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Rainbow Step Barrier Options

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Abstract: This article provides exact analytical formulae for various kinds of rainbow step barrier options. These are highly flexible and sophisticated multi-asset barrier options based on the following principle: the option life is divided into several time intervals on which different barriers are monitored w.r.t. different underlying assets. From a mathematical point of view, new results are provided for the first passage time of a multidimensional geometric Brownian motion to a boundary defined as a step function. The article shows how to implement the obtained option valuation formulae in a simple and very efficient manner. Numerical results highlight a strong sensitivity of rainbow step barrier options to the correlations between the underlying assets.

Keywords: rainbow step barrier option; rainbow option; step barrier option; barrier option; multiasset option; multiasset barrier option; first passage time; boundary crossing probability; multidimensional Brownian motion

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1. Introduction

Barrier options are characterised by the introduction in the option contract of a parameter called the "barrier", which, in the most standard form, is a predefined reference value of the underlying asset S that may be located above the spot value of the underlying (upward barrier or "up-barrier") or below it (downward barrier or "down-barrier"). The barrier may be of a "knock-out" type, i.e., the option expires worthless if the barrier is hit by S at any time during the option life (in which case the barrier is called "continuous") or at one or a few predefined times (in which case the barrier is called "discrete"). The first passage time of an underlying to a knock-out barrier, before the option expiry, triggers the "deactivation" of the option. Alternatively, the barrier may be of a "knock-in" type, i.e., the option expires worthless unless the barrier has been hit at least once before expiry, an event called "activation".

Barrier options are the oldest and the most widely traded non-vanilla options. They are embedded in a lot of popular structured derivatives in stock and interest rate markets (see, e.g., Bouzoubaa and Osseiran 2010). They are also extensively used as analytical tools in financial modelling, for instance, in the so-called "structural models" of default risk (see, e.g., Bielecki and Rutkowski 2004) or in the valuation of investments (theory of "real options"). Since their first appearance in the financial markets during the 1970s, there have been a huge number of variations in their original payoff, leading to an extraordinary variety of non-standard barrier options. Among the most well-known of them are the step barrier options, which divide the option lifetime into several time intervals on which the barrier takes on different values. In its standard form, a step barrier option features a piecewise constant barrier, i.e., a barrier defined as a step function. This allows modulation of the barrier level during the option life, thus offering increased flexibility and enhanced risk management capabilities, relative to a traditional barrier option. For instance, up-and-out barrier levels can be raised and down-and-out barrier levels can be lowered during time intervals when more protection is required, thus reducing the risk of deactivation. Likewise, up-and-in barrier levels can be lowered and down-and-in barrier levels can be raised in time intervals during which the implicit volatility of the underlying rises, thus, increasing the chances of activation.

The exact analytical valuation of step barrier functions was first achieved by Guillaume (2001). Further mathematical details on how closed-form valuation can be achieved, as well as exact results for more general deterministic step barriers, are provided in Guillaume (2015, 2016). In the last couple of years, there has been a renewed interest in step barrier options, as they stand out as an essential component of innovative forms of investment, such as autocallable structured products and other equity-linked products. This has led to a new series of academic contributions, such as Lee et al. (2019a, 2019b), Lee et al. (2021), Lee et al. (2022). Unlike previously cited references, these articles do not provide explicit formulae, except for a few simple and already known cases, nor do they handle outside step barriers as in Guillaume (2001), alternatively upward and downward steps as in Guillaume (2015) and exponentially moving step barriers as in Guillaume (2016). These recent contributions ignore previous results given in references they do not cite, such as Guillaume (2001), that actually solve the problems they discuss. They also claim to be able to analytically value a step barrier option with an arbitrary number of steps, but without explaining how they intend to solve the difficult problem known in numerical integration as the "curse of dimensionality", nor even beginning to discuss the numerical implementation of their approach, which constitutes the main issue, though.

There are still a number of unsolved problems related to the valuation of step barrier options. In particular, multi-asset step barrier options are barely touched upon in the existing literature, apart from an isolated formula for an "outside" step barrier option given in Guillaume (2001), also called an "external" step barrier option, featuring one underlying asset w.r.t. which barrier crossing is monitored and another underlying asset w.r.t. which the moneyness of the option is measured at expiry (the reader may refer to Heynen and Kat 1994, or to Kwok et al. 1998, for background on outside barrier options in general). Yet, multi-asset contracts with step barriers are actively traded in today's financial markets, as they allow investors to benefit from the advantages of diversification in terms of risk control and expansion of investment opportunities. A particularly important subset of these contracts is the so-called "rainbow" step barrier option. Broadly speaking, in the literature on options, the denomination "rainbow" applies to payoffs linked to the performances of two or more underlying assets (Chang et al. 2005; Gao and Wu 2022); metaphorically, each underlying represents a different color, so that the association of all of these factors makes up a rainbow. In the realm of barrier options, the rainbow step barrier option is characterised by the property that, at each time interval, the barrier is monitored w.r.t. a different underlying asset. A contract featuring a number $n \in \mathbb{N}$ of underlying assets associated with n time intervals $[t_0 = 0, t_1], \ldots, [t_{n-1}, t_n]$, on which n steps of a piecewise constant barrier are monitored, is called an $n-$the colour rainbow barrier option. Each time interval is matched with a specific step of the barrier and a specific underlying asset. In the standard form of the contract, it is the $n-$th asset associated with the $n-$th last step of the barrier that is used to determine the moneyness of the contract at expiry. Rainbow step barrier options are typically priced by Monte Carlo simulation, even in a standard Black–Scholes model, because of the difficulties of the entailed analytical calculations and also because the dimension of the valuation problem quickly increases with the number of "colors", leading to non-trivial issues of numerical evaluation of high-dimensional integrals. Due to these obstacles, the present article is restricted to two-colour rainbow step barrier options. Closed-form valuation is achieved not only for standard two-colour contracts but also for two-colour outside step barrier options involving a third correlated asset at expiry, and for two-colour contracts featuring a two-sided barrier (also known as a double barrier), i.e., both an upward and a downward barrier on each time interval. Numerical evaluation of the obtained analytical solutions is dealt with so that the valuation formulae derived in this paper can be immediately implemented and yield extremely accurate results in a few tenths of one second. Numerical results are provided, which reveal a strong and stable dependency of rainbow step barrier options on the correlations between the underlying assets, as well as the importance of the volatility of the asset used to measure moneyness when dealing with rainbow outside step barrier options. These findings suggest clear application ideas to traders and investors, whether for a hedging or speculative purposes. They highlight the specificity of rainbow step barrier options as instruments highly sensitive to correlation, in contrast to standard step barrier options, which are sensitive to volatility but, by constuction, cannot be sensitive to correlation.

This article is organised as follows: Section 2 states the main analytical results, provides numerical results, and discusses their implications; Section 3 gives the mathematical proofs of the analytical results presented in Section 2.

2. Formulae and Numerical Results

Let us begin with a few definitions. Let S_1 and S_2 be two GBMs (geometric Brownian motions) modelling two asset prices, whose differentials, under a given probability measure P, are given by:

$$dS_1(t) = v_1 S_1(t)dt + \sigma_1 S_1(t)dB_1(t) \tag{1}$$

$$dS_2(t) = v_2 S_2(t)dt + \sigma_2 S_2(t)dB_2(t) \tag{2}$$

where $v_1, v_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_+$, and B_1 and B_2 are two standard Brownian motions whose correlation coefficient is denoted by $\rho_{1.2}$.

The measure P is characterised by the pair (v_1, v_2) or, equivalently, by the pair:

$$\left(\mu_1 = v_1 - \sigma_1^2/2, \mu_2 = v_2 - \sigma_2^2/2\right) \tag{3}$$

If we refer to the log-return processes $X_i(t) = \ln(S_i(t)/S_i(0))$, $i \in \{1, 2\}$, whose differentials under P are given by:

$$dX_i(t) = \mu_i dt + \sigma_i dB_i(t) \tag{4}$$

Let H_1, H_2, K_1, K_2, K_3 be positive real numbers. The numbers H_1, H_2 are the values of two knock-out continuous barriers. H_1 is monitored w.r.t. S_1 on a time interval $[t_0 = 0, t_1]$, while H_2 is monitored w.r.t. S_2 on a time interval $[t_1, t_2]$. The numbers K_1, K_2, K_3 are the values of three discrete knock-out barriers; K_1 is monitored w.r.t. S_1 at time t_1, while K_2 and K_3 are monitored w.r.t. S_2 at times t_1 and t_2, respectively.

We can now begin to value two-colour step barrier options in the following order:

- both steps either upward or downward (Section 2.1);
- one upward step and one downward step (Section 2.2);
- reverse-type contract (Section 2.3);
- outside or external two-colour step barrier (Section 2.4);
- two-colour step double barrier (Section 2.5).

2.1. Valuation of Two-Colour Step Barrier Options When the Steps of the Barrier Are on the Same Side in Each Time Interval

Section 2.1 deals with the valuation of two-colour step barrier options when the steps of the barrier are either both upward or both downward. Our objective is to find the value of the joint cumulative distribution function $\mathrm{P}_{[RUU]}(\mu_1, \mu_2)$ defined by:

$$\mathrm{P}_{[RUU]}(\mu_1, \mu_2) \triangleq P\left(\sup_{0 \le t \le t_1} S_1(t) \le H_1, S_1(t_1) \le K_1, S_2(t_1) \le K_2, \sup_{t_1 \le t \le t_2} S_2(t) \le H_2, S_2(t_2) \le K_3\right) \tag{5}$$

where the acronym "[RUU]" stands for "Rainbow Up and Up".

The main result of Section 2.1 is given by the following Proposition 1.

Proposition 1. *The exact value of* $\mathrm{P}_{[RUU]}(\mu_1, \mu_2)$ *is given by:*

$$\mathrm{P}_{[RUU]}(\mu_1, \mu_2) = N_3\left[\frac{\min(k_1, h_1) - \mu_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{\min(k_2, h_2) - \mu_2 t_1}{\sigma_2\sqrt{t_1}}, \frac{\min(k_3, h_2) - \mu_2 t_2}{\sigma_2\sqrt{t_2}}; \theta_{1.2}, \theta_{1.3}, \theta_{2.3}\right] \tag{6}$$

$$-\exp\left(\frac{2\mu_1 h_1}{\sigma_1^2}\right)$$
$$\times N_3\left[\begin{array}{l}\frac{\min(k_1,h_1)-2h_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{\min(k_2,h_2)-\mu_2 t_1}{\sigma_2\sqrt{t_1}} - \frac{2\theta_{1.2}h_1}{\sigma_1\sqrt{t_1}}, \frac{\min(k_3,h_2)-\mu_2 t_2}{\sigma_2\sqrt{t_2}} - \frac{2\theta_{1.2}h_1}{\sigma_1\sqrt{t_2}};\\ \theta_{1.2},\theta_{1.3},\theta_{2.3}\end{array}\right] \tag{7}$$

$$-\exp\left(\frac{2\mu_2 h_2}{\sigma_2^2}\right)$$
$$\times N_3\left[\begin{array}{l}\frac{\min(k_1,h_1)-\mu_1 t_1}{\sigma_1\sqrt{t_1}} + \frac{2\theta_{1.2}\mu_2\sqrt{t_1}}{\sigma_2}, \frac{\min(k_2,h_2)+\mu_2 t_1}{\sigma_2\sqrt{t_1}}, \frac{\min(k_3,h_2)-2h_2-\mu_2 t_2}{\sigma_2\sqrt{t_2}};\\ \theta_{1.2},-\theta_{1.3},-\theta_{2.3}\end{array}\right] \tag{8}$$

$$+\exp\left(\left(\frac{2\mu_1}{\sigma_1^2} - \frac{4\mu_2\theta_{1.2}}{\sigma_1\sigma_2}\right)h_1 + \frac{2\mu_2 h_2}{\sigma_2^2}\right)$$
$$\times N_3\left[\begin{array}{l}\frac{\min(k_1,h_1)-2h_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}} + \frac{2\theta_{1.2}\mu_2\sqrt{t_1}}{\sigma_2}, \frac{\min(k_2,h_2)+\mu_2 t_1}{\sigma_2\sqrt{t_1}} - \frac{2\theta_{1.2}h_1}{\sigma_1\sqrt{t_1}},\\ \frac{\min(k_3,h_2)-2h_2-\mu_2 t_2}{\sigma_2\sqrt{t_2}} + \frac{2\theta_{1.2}h_1}{\sigma_1\sqrt{t_2}};\theta_{1.2},-\theta_{1.3},-\theta_{2.3}\end{array}\right] \tag{9}$$

where the μ_i*'s are given by (3) and:*

- $N_3[b_1,b_2,b_3;c_{12},c_{13},c_{23}]$ *is the trivariate standard normal cumulative distribution function with correlation coefficients* c_{12},c_{13},c_{23}

$$h_1=\ln\left(\frac{H_1}{S_1(0)}\right), h_2=\ln\left(\frac{H_2}{S_2(0)}\right), k_1=\ln\left(\frac{K_1}{S_1(0)}\right), k_2=\ln\left(\frac{K_2}{S_2(0)}\right), k_3=\ln\left(\frac{K_3}{S_2(0)}\right) \tag{10}$$

$$\theta_{1.2}=\rho_{1.2}, \theta_{1.3}=\sqrt{\frac{t_1}{t_2}}\rho_{1.2}, \theta_{2.3}=\sqrt{\frac{t_1}{t_2}} \tag{11}$$

Corollary 1. *It suffices to multiply by* (-1) *all the first three arguments of each* $N_3[.,.,.;.,.,.]$ *function and to substitute each* min *operator by a* max *operator in Proposition 1 to obtain an exact formula for* $\mathrm{P}_{[\mathrm{RDD}]}(\mu_1,\mu_2)$ *defined as:*

$$\mathrm{P}_{[\mathrm{RDD}]}(\mu_1,\mu_2) \triangleq P\left(\inf_{0\le t\le t_1} S_1(t)\ge H_1, S_1(t_1)\ge K_1, S_2(t_1)\ge K_2, \inf_{t_1\le t\le t_2} S_2(t)\ge H_2, S_2(t_2)\ge K_3\right) \tag{12}$$

where the acronym "[RDD]*" stands for "Rainbow Down and Down".*

Corollary 2. The term numbered (9) in Proposition 1 gives the value of the corrresponding knock-in probability denoted by $\mathrm{P}^{(\mathrm{I})}_{[\mathrm{RUU}]}(\mu_1,\mu_2)$ *and defined by:*

$$\mathrm{P}^{(\mathrm{I})}_{[\mathrm{RUU}]}(\mu_1,\mu_2) \triangleq P\left(\sup_{0\le t\le t_1} S_1(t)\ge H_1, S_1(t_1)\le K_1, S_2(t_1)\le K_2, \sup_{t_1\le t\le t_2} S_2(t)\ge H_2, S_2(t_2)\le K_3\right) \tag{13}$$

Corollary 3. *It suffices to substitute each argument* $\theta_{2.3}$ *in each* $N_3[.,.,.;.,.,.]$ *function of Proposition 1 by* $\sqrt{\frac{t_2}{t_3}}, \forall t_3\ge t_2$*, to obtain an exact formula for the early-ending variant* $\mathrm{P}_{[\mathrm{EERUU}]}(\mu_1,\mu_2)$ *defined by:*

$$\mathrm{P}_{[\mathrm{EERUU}]}(\mu_1,\mu_2) \triangleq P\left(\sup_{0\le t\le t_1} S_1(t)\le H_1, S_1(t_1)\le K_1, S_2(t_1)\le K_2, \sup_{t_1\le t\le t_2} S_2(t)\le H_2, S_2(t_3)\le K_3\right) \tag{14}$$

Corollary 4. *Let* $\hat{p}$ *be the value of* $\mathrm{P}_{[\mathrm{RUU}]}(\mu_1,\mu_2)$ *when the value of* K_3 *becomes "very high", i.e., high enough for the probability* $P(S_2(t_2)\le K_3)$ *to tend to zero; then, the difference* $\hat{p}-\mathrm{P}_{[\mathrm{RUU}]}(\mu_1,\mu_2)$ *provides the value of the following minor variant:*

$$\hat{p} - \mathrm{P}_{[\mathrm{RUU}]}(\mu_1, \mu_2) = P\left(\sup_{0\le t\le t_1} S_1(t) \le H_1, S_1(t_1) \le K_1, S_2(t_1) \le K_2, \sup_{t_1\le t\le t_2} S_2(t) \le H_2, S_2(t_2) > K_3\right) \tag{15}$$

End of Proposition 1.

Equipped with Proposition 1, one can value in closed form a two-colour step barrier option with two successive upward or two successive downward steps. Applying the theory of non-arbitrage pricing in a complete market (Harrison and Kreps 1979; Harrison and Pliska 1981), the value of a two-colour up-and-up knock-out put, denoted by $V_{[\mathrm{RUU}]}$, is given by:

$$V_{[\mathrm{RUU}]} = e^{-rt_2}\left(E_Q\left[K_3 \mathbf{1}_{\{A\}} - S_2(t_2)\mathbf{1}_{\{A\}}\right]\right) \tag{16}$$

where r is the riskless interest rate assumed to be constant, $\mathbf{1}_{\{.\}}$ is the indicator function and A is the set constructed by the intersection of elements of the $\sigma-$algebra generated by the pair of processes $(S_1(t), S_2(t))$ that characterises the probability $\mathrm{P}_{[\mathrm{RUU}]}(\mu_1, \mu_2)$ as given by the arguments of the probability operator in (5).

A simple application of the Cameron–Martin–Girsanov theorem yields:

$$V_{[\mathrm{RUU}]} = e^{-rt_2} K_3 \mathrm{P}_{[\mathrm{RUU}]}\left(\mu_1^{(Q)}, \mu_2^{(Q)}\right) - S_2(0)\mathrm{P}_{[\mathrm{RUU}]}\left(\mu_1^{(P_2)}, \mu_2^{(P_2)}\right) \tag{17}$$

where

$$\mu_i^{(Q)} = r - \frac{\sigma_i^2}{2},\ \mu_1^{(P_2)} = r - \frac{\sigma_1^2}{2} + \sigma_1\sigma_2\rho_{1.2},\ \mu_2^{(P_2)} = r + \frac{\sigma_2^2}{2}, \tag{18}$$

Q is the measure under which $\left\{B_i(t) + \frac{\mu_i - r}{\sigma_i}t, t \ge 0\right\}$ is a standard Brownian motion (the classical so-called risk-neutral measure), while P_2 is the measure under which $\{B_1(t) - \sigma_2\rho_{1.2}t, t \ge 0\}$ and $\left\{B_2(t) - \sigma_2\sqrt{1-\rho_{1.2}^2}t, t \ge 0\right\}$ are two independent standard Brownian motions.

To factor in a continuous dividend rate δ_i associated with each asset S_i, simply replace r by $r - \delta_i$.

All the other two-colour rainbow barrier options subsequently mentioned in Sections 2.1 and 2.2, whether they be knock-in or feature a mixture of a downward and an upward barrier, are identically valued, by taking the relevant $\mathrm{P}_{[.]}$ probability along with the pairs $\left(\mu_1^{(Q)}, \mu_2^{(Q)}\right)$ and $\left(\mu_1^{(P_2)}, \mu_2^{(P_2)}\right)$.

The numerical implementation of Proposition 1 is easy. Using Genz's (2004) algorithm to evaluate the trivariate standard normal cumulative distribution function, the accuracy and efficiency required for all practical purposes can be achieved in computational times in the order of 0.1 s. Table 1 provides the prices of a few two-colour up-and-up knock-out put options, for various levels of the volatility and correlation parameters of the underlying assets S_1 and S_2, and different values of the knock-out barriers. All the initial values of the underlying assets $S_i(0)$ and the strike prices K_i are set at 100. Expiry is 1 year. The two time intervals $[t_0, t_1]$ and $[t_1, t_2]$ have equal length, i.e., t_1 =6 months, but unequal time lengths can be handled just as well by the formulae. The riskless interest rate is assumed to be 2.5%.

In each cell, four prices are reported: the first one is the exact analytical value as obtained by implementing Proposition 1, while the prices in brackets are three successive approximations obtained by performing increasingly large Monte Carlo simulations. More specifically, these approximations rely on the conditional Monte Carlo method, which is well known for its accuracy and efficiency (Glasserman 2003). The number of simulations performed is 500,000 for the first approximation, 2,000,000 for the second approximation, and 10,000,000 for the third approximation. The pseudo-random numbers are drawn from the reliable Mersenne Twister generator (Matsumoto and Nishimura 1998).

Table 1. Two-colour up-and-up knock-out put.

	$\rho_{1.2} = -0.6$	$\rho_{1.2} = -0.2$	$\rho_{1.2} = 0.2$	$\rho_{1.2} = 0.6$
$\sigma_1 = \sigma_2 = 20\%$ $H_1 = H_2 = 115$	1.189 (1.123,1.168, 1.182)	2.163 (2.237, 2.141, 2.161)	3.134 (3.082, 3.116, 3.132)	4.141 (4.174, 4.127, 4.140)
$\sigma_1 = \sigma_2 = 60\%$ $H_1 = H_2 = 125$	4.065 (4.109, 4.072, 4.061)	6.440 (6.392, 6.449, 6.442)	8.769 (8.734, 8.755, 8.763)	11.213 (11.278, 11.196, 11.218)
$\sigma_1 = 20\%, \sigma_2 = 60\%$ $H_1 = 115, H_2 = 125$	4.454 (4.411, 4.443, 4.454)	7.260 (7.355, 7.301, 7.264)	9.979 (9.912, 9.101, 9.975)	12.844 (12.957, 12.892, 12.848)
$\sigma_1 = 60\%, \sigma_2 = 20\%$ $H_1 = 125, H_2 = 115$	1.096 (1.082, 1.108, 1.091)	1.921 (1.107, 1.953, 1.928)	2.753 (2.796, 2.745, 2.752)	3.617 (3.692, 3.599, 3.614)

In purely numerical terms, it can be clearly observed that the conditional Monte Carlo approximations gradually converge to the analytical values as more and more simulations are performed. A minimum of 10,000,000 simulations are necessary to guarantee a modest 10^{-3} convergence. This requires a computational time of approximately 35 s on a computer equipped with a Core i7 CPU. Much more accurate values can be obtained by means of Proposition 1 in only two-tenths of a second. This gap in accuracy and efficiency makes a particularly valuable difference when pricing large portfolios of options.

From a financial point of view, the most striking phenomenon observed in Table 1 is that the option price regularly and significantly increases with the value of the correlation coefficient between assets S_1 and S_2, whatever the volatilities and the levels of the barriers. Roughly speaking, the price of an at-the-money two-colour up-and-up knock-out put option when $\rho_{1.2} = 0.6$ is three times greater than when $\rho_{1.2} = -0.6$. This property can be exploited by traders who take positions on correlation, as the prices of these options will substantially increase if implicit correlation turns out to be underestimated by the markets. This property can also be harnessed by traders to construct hedges on sold derivatives that are sensitive to pairwise correlation. From an investor's perspective, the observed phenomenon allows to define effective strategies to reduce the cost of hedging by tapping into negative correlation. Such a significant functional relation w.r.t. correlation is a major attraction of rainbow step barrier options relative to non-rainbow step barrier options, as the latter can only handle volatility effects.

Another noticeable fact in Table 1 is that lowering the up-and-out barriers seems much more effective in reducing the option's price than lowering the volatilities of assets S_1 and S_2, regardless of the sign and the magnitude of correlation. Indeed, looking at row 1 in Table 1, one can see that the options are relatively cheap, although the volatilities of both assets S_1 and S_2 are low, because the knock-out barriers are located quite near the spot prices of the underlying assets; and looking at row 2 in Table 1, one can see that the options are relatively expensive, although the volatilities of both assets S_1 and S_2 are high because the knock-out barriers are more distant. This shows that the barrier effect, which drives prices down as up-and-out barriers become lower and conversely drives prices up as the up-and-out barrier becomes higher, and prevails over the volatility effect, which exerts its influence in the opposite direction, i.e., a lower volatility pushes prices up by decreasing the probability of knocking out before expiry and a higher volatility pushes prices down by increasing the latter probability. This phenomenon can be explained by the ambivalent nature of volatility: on the one hand, less volatility means less risk of being deactivated before expiry, but on the other hand, it also means fewer chances of ending in-the-money at expiry; whichever of this positive and this negative effect weighs more on the option price depends on the relative values of barrier, strike, volatility and expiry parameters in a complex manner.

2.2. Valuation of Two-Colour Step Barrier Options Involving One Upward Step and One Downward Step

Section 2.2 deals with the case when the steps of the barrier are not on the same side in each time interval, i.e., either first downward, then upward, or first upward, then downward.

The main result of Section 2.2 is given by the following Proposition 2.

Proposition 2. *Let* $\mathrm{P}_{[\mathrm{RUD}]}(\mu_1,\mu_2)$ *denote the joint cumulative distribution function defined by:*

$$\mathrm{P}_{[\mathrm{RUD}]}(\mu_1,\mu_2) \triangleq P\left(\sup_{0\le t\le t_1} S_1(t) \le H_1, S_1(t_1) \le K_1, S_2(t_1) \ge K_2, \inf_{t_1\le t\le t_2} S_2(t) \ge H_2, S_2(t_2) \ge K_3\right) \tag{19}$$

where the acronym "[RUD]*" stands for "Rainbow Up and Down".*

Then, the exact value of $\mathrm{P}_{[\mathrm{RUD}]}(\mu_1,\mu_2)$ is given by:

$$\begin{aligned}&\mathrm{P}_{[\mathrm{RUD}]}(\mu_1,\mu_2)\\ &= N_3\left[\frac{\min(k_1,h_1)-\mu_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-\max(k_2,h_2)+\mu_2 t_1}{\sigma_2\sqrt{t_1}}, \frac{-\max(k_3,h_2)+\mu_2 t_2}{\sigma_2\sqrt{t_2}}; -\theta_{1.2}, -\theta_{1.3}, \theta_{2.3}\right]\end{aligned} \tag{20}$$

$$\begin{aligned}&-\exp\left(\frac{2\mu_1 h_1}{\sigma_1^2}\right)\\ &\times N_3\left[\begin{array}{l}\frac{\min(k_1,h_1)-2h_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-\max(k_2,h_2)+\mu_2 t_1}{\sigma_2\sqrt{t_1}} + \frac{2\theta_{1.2}h_1}{\sigma_1\sqrt{t_1}},\\ \frac{-\max(k_3,h_2)+\mu_2 t_2}{\sigma_2\sqrt{t_2}} + \frac{2\theta_{1.2}h_1}{\sigma_1\sqrt{t_2}}; -\theta_{1.2}, -\theta_{1.3}, \theta_{2.3}\end{array}\right]\end{aligned} \tag{21}$$

$$\begin{aligned}&-\exp\left(\frac{2\mu_2 h_2}{\sigma_2^2}\right)\\ &\times N_3\left[\begin{array}{l}\frac{\min(k_1,h_1)-\mu_1 t_1}{\sigma_1\sqrt{t_1}} + \frac{2\theta_{1.2}\mu_2\sqrt{t_1}}{\sigma_2}, \frac{-\max(k_2,h_2)-\mu_2 t_1}{\sigma_2\sqrt{t_1}}, \frac{-\max(k_3,h_2)+2h_2+\mu_2 t_2}{\sigma_2\sqrt{t_2}};\\ -\theta_{1.2}, \theta_{1.3}, -\theta_{2.3}\end{array}\right]\end{aligned} \tag{22}$$

$$\begin{aligned}&+\exp\left(\left(\frac{2\mu_1}{\sigma_1^2} - \frac{4\mu_2\theta_{1.2}}{\sigma_1\sigma_2}\right)h_1 + \frac{2\mu_2 h_2}{\sigma_2^2}\right)\\ &\times N_3\left[\begin{array}{l}\frac{\min(k_1,h_1)-2h_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}} + \frac{2\theta_{1.2}\mu_2\sqrt{t_1}}{\sigma_2}, \frac{-\max(k_2,h_2)-\mu_2 t_1}{\sigma_2\sqrt{t_1}} + \frac{2\theta_{1.2}h_1}{\sigma_1\sqrt{t_1}},\\ \frac{-\max(k_3,h_2)+2h_2+\mu_2 t_2}{\sigma_2\sqrt{t_2}} - \frac{2\theta_{1.2}h_1}{\sigma_1\sqrt{t_2}}; \theta_{1.2}, -\theta_{1.3}, -\theta_{2.3}\end{array}\right]\end{aligned} \tag{23}$$

where all the notations are identical, as in Proposition 1.

Corollary 1. *It suffices to multiply by* (-1) *all the first three arguments of each* $N_3[.,.,.;.,.,.]$ *function and substitute each* min *operator by a* max *operator as well as each* max *operator by a* min *operator in Proposition 2 to obtain an exact formula for* $\mathrm{P}_{[\mathrm{RDU}]}(\mu_1,\mu_2)$ *defined as:*

$$\mathrm{P}_{[\mathrm{RDU}]}(\mu_1,\mu_2) \triangleq P\left(\inf_{0\le t\le t_1} S_1(t) \ge H_1, S_1(t_1) \ge K_1, S_2(t_1) \le K_2, \sup_{t_1\le t\le t_2} S_2(t) \le H_2, S_2(t_2) \le K_3\right) \tag{24}$$

Corollary 2. *The term numbered (23) in Proposition 2 provides the value of the corrresponding up-and-in, then down-and-in probability, denoted as* $\mathrm{P}^{(I)}_{[\mathrm{RUD}]}(\mu_1,\mu_2)$ *and defined by:*

$$\mathrm{P}^{(I)}_{[\mathrm{RUD}]}(\mu_1,\mu_2) \triangleq P\left(\sup_{0\le t\le t_1} S_1(t) \ge H_1, S_1(t_1) \le K_1, S_2(t_1) \ge K_2, \inf_{t_1\le t\le t_2} S_2(t) \le H_2, S_2(t_2) \ge K_3\right) \tag{25}$$

End of Proposition 2.

Equipped with Proposition 2, one can value in closed form a two-colour step barrier option with one upward step and one downward step, by taking the relevant $\mathrm{P}_{[.]}$ or $\mathrm{P}^{(I)}_{[.]}$

probability along with the pairs $\left(\mu_1^{(Q)}, \mu_2^{(Q)}\right)$ and $\left(\mu_1^{(P_2)}, \mu_2^{(P_2)}\right)$ defined in (18), as explained in Section 2.1. Table 2 reports the prices of a few down-and-up two-colour knock-out put options by implementing Proposition 2 to obtain exact analytical values and by computing three successive conditional Monte Carlo approximations in the same way, as in Table 1.

Table 2. Two-colour down-and-up knock-out put.

	$\rho_{1.2} = -0.6$	$\rho_{1.2} = -0.2$	$\rho_{1.2} = 0.2$	$\rho_{1.2} = 0.6$
$\sigma_1 = \sigma_2 = 20\%$ $H_1 = 85, H_2 = 115$	4.299 (4.287, 4.305, 4.297)	3.293 (3.318, 3.286, 3.292)	2.307 (2.282, 2.298, 2.305)	1.303 (1.284, 1.309, 1.304)
$\sigma_1 = \sigma_2 = 60\%$ $H_1 = 75, H_2 = 125$	10.221 (10.142, 10.228, 10.224)	7.610 (7.563, 7.597, 7.612)	5.212 (5.255, 5.204, 5.214)	2.862 (2.834, 2.854, 2.865)
$\sigma_1 = 20\%, \sigma_2 = 60\%$ $H_1 = 85, H_2 = 125$	13.377 (13.385, 13.391, 13.376)	10.496 (10.472, 10.482, 10.948)	7.735 (7.783, 7.717, 7.731)	4.857 (4.894, 4.866, 4.857)
$\sigma_1 = 60\%, \sigma_2 = 20\%$ $H_1 = 75, H_2 = 115$	3.344 (3.387, 3.358, 3.346)	2.401 (2.383, 2.413, 2.402)	1.545 (1.596, 1.530, 1.542)	0.748 (0.884, 0.787, 0.752)

In Table 2, the most salient feature is still the functional dependency of the option's price on the correlation between assets S_1 and S_2, but, this time, the direction is opposite to that in Table 1, i.e., the two-colour down-and-up knock-out put prices steadily decrease as $\rho_{1.2}$ goes from -60% to 60%. In a trader's perspective, one could sum up the argument by saying that two-colour rainbow barrier options are a bet on a positive correlation when both barriers are on the same side (upward or downward), while they are a bet on a negative correlation when the barriers stand on opposite sides (up-and-down or down-and-up).

The barrier effect also prevails over the volatility effect in Table 2. Overall, two-colour down-and-up knock-out puts display maximum values that are a little higher, and minimum values that are a little lower than two-colour up-and-up knock-out puts, although up-and-out barriers and down-and-out barriers are designed with the exact same distance to the spot prices of S_1 and S_2.

2.3. Valuation of Reverse Two-Colour Step Barrier Options

A two-colour rainbow barrier option is said to be reverse when the moneyness of the option is defined w.r.t. the first and former "colour" (i.e., asset S_1) instead of the second and last one (asset S_2): the option, so to speak, reverts back to asset one at expiry, hence the denomination. From a computational standpoint, this is not a trivial difference since it adds an additional dimension to the integral formulation of the problem. Let us define as $\mathrm{P}_{[\mathrm{RRUU}]}(\mu_1, \mu_2)$ the following cumulative joint distribution at the core of reverse rainbow option valuation:

$$\mathrm{P}^{(\mathrm{Rev})}_{[\mathrm{RRUU}]}(\mu_1, \mu_2) = P\left(\sup_{0 \le t \le t_1} S_1(t) \le H_1, S_1(t_1) \le K_1, S_2(t_1) \le K_2, \sup_{t_1 \le t \le t_2} S_2(t) \le H_2, S_1(t_2) \le K_3\right) \tag{26}$$

where the acronym "[RRUU]" stands for "Reverse Rainbow Up and Up".

Then, Proposition 3 provides the exact value of $\mathrm{P}_{[\mathrm{RRUU}]}(\mu_1, \mu_2)$ in the form of a triple integral.

Proposition 3.

$$\mathrm{P}_{[\mathrm{RRUU}]}(\mu_1, \mu_2) = \frac{1}{(2\pi)^{3/2}\sigma_{2|1}\sigma_{3|1.2}\sigma_1^2\sigma_2 t_1\sqrt{t_2}} \int_{-\infty}^{\min(k_1,h_1)} \int_{-\infty}^{h_2} \int_{-\infty}^{\min(k_2,h_2)} \varphi_2(x_1)\varphi_3(x_2, x_3) \tag{27}$$

$$e^{-\frac{1}{2}\left(\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)^2-\frac{1}{2\sigma_{2|1}^2}\left(\frac{x_2-\mu_2 t_1}{\sigma_2\sqrt{t_1}}-\theta_{1.2}\left(\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)\right)^2-\frac{1}{2\sigma_{3|1.2}^2}\left(\frac{x_3-\mu_2 t_2}{\sigma_2\sqrt{t_2}}-\theta_{1.3}\left(\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)-\frac{\theta_{2.3|1}}{\sigma_{2|1}}\left(\frac{x_2-\mu_2 t_1}{\sigma_2\sqrt{t_1}}-\theta_{1.2}\left(\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)\right)\right)^2}$$

$$N\left[\frac{1}{\sigma_{4|1.2.3}}\left(\begin{array}{l}\frac{k_3-\mu_1 t_2}{\sigma_1\sqrt{t_2}}-\theta_{1.4}\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}-\frac{\theta_{2.4|1}}{\sigma_{2|1}}\left(\frac{x_2-\mu_2 t_1}{\sigma_2\sqrt{t_1}}-\theta_{1.2}\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)\\ -\frac{\theta_{3.4|1.2}}{\sigma_{3|1.2}}\left(\frac{x_3-\mu_2 t_2}{\sigma_2\sqrt{t_2}}-\theta_{1.3}\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}-\frac{\theta_{2.3|1}}{\sigma_{2|1}}\left(\frac{x_2-\mu_2 t_1}{\sigma_2\sqrt{t_1}}-\theta_{1.2}\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)\right)\end{array}\right)\right]dx_3 dx_2 dx_1$$

where:

$$\theta_{2.4|1}=\frac{\theta_{2.4}-\theta_{1.2}\theta_{1.4}}{\sqrt{1-\theta_{1.2}^2}},\theta_{3.4|1.2}=\frac{\theta_{3.4}-\theta_{1.3}\theta_{1.4}-\theta_{2.3|1}\theta_{2.4|1}}{\sigma_{3|1.2}},\sigma_{4|1.2.3}=\sqrt{1-\theta_{1.4}^2-\theta_{2.4|1}^2-\theta_{3.4|1.2}^2} \tag{28}$$

- $N[.]$ *is the univariate standard normal cumulative distribution function;*
- *the functions* φ_2 *and* φ_3 *are defined by (80) and (81) in Section 3.*

All the other notations in Proposition 3 have been previously defined.

Remark 1. *Other types of reverse two-colour knock-out or knock-in barrier probability distributions are handled similarly by modifying the upper bounds of the integral and, possibly, the* φ_i *functions, according to the considered combination of events.*

Remark 2. $\theta_{2.4|1}$ *is the partial correlation between* $X_2(t_1)$ *and* $X_1(t_2)$ *conditional on* $X_1(t_1)$*, while* $\theta_{3.4|1.2}$ *is the partial correlation between* $X_2(t_2)$ *and* $X_1(t_2)$ *conditional on* $X_1(t_1)$ *and* $X_2(t_1)$*, and* $\sigma_{4|1.2.3}$ *is the conditional standard deviation of* $X_1(t_2)$ *given* $X_1(t_1)$*,* $X_2(t_1)$ *and* $X_2(t_2)$*.*

End of Proposition 3.

The application of Proposition 3 to value a reverse two-colour step barrier option is now discussed. The no-arbitrage price of a reverse two-colour rainbow up-and-up knock-out put, denoted by $V_{[RRUU]}$, is given by:

$$V_{[RRUU]}^{(R)}=e^{-rt_2}\left(E_Q\left[K_3\mathbf{1}_{\{A\}}-S_1(t_2)\mathbf{1}_{\{A\}}\right]\right)=e^{-rt_2}K_3\mathrm{P}_{[RRUU]}^{(R)}\left(\mu_1^{(Q)},\mu_2^{(Q)}\right)-S_2(0)\mathrm{P}_{[RRUU]}^{(R)}\left(\mu_1^{(P_1)},\mu_2^{(P_1)}\right) \tag{29}$$

where

$$\mu_1^{(P_1)}=r+\frac{\sigma_1^2}{2},\ \mu_2^{(P_1)}=r-\frac{\sigma_2^2}{2}+\sigma_1\sigma_2\rho_{1.2} \tag{30}$$

- A is the set constructed by the intersection of elements of the $\sigma-$algebra generated by the pair of processes $(S_1(t),S_2(t))$ that characterises the probability $\mathrm{P}_{[RRUU]}(\mu_1,\mu_2)$ as given by the arguments of the probability operator in (26);
- P_1 is the measure under which $B_1(t)-\sigma_1 t$ is a standard Brownian motion.

However, it is less easy to evaluate Proposition 3 than to evaluate Proposition 1 and Proposition 2. The problem at hand has two "nice" features from the standpoint of numerical integration: first, the dimension, equal to 3, is moderate; second, the integrand is continuous. The snag is the large number of parameters in each evaluation of the integrand in a quadrature process, especially the various conditional standard deviations at the denominators of the fractions, that may hinder fast convergence when they take on absolute values that become smaller and smaller. That is why it is recommended to use a subregion adaptive algorithm of numerical integration, as explained by Berntsen et al. (1991), that adapts the number of integrand evaluations in each subregion according to the rate of change of the integrand. Although more time-consuming than a fixed degree rule, it is more accurate to control the approximation error, as the subdivision of the integration domain stops only when the sum of the local error deterministic estimates becomes smaller than some prespecified requested accuracy. Adaptive integration can be enhanced by a Kronrod rule to reduce the number of required iterations (see, e.g., Davis and Rabinowitz 2007). These techniques are widely used in numerical integration, and it is easy to find available code or built-in functions in the usual scientific computing software.

2.4. Valuation of Two-Colour Outside Step Barrier Options

In this section, a third correlated asset S_3 is introduced, w.r.t. which the option's moneyness is measured at expiry, while the assets S_1 and S_2 serve exclusively as the underlyings w.r.t. which barrier crossing is monitored. This is an important extension, as outside barrier options allow to manage volatility more consistently than standard (non-outside) barrier options, as explained, e.g., by Das (2006).

Let us consider a third asset S_3 with the following differential:

$$dS_3(t) = v_3 S_3(t)dt + \sigma_3 S_3(t)dB_3(t) \tag{31}$$

The instantaneous pairwise correlations between the Brownian motions $B_i's$ are denoted as $\rho_{i.j}$.

The objective is to compute the probabilities $p_m(\mu_1, \mu_2, \mu_3), m \in \{1,2,3,4\}$ defined by:

$$p_1(\mu_1,\mu_2,\mu_3) = P\begin{pmatrix} \sup\limits_{0\le t\le t_1} S_1(t) \le H_1, S_1(t_1) \le K_1, S_2(t_1) \le H_2, \\ \sup\limits_{t_1\le t\le t_2} S_2(t) \le H_2, S_2(t_2) \le K_2, S_3(t_2) \le K_3 \end{pmatrix} \tag{32}$$

$$p_2(\mu_1,\mu_2,\mu_3) = P\begin{pmatrix} \inf\limits_{0\le t\le t_1} S_1(t) \ge H_1, S_1(t_1) \ge K_1, S_2(t_1) \ge H_2, \\ \inf\limits_{t_1\le t\le t_2} S_2(t) \ge H_2, S_2(t_2) \ge K_2, S_3(t_2) \ge K_3 \end{pmatrix} \tag{33}$$

$$p_3(\mu_1,\mu_2,\mu_3) = P\begin{pmatrix} \sup\limits_{0\le t\le t_1} S_1(t) \le H_1, S_1(t_1) \le K_1, S_2(t_1) \ge H_2, \\ \inf\limits_{t_1\le t\le t_2} S_2(t) \ge H_2, S_2(t_2) \ge K_2, S_3(t_2) \ge K_3 \end{pmatrix} \tag{34}$$

$$p_4(\mu_1,\mu_2,\mu_3) = P\begin{pmatrix} \inf\limits_{0\le t\le t_1} S_1(t) \ge H_1, S_1(t_1) \ge K_1, S_2(t_1) \le H_2, \\ \sup\limits_{t_1\le t\le t_2} S_2(t) \le H_2, S_2(t_2) \le K_2, S_3(t_2) \le K_3 \end{pmatrix} \tag{35}$$

Let $\mathrm{x} = [c_1, c_2, c_3, c_4, c_5]$ be a vector of five coordinates where each $c_i \in \,]-1, 1[$, $\forall i \in \{1, \ldots, 5\}$.

Let the function $\Psi_4[b_1, b_2, b_3, b_4; \mathrm{x}]$, $\forall b_1, b_2, b_3, b_4 \in \mathbb{R}$, be defined by:

$$\Psi_4[b_1, b_2, b_3, b_4; \mathrm{x}]$$

$$= \int\limits_{x_1=-\infty}^{b_1} \int\limits_{x_2=-\infty}^{\frac{b_2 - c_1 x_1}{\sqrt{1-c_1^2}}} \int\limits_{x_3=-\infty}^{\frac{b_3 - c_4 x_2\sqrt{1-c_1^2} - c_4 c_1 x_1}{\sqrt{1-c_2^2}}} \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{x_1^2}{2} - \frac{x_2^2}{2} - \frac{x_3^2}{2}\right) \tag{36}$$
$$N\left[\frac{b_4 - \frac{c_5 - c_2 c_3}{1-c_2^2}\left(x_3\sqrt{1-c_4^2} + x_2 c_4\sqrt{1-c_1^2}\right) - x_1\left(c_3 + \frac{c_5 - c_2 c_3}{1-c_2^2}(c_1 c_4 - c_2)\right)}{\sqrt{1 - c_3^2 - \left(\frac{c_5 - c_2 c_3}{1-c_2^2}\right)^2}}\right] dx_3 dx_2 dx_1$$

The following Proposition 4 combines all the probabilities defined in (32)–(35) into a single formula.

Proposition 4. *The exact values of the probabilities $p_m(\mu_1, \mu_2, \mu_3), m \in \{1,2,3,4\}$, written in shorter notation as p_m, are given by:*

$$p_m = \Psi_4\left[\delta_1\left(\frac{G_1(k_1,h_1) - \mu_1 t_1}{\sigma_1\sqrt{t_1}}\right), \delta_2\left(\frac{h_2 - \mu_2 t_1}{\sigma_2\sqrt{t_1}}\right), \delta_2\left(\frac{G_2(k_2,h_2) - \mu_2 t_2}{\sigma_2\sqrt{t_2}}\right), \delta_2\left(\frac{k_3 - \mu_3 t_2}{\sigma_3\sqrt{t_2}}\right); \mathrm{x}_1\right] \tag{37}$$

$$-\exp\left(\frac{2\mu_1 h_1}{\sigma_1^2}\right)\times\Psi_4\left[\begin{array}{l}\delta_1\left(\frac{G_1(k_1,h_1)-2h_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right),\delta_2\left(\frac{h_2-\mu_2 t_1}{\sigma_2\sqrt{t_1}}-\theta_{1.2}\frac{2h_1}{\sigma_1\sqrt{t_1}}\right),\\ \delta_2\left(\frac{G_2(k_2,h_2)-\mu_2 t_2}{\sigma_2\sqrt{t_2}}-\theta_{1.2}\frac{2h_1}{\sigma_1\sqrt{t_2}}\right),\\ \delta_2\left(\frac{k_3-\mu_3 t_2}{\sigma_3\sqrt{t_2}}-\theta_{1.4}\frac{2h_1}{\sigma_1\sqrt{t_1}}-\theta_{3.4|1}\left(\theta_{1.2}\frac{2h_1}{\sigma_1\sqrt{t_2}}-\theta_{1.3}\frac{2h_1}{\sigma_1\sqrt{t_1}}\right)\right);\mathrm{x}_1\end{array}\right] \tag{38}$$

$$-\exp\left(\frac{2\mu_2 h_2}{\sigma_2^2}\right)\times\Psi_4\left[\begin{array}{l}\delta_1\left(\frac{G_1(k_1,h_1)-\mu_1 t_1}{\sigma_1\sqrt{t_1}}+\theta_{1.2}\frac{2\mu_2 t_1}{\sigma_2\sqrt{t_1}}\right),\delta_2\left(\frac{h_2+\mu_2 t_1}{\sigma_2\sqrt{t_1}}\right),\\ \delta_2\left(\frac{G_2(k_2,h_2)-2h_2-\mu_2 t_2}{\sigma_2\sqrt{t_2}}\right),\\ \delta_2\left(\frac{k_3-\mu_3 t_2}{\sigma_3\sqrt{t_2}}-\theta_{3.4|1}\left(\frac{2h_2}{\sigma_2\sqrt{t_2}}+\theta_{1.2}\theta_{1.3}\frac{2\mu_2 t_1}{\sigma_2\sqrt{t_1}}\right)+\theta_{1.2}\theta_{1.4}\frac{2\mu_2 t_1}{\sigma_2\sqrt{t_1}}\right);\mathrm{x}_2\end{array}\right] \tag{39}$$

$$+\exp\left(\left(\frac{2\mu_1}{\sigma_1^2}-\frac{4\mu_2\rho_{1.2}}{\sigma_1\sigma_2}\right)h_1+\frac{2\mu_2 h_2}{\sigma_2^2}\right)$$
$$\times\Psi_4\left[\begin{array}{l}\delta_1\left(\frac{G_1(k_1,h_1)-2h_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}+\theta_{1.2}\frac{2\mu_2 t_1}{\sigma_2\sqrt{t_1}}\right),\\ \delta_2\left(\frac{h_2+\mu_2 t_1}{\sigma_2\sqrt{t_1}}-\theta_{1.2}\frac{2h_1}{\sigma_1\sqrt{t_1}}\right),\\ \delta_2\left(\frac{G_2(k_2,h_2)-2h_2-\mu_2 t_2}{\sigma_2\sqrt{t_2}}+\theta_{1.2}\frac{2h_1}{\sigma_1\sqrt{t_2}}\right),\\ \delta_2\left(\begin{array}{l}\frac{k_3-\mu_3 t_2}{\sigma_3\sqrt{t_2}}+\theta_{1.4}\left(\theta_{1.2}\frac{2\mu_2\sqrt{t_1}}{\sigma_2}-\frac{2h_1}{\sigma_1\sqrt{t_1}}\right)\\ -\theta_{3.4|1}\left(\frac{2h_2}{\sigma_2\sqrt{t_2}}-\theta_{1.2}\frac{2h_1}{\sigma_1\sqrt{t_2}}+\theta_{1.3}\left(\theta_{1.2}\frac{2\mu_2\sqrt{t_1}}{\sigma_2}-\frac{2h_1}{\sigma_1\sqrt{t_1}}\right)\right)\end{array}\right);\mathrm{x}_2\end{array}\right] \tag{40}$$

where k_1,k_2,h_1,h_2 are as in Proposition 2, $k_3=\ln\left(\frac{K_3}{S_3(0)}\right)$, and we have:

$$\theta_{1.2}=\rho_{1.2},\theta_{1.3}=\sqrt{\frac{t_1}{t_2}}\rho_{1.2},\theta_{1.4}=\sqrt{\frac{t_1}{t_2}}\rho_{1.3},\theta_{2.3}=\sqrt{\frac{t_1}{t_2}},\theta_{3.4}=\rho_{2.3},\ \theta_{3.4|1}=\frac{\theta_{3.4}-\theta_{1.3}\theta_{1.4}}{\sqrt{1-\theta_{1.3}^2}} \tag{41}$$

$$\delta_1=\begin{cases}1 \text{ if } p_m=p_1 \text{ or } p_m=p_3\\ -1 \text{ if } p_m=p_2 \text{ or } p_m=p_4\end{cases},\ \delta_2=\begin{cases}1 \text{ if } p_m=p_1 \text{ or } p_m=p_4\\ -1 \text{ if } p_m=p_2 \text{ or } p_m=p_3\end{cases} \tag{42}$$

$$G_1(.,.)=\begin{cases}\min(.,.) \text{ if } p_m=p_1 \text{ or } p_m=p_3\\ \max(.,.) \text{ if } p_m=p_2 \text{ or } p_m=p_4\end{cases},\ G_2(.,.)=\begin{cases}\max(.,.) \text{ if } p_m=p_1 \text{ or } p_m=p_3\\ \min(.,.) \text{ if } p_m=p_2 \text{ or } p_m=p_4\end{cases} \tag{43}$$

$$\mathrm{x}_1=\begin{cases}[\theta_{1.2},\theta_{1.3},\theta_{1.4},\theta_{2.3},\theta_{3.4}] \text{ if } p_m=p_1 \text{ or } p_m=p_2\\ [-\theta_{1.2},-\theta_{1.3},-\theta_{1.4},\theta_{2.3},\theta_{3.4}] \text{ if } p_m=p_3 \text{ or } p_m=p_4\end{cases} \tag{44}$$

$$\mathrm{x}_2=\begin{cases}[\theta_{1.2},\theta_{1.3},\theta_{1.4},-\theta_{2.3},\theta_{3.4}] \text{ if } p_m=p_1 \text{ or } p_m=p_2\\ [-\theta_{1.2},-\theta_{1.3},-\theta_{1.4},-\theta_{2.3},\theta_{3.4}] \text{ if } p_m=p_3 \text{ or } p_m=p_4\end{cases} \tag{45}$$

Corollary 1. *The corresponding knock-in probabilities can be inferred in the same way as in Proposition 1 and Proposition 2. Let the probabilities $p_m^{(I)}(\mu_1,\mu_2,\mu_3), m\in\{1,2,3,4\}$ be defined by:*

$$p_1^{(I)}(\mu_1,\mu_2,\mu_3)=P\left(\begin{array}{l}\sup\limits_{0\le t\le t_1} S_1(t)\ge H_1, S_1(t_1)\le K_1, S_2(t_1)\le H_2,\\ \sup\limits_{t_1\le t\le t_2} S_2(t)\ge H_2, S_2(t_2)\le K_2, S_3(t_2)\le K_3\end{array}\right) \tag{46}$$

$$p_2^{(I)}(\mu_1,\mu_2,\mu_3)=P\left(\begin{array}{l}\inf\limits_{0\le t\le t_1} S_1(t)\le H_1, S_1(t_1)\ge K_1, S_2(t_1)\ge H_2,\\ \inf\limits_{t_1\le t\le t_2} S_2(t)\le H_2, S_2(t_2)\ge K_2, S_3(t_2)\ge K_3\end{array}\right) \tag{47}$$

$$p_3^{(I)}(\mu_1,\mu_2,\mu_3)=P\left(\begin{array}{l}\sup\limits_{0\le t\le t_1} S_1(t)\ge H_1, S_1(t_1)\le K_1, S_2(t_1)\ge H_2,\\ \inf\limits_{t_1\le t\le t_2} S_2(t)\le H_2, S_2(t_2)\ge K_2, S_3(t_2)\ge K_3\end{array}\right) \tag{48}$$

$$p_4^{(I)}(\mu_1,\mu_2,\mu_3)=P\left(\begin{array}{l}\inf\limits_{0\le t\le t_1} S_1(t)\le H_1, S_1(t_1)\ge K_1, S_2(t_1)\le H_2,\\ \sup\limits_{t_1\le t\le t_2} S_2(t)\ge H_2, S_2(t_2)\le K_2, S_3(t_2)\le K_3\end{array}\right) \tag{49}$$

Then, $p_m^{(I)}(\mu_1, \mu_2, \mu_3)$ *is given by (40).*

Corollary 2. *It suffices to substitute each argument* $\theta_{3.4}$ *in each* $\Psi_4[.,.,.,.;.,.,.,.,.]$ *function of Proposition 4 by* $\rho_{2.3}\sqrt{\frac{t_2}{t_3}}$, $\forall t_3 \geq t_2$, *to obtain an exact formula for the early-ending variant of* $p_m(\mu_1, \mu_2, \mu_3)$.

End of Proposition 4.

Equipped with Proposition 4, one can value in closed form a two-colour outside step barrier option. More precisely, the value of a two-colour outside up-and-out put, denoted by $V_{[\text{ORUU}]}$, is given by:

$$V_{[\text{ORUU}]} = e^{-rt_2}\left(E_Q\left[K_3\mathbf{1}_{\{A\}} - S_3(t_2)\mathbf{1}_{\{A\}}\right]\right) \tag{50}$$

where A is the set constructed by the intersection of elements of the $\sigma-$algebra generated by the pair of processes $(S_1(t), S_2(t))$ that characterises the probability $p_1(\mu_1, \mu_2)$ as given by the arguments of the probability operator in (32), and the acronym "[ORUU]" stands for "Outside Rainbow Up and Up".

Using the following orthogonal decomposition of Brownian motion $B_3(t)$:

$$B_3(t) = \rho_{1.3}W_1(t) + \rho_{2.3|1}W_2(t) + \sigma_{3|1.2}W_3(t) \tag{51}$$

where:

$$\rho_{2.3|1} = \frac{\rho_{2.3} - \rho_{1.2}\rho_{1.3}}{\sqrt{1-\rho_{1.2}^2}},\ \sigma_{3|1.2} = \sqrt{1-\rho_{1.3}^2-\rho_{2.3|1}^2} \tag{52}$$

and $(W_1(t), W_2(t), W_3(t))$ is a basis of three independent Brownian motions (Guillaume 2018), the multidimensional Cameron-Martin-Girsanov theorem yields:

$$V_{[\text{ORUU}]} = e^{-rt_2}K_3p_1\left(\mu_1^{(Q)}, \mu_2^{(Q)}, \mu_3^{(Q)}\right) - S_3(0)p_1\left(\mu_1^{(P_3)}, \mu_2^{(P_3)}, \mu_3^{(P_3)}\right) \tag{53}$$

where:

$$\mu_1^{(P_3)} = r - \frac{\sigma_1^2}{2} + \sigma_1\sigma_3\rho_{1.3},\ \mu_2^{(P_3)} = r - \frac{\sigma_2^2}{2} + \sigma_2\sigma_3\rho_{2.3},\ \mu_3^{(P_3)} = r + \frac{\sigma_3^2}{2} \tag{54}$$

The measure P_3 is the measure under which $B_1(t) - \sigma_3\rho_{1.3}t$, $B_2(t) - \sigma_3\rho_{2.3|1}t$ and $B_3(t) - \sigma_3\sigma_{3|1.2}t$ are three independent standard Brownian motions.

A simple and robust numerical evaluation of the function Ψ_4 consists in selecting an appropriate cutoff value for the negative infinity lower bounds and then applying a fixed-degree quadrature rule. Given the smoothness of the integrand, even a low-degree rule will perform well. Table 3 provides the prices of a few two-colour outside up-and-down knock-out call options for various levels of the volatility and correlation parameters of the underlying assets S_1, S_2, and S_3, and different values of the knock-out barriers. The parameters $S_i(0)$, K_i, t_1, t_2, and r are identical as those as in Tables 1 and 2. In each cell, the first reported value is the exact analytical price, as obtained by implementing Proposition 4 by means of a classical 16-point Gauss–Legendre quadrature, while the numbers in the brackets are three successive Monte Carlo approximations, as explained in Section 2.1.

From a purely numerical standpoint, the pattern of convergence of conditional Monte Carlo approximations to the analytical values is as clear in Table 3 as in Tables 1 and 2. This illustrates the robustness of our numerical integration scheme for the Ψ_4 function. The efficiency gap between Monte Carlo pricing and analytical pricing is even more pronounced than for non-outside rainbow step barrier options due to the presence of an additional stochastic process to simulate: the average computational time required by simulation is 42 s, whereas the evaluation of the analytical formula based on Proposition 4 only takes a few tenths of a second.

Table 3. Outside two-colour up-and-down knock-out call.

	$\rho_{1.2} = -0.6$, $\rho_{1.3} = \rho_{2.3} = -0.4$	$\rho_{1.2} = -0.6$, $\rho_{1.3} = \rho_{2.3} = 0.4$	$\rho_{1.2} = 0.6$ $\rho_{1.3} = \rho_{2.3} = 0.4$	$\rho_{1.2} = 0.6$ $\rho_{1.3} = \rho_{2.3} = -0.4$
$\sigma_1 = \sigma_2 = 20\%$ $\sigma_3 = 20\%$ $H_1 = 115, H_2 = 85$	2.378 (2.452, 2.361, 2.375)	2.772 (2.914, 2.812, 2.779)	1.717 (1.585, 1.731, 1.720)	1.182 (1.193, 1.178, 1.180)
$\sigma_1 = \sigma_2 = 20\%$ $\sigma_3 = 60\%$ $H_1 = 115, H_2 = 85$	5.769 (5.728, 5.781, 5.764)	7.053 (7.137, 7.036, 7.054)	4.522 (4.534, 4.541, 4.524)	2.897 (2.852, 2.923, 2.893)
$\sigma_1 = \sigma_2 = 60\%$ $\sigma_3 = 20\%$ $H_1 = 125, H_2 = 75$	1.627 (1.592, 1.614, 1.628)	2.783 (2.848, 2.767, 2.788)	1.351 (1.320, 1.365, 1.353)	0.849 (0.915, 0.828, 0.842)
$\sigma_1 = \sigma_2 = 60\%$ $\sigma_3 = 60\%$ $H_1 = 125, H_2 = 75$	3.864 (3.814, 3.872, 3.865)	7.554 (7.518, 7.535, 7.558)	3.823 (3.856, 3.829, 3.827)	2.076 (2.011, 2.091, 2.072)
$\sigma_1 = 20\%, \sigma_2 = 60\%$ $\sigma_3 = 20\%$ $H_1 = 115, H_2 = 75$	1.534 (1.502, 1.526, 1.535)	2.573 (2.495, 2.556, 2.577)	1.188 (1.207, 1.179, 1.185)	0.621 (0.774, 0.684, 0.613)
$\sigma_1 = 20\%, \sigma_2 = 60\%$ $\sigma_3 = 60\%$ $H_1 = 115, H_2 = 75$	3.629 (3.787, 3.662, 3.621)	6.697 (6.724, 6.684, 6.692)	3.217 (3.051, 3.252, 3.221)	1.518 (1.586, 1.476, 1.511)
$\sigma_1 = 60\%, \sigma_2 = 20\%$ $\sigma_3 = 20\%$ $H_1 = 125, H_2 = 85$	2.572 (2.734, 2.548, 2.567)	2.989 (3.125, 3.016, 2.994)	1.931 (2.071, 1.965, 1.938)	1.496 (1.634, 1.454, 1.489)
$\sigma_1 = 60\%, \sigma_2 = 20\%$ $\sigma_3 = 60\%$ $H_1 = 125, H_2 = 85$	6.248 (6.304, 6.237, 6.241)	7.953 (8.060, 7.984, 7.961)	5.323 (5.212, 5.348, 5.324)	3.707 (3.569, 3.726, 3.702)

From a financial point of view, the prices in Table 3 display a very different pattern from those in Tables 1 and 2. With regard to correlation, the highest option values attained are when the correlation between S_1 and S_2 is negative and the correlation between S_3 and both S_1 and S_2 is positive. The lowest option values are when the correlation between S_1 and S_2 is positive and the correlation between S_3 and both S_1 and S_2 is negative. On average across all volatilities and barrier levels in Table 3, options are approximately three times more expensive under the former correlation structure than under the latter one. In terms of volatility, the highest option values attained are when the volatility of asset S_3 is high. This remains true under very different combinations of values for all the other parameters (volatilities of S_1 and S_2, barrier levels and correlation structure). Such an observation highlights the prominent role of the volatility of the asset chosen to determine the moneyness of the option at expiry. In particular, the value of a rainbow outside step barrier option is a monotonically increasing function of σ_3, whereas the value of a rainbow step barrier option is not a monotonically increasing function of σ_2, just like the value of a reverse rainbow step barrier option is not a monotonically increasing function of σ_1. This is because a rainbow outside step barrier option allows to make a clear distinction between the functions of each underlying asset: two of them, S_1 and S_2, are only concerned with barrier crossing during the option life, and the third one, S_3, is only concerned with moneyness testing at the option expiry. That distinction is impossible to make when it comes to non-outside rainbow step barrier options, so that the impact of volatility becomes ambiguous and difficult to handle. It should be emphasised that, for the vast majority of parameters, the sensitivities of the rainbow outside step knock-out barrier options to σ_1 and σ_2 is negative, reflecting an increased risk of being deactivated before expiry. Only for quite specific correlation structures between the underlying assets and quite specific combinations of barrier values can these sensitivities be positive. A major advantage of closed form formulae such as those derived in this article is precisely to allow measurement of such sensitivities with high precision by mere differentiation of the formulae w.r.t. the relevant parameters.

One more noticeable difference in the reported numerical results between outside and non-outside rainbow step barrier options is that the volatility effect prevails over the barrier effect in Table 3, in contrast to Tables 1 and 2. Indeed, in row 2 of Table 3, tight barrier levels do not preclude relatively high option prices thanks to the volatility of asset S_3 set at 60%. Likewise, in row 3 of Table 3, wider barrier levels do not preclude relatively low option prices due to the volatility of asset S_3 set at only 20%.

2.5. Valuation of Two-Sided, Two-Colour Step Barrier Options

In this section, a two-sided barrier is introduced in each time interval, i.e., the valuation of rainbow step double barrier options is handled.

Let H_1 and H_2 denote an upward and a downward barrier, respectively, on the time interval $[t_0 = 0, t_1]$. Similarly, H_3 and H_4 represent an upward and a downward barrier, respectively, on the time interval $[t_1, t_2]$. As in the previous sections, barrier crossing is monitored w.r.t. process S_1 following Equation (1) on $[t_0 = 0, t_1]$ and w.r.t. process S_2 following Equation (2) on $[t_1, t_2]$. Our objective now is to find the value of the joint cumulative distribution function $\mathrm{P}_{[\mathrm{RDKO}]}(\mu_1, \mu_2)$ defined by:

$$\begin{aligned} &\mathrm{P}_{[\mathrm{RDKO}]}(\mu_1, \mu_2) \\ &= P\left(\begin{array}{c} \sup\limits_{0 \le t \le t_1} S_1(t) \le H_1, \inf\limits_{0 \le t \le t_1} S_1(t) \ge H_2, S_1(t_1) \le \min(K_1, H_1), \\ \sup\limits_{t_1 \le t \le t_2} S_2(t) \le H_3, \inf\limits_{t_1 \le t \le t_2} S_2(t) \ge H_4, S_2(t_2) \le \min(H_3, K_2) \end{array} \right) \end{aligned} \tag{55}$$

where the acronym "[RDKO]" stands for "Rainbow Double Knock Out".

The main result of Section 2.5 is given by the following Proposition 5.

Proposition 5. *The exact value of* $\mathrm{P}_{[\mathrm{RDKO}]}(\mu_1, \mu_2)$ *is given by:*

$$\begin{aligned} \mathrm{P}_{[\mathrm{RDKO}]}(\mu_1, \mu_2) = &\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \exp\left(\frac{2\mu_1}{\sigma_1^2} n_1 a_1 + \frac{2\mu_2}{\sigma_2^2} n_2 a_2 \right) \\ &\{ N_3[\mathrm{A}_1(\min(k_1, h_1)), \mathrm{A}_2(h_3), \mathrm{A}_3(\min(h_3, k_2)); \mathrm{x}_1] - N_3\left[\begin{array}{l} \mathrm{A}_1(h_2), \mathrm{A}_2(h_3), \\ \mathrm{A}_3(\min(h_3, k_2)); \mathrm{x}_1 \end{array} \right] \\ &- N_3[\mathrm{A}_1(\min(k_1, h_1)), \mathrm{A}_2(h_4), \mathrm{A}_3(\min(h_3, k_2)); \mathrm{x}_1] + N_3\left[\begin{array}{l} \mathrm{A}_1(h_2), \mathrm{A}_2(h_4), \\ \mathrm{A}_3(\min(h_3, k_2)); \mathrm{x}_1 \end{array} \right] \\ &- N_3[\mathrm{A}_1(\min(k_1, h_1)), \mathrm{A}_2(h_3), \mathrm{A}_3(h_4); \mathrm{x}_1] + N_3[\mathrm{A}_1(h_2), \mathrm{A}_2(h_3), \mathrm{A}_3(h_4); \mathrm{x}_1] \\ &+ N_3[\mathrm{A}_1(\min(k_1, h_1)), \mathrm{A}_2(h_4), \mathrm{A}_3(h_4); \mathrm{x}_1] - N_3[\mathrm{A}_1(h_2), \mathrm{A}_2(h_4), \mathrm{A}_3(h_4); \mathrm{x}_1] \} \end{aligned} \tag{56}$$

$$\begin{aligned} &- \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \exp\left(n_1 a_1 \left(\frac{2\mu_1}{\sigma_1^2} - \frac{4\rho_{1.2}\mu_2}{\sigma_1 \sigma_2} \right) + \frac{2\mu_2}{\sigma_2^2}(h_4 - n_2 a_2) \right) \\ &\{ N_3[\mathrm{A}_4(\min(k_1, h_1)), \mathrm{A}_5(h_3), \mathrm{A}_6(\min(h_3, k_2)); \mathrm{x}_2] - N_3\left[\begin{array}{l} \mathrm{A}_4(h_2), \mathrm{A}_5(h_3), \\ \mathrm{A}_6(\min(h_3, k_2)); \mathrm{x}_2 \end{array} \right] \\ &- N_3[\mathrm{A}_4(\min(k_1, h_1)), \mathrm{A}_5(h_4), \mathrm{A}_6(\min(h_3, k_2)); \mathrm{x}_2] + N_3\left[\begin{array}{l} \mathrm{A}_4(h_2), \mathrm{A}_5(h_4), \\ \mathrm{A}_6(\min(h_3, k_2)); \mathrm{x}_2 \end{array} \right] \\ &- N_3[\mathrm{A}_4(\min(k_1, h_1)), \mathrm{A}_5(h_3), \mathrm{A}_6(h_4); \mathrm{x}_2] + N_3[\mathrm{A}_4(h_2), \mathrm{A}_5(h_3), \mathrm{A}_6(h_4); \mathrm{x}_2] \\ &+ N_3[\mathrm{A}_4(\min(k_1, h_1)), \mathrm{A}_5(h_4), \mathrm{A}_6(h_4); \mathrm{x}_2] - N_3[\mathrm{A}_4(h_2), \mathrm{A}_5(h_4), \mathrm{A}_6(h_4); \mathrm{x}_2] \} \end{aligned} \tag{57}$$

$$
\begin{aligned}
&-\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\exp\left(\frac{2\mu_1}{\sigma_1^2}(h_2-n_1a_1)+\frac{2\mu_2}{\sigma_2^2}n_2(h_3-a_2)\right)\\
&\{N_3[\mathrm{A}_7(\min(k_1,h_1)),\mathrm{A}_8(h_3),\mathrm{A}_9(\min(h_3,k_2));\mathrm{x}_1]-N_3\left[\begin{array}{l}\mathrm{A}_7(h_2),\mathrm{A}_8(h_3),\\ \mathrm{A}_9(\min(h_3,k_2));\mathrm{x}_1\end{array}\right]\\
&-N_3[\mathrm{A}_7(\min(k_1,h_1)),\mathrm{A}_8(h_4),\mathrm{A}_9(\min(h_3,k_2));\mathrm{x}_1]+N_3\left[\begin{array}{l}\mathrm{A}_7(h_2),\mathrm{A}_8(h_4),\\ \mathrm{A}_9(\min(h_3,k_2));\mathrm{x}_1\end{array}\right]\\
&-N_3[\mathrm{A}_7(\min(k_1,h_1)),\mathrm{A}_8(h_3),\mathrm{A}_9(h_4);\mathrm{x}_1]+N_3[\mathrm{A}_7(h_2),\mathrm{A}_8(h_3),\mathrm{A}_9(h_4);\mathrm{x}_1]\\
&+N_3[\mathrm{A}_7(\min(k_1,h_1)),\mathrm{A}_8(h_4),\mathrm{A}_9(d_2);\mathrm{x}_1]-N_3[\mathrm{A}_7(h_2),\mathrm{A}_8(h_4),\mathrm{A}_9(d_2);\mathrm{x}_1]\}
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
&+\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\exp\left((h_2-n_1a_1)\left(\frac{2\mu_1}{\sigma_1^2}-\frac{4\rho_{1.2}\mu_2}{\sigma_1\sigma_2}\right)+\frac{2\mu_2}{\sigma_2^2}(h_4-n_2a_2)\right)\\
&\{N_3[\mathrm{A}_{10}(\min(k_1,h_1)),\mathrm{A}_{11}(h_3),\mathrm{A}_{12}(\min(h_3,k_2));\mathrm{x}_2]-N_3\left[\begin{array}{l}\mathrm{A}_{10}(h_2),\mathrm{A}_{11}(h_3),\\ \mathrm{A}_{12}(\min(h_3,k_2));\mathrm{x}_2\end{array}\right]\\
&-N_3[\mathrm{A}_{10}(\min(k_1,h_1)),\mathrm{A}_{11}(h_4),\mathrm{A}_{12}(\min(h_3,k_2));\mathrm{x}_2]+N_3\left[\begin{array}{l}\mathrm{A}_{10}(h_2),\mathrm{A}_{11}(h_4),\\ \mathrm{A}_{12}(\min(h_3,k_2));\mathrm{x}_2\end{array}\right]\\
&-N_3[\mathrm{A}_{10}(\min(k_1,h_1)),\mathrm{A}_{11}(h_3),\mathrm{A}_{12}(h_4);\mathrm{x}_2]+N_3[\mathrm{A}_{10}(h_2),\mathrm{A}_{11}(h_3),\mathrm{A}_3(h_4);\mathrm{x}_2]\\
&+N_3[\mathrm{A}_{10}(\min(k_1,h_1)),\mathrm{A}_{11}(h_4),\mathrm{A}_{12}(h_4);\mathrm{x}_2]-N_3[\mathrm{A}_{10}(h_2),\mathrm{A}_{11}(h_4),\mathrm{A}_{12}(h_4);\mathrm{x}_2]\}
\end{aligned} \tag{59}
$$

where:

$$-\ h_1=\ln\left(\frac{H_1}{S_1(0)}\right)>0,h_2=\ln\left(\frac{H_2}{S_1(0)}\right)<0,h_3=\ln\left(\frac{H_3}{S_2(0)}\right),h_4=\ln\left(\frac{H_4}{S_2(0)}\right) \tag{60}$$

$$-\ a_1=h_1-h_2,a_2=h_3-h_4 \tag{61}$$

$$-\ \mathrm{A}_1(x)=\frac{x-2n_1a_1-\mu_1t_1}{\sigma_1\sqrt{t_1}},\mathrm{A}_2(x)=\frac{x-\mu_2t_1}{\sigma_2\sqrt{t_1}}-\frac{2\rho_{1.2}n_1a_1}{\sigma_1\sqrt{t_1}} \tag{62}$$

$$-\ \mathrm{A}_3(x)=\frac{x-2n_2a_2-\mu_2t_2}{\sigma_2\sqrt{t_2}}-\frac{2\rho_{1.2}n_1a_1}{\sigma_1\sqrt{t_2}} \tag{63}$$

$$-\ \mathrm{A}_4(x)=\mathrm{A}_1(x)-\frac{2\rho_{1.2}\mu_2\sqrt{t_1}}{\sigma_2},\mathrm{A}_5(x)=\frac{x+\mu_2t_1}{\sigma_2\sqrt{t_1}}-\frac{2\rho_{1.2}n_1a_1}{\sigma_1\sqrt{t_1}} \tag{64}$$

$$-\ \mathrm{A}_6(x)=\frac{x-2h_4+2n_2a_2-\mu_2t_2}{\sigma_2\sqrt{t_2}}+\frac{2\rho_{1.2}n_1a_1}{\sigma_1\sqrt{t_2}} \tag{65}$$

$$-\ \mathrm{A}_7(x)=\frac{x-2h_2+2n_1a_1-\mu_1t_1}{\sigma_1\sqrt{t_1}},\mathrm{A}_8(x)=\frac{x-\mu_2t_1}{\sigma_2\sqrt{t_1}}-\frac{2\rho_{1.2}(h_2-n_1a_1)}{\sigma_1\sqrt{t_1}} \tag{66}$$

$$-\ \mathrm{A}_9(x)=\frac{x-2n_2a_2-\mu_2t_2}{\sigma_2\sqrt{t_2}}-\frac{2\rho_{1.2}(h_2-n_1a_1)}{\sigma_1\sqrt{t_2}} \tag{67}$$

$$-\ \mathrm{A}_{10}(x)=\mathrm{A}_7(x)-\frac{2\rho_{1.2}\mu_2\sqrt{t_1}}{\sigma_2},\mathrm{A}_{11}(x)=\frac{x+\mu_2t_1}{\sigma_2\sqrt{t_1}}-\frac{2\rho_{1.2}(h_2-n_1a_1)}{\sigma_1\sqrt{t_1}} \tag{68}$$

$$-\ \mathrm{A}_{12}(x)=\frac{x-2h_4+2n_2a_2-\mu_2t_2}{\sigma_2\sqrt{t_2}}+\frac{2\rho_{1.2}(h_2-n_1a_1)}{\sigma_1\sqrt{t_2}} \tag{69}$$

$$-\ \mathrm{x}_1=\left\{\rho_{1.2},\rho_{1.2}\sqrt{\frac{t_1}{t_2}},\sqrt{\frac{t_1}{t_2}}\right\},\mathrm{x}_2=\left\{\rho_{1.2},-\rho_{1.2}\sqrt{\frac{t_1}{t_2}},-\sqrt{\frac{t_1}{t_2}}\right\} \tag{70}$$

All other notations have been defined in the previous sections.
End of Proposition 5.

Pricing two-colour double knock-out barrier options can be achieved through the same changes of probability measures as those applicable to two-colour single knock-out barrier options, i.e., the value of a two-colour double knock-out put, denoted as $V_{[RDKO]}$, is given by:

$$V_{[RDKO]} = e^{-rt_2} K_3 P_{[RDKO]}\left(\mu_1^{(Q)}, \mu_2^{(Q)}\right) - S_2(0) P_{[RDKO]}\left(\mu_1^{(P_2)}, \mu_2^{(P_2)}\right) \quad (71)$$

where the parameters $\mu_i^{(Q)}$ and $\mu_i^{(P_2)}$ are given by Equation (18).

Table 4 provides the prices of a few two-colour knock-out double barrier puts for various levels of the volatility and correlation parameters of the underlying assets S_1 and S_2, and different values of the knock-out barriers. Expiry is 6 months and t_1 is one quarter of a year. The parameters $S_i(0)$, K_i, and r are identical to those in Tables 1–3. In each cell, the first number is the exact analytical value as derived from (71), while the numbers in the brackets are three successive Monte Carlo approximations, as explained in Section 2.1.

Table 4. Two-colour rainbow double knock-out put.

	$\rho_{1.2} = -0.6$	$\rho_{1.2} = -0.2$	$\rho_{1.2} = 0.2$	$\rho_{1.2} = 0.6$
$\sigma_1 = \sigma_2 = 15\%$ $H_1 = H_3 = 120$ $H_2 = H_4 = 80$	2.919 (2.985, 2.956, 2.916)	2.938 (2.792, 2.883, 2.932)	2.936 (2.974, 2.918, 2.931)	3.027 (3.191, 3.088, 3.032)
$\sigma_1 = \sigma_2 = 30\%$ $H_1 = H_3 = 130$ $H_2 = H_4 = 70$	4.370 (4.529, 4.281, 4.365)	5.791 (5.978, 5.697, 5.786)	4.427 (4.196, 4.443, 4.426)	5.175 (5.329, 5.092, 5.158)
$\sigma_1 = 15\%, \sigma_2 = 30\%$ $H_1 = 120, H_3 = 130$ $H_2 = 80, H_4 = 70$	4.726 (4.594, 4.752, 4.728)	4.744 (4.868, 4.785, 4.749)	4.724 (4.574, 4.771, 4.728)	5.042 (5.226, 5.018, 5.045)
$\sigma_1 = 30\%, \sigma_2 = 15\%$ $H_1 = 130, H_3 = 120$ $H_2 = 70, H_4 = 80$	2.614 (2.429, 2.576, 2.610)	2.683 (2.872, 2.612, 2.679)	2.771 (2.602, 2.742, 2.773)	2.942 (3.165, 3.036, 2.953)

Thanks to the rapidly decaying exponential functions in the integrands, a level of 10^{-7} convergence is attained by stopping at 8, the number of iterations controlled by the absolute values of n_1 and n_2 in the double sum operators, which results in a total computational time of less than 1 s. For higher values of the volatility parameters than those in Table 4, however, the uniform convergence of the double sums in (56)–(59) may require a greater number of iterations and thus take more time. The implementation of Proposition 5 using the Φ_3 function introduced in Section 3 is slightly faster than the one using the trivariate standard normal cumulative distribution function N_3, although the difference is relatively negligible for most practical purposes. Both methods of implementation yield prices equal to at least 4 decimals.

From a financial standpoint, a striking contrast between the numerical results in Table 4 and those of the previous sections is the much weaker dependency of the option value on the correlation structure, as illustrated by the smaller differences between the four option prices associated with each combination of volatility and barrier parameters. It seems that, the more volatility, the more dependency on the correlation structure, as suggested by the comparison between row 1 and row 2. Another noticeable difference is that the functional relation with the correlation structure is not monotonic. This is particularly clear in row 2 where a relatively significant increase in value from $\rho_{1.2} = -0.6$ to $\rho_{1.2} = -0.2$ is followed by a relatively significant decrease in value from $\rho_{1.2} = -0.2$ to $\rho_{1.2} = 0.2$, before a new increase in value from $\rho_{1.2} = 0.2$ to $\rho_{1.2} = 0.6$. This more complex and unstable dependency on correlation structure suggests that two-colour knock-out double barrier options are a less suitable instrument for correlation trading than two-colour knock-out single barrier options. However, one should remain wary of drawing hasty conclusions from the comparison of the results in Table 4 and those in the previous sections, as the option parameters are not identical, especially regarding volatility and expiry.

3. Proofs of Formulae

The proofs of Propositions 2 and 3 are only outlined as they essentially follow the same steps as the proof of Proposition 1.

Proof of Proposition 1. Since the log function is strictly increasing, we have:

$$P_{[RUU]}(\mu_1,\mu_2) = P\left(\sup_{0\le t\le t_1} X_1(t) \le h_1, X_1(t_1) \le k_1, X_2(t_1) \le k_2, \sup_{t_1\le t\le t_2} X_2(t) \le h_2, X_2(t_2) \le k_3\right) \tag{72}$$

Next, it can be noticed that, despite the non-zero correlation between $X_1(t)$ and $X_2(t)$, the law of $\sup_{0\le t\le t_1} X_1(t)$ conditional on $X_1(t_1)$ and $X_2(t_1)$ is equal to the law of $\sup_{0\le t\le t_1} X_1(t_1)$ conditional on $X_1(t_1)$.

Indeed, denoting the density function operator as $f(.)$ and making use of the Markov property of $X_2(t)$ we have:

$$f\left(\sup_{0\le t\le t_1} X_1(t)|X_1(t_1), X_2(t_1)\right) = \frac{f\left(X_2(t_1)\middle|\sup_{0\le t\le t_1} X_1(t), X_1(t_1)\right) f\left(\sup_{0\le t\le t_1} X_1(t), X_1(t_1)\right)}{f(X_1(t_1), X_2(t_1))} \tag{73}$$

$$= \frac{f(X_2(t_1)|X_1(t_1))f\left(\sup_{0\le t\le t_1} X_1(t), X_1(t_1)\right)}{f(X_1(t_1), X_2(t_1))} = \frac{f(X_1(t_1), X_2(t_1))}{f(X_1(t_1))}\frac{f\left(\sup_{0\le t\le t_1} X_1(t), X_1(t_1)\right)}{f(X_1(t_1), X_2(t_1))} \tag{74}$$

$$= \frac{f\left(\sup_{0\le t\le t_1} X_1(t), X_1(t_1)\right)}{f(X_1(t_1))} = f\left(\sup_{0\le t\le t_1} X_1(t)|X_1(t_1)\right) \tag{75}$$

A translation from the time interval $[t_0 = 0, t_1]$ to the time interval $[t_1, t_2]$, through the substitution of $X_1(0)$ with $X_2(t_1)$, of $X_1(t_1)$ with $X_2(t_2)$ and of $X_2(t_1)$ with $X_1(t_2)$, shows similarly that the law of $\sup_{t_1\le t\le t_2} X_2(t)$ conditional on $X_2(t_1)$, $X_2(t_2)$ and $X_1(t_2)$ is equal to the law of $\sup_{t_1\le t\le t_2} X_2(t)$ conditional on $X_2(t_1)$ and $X_2(t_2)$.

Thus, by conditioning w.r.t. the absolutely continuous random variables $X_1(t_1)$, $X_2(t_1)$ and $X_2(t_2)$, we can express the problem as the following integral:

$$P_{[RUU]}(\mu_1,\mu_2) = \int_{-\infty}^{\min(k_1,h_1)}\int_{-\infty}^{\min(k_2,h_2)}\int_{-\infty}^{\min(k_3,h_2)} \varphi_1(x_1,x_2,x_3)\varphi_2(x_1)\varphi_3(x_2,x_3)dx_3dx_2dx_1 \tag{76}$$

where

$$\varphi_1(x_1,x_2,x_3) = P(X_1(t_1) \in dx_1, X_2(t_1) \in dx_2, X_2(t_2) \in dx_3)dx_3dx_2dx_1 \tag{77}$$

$$\varphi_2(x_1) = P\left(\sup_{0\le t\le t_1} X_1(t) \le h_1|X_1(t_1) \in dx_1\right)dx_1 \tag{78}$$

$$\varphi_3(x_2,x_3) = P\left(\sup_{t_1\le t\le t_2} X_2(t) \le h_2|X_2(t_1) \in dx_2, X_2(t_2) \in dx_3\right)dx_2dx_3 \tag{79}$$

The functions φ_2 and φ_3 in (78) and (79) can be expanded by applying known formulae that can be found in Wang and Pötzelberger (1997):

$$\varphi_2(x_1) = 1 - \exp\left(\frac{2h_1(x_1 - h_1)}{\sigma_1^2 t_1}\right) \tag{80}$$

$$\varphi_3(x_2, x_3) = 1 - \exp\left(\frac{2(h_2 - x_2)(x_3 - h_2)}{\sigma_2^2(t_2 - t_1)}\right) \tag{81}$$

The function φ_1 derives from the trivariate normality of the triple $(X_1(t_1), X_2(t_1), X_2(t_2))$. It is elementary to obtain the marginal distributions:

$$X_1(t_1) \sim \mathcal{N}\left(\mu_1 t_1, \sigma_1^2 t_1\right), X_2(t_1) \sim \mathcal{N}\left(\mu_2 t_1, \sigma_2^2 t_1\right), X_2(t_2) \sim \mathcal{N}\left(\mu_2 t_1, \sigma_2^2 t_1\right) \tag{82}$$

where $\mathcal{N}(a, b^2)$ refers to the normal distribution with expectation a and variance b^2.

Denoting by Z_1, Z_2, Z_3 three independent standard normal random variables, the pairwise covariances can be written as follows:

$$\mathrm{cov}[X_1(t_1), X_2(t_1)] = \mathrm{cov}\left[\mu_1 t_1 + \sigma_1\sqrt{t_1}Z_1, \mu_2 t_1 + \sigma_2\sqrt{t_1}\left(\rho_{1.2}Z_1 + \sqrt{1-\rho_{1.2}^2}Z_2\right)\right] = \sigma_1\sigma_2\rho_{1.2}t_1 \tag{83}$$

$$\mathrm{cov}[X_2(t_1), X_2(t_2)] = \mathrm{cov}\left[\begin{array}{l} \mu_2 t_1 + \sigma_2\sqrt{t_1}\left(\rho_{1.2}Z_1 + \sqrt{1-\rho_{1.2}^2}Z_2\right), \\ \mu_2 t_2 + \sigma_2\sqrt{t_1}\left(\rho_{1.2}Z_1 + \sqrt{1-\rho_{1.2}^2}Z_2\right) + \sigma_2\sqrt{t_2 - t_1}Z_3 \end{array}\right] = \sigma_2^2 t_1 \tag{84}$$

$$\begin{aligned} &\mathrm{cov}[X_1(t_1), X_2(t_2)] = \mathrm{cov}\left[\mu_1 t_1 + \sigma_1\sqrt{t_1}Z_1, \mu_2 t_2 + \sigma_2\sqrt{t_1}\left(\rho_{1.2}Z_1 + \sqrt{1-\rho_{1.2}^2}Z_2\right) + \sigma_2\sqrt{t_2 - t_1}Z_3\right] \\ &= \sigma_1\sigma_2\rho_{1.2}t_1 \end{aligned} \tag{85}$$

where we have applied the bilinearity of the covariance operator, the independence of increments of Brownian motion, and the orthogonal decomposition of two-dimensional correlated Brownian motion. The correlation coefficients $\theta_{1.2}, \theta_{1.3}, \theta_{2.3}$ in Proposition 1 ensue. Expanding the trivariate normal density function $\varphi_1(x_1, x_2, x_3)$ as a product of normal conditional densities (Guillaume 2018), we obtain:

$$\begin{aligned} &\varphi_1(x_1, x_2, x_3) \\ &= \frac{e^{-\frac{1}{2}\left(\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)^2 - \frac{1}{2\sigma_{2|1}^2}\left(\frac{x_2-\mu_2 t_1}{\sigma_2\sqrt{t_1}} - \theta_{1.2}\left(\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)\right)^2 - \frac{1}{2\sigma_{3|1.2}^2}\left(\frac{x_3-\mu_2 t_2}{\sigma_2\sqrt{t_2}} - \theta_{1.3}\left(\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right) - \frac{\theta_{2.3|1}}{\sigma_{2|1}}\left(\frac{x_2-\mu_2 t_1}{\sigma_2\sqrt{t_1}} - \theta_{1.2}\left(\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)\right)\right)^2}}{(2\pi)^{3/2}\sigma_{2|1}\sigma_{3|1.2}\sigma_2^2\sigma_1 t_1\sqrt{t_2}} \end{aligned} \tag{86}$$

where:

$$\sigma_{2|1} = \sqrt{1-\theta_{1.2}^2}, \theta_{2.3|1} = \frac{\theta_{2.3} - \theta_{1.2}\theta_{1.3}}{\sqrt{1-\theta_{1.2}^2}}, \sigma_{3|1.2} = \sqrt{1-\theta_{1.3}^2 - \theta_{2.3|1}^2} \tag{87}$$

The terms $\sigma_{2|1}$, $\theta_{2.3|1}$ and $\sigma_{3|1.2}$ have the following precise meanings:

- $\sigma_{2|1}$ is the conditional standard deviation of $X_2(t_1)$ given $X_1(t_1)$;
- $\theta_{2.3|1}$ is the conditional correlation between $X_2(t_1)$ and $X_2(t_2)$ given $X_1(t_1)$;
- $\sigma_{3|1.2}$ is the conditional standard deviation of $X_2(t_2)$ given $X_1(t_1)$ and $X_2(t_1)$.

The rest of the proof, whose cumbersome details are omitted, then consists in solving the four integrals implied by (76). The final result takes the form of the linear combination of four N_3 functions written in Proposition 1.

Corollary 1 comes from the property of symmetry of Brownian paths.

Corollary 2 is a consequence of the fact that:

$$\begin{aligned} &\mathrm{P}_{[\mathrm{RUU}]}^{(\mathrm{I})}(\mu_1, \mu_2) \\ &= \int_{-\infty}^{\min(k_1,h_1)} \int_{-\infty}^{\min(k_2,h_2)} \int_{-\infty}^{\min(k_3,h_2)} \varphi_1(x_1, x_2, x_3) \exp\left(\frac{2h_1(x_1 - h_1)}{\sigma_1^2 t_1}\right) \exp\left(\frac{2(h_2 - x_1)(x_2 - h_2)}{\sigma_2^2(t_2 - t_1)}\right) dx_3 dx_2 dx_1 \end{aligned} \tag{88}$$

Corollary 3 comes from the fact that the correlation coefficient between the random variables $S_2(t_2)$ and $S_2(t_3)$ is equal to $\sqrt{\frac{t_2}{t_3}}$.

Corollary 4 is a straightforward application of the law of total probability. □

Proof of Proposition 2. Using similar steps as in the proof of Proposition 1, one can express the problem at hand as the following integral:

$$\mathrm{P}_{[RUD]}(\mu_1,\mu_2) = \int_{-\infty}^{\min(k_1,h_1)} \int_{\max(k_2,h_2)}^{\infty} \int_{\max(k_3,h_2)}^{\infty} \varphi_1(x_1,x_2,x_3)\varphi_2(x_1)\varphi_4(x_2,x_3)dx_3dx_2dx_1 \quad (89)$$

where the functions φ_1 and φ_2 are given by (86) and (80), respectively, and:

$$\varphi_4(x_2,x_3) = P\left(\inf_{t_1\le t\le t_2} X_2(t) \ge h_2 | X_2(t_1)\in dx_2, X_2(t_2)\in dx_3\right)dx_2dx_3 = \varphi_3(x_2,x_3) \quad (90)$$

where the function φ_3 is given by (81).

Performing the necessary calculations, one can obtain the linear combination of four N_3 functions given in Proposition 2.

As in Proposition 1, Corollary 1 comes from the property of symmetry of Brownian paths.

Corollary 2 is a consequence of the fact that:

$$\mathrm{P}^{(I)}_{[RUD]}(\mu_1,\mu_2) = \int_{-\infty}^{\min(k_1,h_1)} \int_{\max(k_2,h_2)}^{\infty} \int_{\max(k_3,h_2)}^{\infty} \varphi_1(x_1,x_2,x_3)e^{\frac{2\mu_1}{\sigma_1^2}h_1}\frac{e^{-\frac{1}{2}\left(\frac{x_1-2h_1-\mu_1t_1}{\sigma_1\sqrt{t_1}}\right)^2}}{\sigma_1\sqrt{2\pi t_1}}e^{\frac{2\mu_2}{\sigma_2^2}(h_2-x_2)}\frac{e^{-\frac{1}{2}\left(\frac{-x_3-x_2+2h_2+\mu_2(t_2-t_1)}{\sigma_2\sqrt{t_2-t_1}}\right)^2}}{\sigma_2\sqrt{2\pi(t_2-t_1)}}dx_1dx_2dx_3 \quad (91)$$

□

Proof of Proposition 3. One can express the problem at hand as the following integral:

$$\mathrm{P}_{[RRUU]}(\mu_1,\mu_2) = \int_{-\infty}^{\min(k_1,h_1)} \int_{-\infty}^{h_2} \int_{-\infty}^{\min(k_2,h_2)} \int_{-\infty}^{k_3} \varphi_2(x_1)\varphi_3(x_2,x_3)\varphi_5(x_1,x_2,x_3,x_4)dx_4dx_3dx_2dx_1 \quad (92)$$

where

$$\varphi_5(x_1,x_2,x_3,x_4) = P(X_1(t_1)\in dx_1, X_2(t_1)\in dx_2, X_2(t_2)\in dx_3, X_1(t_2)\in dx_4)dx_4dx_3dx_2dx_1 \quad (93)$$

Plugging the quadrivariate normal joint density function of the set of random variables $X_1(t_1)$, $X_2(t_1)$, $X_1(t_2)$ and $X_2(t_2)$, as a product of conditional density functions as explained in Guillaume (2018), and then factoring in the conditional cumulative distribution function of $X_1(t_2)$ given the triple $(X_1(t_1), X_2(t_1), X_2(t_2))$, Proposition 3 ensues. □

Proof of Proposition 4. Proof is given only for $p_1(\mu_1,\mu_2,\mu_3)$ and $p_3(\mu_1,\mu_2,\mu_3)$, as $p_2(\mu_1,\mu_2,\mu_3)$ and $p_4(\mu_1,\mu_2,\mu_3)$ can then be deduced by the same symmetry argument as that already used in Corollary 1 of Proposition 1.

Following steps similar to the beginning of the proof of Proposition 1, one can express the problem at hand as the following two integrals:

$$p_1(\mu_1,\mu_2,\mu_3) = \int_{x_1=-\infty}^{\min(k_1,h_1)} \int_{x_2=-\infty}^{\min(k_2,h_2)} \int_{x_3=-\infty}^{\min(k_3,h_2)} \int_{x_4=-\infty}^{k_4} \varphi_6(x_1)\varphi_7(x_1,x_2)\varphi_8(x_2,x_3)\varphi_9(x_1,x_3,x_4)dx_4dx_3dx_2dx_1 \quad (94)$$

$$p_3(\mu_1,\mu_2,\mu_3) = \int_{x_1=-\infty}^{\min(k_1,h_1)} \int_{x_2=\max(k_2,h_2)}^{\infty} \int_{x_3=\max(k_3,h_2)}^{\infty} \int_{x_4=k_4}^{\infty} \varphi_6(x_1)\varphi_7(x_1,x_2)\varphi_{10}(x_2,x_3)\varphi_9(x_1,x_3,x_4)dx_4dx_3dx_2dx_1 \quad (95)$$

where

$$\varphi_6(x_1) = P\left(\sup_{0\le t\le t_1} X_1(t) \le h_1, X_1(t_1)\in dx_1\right)dx_1 \quad (96)$$

$$\varphi_7(x_1, x_2) = P(X_2(t_1) \in dx_2 | X_1(t_1) \in dx_1) dx_1 dx_2 \tag{97}$$

$$\varphi_8(x_2, x_3) = P\left(\sup_{t_1 \le t \le t_2} X_2(t) \le h_2, X_2(t_2) \in dx_3 | X_2(t_1) \in dx_2\right) dx_2 dx_3 \tag{98}$$

$$\varphi_{10}(x_2, x_3) = P\left(\inf_{t_1 \le t \le t_2} X_2(t) \ge h_2, X_2(t_2) \in dx_3 | X_2(t_1) \in dx_2\right) dx_2 dx_3 \tag{99}$$

$$\varphi_9(x_1, x_3, x_4) = P(X_3(t_2) \in dx_4 | X_2(t_2) \in dx_3, X_1(t_1) \in dx_1) \tag{100}$$

The function φ_6 is obtained by differentiating the classical formula for the joint cumulative distribution of the maximum of a Brownian motion with drift and its endpoint over a closed time interval (see, e.g., Karatzas and Shreve 2000):

$$\varphi_6(x_1) = \frac{e^{-\frac{1}{2}\left(\frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)^2}}{\sigma_1 \sqrt{2\pi t_1}} - e^{\frac{2\mu_1}{\sigma_1^2} h_1} \frac{e^{-\frac{1}{2}\left(\frac{x_1 - 2h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)^2}}{\sigma_1 \sqrt{2\pi t_1}} dx_1 \tag{101}$$

The function φ_7 is easily derived from the bivariate normality of the pair $(X_1(t_1), X_2(t_1))$:

$$\varphi_7(x_1, x_2) = \frac{e^{-\frac{1}{2(1-\rho_{1.2}^2)}\left(\frac{x_2 - \mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \rho_{1.2} \frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)^2}}{\sigma_2 \sqrt{2\pi t_1 (1 - \rho_{1.2}^2)}} dx_1 dx_2 \tag{102}$$

To handle the function φ_8, we notice that, by conditioning w.r.t. the filtration at time t_1 the same classical formula as the one used to derive φ_6, we can obtain:

$$P\left(\sup_{t_1 \le t \le t_2} S_2(t) \le H_2, S_2(t_2) \le S_2(0) e^{x_3} | S_2(t_1) = S_2(0) e^{x_2}\right)$$

$$= N\left[\frac{\ln\left(\frac{S_2(0)e^{x_3}}{S_2(0)e^{x_2}}\right) - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right] - \left(\frac{H_2}{S_2(0)e^{x_2}}\right)^{\frac{2\mu_2}{\sigma_2^2}} N\left[\frac{\ln\left(\frac{S_2(0)e^{x_3}}{S_2(0)e^{x_2}}\right) - 2\ln\left(\frac{H_2}{S_2(0)e^{x_2}}\right) - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right] \tag{103}$$

for any given $(x_2, x_3) \in \mathbb{R}^2$ and $H_2 > S_2(0)e^{x_3}$.

Equation (103) can be rewritten as follows:

$$P\left(\sup_{t_1 \le t \le t_2} X_2(t) \le h_2, X_2(t_2) \le x_3 | X_2(t_1) \in dx_2\right) \tag{104}$$

$$= N\left[\frac{x_3 - x_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right] - \exp\left(\frac{2\mu_2}{\sigma_2^2}(h_2 - x_2)\right) N\left[\frac{x_3 - x_2 - 2(h_2 - x_2) - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right]$$

Therefore, by differentiating (104) w.r.t. x_3, we obtain:

$$\varphi_8(x_2, x_3) = \frac{e^{-\frac{1}{2}\left(\frac{x_3 - x_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right)^2}}{\sigma_2 \sqrt{2\pi(t_2 - t_1)}} - e^{\frac{2\mu_2}{\sigma_2^2}(h_2 - x_2)} \frac{e^{-\frac{1}{2}\left(\frac{x_3 + x_2 - 2h_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right)^2}}{\sigma_2 \sqrt{2\pi(t_2 - t_1)}} dx_2 dx_3 \tag{105}$$

By the symmetry of paths of Brownian motion, we have:

$$P\left(\inf_{t_1 \le t \le t_2} X_2(t) \ge h_2, X_2(t_2) \ge x_3 | X_2(t_1) \in dx_2\right) \tag{106}$$

$$= N\left[\frac{-x_3 + x_2 + \mu_2(t_2 - t_1)}{\sigma_2\sqrt{t_2 - t_1}}\right] - \exp\left(\frac{2\mu_2}{\sigma_2^2}(h_2 - x_2)\right)N\left[\frac{-x_3 + x_2 + 2(h_2 - x_2) + \mu_2(t_2 - t_1)}{\sigma_2\sqrt{t_2 - t_1}}\right]$$

Hence,

$$\varphi_{10}(x_2, x_3) = \varphi_8(x_2, x_3) \tag{107}$$

The function φ_9 derives from the joint trivariate normality of the triple $(X_1(t_1), X_2(t_2), X_3(t_2))$. The marginal distributions of the elements of this triple come from the known marginal distributions of $S_1(t_1)$, $S_2(t_2)$ and $S_3(t_2)$. The pairwise correlations, as given by $\theta_{1.3}, \theta_{1.4}$ and $\theta_{3.4}$ in Proposition 4 can be easily determined using the same method as in the proof of Proposition 3. We obtain:

$$\varphi_9(x_1, x_3, x_4) = \frac{e^{-\frac{1}{2\phi_{4|1.3}^2}\left(\frac{x_4-\mu_3 t_2}{\sigma_3\sqrt{t_2}} - \theta_{1.4}\left(\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right) - \theta_{3.4|1}\left(\frac{x_3-\mu_2 t_2}{\sigma_2\sqrt{t_2}} - \theta_{1.3}\left(\frac{x_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)\right)\right)^2}}{\phi_{4|1.3}\sigma_3\sqrt{2\pi t_2}} \tag{108}$$

where $\sigma_{3|1} = \sqrt{1 - \theta_{1.3}^2}$.

The rest of the proof, whose cumbersome details are omitted, then consists in solving the integrals implied by (94) and (95). The final result can be expressed as the linear combination of four Ψ_4 functions written in Proposition 4. The origin of the function Ψ_4, which is a special form of quadrivariate normal cumulative distribution, lies in the FDD (Finite Dimensional Distribution) of the quadruple $[S_1(t_1), S_2(t_1), S_2(t_2), S_3(t_2)]$. Indeed, a little algebra shows that, $\forall D_1, D_2, D_3, D_4 \in \mathbb{R}^+$, we have:

$$P(S_1(t_1) \le D_1, S_2(t_1) \le D_2, S_2(t_2) \le D_3, S_3(t_2) \le D_4)$$

$$= \Psi_4\left[\begin{array}{l}\frac{\ln\left(\frac{D_1}{S_1(0)}\right)-\mu_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{\ln\left(\frac{D_2}{S_2(0)}\right)-\mu_2 t_1}{\sigma_2\sqrt{t_1}}, \frac{\ln\left(\frac{D_3}{S_2(0)}\right)-\mu_2 t_2}{\sigma_2\sqrt{t_2}}, \frac{\ln\left(\frac{D_4}{S_3(0)}\right)-\mu_3 t_2}{\sigma_3\sqrt{t_2}}; \\ \theta_{1.2}, \theta_{1.3}, \theta_{1.4}, \theta_{2.3}, \theta_{3.4}\end{array}\right] \tag{109}$$

Corollary 1 is a consequence of the fact that:

$$\begin{aligned} p_1^{(I)}(\mu_1, \mu_2) = & \int_{x_1=-\infty}^{\min(k_1,h_1)} \int_{x_2=-\infty}^{\min(k_2,h_2)} \int_{x_3=-\infty}^{\min(k_3,h_2)} \int_{x_4=-\infty}^{k_4} e^{\frac{2\mu_1}{\sigma_1^2}h_1} \frac{e^{-\frac{1}{2}\left(\frac{x_1 - 2h_1 - \mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)^2}}{\sigma_1\sqrt{2\pi t_1}}\varphi_7(x_1, x_2) \\ & e^{\frac{2\mu_2}{\sigma_2^2}(h_2-x_2)} \frac{e^{-\frac{1}{2}\left(\frac{x_3 + x_2 - 2h_2 - \mu_2(t_2 - t_1)}{\sigma_2\sqrt{t_2 - t_1}}\right)^2}}{\sigma_2\sqrt{2\pi(t_2 - t_1)}}\varphi_9(x_1, x_3, x_4)dx_4dx_3dx_2dx_1 \end{aligned} \tag{110}$$

and:

$$\begin{aligned} p_3^{(I)}(\mu_1, \mu_2) = & \int_{x_1=-\infty}^{\min(k_1,h_1)} \int_{x_2=\max(k_2,h_2)}^{\infty} \int_{x_3=\max(k_3,h_2)}^{\infty} \int_{x_4=-\infty}^{k_4} e^{\frac{2\mu_1}{\sigma_1^2}h_1} \frac{e^{-\frac{1}{2}\left(\frac{x_1 - 2h_1 - \mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)^2}}{\sigma_1\sqrt{2\pi t_1}}\varphi_7(x_1, x_2) \\ & e^{\frac{2\mu_2}{\sigma_2^2}(h_2-x_2)} \frac{e^{-\frac{1}{2}\left(\frac{-x_3 - x_2 + 2h_2 + \mu_2(t_2 - t_1)}{\sigma_2\sqrt{t_2 - t_1}}\right)^2}}{\sigma_2\sqrt{2\pi(t_2 - t_1)}}\varphi_9(x_1, x_3, x_4)dx_4dx_3dx_2dx_1 \end{aligned} \tag{111}$$

Corollary 2 comes from the fact that the correlation coefficient between the random variables $S_2(t_2)$ and $S_3(t_3)$ is equal to $\rho_{2.3}\sqrt{\frac{t_2}{t_3}}$. □

Proof of Proposition 5. Following steps similar to the beginnings of the previous proofs, one can express the problem at hand as the following integral:

$$\mathrm{P}_{[\mathrm{RDKO}]}(\mu_1, \mu_2) \tag{112}$$

$$= \int_{x_1=h_2}^{\min(k_1,h_1)} \int_{x_2=h_4}^{h_3} \int_{x_3=h_4}^{\min(k_2,h_3)} \varphi_1(x_1,x_2,x_3)\varphi_{11}(x_1,x_2)\varphi_{12}(x_2,x_3)dx_3dx_2dx_1$$

where the function φ_1 is given by Equation (86) and:

$$\varphi_{11}(x_1,x_2) = P\left(\sup_{0\le t\le t_1} X_1(t) \le h_1, \inf_{0\le t\le t_1} X_1(t) \ge h_2 | X_1(t_1) \in dx_1\right)dx_2dx_1 \tag{113}$$

$$\varphi_{12}(x_2,x_3) \tag{114}$$

$$= P\left(\sup_{t_1\le t\le t_2} X_2(t) \le h_3, \inf_{t_1\le t\le t_2} X_2(t) \ge h_4 | X_2(t_1) \in dx_2, X_2(t_2) \in dx_3\right)dx_3dx_2$$

From Pötzelberger and Wang (2001), one can plug:

$$\varphi_{11}(x_1,x_2) = \sum_{n=-\infty}^{\infty} e^{\frac{2na_1(x_1-na_1)}{\sigma_1^2 t_1}} - e^{\frac{2(h_1-na_1)(x_1-h_1+na_1)}{\sigma_1^2 t_1}} \tag{115}$$

$$\varphi_{12}(x_2,x_3) = \sum_{n=-\infty}^{\infty} e^{\frac{2na_2(x_3-x_2-na_2)}{\sigma_2^2 (t_2-t_1)}} - e^{\frac{2(h_3-x_2-na_2)(x_3-h_3+na_2)}{\sigma_2^2 (t_2-t_1)}} \tag{116}$$

The bulk of the proof, whose cumbersome details are omitted, then consists of solving the sixteen integrals implied by (112). The final result takes the form of the linear combinations of double sums of N_3 functions in Proposition 5.

An elementary adjustment identical to the one in Corollary 3 of Proposition 1 allows to value an early-ending variant of $\mathrm{P}_{[\mathrm{RDKO}]}(\mu_1,\mu_2)$.

Alternatively, one can also expand the problem as the following integral:

$$\mathrm{P}_{[\mathrm{RDKO}]}(\mu_1,\mu_2) = \int_{x_1=h_2}^{\min(k_1,h_1)} \int_{x_2=h_4}^{h_3} \int_{x_3=h_4}^{\min(k_2,h_3)} \varphi_{13}(x_1)\varphi_7(x_1,x_2)\varphi_{14}(x_2,x_3)dx_3dx_2dx_1 \tag{117}$$

where the function φ_7 is given by Equation (102) and:

$$\varphi_{13}(x_1) = P\left(\sup_{0\le t\le t_1} X_1(t) \le h_1, \inf_{0\le t\le t_1} X_1(t) \ge h_2, X_1(t_1) \in dx_1\right)dx_1 \tag{118}$$

$$\varphi_{14}(x_2,x_3) \tag{119}$$

$$= P\left(\sup_{t_1\le t\le t_2} X_2(t) \le h_3, \inf_{t_1\le t\le t_2} X_2(t) \ge h_4, X_2(t_2) \in dx_3 | X_2(t_1) \in dx_2\right)dx_3dx_2$$

According to the classical formula for the distribution of the maximum, the minimum and the endpoint of a Brownian motion with drift over a closed time interval, which can be traced back to Anderson (1960), we have:

$$\mathrm{P}\left(\sup_{0\le t\le t_1} S_1(t) < H_1, \inf_{0\le t\le t_1} S_1(t) > H_2, S_1(t_1) < S_1(0)e^{x_1} | S_1(0)\right)$$

$$= \sum_{n_1=-\infty}^{\infty} \exp\left(\frac{2\mu_1}{\sigma_1^2} n_1a_1\right)\left\{N\left[\frac{x_1-2n_1a_1-\mu_1t_1}{\sigma_1\sqrt{t_1}}\right] - N\left[\frac{h_2-2n_1a_1-\mu_1t_1}{\sigma_1\sqrt{t_1}}\right]\right\} \tag{120}$$

$$- \sum_{n_1=-\infty}^{\infty} \exp\left(\frac{2\mu_1}{\sigma_1^2}(h_2-n_1a_1)\right)\left\{N\left[\frac{x_1-2h_2+2n_1a_1-\mu_1t_1}{\sigma_1\sqrt{t_1}}\right] - N\left[\frac{-h_2+2n_1a_1-\mu_1t_1}{\sigma_1\sqrt{t_1}}\right]\right\}$$

for a given $x_1 \in \mathbb{R}$, and $\forall H_1 > S_1(0)e^{x_1}$.

Mere differentiation of (120) w.r.t. x_1 yields the function φ_{13}:

$$\varphi_{13}(x_1) = \sum_{n_1=-\infty}^{\infty} \frac{e^{\frac{2\mu_1}{\sigma_1^2}n_1a_1 - \frac{1}{2\sigma^2 t_1}(x_1-\mu_1t_1-2n_1a_1)^2}}{\sigma_1\sqrt{2\pi t_1}} - \sum_{n_1=-\infty}^{\infty} \frac{e^{\frac{2\mu}{\sigma_1^2}(h_2-n_1a_1) - \frac{1}{2\sigma_1^2 t_1}(x_1-2h_2-\mu_1t_1+2n_1a_1)^2}}{\sigma_1\sqrt{2\pi t_1}} \tag{121}$$

To handle the function φ_{14}, we notice that, by conditioning w.r.t. the filtration at time t_1 the same classical formula as the one used to derive φ_{13}, we can obtain:

$$\begin{aligned}
&\mathrm{P}\left(\sup_{t_1\le t\le t_2} S_2(t) < H_3, \inf_{t_1\le t\le t_2} S_2(t) > H_4, S_2(t_2) \le S_2(0)e^{x_3} | S_2(t_1) = S_2(0)e^{x_2}\right) \\
&= \sum_{n_2=-\infty}^{\infty} \exp\left(\frac{2\mu_2}{\sigma_2^2}n_2a_2\right)\left\{\begin{array}{l} N\left[\dfrac{\ln\left(\frac{S_2(0)e^{x_3}}{S_2(0)e^{x_2}}\right) - 2n_2a_2 - \mu_2(t_2-t_1)}{\sigma_2\sqrt{t_2-t_1}}\right] \\ -N\left[\dfrac{\ln\left(\frac{H_4}{S_2(0)e^{x_2}}\right) - 2n_2a_2 - \mu_2(t_2-t_1)}{\sigma_2\sqrt{t_2-t_1}}\right]\end{array}\right\}
\end{aligned} \tag{122}$$

$$\begin{aligned}
&- \sum_{n_2=-\infty}^{\infty} \exp\left(\frac{2\mu_2}{\sigma_2^2}\left(\ln\left(\frac{H_4}{S_2(0)e^{x_2}}\right) - n_2a_2\right)\right) \\
&\left\{\begin{array}{l} N\left[\dfrac{\ln\left(\frac{S_2(0)e^{x_3}}{S_2(0)e^{x_2}}\right) - 2\ln\left(\frac{H_4}{S_2(0)e^{x_2}}\right) + 2n_2a_2 - \mu_2(t_2-t_1)}{\sigma_2\sqrt{t_2-t_1}}\right] \\ -N\left[\dfrac{-\ln\left(\frac{H_4}{S_2(0)e^{x_2}}\right) + 2n_2a_2 - \mu_2(t_2-t_1)}{\sigma_2\sqrt{t_2-t_1}}\right]\end{array}\right\}
\end{aligned} \tag{123}$$

for any given $(x_2, x_3) \in \mathbb{R}^2$ and $H_3 > S_2(0)e^{x_3}$.

Equations (122) and (123) can be rewritten as follows:

$$\begin{aligned}
&\mathrm{P}\left(\sup_{t_1\le t\le t_2} X_2(t) < h_3, \inf_{t_1\le t\le t_2} X_2(t) > h_4, X_2(t_2) \le x_3 | X_2(t_1) \in dx_2\right) \\
&= \sum_{n_2=-\infty}^{\infty} \exp\left(\frac{2\mu_2}{\sigma_2^2}n_2a_2\right)\left\{\begin{array}{l} N\left[\frac{x_3 - x_2 - 2n_2a_2 - \mu_2(t_2-t_1)}{\sigma_2\sqrt{t_2-t_1}}\right] \\ -N\left[\frac{h_4 - x_2 - 2n_2a_2 - \mu_2(t_2-t_1)}{\sigma_2\sqrt{t_2-t_1}}\right]\end{array}\right\}
\end{aligned} \tag{124}$$

$$- \sum_{n_2=-\infty}^{\infty} \exp\left(\frac{2\mu_2}{\sigma_2^2}((h_4-x_2) - n_2a_2)\right)\left\{\begin{array}{l} N\left[\frac{x_3 - x_2 - 2(h_4-x_2) + 2n_2a_2 - \mu_2(t_2-t_1)}{\sigma_2\sqrt{t_2-t_1}}\right] \\ -N\left[\frac{-(h_4-x_2) + 2n_2a_2 - \mu_2(t_2-t_1)}{\sigma_2\sqrt{t_2-t_1}}\right]\end{array}\right\} \tag{125}$$

Therefore, by differentiating (124) and (125) w.r.t. x_3, we obtain:

$$\varphi_{14}(x_2, x_3) = \sum_{n_2=-\infty}^{\infty} e^{(\frac{2\mu_2}{\sigma_2^2}n_2a_2)} \frac{e^{-\frac{1}{2\sigma_2^2(t_2-t_1)}(x_3-x_2-2n_2a_2-\mu_2(t_2-t_1))^2}}{\sigma_2\sqrt{2\pi(t_2-t_1)}} \tag{126}$$

$$-\sum_{n_2=-\infty}^{\infty} e^{\frac{2\mu_2}{\sigma_2^2}(h_4-x_2-n_2a_2)} \frac{e^{-\frac{1}{2\sigma_2^2(t_2-t_1)}(x_3+x_2-2h_4+2n_2a_2-\mu_2(t_2-t_1))^2}}{\sigma_2\sqrt{2\pi(t_2-t_1)}} \tag{127}$$

This second formulation leads to a formula identical to Proposition 5 except for the fact that the N_3 functions are replaced by Φ_3 functions defined as follows:

$$\Phi_3[b_1, b_2, b_3; \mathrm{x}] = \int_{x=-\infty}^{b_2} \frac{\exp(-x^2/2)}{\sqrt{2\pi}} N\left[\frac{b_1 - c_1 x}{\sqrt{1-c_1^2}}\right] N\left[\frac{b_3 - c_2 x}{\sqrt{1-c_2^2}}\right] dx \tag{128}$$

where $(b_1, b_2, b_3) \in \mathbb{R}^3$ and x is a vector with two real coordinates $c_1, c_2 \in \,]-1,1[$.

The vectors of correlation coefficients x_1 and x_2 become:

$$\mathrm{x}_1 = \left\{\rho_{1.2}, \sqrt{\frac{t_1}{t_2}}\right\}, \mathrm{x}_2 = \left\{\rho_{1.2}, -\sqrt{\frac{t_1}{t_2}}\right\} \tag{129}$$

The origin of the function Φ_3, which is a special form of trivariate normal cumulative distribution, lies in the FDD (finite dimensional distribution) of the triple $[S_1(t_1), S_2(t_1), S_2(t_2)]$. Indeed, a little algebra shows that, $\forall D_1, D_2, D_3 \in \mathbb{R}^+$, we have:

$$P(S_1(t_1) \le D_1, S_2(t_1) \le D_2, S_2(t_2) \le D_3)$$

$$= \Phi_3\left[\frac{\ln\left(\frac{D_1}{S_1(0)}\right) - \mu_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{\ln\left(\frac{D_2}{S_2(0)}\right) - \mu_2 t_1}{\sigma_2\sqrt{t_1}}, \frac{\ln\left(\frac{D_3}{S_2(0)}\right) - \mu_2 t_2}{\sigma_2\sqrt{t_2}}; \rho_{1.2}, \sqrt{\frac{t_1}{t_2}}\right] \tag{130}$$

Notice that the two-colour probability distributions of Sections 2.1 and 2.2 can also be written as linear combinations of functions Φ_3. □

4. Conclusions

This article has shown how to value in closed form an important kind of multi-asset step barrier option known as a rainbow step barrier option, under the condition that the number of "colours" is restricted to two, along with widespread variants such as a two-colour outside step barrier and a two-colour step double barrier. It may be feasible, albeit tedious, to find an analytical solution to an extended valuation problem with three or four colours, but the expected benefits, compared with a conditional Monte Carlo approximation method, would greatly depend on the degree of the quadrature required to numerically evaluate the resulting multidimensional integrals. It should be emphasised that, even if more sophisticated models (allowing, e.g., for stochastic volatility) or a greater number of underlying assets are needed, closed form solutions obtained in a low-dimensional Black–Scholes framework remain useful as fast and accurate benchmarks that can: (i) serve as control variates in a simulation; (ii) speed up the calibration process; (iii) facilitate the analysis and the understanding of the interactions between the variables, as well as of the sensitivities of the option value w.r.t. its main parameters, which is instrumental in devising appropriate hedging techniques or trading strategies.

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