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Diagnostic for Volatility and Local Influence Analysis for the Vasicek Model

Manuel Galea [1,*], Alonso Molina [1] and Isabelle S. Beaudry [2]

[1] Departamento de Estadística, Pontificia Universidad Católica de Chile, Santiago 7820436, Chile; aomolina@uc.cl
[2] Department of Mathematics and Statistics, Mount Holyoke College, South Hadley, MA 01075, USA; ibeaudry@mtholyoke.edu
* Correspondence: manuel.galea@uc.cl; Tel.: +56-9-5504-5973

Abstract: The Ornstein–Uhlenbeck process is widely used in modeling biological systems and, in financial engineering, is commonly employed to describe the dynamics of interest rates, currency exchange rates, and asset price volatilities. As in any stochastic model, influential observations, such as outliers, can significantly influence the accuracy of statistical analysis and the conclusions we draw from it. Identifying atypical data is, therefore, an essential step in any statistical analysis. In this work, we explore a set of methods called local influence, which helps us understand how small changes in the data or model can affect an analysis. We focus on deriving local influence methods for models that predict interest or currency exchange rates, specifically the stochastic model called the Vasicek model. We develop and implement local influence diagnostic techniques based on likelihood displacement, assessing the impact of the perturbation of the variance and the response. We also introduce a novel and simple way to test whether the model's variability stays constant over time based on the Gradient test. The purpose of these methods is to identify potential risks of reaching incorrect conclusions from the model, such as the inaccurate prediction of future interest rates. Finally, we illustrate the methodology using the monthly exchange rate between the US dollar and the Swiss franc over a period exceeding 20 years and assess the performance through a simulation study.

Keywords: influence diagnostics; Ornstein–Uhlenbeck processes; likelihood inference; stochastic interest rate models; gradient test

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1. Introduction

Understanding and predicting interest rate movements plays a significant role in investment decisions and financial risk management. The main objective of this paper is to diagnose certain assumption violations in classical stochastic interest rate models. In particular, we derive closed-form expressions of the local influence methodology for the one-factor models, a popular class of interest rate models mainly used in pricing interest rate derivatives. These models are represented by the following stochastic differential equation (Black & Scholes, 1973; Hassler, 2016; Rémillard, 2013):

$$dr(t) = m(t, r(t))dt + v(t, r(t))dW(t), \tag{1}$$

where m is the drift function, v is the volatility function, $r(t)$ denotes the interest rate at time $t \geq 0$, and W denotes a Brownian motion or Wiener process. Various models for the interest rates $\{r(t); t \geq 0\}$ have been proposed in the literature. Overviews may be found in the books by Brigo and Mercurio (2006), Filipović (2009), McDonald (2013), and Rémillard (2013), and in the papers by Merton (1973), Vasicek (1977), and Cox et al. (1985), among others. Equation (2) presents a common class of drift and volatility functions (Chan et al., 1992; Nowman, 1997):

$$m(t, r(t)) = \alpha(\beta - r(t)) \ \ v(t, r(t)) = \sigma r^{\tau}(t), \tag{2}$$

where α is the mean reversion parameter and β is the long-term mean rate of the process, σ is the volatility of the short-term rate and τ is the proportional volatility exponent (the symbols appearing in the interest rate models are summarized in Appendix A). A class of well-known interest rate processes is obtained by imposing restrictions on the parameters of the stochastic model (2). For more details on this family of models, see, for example, Chan et al. (1992) and Nowman (1997, 1998).

Remark 1. *The drift function in Equation (2) may also be written as $m(t, r(t)) = a + br(t)$ (Chan et al., 1992). However, this paper follows the parametrization presented in Rémillard (2013) and Mazzoni (2018). The advantage of Equation (2) is that the parameters have straightforward economic interpretations.*

Estimating the parameters in the process (2) has been vastly discussed in the literature. Lo (1986), Duan (1994), Valdivieso et al. (2009), Rémillard (2013) and Fergusson and Platen (2015), among others, present maximum likelihood estimation in processes of the Ornstein–Uhlenbeck type. Chan et al. (1992) estimate the parameters of several continuous-time models for short-term rates using the generalized method of moments. In this work, we derive the maximum likelihood (ML) estimators for the Vasicek model, which corresponds to the special case when $\tau = 0$. Subsequently, we develop the local influence methods for stochastic models defined in (2). To our knowledge, the local influence methodology has not been applied to this type of stochastic models.

As mentioned in Galea and Giménez (2019), influence analysis is a group of techniques designed to evaluate the sensitivity of some statistics to perturbations in the data or model assumptions. Detecting atypical (outliers) and influential observations is an essential step in any financial model's econometric analysis. Also, this is important to evaluate the sensitivity (robustness) of the results obtained using the available data set, since atypical interest rates can distort predictions and statistics of interest, leading to, in some cases, wrong decisions. In this direction, van der Hart et al. (2003) showed that outliers are one of the important factors in the selection of stocks in emerging markets. In addition, decisions based on distorted inferences may be incorrect or suboptimal. Several approaches exist to assess the influence of data and model perturbations on parameter estimates. Overviews may be found in the books by Cook and Weisberg (1982) and Chatterjee and Hadi (1988) and the papers by Cook (1986), Escobar and Meeker (1992) and Zhu et al. (2007). Case deletion is a popular way to assess the impact of individual cases on the estimation process. This approach, referred to as global influence analysis, consists of quantifying the effect of a given observation by completely removing it. However, this approach is impractical for data collected at regular time intervals, as eliminating certain observations would disrupt the consistency of those intervals.

Local influence is an alternative approach based on differential geometry rather than complete deletion. This method employs a differential comparison of parameter estimates before and after a given perturbation to the data or model assumptions. Recently, Galea and Giménez (2019) applied the local influence approach to the capital asset pricing model under the multivariate normal distribution for modeling asset returns. However, the application of this methodology to the Vasicek model has not been considered in the literature. The main contribution of this paper is to apply the local influence methodology under various perturbation schemes to identify influential outliers among the observations or assumption violations of the Vasicek model. We consider a parametrization of the model that substantially simplifies the development of the influence measures.

We apply the methodology to the monthly exchange rates of the US dollar and the Swiss franc from January 2001 to November 2024. This application is particularly relevant, as both currencies are regarded as "safe-haven" assets. Understanding the relationship between these two currencies is of significant interest to financial market stakeholders, including investors seeking stability during periods of economic uncertainty. Moreover, the Swiss National Bank implemented a notable shift in its monetary policies during the analyzed period, which likely influenced the volatility of the Swiss franc. By utilizing the methods developed in this paper, we aim to uncover meaningful patterns in the volatility of this exchange rate over time. We also assess the performance of the proposed methodology through a simulation study.

The paper is structured as follows. In Section 2, we describe the Vasicek model for the modeling of interest rates. In Section 3, we present the ML estimators for the classical Ornstein–Uhlenbeck process used in the Vasicek model. In that section, we also propose a statistic to test the hypothesis of constant volatility based on the Gradient test. The local influence methodology and explicit calculations of the influence measures are then derived in Section 4. Section 5 presents a simulation study and an application based on the monthly exchange rate between the US dollar and the Swiss franc after 31 December 2000. Finally, we discuss some concluding remarks in Section 6.

2. The Vasicek Model

Interest rate modeling finds most of its applications in investment and financial decisions, portfolio management, and insurance. The Vasicek model (Vasicek, 1977) is a stochastic model describing the evolution of interest rates and is based on an arithmetic Brownian motion with mean reversion. The Vasicek model was developed assuming an efficient market economy, that is, based on the assumptions that (1) complete information is available to all the economic agents acting in the markets, and (2) privileged information providing any advantage does not exist. Furthermore, the Vasicek model assumes that interest rates have a regressive behavior towards a fixed value representing the long-term value of the interest rate. This model is known as the mean regression model and is based on the Wiener and Ornstein–Uhlenbeck processes, which we define below. The Wiener process, also referred to as a Brownian motion, is a continuous-time stochastic process with independent increments. In finance, the Wiener process is used to describe changes in stock prices or interest rates.

Definition 1. *A stochastic process* $W = \{W(t); t \geq 0\}$ *is a standard Wiener process or standard Brownian motion if* (i) $W(0) = 0$, (ii) *W has independent increments, that is,* $W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots, W(t_n) - W(t_{n-1})$ *are independent random variables for* $t_1 \leq t_2 \leq \ldots \leq t_n$, (iii) *for each time interval* (t_i, t_{i-1}), $W(t_i) - W(t_{i-1}) \sim N(0, t_i - t_{i-1})$, *and* (iv) *the realizations of W are continuous; that is,* $t \to W(t)$ *is a continuous function of* $t \geq 0$.

Also, we know that the Wiener process is a Gaussian process, and its covariance function is given by $\mathrm{Cov}\{W(s), W(t)\}$=$\min(s,t)$, for $s, t \geq 0$.

Definition 2. *Let $\{r(t); t \geq 0\}$ be a stochastic process. The process defined by $dr(t) = \alpha(\beta - r(t))dt + \sigma dW(t)$ is called an Ornstein–Uhlenbeck process, where W is Brownian motion and $\sigma > 0$. The solution of this equation is:*

$$r(t) = \beta + e^{-\alpha t}(r_0 - \beta) + \sigma \int_0^t e^{-\alpha(t-u)} dW(u), \quad t \geq 0, \tag{3}$$

with known $r(t_0) = r_0$, and where α, β, and σ are parameters of the process. The Ornstein–Uhlenbeck process is a Gaussian and Markovian process. For properties of stochastic integrals of the form $\int_0^t f(u)dW(u)$, see Appendix B.

Definition 3. *If the interest rate follows an Ornstein–Uhlenbeck mean-reverting process, defined by (3), where α is the mean speed of reversion, β is the mean interest rate level, σ is the instantaneous volatility, and W is a standard Brownian motion, then the resulting process is referred to as the Vasicek model.*

From the properties of the integral of a deterministic function relative to a Brownian motion, see Appendix B, it follows that the interest rate $r(t)$, conditional on $\mathcal{F}_s$, the information set at time s, for $s \leq t$, is normally distributed with mean and variance functions given by:

$$\begin{aligned} E\{r(t)|\mathcal{F}_s\} &= r(s)\mathrm{e}^{-\alpha(t-s)} + \beta\{1 - \mathrm{e}^{-\alpha(t-s)}\}, \\ \mathrm{Var}\{r(t)|\mathcal{F}_s\} &= \frac{\sigma^2}{2\alpha}\{1 - \mathrm{e}^{-2\alpha(t-s)}\}, \end{aligned} \tag{4}$$

respectively. Hence $P(r(t) < 0) > 0$, which is inappropriate when $r(t)$ represents the rate of interest at time t. However, this probability is typically negligible. On the other hand, the long-term distribution of the Ornstein–Uhlenbeck process is stationary and is Gaussian with mean β and variance $\sigma^2/2\alpha$, which we denote as $r_\infty \sim N(\beta, \sigma^2/2\alpha)$. So, in the long-term equilibrium, the probability of negative values of r_∞ is given by:

$$P(r_\infty < 0) = \Phi\Big(-\beta\sqrt{2\alpha/\sigma^2}\Big), \tag{5}$$

where $\Phi(x)$ denotes the cumulative distribution function of a standard normal distribution.

Note that if observations are collected at regular intervals of length $h > 0$, we can write:

$$r(t+h) = \beta + \mathrm{e}^{-\alpha h}(r(t) - \beta) + \sigma \int_t^{t+h} \mathrm{e}^{-\alpha(t+h-u)} dW(u). \tag{6}$$

Remark 2. *To simulate n observations from a Vasicek model, we can use the following equation:*

$$r(k+h) = \beta + e^{-\alpha h}(r(k) - \beta) + \sigma\sqrt{\frac{1 - exp\{-2\alpha h\}}{2\alpha}} Z_k, \tag{7}$$

where, $Z_k, k = 0, 1, ..., n$ are random variables independent and identically distributed $N(0, 1)$.

Remark 3. *Another important application of the Ornstein–Uhlenbeck process, and in particular of the Vasicek model, is the modeling of exchange rates. The exchange rate is a very important macroeconomic variable that plays a crucial role in international trade. See, for example, Mostafa and Mohammed (2016), Sikora et al. (2019) and Serafin et al. (2020).*

Let $\theta = (\alpha, \beta, \sigma^2)^T$. To simplify the statistical analysis of the Vasicek model given in Definition 3, we consider the following parametrization (Brigo & Mercurio, 2006; Rémillard, 2013). Let $\gamma = (\gamma_0, \gamma_1, \gamma_2)^T$, where $\gamma_0 = \beta(1-\gamma_1)$, $\gamma_1 = \exp\{-\alpha h\}$, and $\gamma_2 = \sigma^2(1-\gamma_1^2)/2\alpha$. We also assume n observations of $r(t)$ at times $\{t_1, t_2, ..., t_n\}$, where $0 < t_1 < t_2 < ... < t_n$. Let $r = (r_1, ..., r_n)^T$, where $r_i = r(t_i)$ for $i = 1, ..., n$ and $r(t_0) = r_0$ is known. Usually, interest rates are observed at equally spaced time points $\{h, 2h, ..., nh\}$ over a given time interval $[0, T]$, where $T = nh$. Therefore, we may rewrite Equation (7) with the above parametrization, such that:

$$r_i = \gamma_0 + \gamma_1 r_{i-1} + \epsilon_i, \tag{8}$$

where $\epsilon_i \sim N(0, \gamma_2)$, for $i = 1, ..., n$.

Let $p(r|\gamma, r_0)$ denote the density function of r given r_0. As described in Lo (1986), the density p may be written as the product of conditional densities:

$$p(r|\gamma, r_0) = p_1(r_1|\gamma, r_0)p_2(r_2|\gamma, r_1, r_0)....p_n(r_n|\gamma, r_{n-1}, ..., r_0). \tag{9}$$

However, because the interest rate $r(t)$ follows a Markovian process, Equation (9) reduces to:

$$p(r|\gamma, r_0) = p_1(r_1|\gamma, r_0)p_2(r_2|\gamma, r_1)....p_n(r_n|\gamma, r_{n-1}),$$

and since the increments of the Wiener process are normally distributed, we have that:

$$p(r_i|\gamma, r_{i-1}) = \frac{1}{\sqrt{2\pi\gamma_2}}\exp\Big\{-\frac{(r_i - \gamma_1 r_{i-1} - \gamma_0)^2}{2\gamma_2}\Big\},$$

which corresponds to the probability density function of a normal distribution with mean $\gamma_1 r_{i-1} + \gamma_0$ and variance γ_2, for $i = 1, ..., n$. Finally, the joint density function of the sample $r = (r_1, ..., r_n)^T$ takes the following form:

$$p(r|\gamma, r_0) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\gamma_2}}\exp\Big\{-\frac{(r_i - \gamma_1 r_{i-1} - \gamma_0)^2}{2\gamma_2}\Big\}, \tag{10}$$

that is, $r|r_0 \sim N_n(X\gamma^*, \gamma_2 I_n)$, with $X^T = \begin{pmatrix} 1 & . & . & . & 1 \\ r_0 & . & . & . & r_{n-1} \end{pmatrix}$ and $\gamma^* = (\gamma_0, \gamma_1)^T$, a simple linear regression model.

3. Likelihood Inference

In this section, we briefly discuss the likelihood inference for the model parameters and their associated standard errors, as well as hypothesis testing for constant volatility.

3.1. Maximum Likelihood Estimation

Maximum likelihood estimation in processes of theOrnstein–Uhlenbeck type has been discussed in the literature, for instance, see Lo (1986), Duan (1994), Brigo and Mercurio (2006), Valdivieso et al. (2009), Rémillard (2013) and Fergusson and Platen (2015). Chan et al. (1992) estimate the parameters of several continuous-time models for the short-term interest rates using the Generalized Method of Moments. In this section, we present some results for the ML estimation following Rémillard (2013) and Fergusson and Platen (2015), using the parametrization of the Vasicek model discussed in the previous section.

From Equation (10), it follows that the log-likelihood function for γ takes the following form:

$$\mathcal{L}(\gamma) = \sum_{i=1}^{n} \mathcal{L}_i(\gamma), \tag{11}$$

where $\mathcal{L}_i(\gamma) = -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\gamma_2 - \frac{1}{2\gamma_2}(r_i - \gamma_1 r_{i-1} - \gamma_0)^2$, for $i = 1, \ldots, n$. The score functions for γ then follow, which are given by:

$$\begin{aligned}
\frac{\partial\mathcal{L}(\gamma)}{\partial\gamma_0} &= \frac{1}{\gamma_2}\sum_{i=1}^{n}(r_i - \gamma_1 r_{i-1} - \gamma_0), \\
\frac{\partial\mathcal{L}(\gamma)}{\partial\gamma_1} &= \frac{1}{\gamma_2}\sum_{i=1}^{n}(r_i - \gamma_1 r_{i-1} - \gamma_0) r_{i-1}, \\
\frac{\partial\mathcal{L}(\gamma)}{\partial\gamma_2} &= \sum_{i=1}^{n}\Big\{ -\frac{1}{2\gamma_2} + \frac{1}{2\gamma_2^2}(r_i - \gamma_1 r_{i-1} - \gamma_0)^2 \Big\}.
\end{aligned} \tag{12}$$

From Equation (12), we obtain the following ML estimators for γ_0, γ_1 and γ_2:

$$\begin{aligned}
\hat{\gamma}_0 &= \frac{1}{n}\sum_{i=1}^{n}(r_i - \hat{\gamma}_1 r_{i-1}), \\
\hat{\gamma}_1 &= \frac{n\sum_{i=1}^{n} r_i r_{i-1} - \sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_{i-1}}{n\sum_{i=1}^{n} r_{i-1}^2 - \{\sum_{i=1}^{n} r_{i-1}\}^2}, \\
\hat{\gamma}_2 &= \frac{1}{n}\sum_{i=1}^{n}(r_i - \hat{\gamma}_1 r_{i-1} - \hat{\gamma}_0)^2.
\end{aligned} \tag{13}$$

As described in Appendix C, we may also derive an estimator of the covariance matrix for $\hat{\gamma}$, which leads to the following result:

$$D(\hat{\gamma}) = \begin{pmatrix} \hat{\gamma}_2(X^T X)^{-1} & 0 \\ 0 & 2\hat{\gamma}_2^2/n \end{pmatrix}.$$

Due to the invariance property of the ML estimators, we also obtain the ML estimators for α, β and σ^2:

$$\begin{aligned}
\hat{\alpha} &= -\frac{1}{h}\ln\hat{\gamma}_1, \\
\hat{\beta} &= \hat{\gamma}_0/(1 - \hat{\gamma}_1), \\
\hat{\sigma}^2 &= -2\hat{\gamma}_2\ln\hat{\gamma}_1/h(1 - \hat{\gamma}_1^2),
\end{aligned} \tag{14}$$

and an estimator of the covariance matrix of $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\sigma}^2)^T$ is given by $D(\hat{\theta}) = L_\theta^{-1}(\hat{\theta})$, as shown in Appendix C.

3.2. Diagnostic for Constant Volatility

An important feature of the class of stochastic models defined in Equation (2) is the evolution of the volatility, $v(t, r(t)) = \sigma r^\tau(t)$. The dynamics of volatility can be modeled using the parameter τ. For example, if $\tau = 0$, the volatility is constant and Equation (2) corresponds to the Vasicek model (Vasicek, 1977), while $\tau = 1/2$ corresponds to the CIR model (Cox et al., 1985). In the latter case, the volatility is not constant. In this section, we propose a statistic for testing the hypothesis of constant volatility. The hypothesis that $\tau = 1/2$ may be tested similarly.

To test the null hypothesis $H_0 : \tau = 0$, we use the Gradient test (Lemonte, 2016; Terrell, 2002), defined as:

$$Ga = \mathcal{U}^T(\tilde{\gamma})(\hat{\gamma} - \tilde{\gamma}), \tag{15}$$

where $\hat{\gamma}$ and $\tilde{\gamma}$ are the ML estimators of $\gamma = (\gamma_0, \gamma_1, \gamma_2, \tau)^T$ based on the full and reduced ($\tau = 0$) models, respectively, and $\mathcal{U}(\gamma)$ is the score function for a sample of the process shown in Equation (2). Under the null hypothesis, Ga asymptotically follows a χ^2-distribution with $p = 1$ degree of freedom.

Following Nowman (1997), the discrete model we use for estimation is given by:

$$r_i = \gamma_0 + \gamma_1 r_{i-1} + \epsilon_i, \tag{16}$$

where $\epsilon_i \sim N(0, \gamma_2 r_{i-1}^{2\tau})$, for $i = 1, \ldots, n$, $\gamma_0 = \beta(1 - \exp(-\alpha))$, $\gamma_1 = \exp(-\alpha)$ and $\gamma_2 = \dfrac{\sigma^2}{2\alpha}(1 - \exp(-2\alpha))$.

Remark 4. *Using the approximation that* $\exp(-2\alpha) \approx 1 - 2\alpha$*, we have that* $\gamma_2 \approx \sigma^2$*, and, in this case, the models proposed by Chan et al. (1992) and Nowman (1997, 1998) coincide.*

From Equation (16), it follows that the log-likelihood function for γ under the full model takes the form:

$$\mathcal{L}(\gamma) = \sum_{i=1}^{n} \mathcal{L}_i(\gamma), \tag{17}$$

where $\mathcal{L}_i(\gamma) = -\dfrac{1}{2}\ln 2\pi - \dfrac{1}{2}\ln \gamma_2 - \tau \ln r_{i-1} - \dfrac{1}{2\gamma_2 r_{i-1}^{2\tau}}(r_i - \gamma_1 r_{i-1} - \gamma_0)^2$, for $i = 1, \ldots, n$. The associated score functions for γ are given by:

$$\begin{aligned}
\frac{\partial \mathcal{L}(\gamma)}{\partial \gamma_0} &= \sum_{i=1}^{n} \left(\frac{1}{\gamma_2 r_{i-1}^{2\tau}}\right)(r_i - \gamma_1 r_{i-1} - \gamma_0), \\
\frac{\partial \mathcal{L}(\gamma)}{\partial \gamma_1} &= \sum_{i=1}^{n} \left(\frac{1}{\gamma_2 r_{i-1}^{2\tau}}\right)(r_i - \gamma_1 r_{i-1} - \gamma_0) r_{i-1}, \\
\frac{\partial \mathcal{L}(\gamma)}{\partial \gamma_2} &= \sum_{i=1}^{n} \left\{ -\frac{1}{2\gamma_2} + \frac{1}{2\gamma_2^2 r_{i-1}^{2\tau}}(r_i - \gamma_1 r_{i-1} - \gamma_0)^2 \right\}, \\
\frac{\partial \mathcal{L}(\gamma)}{\partial \tau} &= \sum_{i=1}^{n} \left\{ -\ln r_{i-1} + \frac{\ln r_{i-1}}{\gamma_2 r_{i-1}^{2\tau}}(r_i - \gamma_1 r_{i-1} - \gamma_0)^2 \right\}.
\end{aligned} \tag{18}$$

Let $c_i = c_i(\tau) = 1/r_{i-1}^{2\tau}$, for $i = 1, \ldots, n$. If we set $\mathcal{U}(\gamma) = 0_4$, then it follows that:

$$\begin{aligned}
\hat{\gamma}_0 &= \sum_{i=1}^{n} c_i(r_i - \hat{\gamma}_1 r_{i-1}) / \sum_{i=1}^{n} c_i, \\
\hat{\gamma}_1 &= \frac{\sum_{i=1}^{n} c_i r_i r_{i-1} - (\sum_{i=1}^{n} c_i r_i)(\sum_{i=1}^{n} c_i r_{i-1}) / \sum_{i=1}^{n} c_i}{\sum_{i=1}^{n} c_i r_{i-1}^2 - (\sum_{i=1}^{n} c_i r_{i-1})^2 / \sum_{i=1}^{n} c_i}, \\
\hat{\gamma}_2 &= \frac{1}{n} \sum_{i=1}^{n} c_i (r_i - \hat{\gamma}_1 r_{i-1} - \hat{\gamma}_0)^2,
\end{aligned} \tag{19}$$

$$g(\tau) = 0, \tag{20}$$

where $g(\tau) = \hat{\gamma}_2 \sum_{i=1}^{n} \ln r_{i-1} - \sum_{i=1}^{n} \ln r_{i-1} \left[c_i (r_i - \hat{\gamma}_1 r_{i-1} - \hat{\gamma}_0)^2\right]$. The parameter vector $\gamma = (\gamma_0, \gamma_1, \gamma_2, \tau)^T$ may be estimated through the following iterative process:

Step 0: Set an initial value for $\hat{\gamma}^{(0)}$ and $k = 0$.

Step 1: Calculate the weights $c_i(\hat{\tau}^{(k)})$, for $i = 1, \ldots, n$.

Step 2: Using Equation (19), calculate the estimates $\hat{\gamma}_0{}^{(k+1)}$, $\hat{\gamma}_1{}^{(k+1)}$, and $\hat{\gamma}_2{}^{(k+1)}$.

Step 3: Solve Equation (20) to obtain $\hat{\tau}^{(k+1)}$. The Newton-type algorithm provides a recursive expression for $\hat{\tau}^{(k+1)}$ given by:

$$\hat{\tau}^{(k+1)} = \hat{\tau}^{(k)} - g(\hat{\tau}^{(k)}) / g'(\hat{\tau}^{(k)}),$$

where $g'(\hat{\tau}^{(k)}) = 2\sum_{i=1}^{n} c_i(\hat{\tau}^{(k)})(\ln r_{i-1})^2 (r_i - \hat{\gamma}_1^{(k+1)} r_{i-1} - \hat{\gamma}_0^{(k+1)})^2$, for $k = 0, 1, \ldots$

Step 4: If $\|\hat{\gamma}^{(k+1)} - \hat{\gamma}^{(k)}\| < \varepsilon$, then stop and set $\hat{\gamma} = \hat{\gamma}^{(k+1)}$. Otherwise, set $k = k+1$ and return to Step 1. Typical values for the constant ε are 10^{-4} or 10^{-6}.

Cook and Weisberg (1983) also proposed an alternative diagnostic test for heteroskedasticity based on the score statistic as well as a graphical procedure to complement the score test. The authors suggest visually inspecting the plots of the squared studentized residuals $e_i^2/s^2(1-h_{ii})$ against $-2(1-h_{ii})\ln r_{i-1}$, where $e_i = r_i - \hat{\gamma}_0 - \hat{\gamma}_1 r_{i-1}$, $s^2 = \sum_{i=1}^n e_i^2/(n-p)$, and h_{ii} are the diagonal elements of the hat matrix $H = X(X^TX)^{-1}X^T$, for $i = 1, ..., n$.

Finally, to assess the dynamic specification of the model, we may also compute the Box Pierce portmanteau statistic (Nowman, 1997). Let $r_i^* = e_i/\sqrt{\gamma_2 r_{i-1}^{2\tau}}$, for $i = 1, ..., n$. If the model is well-specified, these residuals are independent, and their variance is equal to one. The statistic is given by:

$$S = \frac{1}{n-k}\sum_{i=1}^{k} S_i^2, \tag{21}$$

where $S_i = \sum_{t=k+1}^n r_t^* r_{t-i}^*$, $i = 1, ..., k$. Under the null hypothesis of white noise, S follows approximately a χ^2-distribution with k degrees of freedom. In practice, a $k = 12$ is used to calculate the $S-$statistic (Nowman, 1997).

4. Influence Diagnostics for Parameter Estimates

Detecting outliers and influential observations is an important step in the analysis of data sets. Several approaches exist to assess the influence of data and model perturbations on parameter estimates. Overviews may be found in the books by Cook and Weisberg (1982) and Chatterjee and Hadi (1988), and the papers by Cook (1986) and Zhu et al. (2007).

Case deletion is a popular way to assess the impact of individual cases on the estimation process. This approach is referred to as global influence analysis and consists of quantifying the effect of a given observation by completely removing it. Local influence is an alternative approach that is based on differential geometry rather than complete deletion. This method employs a differential comparison of parameter estimates before and after a given perturbation to the data or model assumptions. In this section, we derive closed-form expressions of the local influence methods for the Vasicek model.

4.1. Description of the Local Influence Method

The local influence method was proposed by Cook (1986) as a general tool for assessing the influence of local departures from the assumptions underlying the statistical models. A perturbation scheme is introduced into the postulated model $\{p(r|\gamma, r_0) : \gamma \in \Gamma\}$ through a perturbation vector $\omega = (\omega_1, \ldots, \omega_n)^T$, thus generating the perturbed model $\mathcal{M} = \{p(r|\gamma, \omega, r_0) : \omega \in \Omega\}$, where $p(r|\gamma, \omega, r_0)$ is the pdf of r given in Equation (10) when perturbed by ω, and $\mathcal{L}(\gamma|\omega) = \log p(r|\gamma, \omega, r_0)$ is the corresponding log-likelihood function. We assume that $p(r|\gamma, \omega_0, r_0) = p(r|\gamma, r_0)$ for all $\gamma \in \Gamma$, where ω_0 represents the vector without any perturbation. To assess the influence of the perturbations on the ML estimate of γ, we consider the likelihood displacement $LD(\omega) = 2\{\mathcal{L}(\hat{\gamma}) - \mathcal{L}(\hat{\gamma}_\omega)\}$, where $\hat{\gamma}$ is the ML estimator of γ in the postulated model, and $\hat{\gamma}_\omega$ is the ML estimator of γ in the perturbed model $\mathcal{M}$. Cook (1986) proposed to study the local behavior of $LD(\omega)$ around ω_0 and showed that the normal curvature C_d of $LD(\omega)$ at ω_0 in the direction of some unit vector d is given by $C_d = C_d(\gamma) = 2|d^T\Delta^T L_\gamma^{-1}(\gamma)\Delta d|$, with $||d|| = 1$, where $L_\gamma(\gamma)$ is the observed information matrix, given in Appendix C, evaluated at $\gamma = \hat{\gamma}$, and $\Delta = \partial^2\mathcal{L}(\gamma|\omega)/\partial\gamma\partial\omega^T$ is evaluated at $(\gamma, \omega) = (\hat{\gamma}, \omega_0)$.

Let d_{max} be the direction of maximum normal curvature, the perturbation that produces the greatest local change in $\hat{\gamma}$. The most influential elements of the data may be identified by their large component of the d_{max} vector. Furthermore, d_{max} corresponds to the eigenvector associated with the largest eigenvalue of $F = \Delta^T L_\gamma^{-1}(\gamma)\Delta$. The plot of the elements $|d_{max}|$ versus i (order of the data) can reveal the type of perturbations that has the most influence on $LD(\omega)$ in the neighborhood of ω_0 (Cook, 1986). The index plot of C_i may also be used to detect the presence of influential observations, where $C_i = 2|f_{ii}|$, and f_{ii} are the elements on the main diagonal of the matrix $F = \Delta^T L_{\gamma}^{-1}(\gamma)\Delta$.

Recently, Zhu et al. (2007) proposed a method for selecting appropriate perturbation schemes for a model $\mathcal{M}$. The method is based on the expected value of the Fisher information matrix for $\mathcal{M}$ with respect to the perturbation vector ω, assuming that the vector γ is fixed. The resulting matrix is $G(\omega) = (g_{ij}(\omega))$, such that:

$$g_{ij}(\omega) = E_{\omega}\left[\frac{\partial}{\partial \omega_i}\mathcal{L}(\gamma|\omega)\frac{\partial}{\partial \omega_j}\mathcal{L}(\gamma|\omega)\right], \quad i, j = 1, \ldots, n,$$

where E_ω denotes the expectation taken with respect to the density of the perturbed model $p(r|\gamma, \omega, r_0)$. The elements $g_{ii}(\omega)$ of $G(\omega)$ are the variances in scores with respect to the components of ω, and indicate the amount of perturbation introduced by the ith component of ω. The off-diagonal elements of $G(\omega)$ are the covariances of scores with respect to the components of ω and, hence, represent the linear association between the components of ω. Note that if $G(\omega)$ is a diagonal matrix, the components of ω are orthogonal in the sense of Cox and Reid (1987). Consequently, a perturbation scheme is deemed appropriate if it satisfies the condition $G(\omega_0) = cI_n$, where $c > 0$. This condition ensures that there are no redundancies in the components of ω and allows us to determine the orthogonality between the components of ω so we can identify the cause of a large effect. For more details, see Zhu et al. (2007). For an application of local influence diagnostics to test mean-variance efficiency and systematic risks in the capital asset pricing model, CAPM, see Galea and Giménez (2019).

Since C_d is not invariant under a uniform change of scale, Poon and Poon (1999) proposed the conformal normal curvature $B_d = C_d/\mathrm{tr}(2F)$ (Zhu & Lee, 2005). An interesting property of the conformal curvature is that for any direction unit d, $0 \leq B_d \leq 1$. We denote by $B_i = 2|f_{ii}|/\mathrm{tr}(2F)$ the conformal curvature in the unit direction with the ith entry equal to one and all other entries equal to zero.

According to Zhu and Lee (2005), the ith observation is potentially influential if $B_i > \bar{B} + 2\mathrm{sd}(B)$, where $\bar{B} = \sum_{i=1}^{n} B_i/n$ and $\mathrm{sd}(B)$ is the standard deviation of $B_1, \ldots, B_n$.

An important property of C_d and B_d is that they are invariant with respect to any parametrization of θ. Thus, the results obtained using C_d or B_d do not depends on the parametrization of the statistical model. Moreover, B_d, is invariant to conformal parametrizations of ω (Zhu & Lee, 2005). Consequently, we use the parametrization $\gamma = a(\theta)$, discussed in the previous sections, for the local influence analysis on the Vasicek model, see Appendix D.

4.2. Calculation of the Influence Measures

In this section, we discuss two perturbation schemes: the perturbation of the variance and the perturbation of the response. For both schemes, we present the matrices needed to implement the diagnostic measures.

4.2.1. Perturbation of the Variance

The postulated model in Equation (10) is assumed to be homoscedastic, that is, the variance is assumed to be constant and is equal to γ_2. If we consider the perturbation of the variance given by $\omega_i^{-1}\gamma_2$, for $i = 1, \ldots, n$, the density function of the perturbed model is given by:

$$p(r|\gamma, \omega, r_0) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\gamma_2\omega_i^{-1}}} \exp\left\{ -\frac{(r_i - \gamma_1 r_{i-1} - \gamma_0)^2}{2\gamma_2\omega_i^{-1}} \right\}, \tag{22}$$

and the corresponding perturbed log-likelihood function is given by:

$$\mathcal{L}(\gamma|\omega) = \sum_{i=1}^{n} \mathcal{L}_i(\gamma|\omega_i),$$

where $\mathcal{L}_i(\gamma|\omega_i) = -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\gamma_2 + \frac{1}{2}\ln\omega_i - \frac{\omega_i}{2\gamma_2}\{r_i - \gamma_1 r_{i-1} - \gamma_0\}^2$, for $i = 1, ..., n$.

In this case, the elements $g_{ij}(\omega) = (1/2\omega_i^2)\delta_{ij}$ for $i, j = 1, ..., n$, where δ_{ij} is the Kronecker delta, see Appendix E for more details. Therefore, the metric matrix $G(\omega_0) = (1/2)I_n$, with $\omega_0 = (1, ..., 1)^T$. Thus, the perturbation scheme of the variance is appropriate and allows us to assess the effect of the heteroskedasticity on the ML estimators. The Δ matrix for this perturbation scheme takes the form of $\Delta = (\Delta_1, ..., \Delta_n)$ where $\Delta_i = (1/\hat{\gamma}_2)(e_i, e_i r_{i-1}, e_i^2/2\hat{\gamma}_2)^T$, where $e_i = r_i - \hat{\gamma}_1 r_{i-1} - \hat{\gamma}_0$, for $i = 1, ..., n$.

4.2.2. Perturbation of the Interest Rate

In this section, we discuss the perturbation of the response. Specifically, we consider an additive perturbation scheme $r_{i\omega} = r_i + \omega_i$, $i = 1, ..., n$. In this case, $\omega = (\omega_1, ..., \omega_n)^T$ and $\omega_0 = 0_n$, where 0_n represents an $n \times 1$ vector of zeros. After some algebraic manipulations, we obtain that the metric matrix is $G(\omega) = (1/\gamma_2)I_n$. Thus, the additive perturbation of the response is appropriate too. With this perturbation scheme, we can assess the effect of outliers on the ML estimators. The Δ matrix is given by $\Delta^T = (1/\hat{\gamma}_2)(X, e/\hat{\gamma}_2)$, where $e = (e_1, ..., e_n)^T$ is the vector of residuals (Schwarzmann, 1991). In this case, the influence matrix $F = \Delta^T L_\gamma^{-1}(\gamma)\Delta = (1/\hat{\gamma}_2)\{H + 2hh^T\}$, where the hat matrix $H = X(X^TX)^{-1}X^T$ and the normalized residual vector $h = e/\|e\|$, with $Hh = 0_n$. The direction of the maximum curvature is proportional to the residual vector, $d_{max} = h$. For more details, see Schwarzmann (1991). Finally, since $\mathrm{tr}(F) = 4/\hat{\gamma}_2$ and $f_{ii} = (h_{ii} + 2h_i^2)/\hat{\gamma}_2$, we have $B_i = (h_{ii} + 2h_i^2)/4$, for $i = 1, ..., n$. Note that $f_{ii} = 4B_i/\hat{\gamma}_2$ and therefore, both diagnostic measures highlight the same potentially influential observations, if any.

5. Applications and Simulation Study

5.1. Historical USD to CHF Exchange Rate

This section describes the results we obtained when applying the methods discussed in this paper to the monthly exchange rate between the US dollar (USD) and the Swiss franc (CHF). The data cover the period from January 2001 to November 2024 and therefore include 287 monthly observations. Table 1 presents a descriptive summary of the series. We see that in the period considered, the exchange rate fluctuates between a minimum of 0.779 and a maximum of 1.784, with an average of 1.080. We also observe that the exchange rate is right-skewed (skewness = 1.448) and the observed kurtosis (4.479) is greater than what would be expected under the normal distribution.

Table 1. Summary statistics for monthly USD to CHF exchange rate over the period from January 2001 to November 2024.

n	Min	Median	Mean	Max	SD	Skewness	Kurtosis
287	0.779	0.989	1.080	1.784	0.214	1.448	4.479

Figure 1a shows the fluctuation in the exchange rates in the covered period and the fitted Vasicek model in red. In addition to the overall downward trend in the exchange rate over that period, we note the close fit of the model to the data. Figure 1b displays the exchange rates in the current period against the value in the previous one, (r_{i-1}, r_i), $i = 1, \ldots, n$. From this figure, we observe a linear relationship between the exchange rates of consecutive months. Consequently, both plots suggest that the Vasicek model appears to be reasonable for modeling these exchange rates.

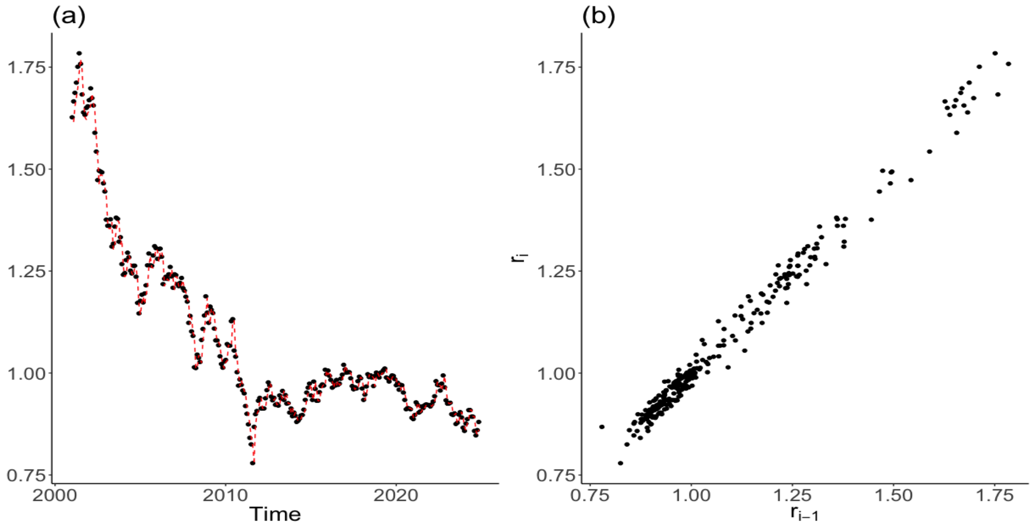

Figure 1. (**a**) Monthly exchange rate between the US dollar and the Swiss franc from January 2001 to November 2024 and the fitted Vasicek model in red. and (**b**) Dispersion plot of the exchange rates of the current period (r_i) against those of the previous one (r_{i-1}).

The maximum likelihood (ML) estimates and their standard errors are shown in Table 2, and the resulting estimated Vasicek model is as follows:

$$\hat{r}_i = 0.92120 + \mathrm{e}^{-0.1983}(r_{i-1} - 0.92120), \quad \text{for } i = 1, \ldots, n. \tag{23}$$

Using the Wald test, we determined that the coefficients α and β and the volatility σ^2 are all statistically significant at a significance level of 0.05. Furthermore, from Equation (5), the estimated probability (in the long-term equilibrium) of a negative exchange rate is $\hat{P}(r_\infty < 0) = 2.19 \times 10^{-12}$. These results do not provide us with evidence against the adequacy of the Vasicek model.

Table 2. ML estimates of the Vasicek model and their respective standard errors.

Parameter	ML Estimate	Standard Error
α	0.19833	(0.08095)
β	0.92120	(0.10793)
σ^2	0.00702	(0.00059)

Panel (a) of Figure 2 shows a Q−Q plot of the studentized residuals $r_1^*, \ldots, r_n^*$ and panel (b), a plot of the studentized residuals against the fitted values of the USD to CHF exchange rate. Figure 2a shows a moderate departure from the normality assumption. Figure 2b does not display any distinctive patterns indicating violations of the model assumptions. Again, both plots indicate that the model assumptions are reasonably met.

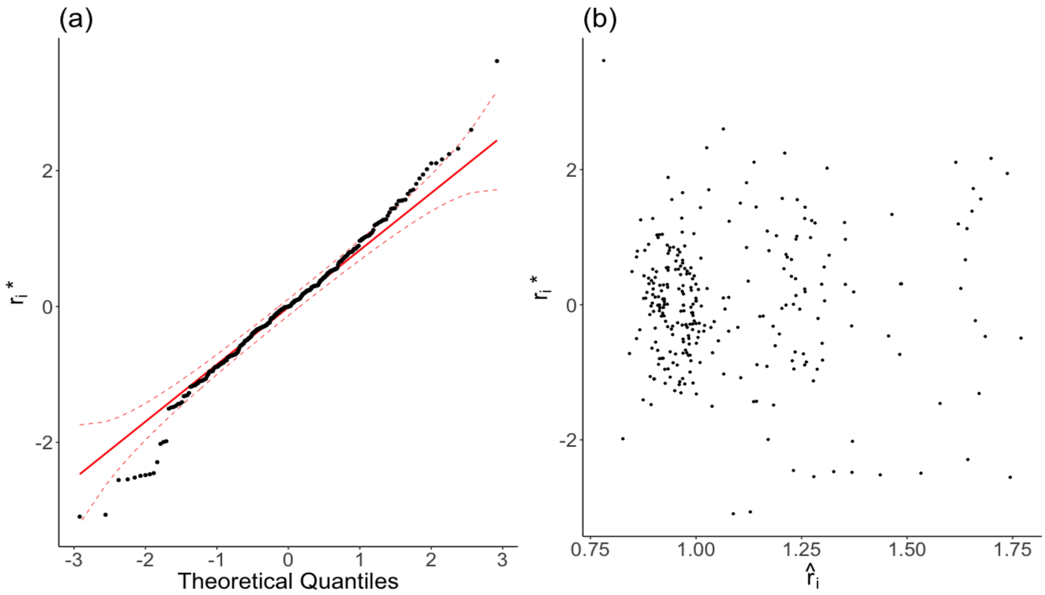

Figure 2. (**a**) Q−Q plots of the studentized residuals and (**b**) of the studentized residuals against the fitted values of the USD to CHF exchange rate.

Figure 3 displays diagnostic plots for the constant volatility assumption. Figure 3a presents the index plot of the B_i statistics of the local influence method for the perturbation of the variance scheme. The dashed line represents the threshold of $\bar{B} + 2\text{sd}(B)$, signaling that the observation is potentially influential. As we may observe, several observations fall above the threshold, but the observations 2011–09 and 2001–08 are the most significant influential observations for this perturbation scheme. Figure 3b shows the Cook–Weisberg plots (Cook & Weisberg, 1983) to assess the constant volatility assumption in regression models. Interestingly, this plot also identifies observation 2011–09 as the most potentially influential under the constant volatility assumption. It is interesting to note that this influential observation happened shortly after the National Swiss Bank adopted a minimum exchange rate at CHF 1.20 per euro in an attempt to limit the overvaluation of the Swiss franc (Swiss National Bank, 2011) and that both methods detect its influence. Also, none of the influential observations occurred after 2012. This may indicate that the monetary policy reduced the volatility between the two currencies and that using the Vasicek model for the entire period from 2001 to 2024 is inappropriate. Finally, both graphs suggest that $\tau \neq 0$.

The Gradient statistics test that $H_0 : \tau = 0$ is $Ga = 23.687$ with a $p-$value of 1.13×10^{-6} so we reject the hypothesis that $H_0 : \tau = 0$ at a 5% significance level. Finally, we also calculated the Box Pierce portmanteau statistic (Nowman, 1997) to assess the dynamic specification of the model. We obtained an S statistic equal to 18.944, corresponding to a p-value of 0.0899. Thus, we have moderate evidence against the null hypothesis of white noise. To some extent, both tests indicate that the variance is not constant across the full duration of the data. Since the perturbation of the variance also suggests that using the Vasicek model may not be appropriate over the entire period, we broke the data into two segments: before 2012 and after 2011, and repeated the Gradient and Box Pierce Portmanteau tests. After splitting the data, both tests failed to reject the null hypothesis of constant variance within each segment, contradicting the initial conclusion of the tests. This outcome could be attributed to the reduced statistical power from using fewer observations to fit the model, or it may indicate that separately modeling the two time periods is more appropriate.

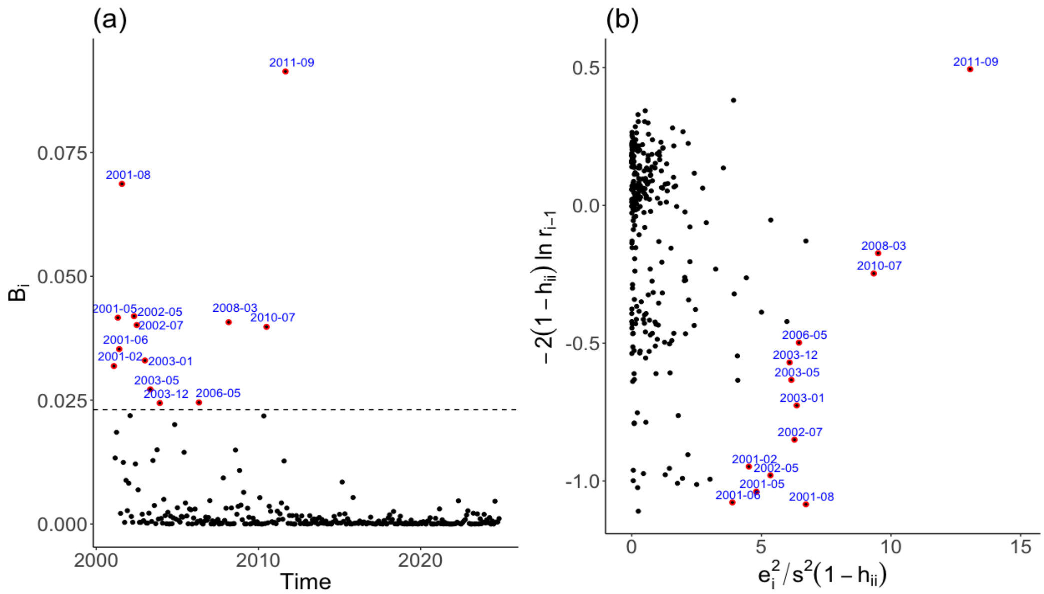

Figure 3. (**a**) Index plot of the B_i statistics of the local influence method for the perturbation of the variance scheme. (**b**) Cook−Weisberg plot. The detected influential observations are shown in red, and their dates of occurrence are indicated next to them in both plots.

Finally, Figure 4 displays the diagnostic plot for the additive perturbation of the exchange rates. Figure 4a presents the index plot of the B_i statistics of the local influence method for that perturbation scheme. The dashed line represents the threshold of $\bar{B} + 2\text{sd}(B)$, indicating that the observation is potentially influential. Again, the most influential observations correspond to 2011–09 and 2001–08. This result suggests that the Swiss National Bank monetary policy adopted in September 2011 might have significantly impacted that year's exchange rate. Figure 4b shows the monthly exchange rate between the US dollar and the Swiss franc from January 2001 to November 2024 and the fitted Vasicek model in red. The blue points represent the detected influential observations. We again note that none of the influential observations occur after 2011. The Vasicek model may be more appropriate for the USD-CHF exchange rates after the introduction of the monetary policy.

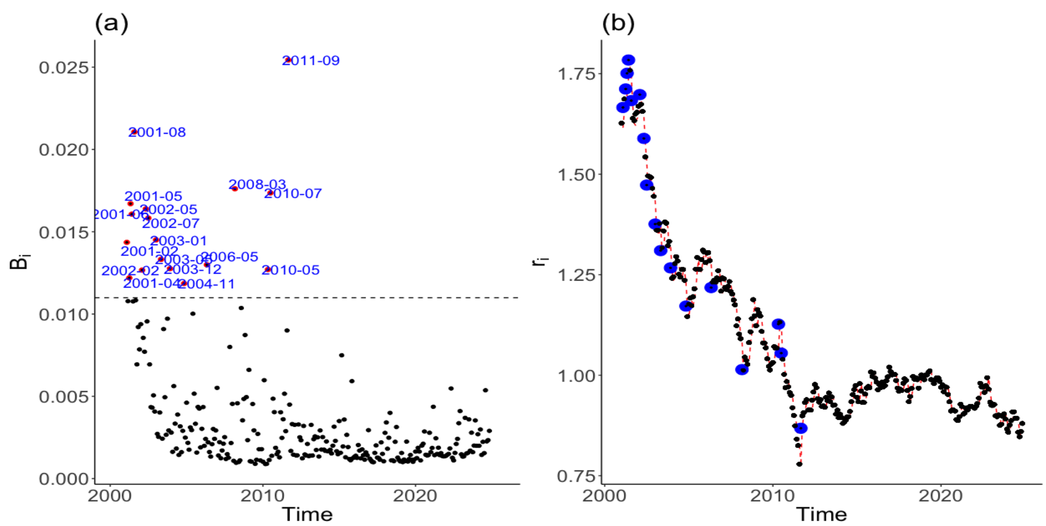

Figure 4. (**a**) Index plot of the B_i statistics for the perturbation of the exchange rate. The detected influential observations are shown in red, and their dates of occurrence are indicated next to them. (**b**) Monthly exchange rate between the US dollar and the Swiss franc from January 2001 to November 2024 and the fitted Vasicek model in red. The blue points represent the detected influential observations.

5.2. Simulation Study

To evaluate the Gradient test's performance in finite samples, we conducted a simulation study to assess the empirical level and the power of the test at a nominal significance level of 5% when testing the hypothesis H_0 that $\tau = 0$. We simulated 5000 series of returns from the stochastic model (2), with $\alpha = 0.10$, $\beta = 0.20$, $\sigma^2 = 0.001$, $\tau = 0$, $\tau = 1/2$, $h = 1/12$ and $r_0 = 0.50$.

Table 3 summarizes some of the results of the simulations. Rejection rates from the Gradient test are close to 5%, independently of the sample size. As expected, when $\tau \neq 0$, the rejection rates tend to 1.00 as the sample size increases. These results suggest that the Gradient test detects with a high probability non-constant interest rate when it is indeed non-constant, as long as the sample size is sufficiently large. Therefore, investors may reconsider their exposure to a financial product if the Gradient test identifies non-constant variability, particularly if maintaining stability in variability is among their investment objectives. However, while doing so, they must recall that, by design, the Gradient test has a probability, 5% in this case, of incorrectly detecting a non-constant variance when the variance is, in fact, constant, irrespective of the sample size.

Table 3. Rejection rates of the Gradient test for the hypothesis of constant volatility at a 5% significance level.

n	$\tau = 0$ (Vasicek Model)	$\tau = 0.5$ (CIR Model)
50	0.0554	0.0692
120	0.0488	0.2170
200	0.0480	0.5210
300	0.0514	0.8074
1000	0.0542	0.9988

Figure 5 shows the empirical histograms of the Gradient test values for the simulated data under different scenarios, where we see clear evidence of the agreement between the empirical and theoretical distributions under the null hypothesis.

In response to reviewers' feedback, we conducted additional simulations using two alternative models to assess their impact on the Gradient test's ability to detect heteroskedasticity. We chose the parameters of the alternative models to ensure that the long-term expected value of the interest rate and their volatility were similar to those of the Vasicek and CIR models. The results showed that the proposed Gradient test has higher power for the alternative models than the CIR model for the selected sample sizes. This can be attributed to the fact that, while the average volatility is comparable across models, the alternative models exhibit greater variability in volatility across samples. It thus suggests that the data are noisier, making it easier for the Gradient test to detect the non-constant variance. In addition, these simulations revealed that the Gradient test performs better for the four models we considered when the initial interest rate, r_0, is higher than the long-term equilibrium rate.

We further investigated the proposed influence measures' empirical performance, based on one simulated data set, with two added outliers; the observation 100, r_{100} was changed to $1.10r_{100}$ and r_{200} was changed to $1.15r_{200}$. Figure 6 presents the index plots of B_i and the Cook–Weisberg plot for this simulated data set. We see that both influence measures can effectively detect the two perturbed observations.

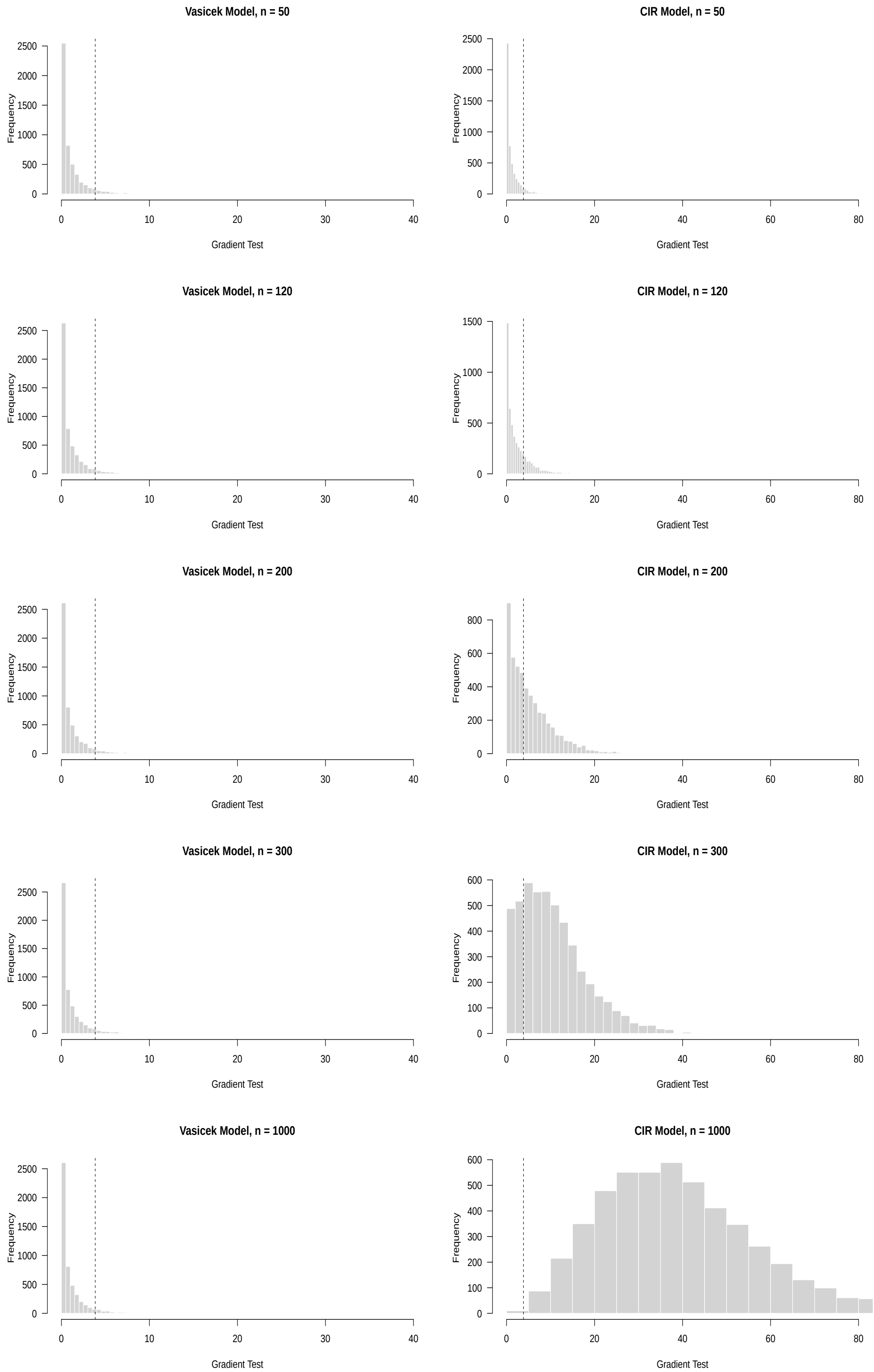

Figure 5. Empirical histograms of the Gradient test for the simulated data.

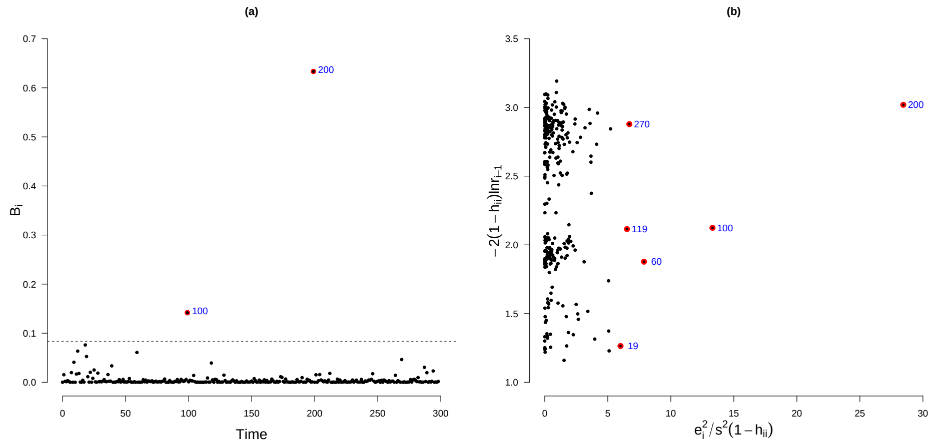

Figure 6. (**a**) Index plots of B_i. (**b**) Cook–Weisberg plot for a simulated data set.

6. Conclusions

Understanding interest rates' fluctuations play an essential role in investment decisions and financial markets' risk management. One approach to understanding the behavior of interest rates is through stochastic modelling. However, inferences for stochastic models may be sensitive to the presence of atypical rates. Since outliers may significantly distort estimators and statistical tests, potentially leading to incorrect or suboptimal decision making, assessing the results' sensitivity to such observations is an essential step to stochastic model inference. The local influence methodology is an approach to identify outliers that may significantly affect the value of statistics of interest in a given model. This work's main contribution is to apply the local influence methodology in stochastic interest rate models. In particular, the proposed local influence measures we present in this work aim at detecting atypical rates for the Vasicek model.

After describing the parametrization of the Vasicek model, we discussed the likelihood inference for the model parameters. We also presented methods for the diagnostic of constant volatility, which, in some instances, we adapted for the Vasicek model. Then, we explained how the local influence approach could also be applied to this stochastic model. To do so, we derived closed-form expressions for the delta matrices under the perturbation of the variance. Subsequently, we illustrated the methods in an application using the monthly exchange rate between the US dollar and the Swiss franc from January 2001 to November 2024. We found similarities in the detection of influential observations among the methodologies.

We also developed the local influence diagnostic for the perturbation in the interest rates and the associated delta matrices. We evaluated the performance of the method through a simulation study in which we artificially added outliers. Empirical results show that the proposed influence measures are able to detect outliers' rates.

This paper's proposed diagnostic measures may be extended to other stochastic models (Brigo & Mercurio, 2006; Chan et al., 1992; Nowman, 1997; Rémillard, 2013). For example, they may be applied to (i) more complex Ornstein–Uhlenbeck Process, (ii) the Black-Scholes Model, (iii) Stochastic Volatility Models, and (iv) stochastic models for selecting investment portfolios. However, the normality assumption is key for these types of stochastic models, and we know that this assumption is not reasonable in many cases. Therefore, another interesting area of research is the development of statistical methodology using alternative distributions, such as elliptical and/or elliptical skew distributions (Cambanis et al., 1981; Galea et al., 2008; Kelker, 1970) which offer a more flexible framework for modeling interest and exchange rates.

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Interest Rate Model Symbols and Their Interpretation

Table A1. Interpretation of the symbols appearing in the interest rate models.

Symbol	Interpretation
r_i	Response variable (e.g., interest) at time i
α	Mean speed of the interest rate reversion
β	Mean interest rate level
σ	Volatility of the short-term rate
τ	Proportional volatility exponent
γ_0	Intercept of the relation between r_i and r_{i-1}
γ_1	Slope of the relation between r_i and r_{i-1}
γ_2	Variance of the error term of the relation between r_i and r_{i-1}

Appendix B. The Stochastic Integral

If f is a deterministic square integrable function, and $W(t)$ is a Wiener process, then the stochastic integral

$$M(t) = \int_0^t f(u)dW(u),$$

has a normal distribution with mean 0 and variance $\int_0^t f^2(u)du$. For instance, if $f(u) = \sigma\exp\{-\alpha(t-u)\}$, then $\int_0^t f^2(u)du = \int_0^t \sigma^2\exp\{-2\alpha(t-u)\}du = \dfrac{\sigma^2}{2\alpha}\{1-\exp(-2\alpha t)\}$, for $t > 0$. Also, the covariance function of $\{M(t), t \geq 0\}$ is given by:

$$\mathrm{Cov}\{M(t), M(s)\} = \mathrm{Cov}\Big\{\int_0^t f_1(u)dW(u), \int_0^s f_2(u)dW(u)\Big\} = \int_0^{t\wedge s} f_1(u)f_2(u)du,$$

where $t \wedge s = \min\{t, s\}$.

Appendix C. The Observed Information Matrix

The second derivatives of the log-likelihood function are given by:

$$
\begin{aligned}
\frac{\partial^2 \mathcal{L}(\gamma)}{\partial \gamma_0^2} &= -\frac{n}{\gamma_2}, \\
\frac{\partial^2 \mathcal{L}(\gamma)}{\partial \gamma_0 \partial \gamma_1} &= -\frac{1}{\gamma_2} \sum_{i=1}^{n} r_{i-1}, \\
\frac{\partial^2 \mathcal{L}(\gamma)}{\partial \gamma_0 \partial \gamma_2} &= -\frac{1}{\gamma_2^2} \sum_{i=1}^{n} (r_i - \gamma_1 r_{i-1} - \gamma_0), \\
\frac{\partial^2 \mathcal{L}(\gamma)}{\partial \gamma_1^2} &= -\frac{1}{\gamma_2} \sum_{i=1}^{n} r_{i-1}^2, \\
\frac{\partial^2 \mathcal{L}(\gamma)}{\partial \gamma_1 \partial \gamma_2} &= -\frac{1}{\gamma_2^2} \sum_{i=1}^{n} (r_i - \gamma_1 r_{i-1} - \gamma_0) r_{i-1}, \\
\frac{\partial^2 \mathcal{L}(\gamma)}{\partial \gamma_2^2} &= \sum_{i=1}^{n} \left\{ \frac{1}{2\gamma_2^2} - \frac{1}{\gamma_2^3} (r_i - \gamma_1 r_{i-1} - \gamma_0)^2 \right\}.
\end{aligned}
$$

This leads to the following observed information matrix for γ:

$$
L_\gamma(\gamma) = \frac{1}{\gamma_2} \begin{pmatrix} X^T X & 0 \\ 0 & n/2\gamma_2 \end{pmatrix},
$$

and the following estimator of the covariance matrix of $\hat{\gamma}$:

$$
D(\hat{\gamma}) = \begin{pmatrix} \hat{\gamma}_2 (X^T X)^{-1} & 0 \\ 0 & 2\hat{\gamma}_2^2/n \end{pmatrix}.
$$

The Fisher information depends on the parametrization of the statistical model. If $\gamma = a(\theta)$ is a continuously differentiable function of θ, then $L_\theta(\theta) = J^T L_\gamma(\theta) J$, where the (i, j)th element of the 3×3 Jacobian matrix J is defined by $J_{ij} = \partial a_i(\theta)/\partial \theta_j$, for $i, j = 1, 2, 3$. In our case:

$$
J = \begin{pmatrix} h\beta\gamma_1 & 1 - \gamma_1 & 0 \\ -h\gamma_1 & 0 & 0 \\ \sigma^2(2\alpha h \gamma_1^2 + \gamma_1^2 - 1)/2\alpha^2 & 0 & (1 - \gamma_1^2)/2\alpha \end{pmatrix},
$$

with $\gamma_1 = \exp\{-\alpha h\}$.

Appendix D. Invariant of C_d and B_d

The conformal normal curvature has two important properties. For details and applications see Poon and Poon (1999) and Zhu and Lee (2005).

(i) The normal curvatures C_d and B_d do not depend on the parametrization. Then, C_d and B_d are invariant with respect to any parametrization $\gamma = a(\theta)$ where a is a differentiable function of θ. (ii) The conformal normal curvature B_d in any direction unit d at ω_0 is invariant with respect to the conformal parametrization $\omega^* = \omega^*(\omega)$.

Appendix E. Determination of $g_{ij}(\omega)$ for the Variance Perturbation Scheme

As described in Section 4.1, $g_{ij}(\omega)$ is defined as:

$$g_{ij}(\omega) = E_{\omega}\Big[\frac{\partial}{\partial\omega_i}\mathcal{L}(\gamma|\omega)\frac{\partial}{\partial\omega_j}\mathcal{L}(\gamma|\omega)\Big], \quad \text{for } i,j = 1,\ldots,n.$$

For the perturbation of the variance, we have that:

$$\begin{aligned}
g_{ii}(\omega) &= E_{\omega}\Big[\Big(\frac{\partial}{\partial\omega_i}\mathcal{L}(\gamma|\omega)\Big)^2\Big] \\
&= E_{\omega}\Big[\Big(\frac{1}{2\omega_i} - \frac{(r_i - \gamma_1 r_{i-1} - \gamma_0)^2}{2\gamma_2}\Big)^2\Big] \\
&= E_{\omega}\Big[\frac{1}{4\omega_i^2} - \frac{1}{2\omega_i^2}\Big(\frac{r_i - \gamma_1 r_{i-1} - \gamma_0}{(\gamma_2\omega_i^{-1})^{0.5}}\Big)^2 + \frac{1}{4\omega_i^2}\Big(\frac{r_i - \gamma_1 r_{i-1} - \gamma_0}{(\gamma_2\omega_i^{-1})^{0.5}}\Big)^4\Big] \\
&= \frac{1}{4\omega_i^2}E_{\omega}\Big[1 - 2Z_i^2 + Z_i^4\Big] \qquad \text{(where } Z_i \sim N(0,1)\text{)} \\
&= \frac{1}{2\omega_i^2},
\end{aligned}$$

for $i = 1, 2, ..., n$. Also, for $i \neq j \in \{1, 2, ..., n\}$, we have that:

$$\begin{aligned}
g_{ij}(\omega) &= E_{\omega}\Big[\frac{\partial}{\partial\omega_i}\mathcal{L}(\gamma|\omega)\frac{\partial}{\partial\omega_j}\mathcal{L}(\gamma|\omega)\Big] \\
&= E_{\omega}\Big[\Big(\frac{1}{2\omega_i} - \frac{(r_i - \gamma_1 r_{i-1} - \gamma_0)^2}{2\gamma_2}\Big)\Big(\frac{1}{2\omega_j} - \frac{(r_j - \gamma_1 r_{j-1} - \gamma_0)^2}{2\gamma_2}\Big)\Big] \\
&= \frac{1}{4\omega_i\omega_j}E_{\omega}\Big[1 - \Big(\frac{r_i - \gamma_1 r_{i-1} - \gamma_0}{(\gamma_2\omega_i^{-1})^{0.5}}\Big)^2 - \Big(\frac{r_j - \gamma_1 r_{j-1} - \gamma_0}{(\gamma_2\omega_j^{-1})^{0.5}}\Big)^2 \\
&\quad + \Big(\frac{(r_i - \gamma_1 r_{i-1} - \gamma_0)(r_j - \gamma_1 r_{j-1} - \gamma_0)}{(\gamma_2\omega_i^{-1}\gamma_2\omega_j^{-1})^{0.5}}\Big)^2\Big] \\
&= \frac{1}{4\omega_i\omega_j}E_{\omega}\Big[1 - Z_i^2 - Z_j^2 + Z_i^2 Z_j^2\Big] \qquad \text{(where } Z_i \text{ and } Z_j \text{ are i.i.d. } N(0,1)\text{)} \\
&= 0.
\end{aligned}$$

Therefore, we have that: $g_{ij}(\omega) = (1/2\omega_i^2)\delta_{ij}$ for $i, j = 1, ..., n$, where δ_{ij} is the Kronecker delta.

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