, 105\% S_0]$. (**b**) The $\|\pi_t\|_1$ versus maturity: the strike price of VIX calls is $105\% S_0$. The strike price of VIX puts is $95\% S_0$. The green line shows the smallest $\|\pi_t\|_1$ is achieved by the strangle given the OTM put strike price $K^{Put} \in [95\% S_0, 100\% S_0]$ and the OTM call strike price $K^{Call} \in [100\% S_0, 105\% S_0]$.

According to Figures 1 and 2, the boundary of the OTM call is reached faster as T_{op} increases, and the boundary significantly restricts risk exposure, thus reducing the effect of the strangle. This leads to a steep slope of risk exposure for the strangle in Figure 4b. In summary, depending on the situation, the investor should choose between VIX products and an equity strangle. The strangle is preferable if the investor has access to short-term maturity options. However, when only long-term maturity products are available, investors should choose call options on the VIX for market completion.

5. Additional Discussion

In principle, any financial derivative and the underlying stock can generate a portfolio that maximizes its utility due to the market's completeness, as the derivatives act to hedge the presence of stochastic volatility. However, our findings convey the importance of properly completing a financial portfolio. As illustrated in Figure 1, even though investors can maximize their utility by adding a call option or a put option to their portfolio, that decision could come with exposure to risky assets of over 1000%. Moreover, if investors want to reduce that exposure while using standard calls and puts, they must acquire exotic products (e.g., out-of-the-money), which could be hard to sell in illiquid markets. In reality, most investors would not consider exposures higher than 200% as it would involve a great level of debt in case of a market crash. As we can see in the same figure, the derivative choice is very much related to the horizon for the investment; long maturities could make Straddles as appealing as Strangles, shining a new light on the importance of these derivatives.

As our model involves stochastic volatility and uses the S&P500 as the exemplary underlying, we also explore the importance of adding the Volatility Index (VIX) into the investor's portfolio. As VIX is not directly tradeable, we consider options on the VIX as the source of market completion. Figure 4 reveals that using the VIX is sometimes beneficial. Still, not always; therefore, investors should do their own assessments along the lines of our methodology to select the proper derivatives in the market to complete their portfolios.

6. Conclusions

This paper explored optimal derivatives-based portfolios to complete a market characterized by volatility risk as a state variable. An accurate and high-speed approximation for optimal allocations is proposed for the unsolvable problem of optimal derivative exposure. In addition to the traditional portfolio decision objective (i.e., EUT maximization), we work with an additional criterion, risk exposure minimization, for derivative selection. This aids in selecting a meaningful product out of many that maximizes the utility. We found that strangle options are the best equity option product for managing volatility risk. Moreover, we demonstrated that options based on the VIX are superior to equity strangles in some realistic situations.

There are many interesting potential extensions to this line of research. For instance, investors could incorporate multi-factor models that consider stochastic interest rates, stochastic correlations, jumps, and stochastic market prices of risk, to mention a few. Selecting the proper derivatives in such rich settings is likely a more challenging task. Still, at the same time, failure to act as per our recommendations could lead to either quite high exposures to risky assets or low-performing portfolios. These are more realistic, solvable settings within our numerical method, providing investors valuable insight into optimal high-dimensional portfolios and multi-asset derivatives for sensible, practical investment.

Author Contributions: M.D.: Methodology, Validation, Investigation, Resources, Writing—Review & Editing, Supervision, Project administration. M.E.-A.: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing—Original Draft, Writing—Review & Editing, Supervision, Project administration. Y.Z.: Methodology, Software, Validation, Formal analysis, Investigation, Data Curation, Writing—Original Draft, Writing—Review & Editing, Visualization. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Proofs

Appendix A.1. Proof of Proposition 1

Let $O_{t,n} = [O_t^{(1)}, O_t^{(2)}, ..., O_t^{(n)}]^T$ with variance matrix Σ_t of rank 2 be an optimal subset of options for problem (11). $\pi^*_{t,n}$ is a strategy maximizing the expected utility if and only if $\Sigma_t^T \pi^*_{t,n} = \eta^*_t$. Therefore, $O_{t,n}$ and $\pi^*_{t,n}$ is an optimal pair for (11) when $\pi^*_{t,n}$ is an optimal solution for

$$\begin{aligned} \underset{\pi_t}{\text{minimize}} \quad & ||\pi_t||_1 \\ \text{subject to} \quad & \Sigma_t^T \pi_t = \eta^*_t \end{aligned} \tag{A1}$$

According to principle 4.5 in Rardin and Rardin (1998), problem (A1) is equivalent to

$$\begin{aligned} \underset{\delta_t}{\text{minimize}} \quad & \mathbb{1}^T \delta_t \\ \text{subject to} \quad & \hat{\Sigma}_t^T \delta_t = \eta^*_t, \\ & \delta_t \geq 0 \end{aligned} \tag{A2}$$

where $\delta_t = [\alpha_t^{(1)}, \alpha_t^{(2)}, ..., \alpha_t^{(n)}, \beta_t^{(1)}, \beta_t^{(2)}, ..., \beta_t^{(n)}]^T$ satisfies $\alpha_t^{(i)} = \frac{|\pi_t^{(i)}| + \pi_t^{(i)}}{2}$, and $\beta_t^{(i)} = \frac{|\pi_t^{(i)}| - \pi_t^{(i)}}{2}$, with

$$\hat{\Sigma}_t = \begin{bmatrix} \Sigma_t \\ -\Sigma_t \end{bmatrix} = \begin{bmatrix} f_t^{11} & f_t^{12} \\ ... & ... \\ f_t^{n1} & f_t^{n2} \\ -f_t^{11} & -f_t^{12} \\ ... & ... \\ -f_t^{n1} & -f_t^{n2} \end{bmatrix}. \tag{A3}$$

Theorems 2.3 and 2.4 in Bertsimas and Tsitsiklis (1997) lists the necessary and sufficient conditions for the extreme point δ_t, i.e.

1. $\delta_t = [\delta_t^{(1)}, \delta_t^{(2)}, ..., \delta_t^{(n)}, \delta_t^{(n+1)}, \delta_t^{(n+2)}, ..., \delta_t^{(2n)}]^T$.
2. the $\hat{q}$th and $\hat{p}$th rows in $\hat{\Sigma}_t$ are linear independent, $\delta_t^{(i)} = 0$ if $i \neq \hat{q}$ or $\hat{p}$.
3. δ_t is feasible solution.

Without loss of generality, we assume the pth and qth rows in Σ are linear independent, and we consider 4 cases:

$$\begin{aligned} \delta_t^{[1]} &= \begin{cases} [\delta_t^{[1],(1)}, \delta_t^{[1],(2)}, ..., \delta_t^{[1],(n)}, \delta_t^{[1],(n+1)}, \delta_t^{[1],(n+2)}, ..., \delta_t^{[1],(2n)}]^T \\ \delta_t^{[1],(i)} = 0 \quad \textit{if } i \neq q \textit{ or } p \end{cases} \\ \delta_t^{[2]} &= \begin{cases} [\delta_t^{[2],(1)}, \delta_t^{[2],(2)}, ..., \delta_t^{[2],(n)}, \delta_t^{[2],(n+1)}, \delta_t^{[2],(n+2)}, ..., \delta_t^{[2],(2n)}]^T \\ \delta_t^{[2],(i)} = 0 \quad \textit{if } i \neq q + n \textit{ or } p \end{cases} \\ \delta_t^{[3]} &= \begin{cases} [\delta_t^{[3],(1)}, \delta_t^{[3],(2)}, ..., \delta_t^{[3],(n)}, \delta_t^{[3],(n+1)}, \delta_t^{[3],(n+2)}, ..., \delta_t^{[3],(2n)}]^T \\ \delta_t^{[3],(i)} = 0 \quad \textit{if } i \neq q \textit{ or } p + n \end{cases} \\ \delta_t^{[4]} &= \begin{cases} [\delta_t^{[4],(1)}, \delta_t^{[4],(2)}, ..., \delta_t^{[4],(n)}, \delta_t^{[4],(n+1)}, \delta_t^{[4],(n+2)}, ..., \delta_t^{[4],(2n)}]^T \\ \delta_t^{[4],(i)} = 0 \quad \textit{if } i \neq q + n \textit{ or } p + n \end{cases} \end{aligned} \tag{A4}$$

There is a non-negative strategy in $\delta_t^{[1]}$, $\delta_t^{[2]}$, $\delta_t^{[3]}$ and $\delta_t^{[4]}$ because the ith row in $\hat{\Sigma}$ is the opposite of the $(i+n)$th row, and the non-negative strategy is feasible and an extreme point. This proves the existence of an extreme point for problem (A2). Now, Theorem 2.7 in Bertsimas and Tsitsiklis (1997) guarantees an optimal solution, an extreme point for problem (A2).

With the second necessary and sufficient conditions of the extreme point, we know that an optimal solution δ_t^* for problem (A2) has at most two non-zero elements. This would imply an optimal solution, denoted by $\pi_{t,n}^* = [\pi_{t,n}^{(1)}, \pi_{t,n}^{(2)}, ..., \pi_{t,n}^{(n)}]^T$, for problem (A1) with at most two non-zero elements, which would also be the optimal strategy for (11).

Without loss of generality, we assume $\pi_{t,n}^{(i)} = 0,\ i \neq x, y$. $O_{t,2} = [O_t^{(x)}, O_t^{(y)}]$ and $\pi_{t,2}^* = [\pi_{t,n}^{(x)}, \pi_{t,n}^{(y)}]^T$ is a feasible strategy for problem (11) with $n = 2$. We show that it is an optimal pair by contradiction.

If there is a feasible solution $\hat{O}_{t,n} = [\hat{O}_t^{(1)}, \hat{O}_t^{(2)}]$ and $\hat{\pi}_{t,2}^* = [\hat{\pi}_{t,2}^{(1)}, \hat{\pi}_{t,2}^{(2)}]^T$ such that $||\hat{\pi}_{t,2}^*||_1 < ||\pi_{t,2}^*||_1$, then $\hat{\pi}_{t,n}^* = [\hat{\pi}_{t,2}^{(1)}, \hat{\pi}_{t,2}^{(2)}, 0, ..., 0]^T$ is a feasible strategy for (11) such that $||\hat{\pi}_{t,n}^*||_1 < ||\pi_{t,n}^*||_1$, which is contradiction to our previous conclusion. Note that $||\pi_{t,2}^*||_1 = ||\pi_{t,n}^*||_1$, so problem (11) with $n = 2$ and with $n \geq 2$ have the same minimum ℓ_1 norm of allocation.

Appendix A.2. Proof of Proposition 2

According to the Bellman equation, the value function can be rewritten as,

$$\begin{aligned} V(t, W, H, \ln S) &= \mathbb{E}_t(V(t+dt, W_{t+dt}, H_{t+dt}, \ln S_{t+dt}) \mid W, H, \ln S) \\ &= \max_{\eta_t} \mathbb{E}_t(V(t+dt, W_{t+dt}, H_{t+dt}, \ln S_{t+dt}) \mid W, \eta, H, \ln S). \end{aligned} \tag{A5}$$

We expand $V(t+dt, W_{t+dt}, H_{t+dt}, \ln S_{t+dt})$ at $t+dt$ in terms of all the variables.

$$\begin{aligned} &V(t+dt, W_{t+dt}, H_{t+dt}, \ln S_{t+dt}) = V(t+dt, W_t, \ln S_t, H_t) + V_{W_t}(t+dt, W_t, H_t, \ln S_t)dW_t \\ &+ \frac{1}{2}V_{W_tW_t}(t+dt, W_t, H_t, \ln S_t)(dW_t)^2 + V_{\ln S_t}(t+dt, W_t, H_t, \ln S_t)d\ln S_t + V_{H_t}(t+dt, W_t, H_t, \ln S_t)dH_t \\ &+ \frac{1}{2}V_{\ln S_t \ln S_t}(t+dt, W_t, H_t, \ln S_t)d\ln S_t d\ln S_t + \frac{1}{2}V_{H_tH_t}(t+dt, W_t, \ln S_t, H_t)dH_t dH_t \\ &+ V_{W_t \ln S_t}(t+dt, W_t, H_t, \ln S_t)dW_t d\ln S_t + V_{W_tH_t}(t+dt, W_t, H_t, \ln S_t)dW_t dH_t \\ &+ V_{\ln S_t H_t}(t+dt, W_t, H_t, \ln S_t)d\ln S_t dH_t + o(dt). \end{aligned} \tag{A6}$$

Substituting dW_t, $d\ln S_t$, dH_t which can be found in Equation (1), taking conditional expectation on both sides, and rewriting $V(t, W_t, H_t, \ln S_t)$ in a quadratic form with respect to η leads to

$$\begin{aligned} V(t, W_t, H_t, \ln S_t) &= \max_{\eta_t}\left(\sum_{i,j=1}^{2} f_{i,j}(t, W_t, , H_t \ln S_t)\eta_t^{(i)}\eta_t^{(j)} + \sum_{i=1}^{2} f_i(t, W_t, H_t, \ln S_t)\eta_t^{(i)} + f_0(t, W_t, H_t, \ln S_t)\right) \\ f_{i,j}(t, W_t, H_t, \ln S_t) &= \frac{1}{2}V_{W_tW_t}(t+dt, W_t, H_t, \ln S_t)W_t^2(\Phi\Phi^T)_{i,j}dt \\ f_i(t, W_t, H_t, \ln S_t) &= V_{W_t}(t+dt, W_t, H_t, \ln S_t)W_t\Lambda_i dt + V_{W_t \ln S_t}(t+dt, W_t, H_t, \ln S_t)W_t\sigma_S B_i dt \\ &+ V_{W_tH_t}(t+dt, W_t, H_t, \ln S_t)W_t\sigma_H A_i dt \\ f_0(t, W_t, H_t, \ln S_t) &= V(t+dt, W_t, H_t, \ln S_t) + V_{W_t}(t+dt, W_t, H_t, \ln S_t)W_t r dt \\ &+ V_{\ln S_t}(t+dt, W_t, H_t, \ln S_t)(r + \lambda^S\sigma^S - \frac{1}{2}(\sigma^S)^2)dt + V_{H_t}(t+dt, W_t, H_t, \ln S_t)\mu^H dt \\ &+ \frac{1}{2}V_{H_tH_t}(t+dt, W_t, H_t, \ln S_t)(\sigma^H)^2 dt + \frac{1}{2}V_{\ln S_t \ln S_t}(t+dt, W_t, H_t, \ln S_t)(\sigma^S)^2 dt \\ &+ V_{\ln S_t H_t}(t+dt, W_t, H_t, \ln S_t)\rho_{SH}\sigma^S\sigma^H dt. \end{aligned} \tag{A7}$$

We assume a sufficiently small dt so that $o(dt)$ terms are omitted when taking conditional expectations. The solution to the system of equations gives the optimal allocation:

$$\sum_{j=1}^{2} 2f_{i,j}(t, W_t, H_t, \ln S_t)\eta_t^{(*,j)} = -f_i(t, W_t, H_t, \ln S_t),\ i = 1,2. \tag{A8}$$

With the representation of the value function in Equation (12) and assuming that $f(t, H, \ln S) = exp(P_k(t, H, \ln S))$, the derivatives of value function with respect to each stock and state variable can be rewritten as,

$$\begin{aligned}
V_W(t+dt, W_t, H_t, \ln S_t) &= W_t^{-\gamma} exp(P_k(t+dt, H_t, \ln S_t)) \\
V_{WW}(t+dt, W_t, H_t, \ln S_t) &= -\gamma W_t^{-\gamma-1} exp(P_k(t+dt, H_t, \ln S_t)) \\
V_{W\ln S_t}(t+dt, W_t, H_t, \ln S_t) &= W_t^{-\gamma} exp(P_k(t+dt, H_t, \ln S_t)) \frac{\partial P_k(t+dt, H_t, \ln S_t)}{\partial \ln S_t} \\
V_{WH_t}(t+dt, W, H_t, \ln S_t) &= W_t^{-\gamma} exp(P_k(t+dt, H_t, \ln S_t)) \frac{\partial P_k(t+dt, H_t, \ln S_t)}{\partial H_t}.
\end{aligned} \tag{A9}$$

Substituting (A9) into (A8), the optimal strategy can be approximated as follows:

$$\begin{aligned}
&\sum_{j=1}^{2} g_{i,j}(t, W_t, H_t, \ln S_t)\eta_t^{(*,j)} = g_i(t, W_t, H_t, \ln S_t),\ i = 1,2 \\
&\quad g_{i,j}(t, W_t, H_t, \ln S_t) = \gamma(\Phi\Phi^T)_{i,j} \\
&\quad g_i(t, W_t, H_t, \ln S_t) = \Lambda_i + \frac{\partial P_k(t+dt, H_t, \ln S_t)}{\partial \ln S_t}\sigma^S B_i + \frac{\partial P_k(t+dt, H_t, \ln S_t)}{\partial H_t}\sigma^S A_i,
\end{aligned} \tag{A10}$$

Then, the optimal strategy can be rewritten in matrix form:

$$\eta_t^* = \frac{1}{\gamma}(\Phi\Phi^T)^{-1}(\Lambda + \frac{\partial P_k}{\partial H}\sigma^H A + \frac{\partial P_k}{\partial \ln S}\sigma^S B). \tag{A11}$$

Appendix B. Alternative Approximation Method and Comparison

Appendix B.1. Direct Method

We introduced an alternative method for derivatives-based portfolio strategy, namely the PAMC-direct method, which is a straightforward application of the PAMC. At each re-balancing time, the path-wise option price O_t, Delta $\frac{\partial O_t}{\partial S_t}$ and the sensitivity to the state variable $\frac{\partial O_t}{\partial H_t}$ are approximated, so the instantaneous dynamics of derivatives are obtained. This way, derivatives can be taken as an asset with explicitly identifiable dynamics; the PAMC method is directly applied. The following proposition shows the estimation of optimal strategy π_t^* in PAMC-direct.

Proposition A1. *Given the approximation of the value function at the next re-balancing time $t + \Delta t$ (i.e., $\frac{W^{1-\gamma}}{1-\gamma}\exp\{P_k\}(t+\Delta t, H, \ln S)$), the optimal strategy at time t is given by*

$$\pi_t^* = \frac{1}{\gamma}(\Sigma_t\Phi\Phi^T\Sigma_t^T)^{-1}(\Sigma_t\Lambda + \frac{\partial P_k}{\partial H}\sigma^H\Sigma_t A + \frac{\partial P_k}{\partial \ln S}\sigma^S\Sigma_t B). \tag{A12}$$

Proof. Similar to Appendix A.2. □

We continue to use the notation in Table 1 and describe the step-by-step algorithm of the PAMC-direct in Algorithm A1.

Algorithm A1: PAMC-direct method

Input: S_0,W_0,H_0

Output: Optimal trading strategy π_0^*

1 initialization;

2 Generating n_r paths of B_t^m, $B_t^{m,H}$,S_t^m, H_t^m $\quad for \quad m = 1...n_r$;

3 Apply approximation methods and obtain the price of $O_t(H_t^m, \ln S_t^m)$ as well as its sensitivity $\frac{\partial O_t}{\partial S_t}(H_t^m, \ln S_t^m)$ and $\frac{\partial O_t}{\partial H_t}(H_t^m, \ln S_t^m)$ for $t = 0, \Delta t, ...T$;

4 **while** $t = T - \Delta t$ **do**

5   **for** $m = 1...n_r$ **do**

6   Compute the variance matrix $\Sigma_{T-\Delta t}^m$ with derivatives price and sensitivity obtained in step 3 ;

7   Directly compute optimal allocation $\pi_{T-\Delta t}^m$ with Equation (A12) where the $P_k = 0$ at time T;

8   **for** $n = 1...N$ **do**

9   Generate $\hat{S}_T^{m,n}$ and $\hat{H}_T^{m,n}$ given $S_{T-\Delta}^m$ and $H_{T-\Delta}^m$ and obtain $\hat{O}_T^{m,n}$;

10   Compute wealth $W_T^{m,n}(\pi_{T-\Delta t}^m)$ at the terminal given the wealth at $W_{T-\Delta t} = W_0$, the transformed value function is estimated by $\hat{v}^m = \ln[(1-\gamma)\frac{1}{N}\sum_{n=1}^{N} U(W_T^{m,n}(\pi_{T-\Delta t}^m))] - (1-\gamma)\ln W_0$;

11   Regress $\hat{v}^m$ over the polynomial of $H_{T-\Delta t}^m$ and $\ln S_{T-\Delta t}^m$, and obtain the function $L_{T-\Delta t}(H, \ln S)$;

12 **for** $t = T - 2\Delta t$ to Δt **do**

13   **for** $m = 1...n_r$ **do**

14   Compute the variance matrix Σ_t^m with derivatives price and sensitivity obtained in step 3;

15   Directly compute optimal allocation π_t^m with Equation (A12) where the $P_k = L_{t+\Delta t}(H, \ln S)$;

16   **for** $n = 1...N$ **do**

17   Generate $\hat{S}_{t+\Delta t}^{m,n}$ and $\hat{H}_{t+\Delta t}^{m,n}$ given S_t^m and H_t^m and obtain $\hat{O}_{t+\Delta t}^{m,n}$;

18   Compute wealth $\hat{W}_{t+\Delta t}^{m,n}(\pi_t^m)$ at the terminal given the wealth at $W_t = W_0$, the transformed value function is estimated by $\hat{v}^m = \ln[\frac{1}{N}\sum_{n=1}^{N}(W_{t+\Delta t}^{m,n}(\pi_t^m))^{1-\gamma} exp(L_{t+\Delta t}(\hat{H}_{t+\Delta t}^{m,n}, \ln \hat{S}_{t+\Delta t}^{m,n}))] - (1-\gamma)\ln W_0$;

19   Regress $\hat{v}^m$ over the polynomial of H_t^m and $\ln S_t^m$, and obtain the function $L_t(H, \ln S)$;

20 **while** $t = 0$ **do**

21   Compute the variance matrix Σ_0 with derivatives price and sensitivity obtained in step 3, and the optimal allocation π_0^* is obtained with Equation (A12) and where the $P_k = L_{\Delta t}(H, \ln S)$;

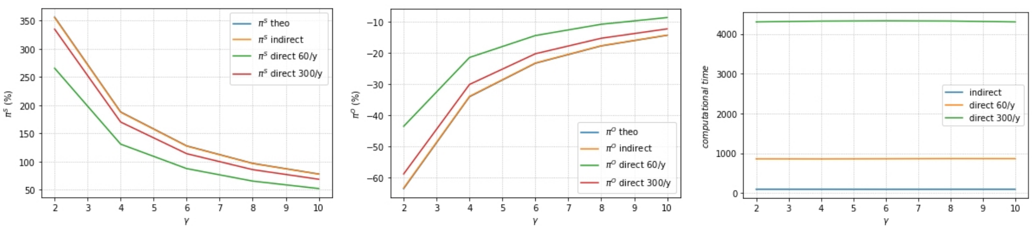
22 return π_0^*

Appendix B.2. Comparison between the PAMC-Direct Method and the PAMC-Indirect Method

In this section, we implement the PAMC-direct and PAMC-indirect methods on the Heston SV model given in (15) for comparison purposes. Given the Heston model, the derivatives-based portfolio was first studied in Liu and Pan (2003), where the author constructed a portfolio with derivative securities and stock to manage volatility risk. The optimal strategy stock-derivatives portfolio is solved in closed form. The accuracy and efficiency of the PAMC-direct and the PAMC-indirect are compared to the analytical solution.

We continue to use the market-calibrated set of parameters in Table 2. For simplicity, we let O_t be a delta-neutral straddle because the delta-neutral position keeps the straddle near-the-money, and the liquidity should not be a concern.

Figure A1a,b compare the optimal allocation on the stock and straddle across different values of risk aversion level γ. We let the re-balancing frequency of the PAMC-indirect method be 60 times per year, i.e., investors roughly adjust their positions weekly. Optimal allocation from the PAMC-indirect method and theoretical solution (re-balancing continuously) are visually overlapped; the PAMC-indirect method exhibits excellent accuracy in this case. The allocation from the PAMC-direct method with 60 re-balances per year is subject to a substantial error; on the other hand, the gap to the theoretical solution shrinks if we let the re-balancing frequency be 300 times per year (roughly daily re-balance). We expect the gap will vanish as the frequency of re-balancing continues to increase. The computational times of the PAMC-direct and PAMC-indirect methods are compared in Figure A1c; the time required for the PAMC-indirect method is significantly smaller than the time for the PAMC-direct method. The PAMC-indirect is superior to the PAMC-direct in accuracy and computational efficiency; hence, we use only the PAMC-indirect in Section 4.



(**a**) Allocation on stock (**b**) Allocation on straddle option (**c**) Computational time

Figure A1. Allocation on straddle versus γ.

Notes

[1] The methodology can be applied to more general utilities like hyperbolic absolute risk aversion (HARA) or S-curves from behavioral finance. The accuracy of the solutions shall be explored in detail.

[2] Other ideas could be explored like minimization of L_2 norm of positions, lifetime or average exposure, or total risk of selected derivatives. All proposals would have limitations, hence the need for future studies and comparison.

[3] The result can be easily extended to a higher dimension. When a model contains $m \geq 2$ independent risk factors (Brownian motions), an optimal strategy exists for Problem (11) such that the number of non-zero allocations is less than or equal to m.

[4] Many other derivatives and their combinations, or even leveraged exchange-traded fund, could be considered. Our work set the ground for further research in this area.

[5] Elements in variance matrix Σ_t, which are functions of option prices and Greeks, can be obtained with numerical integration method (see Rouah 2013, chp. 11). Specifically, we utilized the formula given in Heston (1993) and computed numerical integration with the Newton-Cotes formulas.

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