Zero-Width Quasi-Sliding Mode Band in the Presence of Non-Matched Uncertainties

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Abstract: Sliding mode control strategies are well known for ensuring robustness of the system with respect to disturbance and model uncertainties. For continuous-time plants, they achieve this property by confining the system state to a particular hyperplane in the state space. Contrary to this, discrete-time sliding mode control (DSMC) strategies only drive the system representative point to a certain vicinity of that hyperplane. In established literature on DSMC, the width of this vicinity has always been strictly greater than zero in the presence of uncertainties. Thus, ideal sliding motion was considered impossible for discrete-time systems. In this paper, a new approach to DSMC design is presented with the aim of driving the system representative point exactly onto the sliding hyperplane even in the presence of uncertainties. As a result, the quasi-sliding mode band width is effectively reduced to zero and ideal discrete-time sliding motion is ensured. This is achieved with the proper selection of the sliding hyperplane, using the unique properties of relative degree two sliding variables. It is further demonstrated that, even in cases where selection of a relative degree two sliding variable is impossible, one can use the proposed technique to significantly reduce the quasi-sliding mode band width.

Keywords: robust control; discrete-time systems; sliding mode control

1. Introduction

Effective control of electromechanical systems is crucial to minimize energy consumption and ensure their proper operation. This task can be successfully performed with application of sliding mode control techniques [1,2], which have recently become increasingly popular not only in power electronics [3] and electric drive control [4] but also in robotics and various mechatronic applications. The advantages of these strategies include high computational efficiency and the ability to reject the effect of some uncertainties on the motion of the system. In particular, the disturbance rejection property is achieved by confining the state of the plant to an $n-1$ dimensional hyperplane in finite time [5]. Since most modern control processes are applied digitally, discrete-time sliding mode control (DSMC) strategies are also considered in the literature [6–8]. Unlike continuous-time strategies of this type, DSMC aims to drive the system representative point to a certain vicinity of the sliding hyperplane, usually proportional to the magnitude of the disturbance. In all DSMC strategies in the literature, this width is non-zero in the presence of uncertainties [9].

A significant achievement in sliding mode control was the introduction of the so-called reaching law approach. This approach, proposed by Gao and Hung [10] and Gao et al. [11] for continuous-time and discrete-time systems, respectively, greatly simplifies the process of obtaining a stable sliding motion of the plant. In this method, the desired evolution of the sliding variable is first defined on a step-by-step basis using a recursive function. Then, this function is applied to design the control signal which ensures the desired evolution. Various authors in the control engineering community have proposed new reaching laws for discrete-time systems [12–15]. In particular, the notion of non-switching reaching laws

has been introduced with the aim of eliminating the harmful chattering during quasi-sliding motion of the system [16–19]. Another important development in sliding mode control was the use of sliding variables with arbitrary relative degree. In the continuous-time case, such variables have been used in so-called higher-order sliding modes [20–22]. Some recent works have also proposed the use of arbitrary relative degree sliding variables in DSMC with the aim of reducing the quasi-sliding mode band width [23,24]. Properties of these variables are discussed below.

Robustness property of sliding mode control strategies, in both continuous and discrete time, has motivated many researchers to apply such strategies in control of various electromechanical systems [25–27]. In particular, sliding mode control has served as a building block in control of robotic manipulators [28,29], unmanned aerial vehicles [30,31], bipedal robots [32,33], and many more applications. Although sliding mode control typically produces chattering which can be detrimental in electromechanical systems, one can mitigate this phenomenon by using the aforementioned non-switching type strategies or higher-order sliding modes.

In this paper, we propose a new approach to DSMC design which, for the first time, allows one to reduce the quasi-sliding mode band width to zero even in the presence of non-matched uncertainties. To achieve this property, we select a specific relative degree two sliding variable. Contrary to existing methods, this variable has a higher relative degree with respect to non-matched disturbance instead of the control input. The following novel results can be achieved with the use of the proposed approach:

- Ideal sliding motion of a discrete-time system is achieved in the presence of non-matched uncertainties, which is a result never before seen in literature on DSMC.
- It is demonstrated that, when the quasi-sliding mode band width is reduced to zero, the error of individual state variables is significantly reduced.
- In the case where the considered discrete-time system is subject to complex multidimensional disturbance, the proposed approach can still partially reject the effect of this disturbance and improve system dynamics as a result.

2. Discrete-Time Systems

In this section, we outline the basics of sliding mode control for discrete-time systems and present the new method of sliding hyperplane selection. It will be demonstrated that the proposed method allows one to reduce the width of the quasi-sliding mode band to zero even in the presence of uncertainties, which in turn decreases the absolute values of all state variables.

The dynamics of the considered discrete-time systems are expressed with the following equation:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}u(k) + \boldsymbol{g}f(k), \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $u, f \in \mathbb{R}$ represent the state vector, the control signal, and the disturbance, respectively, and $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{g}$ are of appropriate dimensions. In this paper, systems subject to non-matched uncertainties are considered, which implies $\boldsymbol{b} \neq \alpha \boldsymbol{g}$ for any constant $\alpha \in \mathbb{R}$. In most practical applications, one can feasibly expect the disturbance affecting the plant to be bounded. We express these bounds as

$$|f(k)| \leq f^{\max} = \text{const.} \tag{2}$$

for any time k. In this section, a sliding mode control strategy is designed for such systems.

Remark 1. *In this section, it is assumed that the disturbance affecting the plant is defined by a scalar nonlinear function $f(k)$. Although such an assumption is not particularly restrictive, it may not be satisfied in certain practical applications of sliding mode control. Systems subject to more complex, multidimensional uncertainties are considered below.*

2.1. Discrete-Time Sliding Mode Control

To design a sliding mode control algorithm for the discrete-time plant (1), one must first specify the switching hyperplane onto which the system representative point will be driven. Such a hyperplane is typically chosen as

$$\sigma(k) = c^{\mathrm{T}}x(k) = 0, \tag{3}$$

where the constructed system output $\sigma \in \mathbb{R}$ is referred to as the sliding variable and the vector of constants $c \in \mathbb{R}^n$ is selected to ensure stable closed-loop performance of the plant. The choice of vector c is discussed in detail further in this section.

The objective of discrete-time sliding mode control is to drive the state of the plant to a specified vicinity of the switching hyperplane. The width of that vicinity reflects robustness of the control system with respect to disturbance. In particular, the strategy used in this paper is obtained using the reaching law approach, in which the desired evolution of the sliding variable is first stated and then applied to design the control signal which ensures this exact evolution.

The reaching law considered in this paper is expressed with the following recursive formula:

$$\sigma(k+1) = (1 - k/k^*)_+\sigma(0), \tag{4}$$

where $(\star)_+ = \max\{\star, 0\}$, constant $k^* \in \mathbb{N}$ and $\sigma(0) = c^{\mathrm{T}}x(0)$. The objective of this control scheme is to drive the system representative point onto the switching hyperplane in k^* steps and to confine it to this hyperplane in every subsequent step. In other words, the system representative point is always driven onto the sliding hyperplane in finite time. Furthermore, its convergence rate to this hyperplane can be modified with the choice of parameter k^* in order to adjust the magnitude of the control signal in the initial stages of the control process.

One can now apply the reaching law approach to obtain the control signal. This is done by first substituting the state dynamics into the left-hand side of (4), which gives

$$c^{\mathrm{T}}Ax(k) + c^{\mathrm{T}}bu(k) + c^{\mathrm{T}}gf(k) = (1 - k/k^*)_+\sigma(0). \tag{5}$$

Then, solving this relation for $u(k)$, one gets

$$u(k) = (c^{\mathrm{T}}b)^{-1}\left[(1 - k/k^*)_+\sigma(0) - c^{\mathrm{T}}Ax(k) - c^{\mathrm{T}}gf(k)\right]. \tag{6}$$

When the control signal is designed in such a way, the a priori specified evolution of the sliding variable (4) is ensured. However, such an ideal control signal can hardly be obtained in practice since the formula describing it contains the unknown at time k disturbance term $c^{\mathrm{T}}gf(k)$. To make the reaching law-based strategy applicable to the considered systems, this term needs to be estimated. Most commonly, it is either assumed that the estimate $\hat{f}(k) = f(k-1)$ or that it is a constant equal to the mean value of the disturbance. Consequently, in the presence of uncertainties, reaching law (4) becomes

$$\sigma(k+1) = (1 - k/k^*)_+\sigma(0) + c^{\mathrm{T}}gf(k) - c^{\mathrm{T}}g\hat{f}(k). \tag{7}$$

Thus, instead of being driven directly onto the sliding hyperplane, the system representative point is confined to the following vicinity of the plane:

$$\left\{x : |c^{\mathrm{T}}x| \leq F = \max_{k \geq 0}\left|c^{\mathrm{T}}gf(k) - c^{\mathrm{T}}g\hat{f}(k)\right|\right\}. \tag{8}$$

In discrete-time sliding mode control, this vicinity is referred to as the quasi-sliding mode band. Its width F represents the degree of system robustness with respect to uncertainties.

In this paper, we propose a novel method of sliding hyperplane selection which reduces the width of band (8) to zero. This is a novel achievement, since, in all known literature on discrete-time sliding mode control, this width is always strictly positive in the presence of uncertainties. It is further shown that, by reducing the quasi-sliding mode band width to zero, absolute error of all state variables is also lowered.

2.2. Relative Degree Two Sliding Variable

The objective of this paper is to design a discrete-time sliding mode controller, which ensures that the quasi-sliding mode band width is reduced to zero. This property is obtained with a specific design of the sliding hyperplane (3). In particular, vector c in this hyperplane is selected so that sliding variable σ has a relative degree higher than one with respect to matched uncertainties. Variables with arbitrary relative degrees can be defined in the following way.

Definition 1. *Constructed output σ of a discrete-time plant is said to have relative degree $r \in \mathbb{N}$ with respect to input u if and only if for any $k \geq 0$*
* *$\sigma(k + r) = \varphi_r[x(k), u(k)]$*
* *$\sigma(k + i) = \varphi_i[x(k)]$ for $i = 0, 1, \ldots, r - 1$,*
where φ_i, φ_r are certain functions.

In other words, σ has relative degree r with respect to a certain input if this input only affects the variable after r time instants. Typically, sliding variables are selected so that they have a particular relative degree with respect to control signal $u(k)$. However, in this paper, we propose a novel solution in which the variable has relative degree one with respect to control signal $u(k)$, but relative degree two with respect to non-matched disturbance $f(k)$, which can also be interpreted as an input signal. In other words, vector c in relation (3) is selected so that

$$c^{\mathrm{T}}g = 0, \quad c^{\mathrm{T}}Ag \neq 0 \quad \text{and} \quad c^{\mathrm{T}}b \neq 0. \tag{9}$$

Then, substitution of system dynamics into relation (3) yields

$$\begin{aligned}\sigma(k+1) = &\ c^{\mathrm{T}}Ax(k) + c^{\mathrm{T}}bu(k) + 0 \cdot f(k)\\ = &\ c^{\mathrm{T}}A^2x(k-1) + c^{\mathrm{T}}Abu(k-1) + c^{\mathrm{T}}Agf(k-1) + c^{\mathrm{T}}bu(k).\end{aligned}$$

It can be seen that, with the proposed selection of vector c, the sliding variable is completely unaffected by the disturbance from the previous time instant. According to Definition 1, the selected variable has relative degree two with respect to non-matched disturbance $f(k)$ and relative degree one with respect to control signal $u(k)$.

Although conditions (9) ensure that the sliding variable has the desired relative degree, they are not sufficient to guarantee closed-loop stability of the system. To obtain a stable response of a discrete-time plant, all poles of its closed-loop state matrix

$$A_{\mathrm{CL}} = A - b(c^{\mathrm{T}}b)^{-1}c^{\mathrm{T}}A \tag{10}$$

must be placed inside the unit circle. This condition is not mutually exclusive with (9) as long as the considered plant is of order $n \geq 2$. Proper pole placement ensures that

$$\lim_{k \to \infty} A_{\mathrm{CL}}^k = 0_{n \times n}, \tag{11}$$

which implies that the state of the closed-loop system will always converge to a stable point.

In the next section, it is demonstrated that choice of the switching hyperplane proposed in this paper significantly improves sliding mode performance of the system and, by extension, reduces the absolute error of all state variables.

2.3. Properties of the Proposed Strategy

For any vector c selected according to (9), the control signal obtained from reaching law (4) is simplified to

$$u(k) = (c^{\mathrm{T}}b)^{-1}\left[(1 - k/k^*)_+\sigma(0) - c^{\mathrm{T}}Ax(k)\right]. \tag{12}$$

Since the condition $c^{\mathrm{T}}g = 0$ has eliminated the unknown disturbance term, this control signal can always be calculated exactly without the need to estimate $f(k)$. As a result, it is guaranteed that the sliding variable evolves precisely according to reaching law (4) despite the ultimately unknown non-linear dynamics of the system. This provides very advantageous properties of system sliding motion, which are summarized in the following theorem.

Theorem 1. *If the control strategy for system* (1) *is defined by* (12) *with vector c selected according to* (9), *then for any initial conditions of the system its representative point will arrive at the sliding hyperplane in exactly k^* steps and remain on this hyperplane for any $k \geq k^*$.*

Proof. Let $x(0) \in \mathbb{R}^n$ be any initial state of the plant. We first obtain $\sigma(0) = c^{\mathrm{T}}x(0)$, where c satisfies conditions (9). Since these conditions ensure that the selected sliding variable has relative degree two with respect to disturbance, the single most recent uncertain term $f(k)$ does not affect its evolution. Indeed, taking into account (1), substitution of control signal (12) into (3) yields

$$\sigma(k + 1) = c^{\mathrm{T}}Ax(k) + c^{\mathrm{T}}bu(k) + 0 \cdot f(k) = c^{\mathrm{T}}b(c^{\mathrm{T}}b)^{-1}(1 - k/k^*)_+\sigma(0). \tag{13}$$

One concludes that, when the sliding variable has relative degree two with respect to disturbance, its evolution will directly follow the one specified by reaching law (4).

Let us now select an arbitrary $k^* \in \mathbb{N}$. For any $k = 0, 1, ..., k^* - 1$, reaching law (4) implies

$$\sigma(k) = \frac{k^* - k}{k^*}\sigma(0) = \sigma(k - 1) - \frac{1}{k^*}\sigma(0), \tag{14}$$

which means that the sliding variable monotonically approaches 0 by $\sigma(0)/k^*$ in each step. On the other hand, for $k \geq k^*$, reaching law (4) gives

$$\sigma(k) = \frac{(k^* - k)_+}{k^*}\sigma(0) = \frac{0}{k^*}\sigma(0) = 0. \tag{15}$$

Thus, the system representative point will reach the sliding hyperplane for the first time at time k^* and it will remain on this hyperplane in every subsequent step. □

It is demonstrated above that, with the proposed choice of the sliding hyperplane, one can reduce the width of quasi-sliding mode band (8) to zero. This is possible because the sliding variable is designed so that it is not affected by the most recent disturbance. Although this disturbance naturally still has an effect on the state of the plant (1) itself, reducing the QSMB width to zero significantly decreases the absolute error of each state variable. This valuable property is demonstrated in the next section.

2.4. State Error Estimation

In this section, it is demonstrated that the proposed approach ensures a bounded error of all state variables in the sliding phase. Bartoszewicz and Latosiński [34] showed that this error heavily depends on the quasi-sliding mode band width. Consequently, one can expect that the new approach proposed in this paper will further reduce state error compared to other DSMC strategies.

To develop a formula describing the absolute bound of each state variable, one can first apply relations (1) and (10) to express the state vector as

$$
\begin{aligned}
x(k+1) &= \left[A - b(c^\mathsf{T}b)^{-1}c^\mathsf{T}A + b(c^\mathsf{T}b)^{-1}c^\mathsf{T}A \right] x(k) + bu(k) + gf(k) \\
&= A_{\mathrm{CL}}x(k) + b(c^\mathsf{T}b)^{-1}c^\mathsf{T}Ax(k) + bu(k) + gf(k).
\end{aligned}
\tag{16}
$$

Then, substituting control signal (12) into this relation, one obtains

$$
x(k+1) = A_{\mathrm{CL}}x(k) + b(c^\mathsf{T}b)^{-1}(1 - k/k^*)_+ \sigma(0) + gf(k).
\tag{17}
$$

Reaching law (4) further implies

$$
x(k+1) = A_{\mathrm{CL}}x(k) + b(c^\mathsf{T}b)^{-1}\sigma(k+1) + gf(k).
\tag{18}
$$

Thus, the state of the plant can be expressed as a recursive function of the sliding variable. This is a helpful development, since, in the considered case, values of $\sigma(k)$ are exactly known in advance from reaching law (4), as stated in Theorem 1. With this in mind, one can now form the theorem which explicitly states the absolute bounds of all state variables.

Theorem 2. *If the control strategy for system* (1) *is defined by* (12) *with vector c selected according to* (9), *then, for each $j = 1, ..., n$, the jth state variable will at least asymptotically converge to the vicinity of zero described as*

$$
|x_j| \leq f^{\max} \sum_{i=1}^{\infty} \left| h_j A_{\mathrm{CL}}^{i-1} g \right|,
\tag{19}
$$

where $f^{\max}$ is the upper bound of disturbance (2) *and vector*

$$
h_j = [\underbrace{0 \ \dots \ 0}_{j-1} \ 1 \ \underbrace{0 \ \dots \ 0}_{n-j}]^T.
\tag{20}
$$

Proof. Let k^* in reaching law (4) be arbitrarily selected. Then, since reaching law (4) implies $\sigma(k) = 0$ for all $k > k^*$, for the same k relation, (18) gives

$$
\begin{aligned}
x(k) &= A_{\mathrm{CL}}x(k-1) + b(c^\mathsf{T}b)^{-1} \cdot 0 + gf(k-1) \\
&= A_{\mathrm{CL}}^{k-k^*}x(k^*) + \sum_{i=1}^{k-k^*} A_{\mathrm{CL}}^{i-1}gf(k-i).
\end{aligned}
\tag{21}
$$

Now, let $j = 1, ..., n$. Considering vector h_j defined in this theorem, one easily notices that $x_j(k) = h_j x(k)$. Thus, the jth state variable can be expressed as

$$
x_j(k) = h_j A_{\mathrm{CL}}^{k-k^*}x(k^*) + \sum_{i=1}^{k-k^*} h_j A_{\mathrm{CL}}^{i-1}gf(k-i).
\tag{22}
$$

It is now shown that the absolute value of this variable converges to a specific value as k tends to infinity. One obtains

$$
\limsup_{k \to \infty} |x_j(k)| \leq \limsup_{k \to \infty} \left| h_j A_{\mathrm{CL}}^{k-k^*}x(k^*) \right| + \limsup_{k \to \infty} \left| \sum_{i=1}^{k-k^*} h_j A_{\mathrm{CL}}^{i-1}gf(k-i) \right|.
\tag{23}
$$

Relation (11) implies that $A_{\mathrm{CL}}^{k-k^*}$ converges to $\mathbf{0}$ as k tends to infinity. Furthermore, it is known that, for any k, the disturbance is bounded by $f^{\max}$. With these facts in mind, the jth state variable is bounded in the following way:

$$
\limsup_{k \to \infty} |x_j(k)| \leq f^{\max} \lim_{k \to \infty} \sum_{i=1}^{k-k^*} \left| h_j A_{\mathrm{CL}}^{i-1}g \right| = f^{\max} \sum_{i=1}^{\infty} \left| h_j A_{\mathrm{CL}}^{i-1}g \right|.
\tag{24}
$$

Matrix A_{CL}^k converges to **0** for $k \to \infty$, which implies that the sum on the right-hand side of this inequality is finite. Furthermore, since j is selected arbitrarily, the obtained inequality applies to all state variables. Thus, vector of variables x converges to the band (19) at least asymptotically. $\square$

It is demonstrated that the proposed control strategy confines the system state to a specific area in the state space. Since the sliding variable is effectively reduced to zero, the size of this area depends only on the magnitude of non-matched uncertainties. As a result, the proposed approach ensures smaller state error than other DSMC strategies (e.g., [16–18]) in which the quasi-sliding mode band width is strictly greater than zero.

Remark 2. *Theorem 2 is true for any vector c that ensures stability of the closed-loop system. However, the width of band (19) to which the system state is driven can be further reduced in a relatively common special case. Suppose that c is selected so that all poles of matrix A_{CL} are placed in zero. Then, the so-called dead-beat performance of the closed-loop system is ensured and*

$$A_{CL}^n = \mathbf{0}, \quad where \quad n = dim(A_{CL}). \tag{25}$$

With this in mind, absolute bounds of each state variable can be further simplified. Indeed, when dead-beat performance of the system is ensured, band (19) is reduced to

$$\left\{ x : \forall_{j=1,\ldots,n} |x_j| \leq f^{\max} \sum_{i=1}^{n} \left| h_j A_{CL}^{i-1} g \right| \right\} \tag{26}$$

since $A_{CL}^{i-1} = \mathbf{0}$ for all $i > n$. Absolute bounds of all state variables have therefore been reduced to a finite sum. Furthermore, the state will always converge to this band in finite time $k^* + n$, rather than asymptotically.

In this section, a new approach to discrete-time sliding mode control is introduced. Contrary to existing methods, this approach can drive the system representative point directly onto the sliding hyperplane even in the presence of non-matched uncertainties. This valuable property is achieved by selecting a sliding hyperplane so that $c^T g = 0$ and $c^T b \neq 0$, which in turn eliminates the effect of uncertainties on the sliding variable. However, as stated in Remark 1, some systems may be subject to multidimensional uncertainties that cannot be expressed as $g f(k)$ with a single function $f : \mathbb{R} \to \mathbb{R}$. The performance of such systems is analyzed in the next section.

3. Systems Subject to Multidimensional Disturbance

In the previous section, it is demonstrated that, when system dynamics are expressed as (1), it is possible to reduce the quasi-sliding mode band width to zero. Such a property is unfortunately impossible to achieve when a system is subject to vector disturbance that consists of a number of independent signals. However, it is demonstrated that, even in the presence of multidimensional disturbances, the technique demonstrated in the previous section allows one to significantly reduce the quasi-sliding mode band width.

Let us now consider a broader class of discrete-time systems with dynamics expressed in the following way

$$x(k+1) = Ax(k) + bu(k) + f(k), \tag{27}$$

where x, u, A, and b are the same as in (1) and $f : \mathbb{R} \to \mathbb{R}^n$ is a vector of nonlinear functions representing the uncertainties affecting the system. Since this disturbance can no longer be reduced to $g f(k)$, selection of the sliding hyperplane according to conditions (9) is not feasible. However, this disturbance can always be expressed as

$$f(k) = \sum_{j=1}^{m} g_j f_j(k), \tag{28}$$

where $m \leq n$ and $\boldsymbol{g}_j \in \mathbb{R}^n$, $f_j : \mathbb{R} \to \mathbb{R}$ for each $j = 1, 2, ..., m$. In other words, disturbance $\boldsymbol{f}(k)$ can always be divided into at most n parts, each of which consists of a constant vector and a single scalar function. Further in this paper, we assume that each of these functions is bounded in the following way

$$|f_j(k)| \leq f_j^{\max} \tag{29}$$

for $j = 1, 2, ..., m$ and for all $k \geq 0$. It is now shown that, with the proper selection of the sliding hyperplane, one can partially eliminate the effect of the disturbance on system sliding motion.

3.1. Sliding Hyperplane Selection

As in the previous section, we make use of the fact that if the sliding variable is relative degree two with respect to disturbance, this disturbance will only affect it after two time instants (in line with Definition 1). Because of the complexity of disturbance (28), it is usually impossible to select vector $\boldsymbol{c}$ so that the sliding variable has relative degree two with respect to $\boldsymbol{f}(k)$. However, the variable can still be relative degree two with respect to a part of this disturbance. Since the order of the elements in sum (28) is arbitrary, let us assume that

$$\forall_{j=2,...,m} \ f_1^{\max} \geq f_j^{\max}, \tag{30}$$

which means that the first part of the disturbance is the largest. Then, we define vector

$$\boldsymbol{\varphi}(k) = \sum_{j=2}^{m} \boldsymbol{g}_j f_j(k), \tag{31}$$

which represents the remaining disturbance affecting the system without the first (largest) part. We now select vector $\boldsymbol{c}$ so that the sliding variable has relative degree two with respect to $\boldsymbol{g}_1 f_1(k)$, as described by Definition 1. This property is ensured when

$$\boldsymbol{c}^{\mathrm{T}} \boldsymbol{g}_1 = 0, \quad \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{g}_1 \neq 0 \quad \text{and} \quad \boldsymbol{c}^{\mathrm{T}} \boldsymbol{b} \neq 0, \tag{32}$$

which, considering relation (28), results in

$$\begin{aligned} \sigma(k+1) = \ & \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x}(k) + \boldsymbol{c}^{\mathrm{T}} \boldsymbol{b} u(k) + \boldsymbol{c}^{\mathrm{T}} \sum_{j=1}^{m} \boldsymbol{g}_j f_j(k) \\ = \ & \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x}(k) + \boldsymbol{c}^{\mathrm{T}} \boldsymbol{b} u(k) + 0 + \boldsymbol{c}^{\mathrm{T}} \sum_{j=2}^{m} \boldsymbol{g}_j f_j(k) \\ = \ & \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x}(k) + \boldsymbol{c}^{\mathrm{T}} \boldsymbol{b} u(k) + \boldsymbol{c}^{\mathrm{T}} \boldsymbol{\varphi}(k). \end{aligned} \tag{33}$$

One can observe that the effect of $f_1(k)$ on the sliding variable is nullified and $\sigma(k+1)$ is only affected by the remaining part $\boldsymbol{\varphi}(k)$ of the disturbance $\boldsymbol{f}(k)$. Thus, conditions (32) ensure that this variable has relative degree two with respect to $f_1(k)$. However, as highlighted in the previous section, these conditions alone are not sufficient to ensure a stable response of the closed-loop system. To ensure stability of discrete-time plant (27), vector $\boldsymbol{c}$ must further guarantee that all poles of the closed-loop state matrix (10) are placed inside the unit circle.

In the next section, it is shown that, by partially eliminating the effect of disturbance on the sliding variable, the selected sliding hyperplane improves the dynamic performance of the system.

3.2. Properties of the Sliding Mode Control Strategy

As in the previous section, control signal for system (27) is obtained from (4) using the reaching law approach. This control signal is expressed as

$$u(k) = (\boldsymbol{c}^{\mathrm{T}} \boldsymbol{b})^{-1} \Big[(1 - k/k^*)_+ \sigma(0) - \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x}(k) - \boldsymbol{c}^{\mathrm{T}} \hat{\boldsymbol{\varphi}}(k) \Big], \tag{34}$$

where $\hat{\boldsymbol{\varphi}}(k)$ is the estimated effect of disturbance on the system. In our paper, this estimate is simply equal to mean disturbance. According to (29), this mean $\hat{\boldsymbol{\varphi}} = \boldsymbol{0}$ for all k. As a result, control signal (34) assumes the same form as (12). It is now demonstrated that the proposed strategy drives the system representative point to a specific quasi-sliding mode band.

Theorem 3. *If the control strategy for system* (27) *is defined by* (34) *with vector $\boldsymbol{c}$ selected according to* (32)*, then for any initial conditions of the system its representative point will arrive inside the quasi-sliding mode band*

$$\left\{ \boldsymbol{x} : |\boldsymbol{c}^T\boldsymbol{x}| \leq F = \sum_{j=2}^{m} |\boldsymbol{c}^T\boldsymbol{g}_j| f_j^{\max} \right\} \tag{35}$$

in exactly k^* steps and remain in this band for any $k \geq k^*$.

Proof. Let $\boldsymbol{x}(0) \in \mathbb{R}^n$ be any initial state of the plant and let $\sigma(0) = \boldsymbol{c}^T\boldsymbol{x}(0)$. Considering (27), (28), (32), and (34), as well as the fact that $\hat{\boldsymbol{\varphi}} = \boldsymbol{0}$, for any k relation, (3) yields

$$\begin{aligned} \sigma(k+1) =\ & \boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{c}^T\boldsymbol{b}u(k) + \boldsymbol{c}^T\boldsymbol{f}(k) \\ =\ & \boldsymbol{c}^T\boldsymbol{b}(\boldsymbol{c}^T\boldsymbol{b})^{-1}(1 - k/k^*)_+ \sigma(0) + \boldsymbol{c}^T\boldsymbol{g}_1(k) + \boldsymbol{c}^T\boldsymbol{\varphi}(k) - (\boldsymbol{c}^T\boldsymbol{b})^{-1}\boldsymbol{c}^T\hat{\boldsymbol{\varphi}}(k) \\ =\ & (1 - k/k^*)_+ \sigma(0) + \boldsymbol{c}^T\boldsymbol{\varphi}(k) \end{aligned} \tag{36}$$

Then, considering relation (31), for any $k \geq k^*$ one gets

$$\sigma(k+1) = \boldsymbol{c}^T\boldsymbol{\varphi}(k) = \sum_{j=2}^{m} \boldsymbol{c}^T\boldsymbol{g}_j f_j(k). \tag{37}$$

Then, since all functions $f_j(k)$ satisfy inequality (29) for all k, one further obtains

$$|\sigma(k+1)| \leq \sum_{j=2}^{m} |\boldsymbol{c}^T\boldsymbol{g}_j f_j(k)| \leq \sum_{j=2}^{m} |\boldsymbol{c}^T\boldsymbol{g}_j| f_j^{\max}, \tag{38}$$

which implies that for all $k \geq k^*$ the system representative point will remain inside quasi-sliding mode band (35). $\square$

It is demonstrated that the proposed strategy always drives the system state to a specific band around the sliding hyperplane. Furthermore, the width of this band only partially depends on the disturbance affecting the system, since $f_1(k)$ has no effect on it. One concludes that, with the proper choice of the sliding hyperplane, it is possible to partially negate the effect of disturbance on the sliding motion of the system. In the next section, it is shown that this advantage is reflected in the values of individual state variables.

3.3. State Error Estimation

It is now shown that the proposed sliding mode control scheme ensures boundedness of individual state variables. The obtained bound is proportional to the width of the quasi-sliding mode band (35) as well as to the magnitude of the disturbance affecting system (27). This property is demonstrated in the following theorem.

Theorem 4. *If the control strategy for system* (27) *is defined by* (34) *with vector $\boldsymbol{c}$ selected according to* (32)*, then, for each $j = 1, ..., n$, the jth state variable will at least asymptotically converge to the vicinity of zero described as*

$$|x_j| \leq \sum_{j=1}^{m} f_j^{\max} \sum_{i=1}^{\infty} |\boldsymbol{h}_j \boldsymbol{A}_{CL}^{i-1} \boldsymbol{g}_j| + F|\boldsymbol{c}^T\boldsymbol{b}|^{-1} \sum_{i=1}^{\infty} |\boldsymbol{h}_j \boldsymbol{A}_{CL}^{i-1} \boldsymbol{b}|, \tag{39}$$

where F is the width of the band (35)*, $f_j^{\max}$ for $j = 1, ..., n$ are the constants defined by* (29)*, and vector $\boldsymbol{h}_j$ is consistent with* (20)*.*

Proof. Through the analogy with relation (18), state vector can first be expressed as

$$x(k) = A_{\mathrm{CL}}x(k-1) + b(c^{\mathrm{T}}b)^{-1}\sigma(k) + f(k-1). \tag{40}$$

From Theorem 3, it is known that for any given k^* for all $k \geq k^*$ sliding variable σ satisfies $|\sigma(k)| \leq F$. Thus, for any $k > k^*$, one gets

$$x(k) = A_{\mathrm{CL}}^{k-k^*}x(k^*) + \sum_{i=1}^{k-k^*} A_{\mathrm{CL}}^{i-1}f(k-i) + \sum_{i=1}^{k-k^*} A_{\mathrm{CL}}^{i-1}b(c^{\mathrm{T}}b)^{-1}\sigma(k-i+1). \tag{41}$$

Then, since $x_j(k) = h_j x(k)$, one can express the jth state variable as

$$\begin{aligned} x_j(k) = h_j A_{\mathrm{CL}}^{k-k^*}x(k^*) + \sum_{i=1}^{k-k^*} h_j A_{\mathrm{CL}}^{i-1}f(k-i) \\ + \sum_{i=1}^{k-k^*} h_j A_{\mathrm{CL}}^{i-1}b(c^{\mathrm{T}}b)^{-1}\sigma(k-i+1). \end{aligned} \tag{42}$$

Since vector c is selected to ensure (11), one further obtains

$$\begin{aligned} \limsup_{k\to\infty} |x_j(k)| \leq 0 + \limsup_{k\to\infty} \sum_{i=1}^{k-k^*} |h_j A_{\mathrm{CL}}^{i-1}f(k-i)| \\ + \limsup_{k\to\infty} \sum_{i=1}^{k-k^*} |h_j A_{\mathrm{CL}}^{i-1}b(c^{\mathrm{T}}b)^{-1}\sigma(k-i+1)|. \end{aligned} \tag{43}$$

Considering relation (28), this inequality can be rewritten as

$$\begin{aligned} \limsup_{k\to\infty} |x_j(k)| \leq \limsup_{k\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{k-k^*} |h_j A_{\mathrm{CL}}^{i-1}g_j| \cdot |f_j(k-i)| \\ + |c^{\mathrm{T}}b|^{-1} \limsup_{k\to\infty} \sum_{i=1}^{k-k^*} |h_j A_{\mathrm{CL}}^{i-1}b| \cdot |\sigma(k-i+1)|. \end{aligned} \tag{44}$$

Theorem 3 states that for all $k \geq k^*$ sliding variable satisfies $|\sigma(k)| \leq F$. Further considering relation (29), one obtains the following ultimate bound of the jth state variable

$$\begin{aligned} \limsup_{k\to\infty} |x_j(k)| &\leq \limsup_{k\to\infty} \sum_{j=1}^{m} f_j^{\max} \sum_{i=1}^{k-k^*} |h_j A_{\mathrm{CL}}^{i-1}g_j| + F|c^{\mathrm{T}}b|^{-1} \limsup_{k\to\infty} \sum_{i=1}^{k-k^*} |h_j A_{\mathrm{CL}}^{i-1}b| \\ &= \sum_{j=1}^{m} f_j^{\max} \sum_{i=1}^{\infty} |h_j A_{\mathrm{CL}}^{i-1}g_j| + F|c^{\mathrm{T}}b|^{-1} \sum_{i=1}^{\infty} |h_j A_{\mathrm{CL}}^{i-1}b|. \end{aligned} \tag{45}$$

The obtained absolute bound of the jth variable is consistent with the one given in relation (39). $\square$

It is shown above that the proposed sliding mode control strategy ensures bounded errors of all state variables. Furthermore, the obtained bound is proportional to the quasi-sliding mode band width F as well as to the magnitude of the disturbance. Consequently, since the width of the band (35) is reduced by partially negating the effect of disturbance on the system, the same is true for each individual state variable.

Remark 3. *As in the case outlined in Remark 2, the obtained state error bound (39) can be further reduced if all poles of the closed-loop system state matrix are placed in zero. Then, since $A_{\mathrm{CL}}^{i} = 0$ is ensured for all $i \geq n$, the jth state variable is limited in the following way*

$$|x_j| \leq \sum_{j=1}^{m} f_j^{\max} \sum_{i=1}^{n} |h_j A_{\mathrm{CL}}^{i-1}g_j| + F|c^{\mathrm{T}}b|^{-1} \sum_{i=1}^{n} |h_j A_{\mathrm{CL}}^{i-1}b|. \tag{46}$$

Thus, the infinite sums from relation (39) are reduced to finite ones.

In the next section, the properties of the proposed approach for systems (1) as well as (27) are verified in a simulation example. In particular, it is demonstrated that the effect of disturbance on the sliding variable is (partially) eliminated and that improved sliding mode performance of the system results in smaller state errors.

4. Simulation Results

The proposed control scheme is now applied to a particular discrete-time system in order to verify properties stated in this paper. The simulation consists of two parts. In the first one, it is shown that the proposed approach for system (1) can indeed reduce the quasi-sliding mode band width to zero and ensure bounded state error. In the second part, the system is subject to multidimensional disturbance that we partially reject using the approach proposed in Section 3.

4.1. Zero-Width QSMB

Let us consider a discrete-time system (1), where

$$A = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1/24 \\ 1/6 \\ 1/2 \\ 1 \end{bmatrix}, \quad g = \begin{bmatrix} -1 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}. \tag{47}$$

The system expressed in this way can be seen as a discrete-time representation of a quadruple integrator, which can reflect dynamics of many electromechanical plants. The objective of the control process is to drive the state of the plant from its initial conditions $x(0) = [40 \quad 0 \quad 0 \quad 0]^T$ to zero in the presence of disturbance

$$f(k) = (-1)^{\lfloor k/10 \rfloor}, \tag{48}$$

where $\lfloor \star \rfloor$ is the floor function. To achieve that goal, we select the sliding variable so that it has relative degree two with respect to disturbance, which means it must satisfy (9). One possible choice of c (which also ensures stability of the closed-loop system) is

$$c = [3 \quad 6 \quad 4 \quad 1]^T. \tag{49}$$

We then use control signal (12) with $k^* = 10$, which is meant to drive the sliding variable to zero in 10 time instants. Results of the simulation can be seen is Figures 1–5.
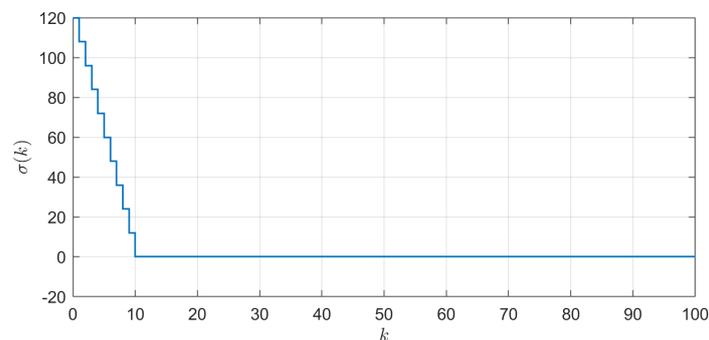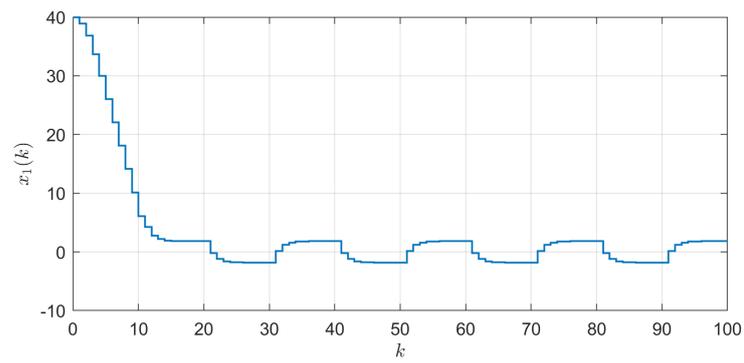

Figure 1. Sliding variable.
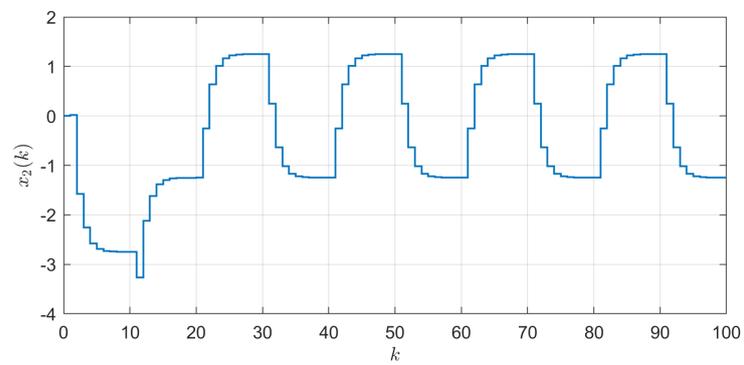
Figure 2. First state variable.

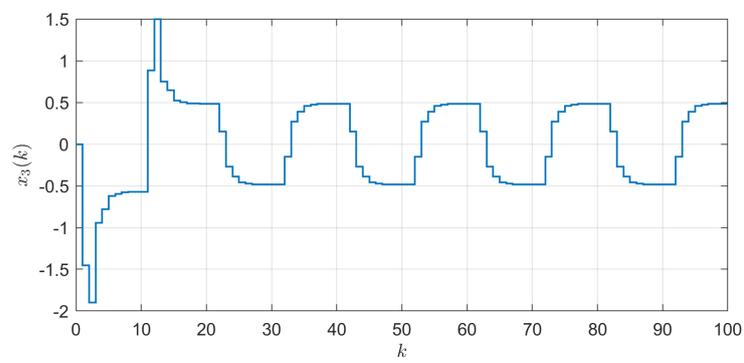**Figure 3.** Second state variable.
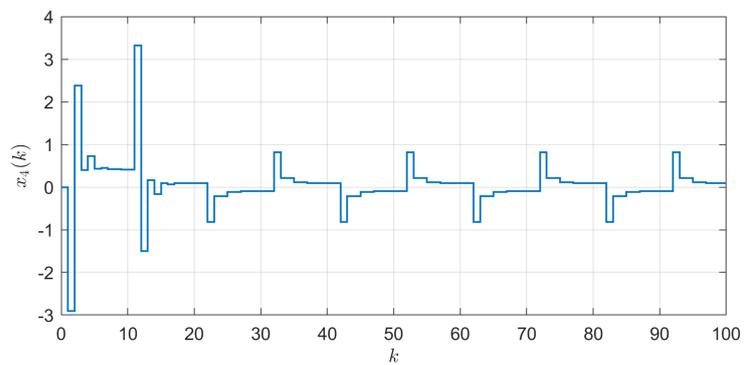

Figure 4. Third state variable.

Figure 5. Fourth state variable.

Figure 1 illustrates that the sliding variable directly follows the trajectory specified by reaching law (4) even in the presence of uncertainties. As a result, the system representative point is driven onto the sliding hyperplane in exactly 10 steps and remains exactly on that hyperplane in each subsequent time instant. Thus, the quasi-sliding mode band width has been effectively reduced to zero.

Figures 2–5 depict the individual state variables of the controlled system. It can be seen that all variables become confined to a specific vicinity of zero in finite time. The width of that vicinity equals:

- 1.823 for x_1;
- 1.250 for x_2;
- 0.484 for x_3; and
- 0.821 for x_4.

All four values obtained in the simulation strictly match the ones obtained from inequality (19). Therefore, state errors are bounded exactly as stated in Theorem 2.

4.2. System Subject to Multidimensional Disturbance

We now consider a system affected by disturbance which cannot be expressed as $\boldsymbol{g}f(k)$ for some scalar function $f(k)$. Suppose that system dynamics are expressed by (27) where $\boldsymbol{A}$ and $\boldsymbol{b}$ are the same as in the previous example. The disturbance affecting the plant

$$\boldsymbol{f}(k) = \boldsymbol{g}_1 f_1(k) + \boldsymbol{g}_2 f_2(k), \tag{50}$$

where

$$f_1(k) = (-1)^{\lfloor k/10 \rfloor}, \quad f_2(k) = \sin(k\pi/50) \tag{51}$$

and

$$\boldsymbol{g}_1 = \begin{bmatrix} -1 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{g}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}. \tag{52}$$

One cannot select vector $\boldsymbol{c}$ that will simultaneously ensure $\boldsymbol{c}^\mathsf{T}\boldsymbol{g}_1 = 0$, $\boldsymbol{c}^\mathsf{T}\boldsymbol{g}_2 = 0$, and place all closed-loop poles inside the unit circle. Thus, obtaining ideal discrete-time sliding motion such as the one in Figure 1 is not possible. However, as shown in Section 3, one can still partially eliminate the effect of this disturbance on system sliding motion. In particular, if vector $\boldsymbol{c}$ is selected as in (49), then $\boldsymbol{c}^\mathsf{T}\boldsymbol{g}_1 = 0$ and part $f_1(k)$ of the disturbance can be rejected. We use the control signal (34) with $k^* = 10$. Since both parts of the disturbance are symmetrical with respect to 0, estimate $\hat{\boldsymbol{\varphi}} = \boldsymbol{0}$. Results of the simulation are depicted in Figures 6–10.
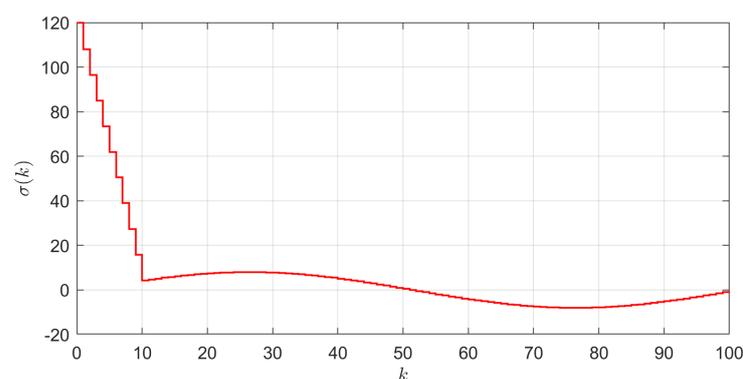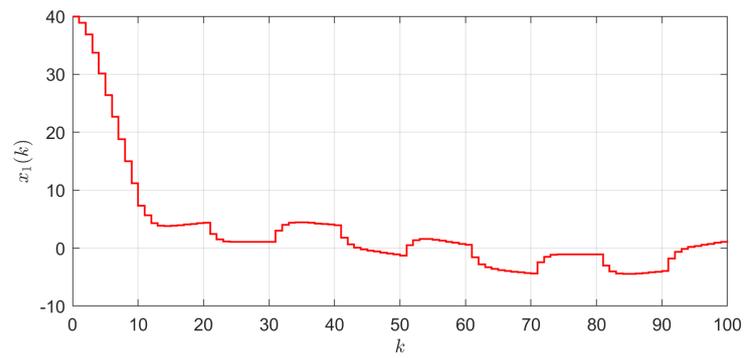

Figure 6. Sliding variable.
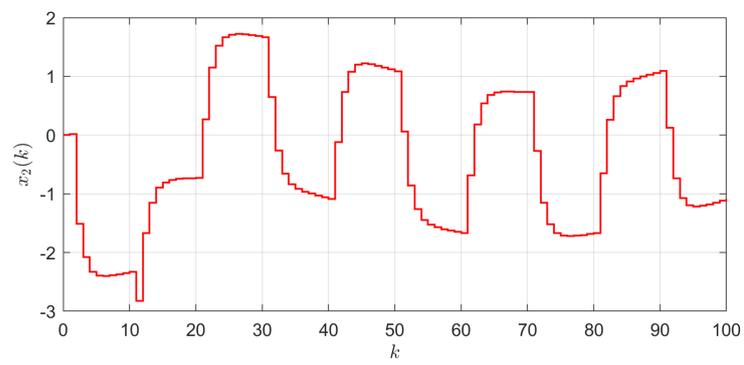
Figure 7. First state variable.

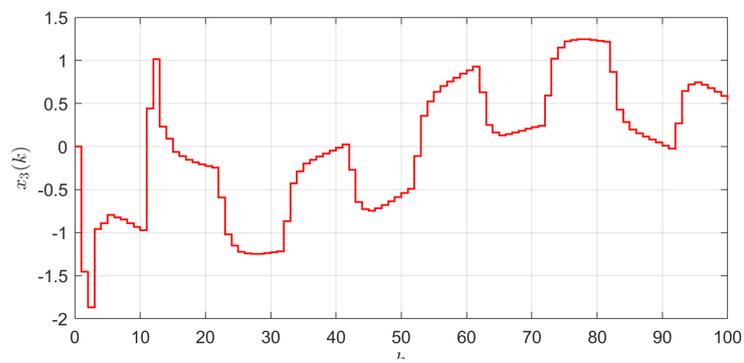
Figure 8. Second state variable.

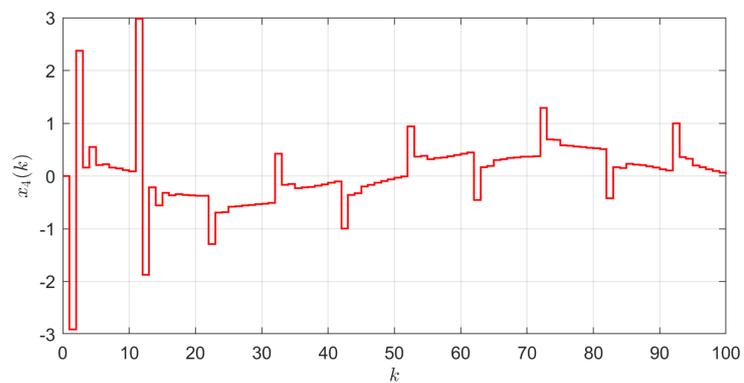**Figure 9.** Third state variable.

Figure 10. Fourth state variable.

It can be seen in Figure 6 that part $f_1(k)$ of the disturbance has no effect on sliding motion of the system and only sinusoidal signal $f_2(k)$ affects the sliding variable. As a result, the system representative point is driven to a quasi-sliding mode band with width equal to $c^{\mathrm{T}} g_2 f_2^{\max} = 10$, which is consistent with Theorem 3.

Figures 7–10 illustrate the evolution of individual state variables. It can be seen that all four variables are driven to a vicinity of zero, and the radius of that vicinity equals

- 4.446 for x_1;
- 1.722 for x_2;
- 1.247 for x_3; and
- 1.295 for x_4.

Considering Theorem 4, one concludes that variable x_1 remains slightly under its maximum projected value, while variables x_2, x_3, x_4 are well below their acceptable bounds.

5. Conclusions

In this paper, we propose a new approach to discrete-time sliding mode controller design. The approach involves a specific selection of a sliding variable, so that it has relative degree two with respect to non-matched disturbance affecting the system. The proposed method allows one to ensure ideal discrete-time sliding motion even in the presence of non-matched disturbance affecting the system. In other words, it ensures that the system representative point is always driven directly onto the specified sliding hyperplane, with the quasi-sliding mode band width effectively reduced to zero. This is an achievement not present in any existing literature on discrete-time sliding mode control. It is further shown that, even if designing a relative degree two sliding variable is impossible (due to multidimensional disturbance), the proposed approach can still be applied to nullify a particular part of this disturbance. This property is particularly desirable when controlling electric drives, electromechanical systems, and mechatronic devices. In the future we hope to extend this result to further improve performance of systems subject to complex multidimensional disturbance. Thus far, in such systems, only partial disturbance rejection is possible and their dynamics can be further improved with properly designed disturbance compensation.

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Abbreviations

The following abbreviations are used in this manuscript:

DSMC	Discrete-time sliding mode control
QSMB	Quasi-sliding mode band

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