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Analysis of Non-Steady Queue-Length Distribution in a Finite-Buffer Model with Group Arrivals and Power Saving Mechanism with Setups

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Abstract: In the manuscript, a probability distribution of the queue length is studied in a model with group Markov arrivals, arbitrarily distributed service times and finite waiting room. After the period of suspension of service due to lack of packets, each new busy period is preceded by a random setup time. Integral equations for time-dependent queue-length distribution are derived by identifying renewal moments in the operation of the system and by applying total probability law. The representation for the solution of the system is found in terms of Laplace transforms. Computational examples illustrating the impact of system parameters on the queue-length distribution are included.

Keywords: finite buffer; integral equations; queue-size distribution; setup time; transient state



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1. Introduction

Queuing systems that employ a mechanism for temporarily shutting down service stations are currently being intensively researched because of their use in modeling real-life processing systems. Particularly important here are models with finite buffers, which can naturally describe systems with losses or untypical delays of different nature due to buffer overflows. Such models are widely used in the analysis and optimization of the operation of network switches, in which incoming data packets are naturally buffered.

One of the key problems of wireless data transmission is the problem of reducing energy consumption. Many systems, therefore, use periodic deactivation of the service station if the buffer for storing incoming messages is empty. The restart of the service station is associated with the arrival of the first packet to be sent after the period of inactivity. Such a mechanism is used, for example, in GSM systems, in which the node is reactivated just before the transmission of the identification frame by a base station.

Most often, it takes a random amount of time, called setup time, for the service station to be fully operational and packet processing. During setup time, transmission is still suspended, so there is an accumulation of incoming packets in the buffer.

The analysis of the $M/M/1$ -type system in steady state with threshold strategies of server waking up and setup times can be found in [1]. A kind of a threshold mechanism in the stationary state of the system with setups is investigated in [2] (see also [3]). In [4] a steady-state behavior of a single batch-arrival queue with random setup times and a vacation period was investigated. This concept was extended to many types of vacations in [5]. The stationary queue-size distribution in the $M^X/G/1$ -type queuing system with vacation time under single vacation policy is derived in [6]. The article [7] contains the steady-state analysis of the queue-size distribution in an infinite-buffer model with batch arrivals, multiple vacations, setup times, N -policy and closedown times. In [8], a recursive method based on the supplementary variable technique is used for development of the

steady-state queue-size distribution in the $GI/M/1$ -type model with finite buffer capacity. A Markovian-type model with machine setups being initialized every time a message arrives into the empty system is studied in [9]. The optimization problem of scheduling in a model of a manufacturing system with finite-sized magazine and the scheme of setup/closedown times is investigated in [10]. One can find some new analytical results for queuing models with N -policy and working breakdowns, and with feedback, balking and maintaining of reneged customers in [11,12], respectively. In [13], a model with active queue management is considered.

Application of setup times in WSNs operation is derived in [14,15]. In particular, in [15], the functioning of the IEEE 802.15.4 is modelled. In [16], queuing systems are used to model the IMS session re-setup delay in WiMAX/LTE heterogeneous networks. Efficient link scheduling based on estimation of the queue size is proposed in [17] for industrial wireless sensor networks. In [18], a queueing model with Poisson arrivals, arbitrarily distributed services, server vacations and setups is proposed for modeling base station operation. In [19], a model of data center with servers restarting the processing with using setups is investigated. In [20], the issue of optimal scheduling strategy is investigated via a queuing model of central air-conditioners. Non-steady analysis of queuing systems with more complex discipline limiting the access to the service station can be found, e.g., in [21,22].

In the paper, we study non-steady operation of the $M/G/1/K$ model in which the service station leaves each idle period by a setup time being a generally distributed random variable. During setup time, packet service is suspended and the service station becomes fully operational.

Transient (time-dependent) analysis is often desired or even necessary, e.g., in the case of low traffic intensity (that is symptomatic for some WSNs), or in the case of the analysis of the system just after implementation of a new control mechanism. Additionally, in practice, the statistical behavior of the system may be destabilized by phenomena such as, e.g., fade-out or interference. New analytical formulae for queuing systems with Poisson input streams and generally distributed service times in the non-stationary case can be found, e.g., in [23,24]. The transient behavior of the $M/G/1$ model with group job arrivals, threshold-type policy and setups is analyzed in [25] (see also [26,27]). In [28], the previous model is extended by “adding” a multiple vacation policy. In [29], the closed-form representation for the queue-size distribution in the model with Poisson arrivals and setup and close-down times is obtained. In [30,31], an auto-correlated input flow is considered in a finite-buffer model.

In the article, as the main analytical result, a formula for the LT (=Laplace transform) of the non-stationary queue-size distribution is obtained. Theoretical apparatus is based on the technique of embedded Markov chains, renewal-theory approach and some auxiliary linear algebraic facts. Applicability of theoretical formulae is illustrated on numerical results.

The rest of the article is organized as follows. In Section 2, we define the considered queuing model mathematically and attach some supplementary algebraic results. In Section 3, we formulate a system of integral equations for non-steady queue-size distribution conditioned by the initial buffer state. Section 4 contains the main analytical result, namely, the representation for the LT of the queue-size distribution. Section 5 is devoted to numerical analysis, while in the last Section 6 one can find a conclusion of the paper.

2. Queuing Model and Preliminaries

In the manuscript, we study a single-machine queuing model with finite waiting-room capacity in that messages (jobs, customers, etc.) enter in groups and are processed one by one, according to FIFO algorithm. Inter-arrival times are exponentially distributed random variables with mean λ^{-1} , while service times have general distribution with a CDF (=cumulative distribution function) $F(\cdot)$. Sizes of arriving groups of messages have general discrete-type distribution described by the sequence (p_k) , where p_k stands for the probability that an arriving group consists of k messages, $\sum_{k=1}^{\infty} p_k = 1$. The total system

capacity equals K , i.e., we have $K - 1$ places in the accumulating buffer (waiting room) and one place “in server”. In consequence, jobs occurring during the buffer overflow period, when the buffer is saturated and the service station is busy with processing (i.e., the number of jobs present equals K), are being lost.

In the considered system, we accept the PBAS (Partial Batch Acceptance Strategy) mechanism: according to the instantaneous number of free places in the buffer, some jobs of the arriving batch can be buffered and some can be lost. Moreover, we assume that, in general, the system may be non-empty before its start, i.e., the waiting room can contain at least one job before $t = 0$. When a job arrives into the empty system, the server leaves the idle time with a generally distributed setup time with a CDF $G(\cdot)$. During the setup time, the processing of messages is still stopped and the service station becomes fully operational. We assume independence of all inter-arrival, processing and setup times, and sizes of the arriving batches in the system operation.

Let us assume that $X(t)$ stands for the number of jobs (messages) present in the system at time t . We define the conditional non-stationary queue-size distribution in the following way:

$$Q_n(t, m) = \mathbf{P}\{X(t) = m \mid X(0) = n\}, \quad t > 0, 0 \leq m, n \leq K. \tag{1}$$

The main goal is to represent in a compact form the LT of $Q_n(t, m)$, so the following function:

$$q_n(s, m) = \int_0^\infty e^{-st} Q_n(t, m) dt, \quad \text{Re}(s) > 0. \tag{2}$$

In the next section, indicating renewal (Markovian) moments in the operation of the studied queuing system, we construct a system of integral equations for $Q_n(t, m)$, $0 \leq n \leq K$. In order to solve a corresponding system written for LTs, we use the following supplementary result from [32]:

Lemma 1. Let $(a_k), k \geq 0$, and $(\psi_k), k \geq 1$, be two given number sequences, where $a_0 \neq 0$. Every solution of the following system of linear equations with respect to $x_n, n \geq 0$:

$$\sum_{k=-1}^n a_{k+1} x_{n-k} - x_n = \phi_n, \quad n \geq 0, \tag{3}$$

can be stated as

$$x_n = CR_{n+1} + \sum_{k=0}^n R_{n-k} \phi_k, \quad n \geq 0, \tag{4}$$

where C is a constant and (R_k) is the so-called potential defined by the sequence (a_k) as follows:

$$\sum_{k=0}^\infty \theta^k R_k = \frac{1}{P_a(\theta) - 1}, \tag{5}$$

where

$$P_a(\theta) = \sum_{k=-1}^\infty \theta^k a_{k+1}, \quad |\theta| < 1. \tag{6}$$

Additionally, successive terms of the sequence (R_n) can be found recursively as follows:

$$\begin{aligned} R_0 &= 0, \quad R_1 = a_0^{-1}, \\ R_{k+1} &= a_0^{-1} \left(R_k - \sum_{i=0}^k a_{i+1} R_{k-i} \right), \quad k \geq 1. \end{aligned} \tag{7}$$

Let us close this section by introducing some necessary notation. In the article, we use the short form $\overline{G}(x) \stackrel{def}{=} 1 - G(x)$ and the nomenclature $I\{\mathbb{A}\}$ for the indicator of the random event \mathbb{A} . Moreover, let

$$f(s) \stackrel{def}{=} \int_0^\infty e^{-st} dF(t), \tag{8}$$

$$g(s) \stackrel{def}{=} \int_0^\infty e^{-st} dG(t), \tag{9}$$

where $\text{Re}(s) > 0$.

Finally, p_i^{j*} stands for the i th term of the j -fold convolution of the sequence $(p_k)_{k=1}^\infty$ with itself defined as

$$p_0^{0*} = 1, \quad p_i^{j*} = \sum_{r=0}^i p_r^{(j-1)*} p_{i-r}, \quad j \geq 1. \tag{10}$$

3. Integral Equations for Conditional Queue-Size Distribution

In this section, we identify renewal (Markovian) moments in the operation of the system to write a system of integral equations for conditional non-steady queue-length distribution $Q_n(t, m)$, for $0 \leq n \leq K$ and for given $0 \leq m \leq K$. In the second step, we obtain from the original equations the system for LTs.

Firstly, let us look at the case in which the waiting room does not contain any job before the start moment. As a consequence, the setup period begins simultaneously with the entering of the first group of jobs. Note that the following three mutually exclusive cases (events) can be distinguished for given t :

- (1) The first group of messages arrives before t and the setup time also completes before t (we denote this event by $E_1(t)$);
- (2) The first group of messages enters before t but the setup time ends after t ($E_2(t)$);
- (3) The first group of messages arrives after t ($E_3(t)$).

Let us introduce the following notation:

$$Q_0^{(i)}(t, m) = \mathbf{P}\{(X(t) = m) \cap E_i(t) \mid X(0) = 0\}, \tag{11}$$

where $t > 0, 0 \leq m \leq K$ and $i = 1, 2, 3$. Thus, e.g., $Q_0^{(1)}(t, m)$ represents the probability that there are exactly m jobs present in the system at time t and the first setup time ends before this moment, on the condition that the buffer is primarily empty. The following equality is a consequence of the formula of total probability:

$$Q_0(t, m) = \mathbf{P}\{X(t) = m \mid X(0) = 0\} = \sum_{i=1}^3 Q_0^{(i)}(t, m). \tag{12}$$

Let us note that for $Q_0^{(1)}(t, m)$ we have the following representation:

$$\begin{aligned} Q_0^{(1)}(t, m) &= \int_{x=0}^t \lambda e^{-\lambda x} \int_{y=0}^{t-x} \left\{ \sum_{i=1}^{K-1} p_i \sum_{j=0}^{K-i-1} \frac{(\lambda y)^j}{j!} e^{-\lambda y} \right. \\ &\times \sum_{r=j}^{K-i-1} p_r^{j*} Q_{i+r}(t-x-y, m) + Q_K(t-x-y, m) \left[\sum_{i=1}^{K-1} p_i \left(\sum_{j=0}^{K-i-1} \frac{(\lambda y)^j}{j!} e^{-\lambda y} \right. \right. \\ &\times \left. \left. \sum_{r=K-i}^\infty p_r^{j*} + \sum_{j=K-i}^\infty \frac{(\lambda y)^j}{j!} e^{-\lambda y} \right) + \sum_{i=K}^\infty p_i \right] \left. \right\} dG(y) dx. \end{aligned} \tag{13}$$

Indeed, the first summand on the right side of (13) describes the situation in which the size of the first arriving batch does not exceed the maximal system capacity K and, moreover, during the setup time, the buffer does not become saturated. Thus, at the completion epoch of the setup time (at time $x + y$), the service station begins the processing with $1 \leq i + r \leq K - 1$ jobs present, where i denotes the size of the first batch and r is the total number of jobs occurring during the setup time. The second summand in (13) presents the case in which before the setup time completion epoch the buffer becomes saturated, so the service process begins with the maximal number K of jobs present.

For $Q_0^{(2)}(t, m)$, we obtain, similarly, the following formula:

$$\begin{aligned}
 Q_0^{(2)}(t, m) = & \int_0^t \lambda e^{-\lambda x} \bar{G}(t-x) \left\{ I\{1 \leq m \leq K-1\} \sum_{i=1}^m p_i \sum_{j=0}^{m-i} p_{m-i}^{j*} \frac{(\lambda(t-x))^j}{j!} \right. \\
 & \times e^{-\lambda(t-x)} + I\{m = K\} \left[\sum_{i=1}^{K-1} p_i \left(\sum_{j=0}^{K-i-1} \sum_{r=K-i}^{\infty} p_r^{j*} \frac{(\lambda(t-x))^j}{j!} e^{-\lambda(t-x)} \right) \right. \\
 & \left. \left. + \sum_{j=K-i}^{\infty} \frac{(\lambda(t-x))^j}{j!} e^{-\lambda(t-x)} \right) + \sum_{i=K}^{\infty} p_i \right] \left. \right\} dx. \tag{14}
 \end{aligned}$$

Let us comment on (14) briefly. If the first setup time completes after t and the queue size measured at t equals $1 \leq m \leq K - 1$, then, if the setup begins at time $0 < x < t$ with the arrival of $1 \leq i \leq m$ jobs in the first batch, during $t - x$, exactly $m - i$ jobs must enter (the first summand on the right side of (14)). If $m = K$, then the buffer must become saturated during time period $(0, t)$ (the second summand in (14)).

Obviously, we also have the following representation:

$$Q_0^{(3)}(t, m) = I\{m = 0\} e^{-\lambda t}. \tag{15}$$

Let us consider now the case of the system being non-empty at the opening (i.e., $1 \leq n \leq K$). Since successive departure epochs are Markov (renewal) moments in the evolution of the $M^X/G/1$ -type system (see, e.g., [33]), then, applying the continuous version of the formula of total probability with respect to the first arrival epoch after $t = 0$, we obtain the following system of integral equations:

$$\begin{aligned}
 Q_n(t, m) = & \int_0^t \left[\sum_{i=0}^{K-n-1} A_i(x) Q_{n+i-1}(t-x, m) \right. \\
 & \left. + Q_{K-1}(t-x, m) \sum_{i=K-n}^{\infty} A_i(x) \right] dF(x) \\
 & + \bar{F}(t) \left[I\{n \leq m \leq K-1\} A_{m-n}(t) + I\{m = K\} \sum_{i=K-n}^{\infty} A_i(t) \right], \tag{16}
 \end{aligned}$$

where $1 \leq n \leq K$ and the functional sequence $(A_i(x))_{i=0}^{\infty}$ describes the number of jobs entering into the system in the time period $(0, x)$ (some of them may be lost due to the buffer saturation), so we have

$$A_i(x) = \sum_{j=0}^i p_i^{j*} \frac{(\lambda x)^j}{j!} e^{-\lambda x}, \quad i \geq 0. \tag{17}$$

Let us clarify the meaning of successive components of the sum in (16). The first component relates to the case in which the waiting room does not become saturated before the first output occurring at $0 < x < t$. The second component represents the situation in

which all places in the waiting room are occupied before the moment t . Finally, in the last component on the right side of (16), the first job leaves the service station after time t .

We define the following functionals:

$$\alpha_j(s) \stackrel{def}{=} \int_0^\infty e^{-(\lambda+s)t} \frac{(\lambda t)^j}{j!} dG(t), \tag{18}$$

$$\beta_j(s, m) \stackrel{def}{=} \int_0^\infty e^{-st} \bar{F}(t) \left[I\{j \leq m \leq K-1\} A_{m-j}(t) + I\{m = K\} \sum_{i=K-j}^\infty A_i(t) \right] dt, \tag{19}$$

$$\gamma_{i,r}(s) \stackrel{def}{=} \frac{\lambda p_i}{\lambda + s} \sum_{j=0}^r p_r^{j*} \alpha_j(s), \tag{20}$$

$$\delta(s) \stackrel{def}{=} \frac{\lambda}{\lambda + s} \left[\sum_{i=1}^{K-1} p_i \left(\sum_{j=0}^{K-i-1} \alpha_j(s) \sum_{r=K-i}^\infty p_r^{j*} + \sum_{j=K-i}^\infty \alpha_j(s) \right) + g(s) \sum_{i=K}^\infty p_i \right] \tag{21}$$

and

$$\eta(s, m) \stackrel{def}{=} \int_0^\infty e^{-st} Q_0^{(2)}(t, m) dt + I\{m = 0\} \frac{1}{\lambda + s}. \tag{22}$$

Since (compare (13))

$$\begin{aligned} & \int_{t=0}^\infty e^{-st} dt \int_{x=0}^t \lambda e^{-\lambda x} dx \int_{y=0}^{t-x} \frac{(\lambda y)^j}{j!} e^{-\lambda y} Q_{i+r}(t-x-y, m) dG(y) \\ &= \frac{\lambda}{\lambda + s} \alpha_j(s) q_{i+r}(s, m), \end{aligned} \tag{23}$$

then, from (12)–(15), (18) and (20)–(22), we obtain

$$q_0(s, m) = \sum_{i=1}^{K-1} \sum_{r=0}^{K-i-1} \gamma_{i,r}(s) q_{i+r}(s, m) + \delta(s) q_K(s, m) + \eta(s, m). \tag{24}$$

Similarly, referring to (16) and (19), we obtain

$$q_n(s, m) = \sum_{i=0}^{K-n-1} a_i(s) q_{n+i-1}(s, m) + q_{K-1}(s, m) \sum_{i=K-n}^\infty a_i(s) + \beta_n(s, m), \tag{25}$$

where $1 \leq n \leq K$ and

$$a_i(s) \stackrel{def}{=} \int_0^\infty e^{-st} A_i(t) dF(t), \quad s > 0. \tag{26}$$

Let us apply to equations of the system (24) and (25) the following substitution:

$$u_n(s, m) \stackrel{def}{=} q_{K-n}(s, m), \quad 0 \leq n \leq K. \tag{27}$$

Now, from (25), we obtain the following system:

$$\sum_{i=-1}^n a_{i+1}(s) u_{n-i}(s, m) - u_n(s, m) = \phi_n(s, m), \tag{28}$$

where $0 \leq n \leq K - 1$, and the functionals $\phi_n(s, m)$ are defined as follows:

$$\phi_n(s, m) \stackrel{def}{=} a_{n+1}(s)u_0(s, m) - u_1(s, m) \sum_{i=n+1}^{\infty} a_i(s) - \beta_{K-n}(s, m). \tag{29}$$

Quite similarly, inserting (27) into (24), we obtain:

$$u_K(s, m) = \sum_{i=1}^{K-1} \sum_{r=0}^{K-i-1} \gamma_{i,r}(s)u_{K-i-r}(s, m) + \delta(s)u_0(s, m) + \eta(s, m). \tag{30}$$

4. Main Results for Transforms

In this section, we find the closed-form solution of the system (28) and (30), applying Lemma 1. Observe that (28) has the same form as (3), but with functional coefficients $a_n(s)$ and $\phi_n(s, m)$, $n \geq 0$. Hence, it is possible to solve (28) by utilizing result (4). Moreover, since the number of equations in (28) is finite compared to (3), the formula for $C = C(s, m)$ can be found explicitly, applying the last Equation (30).

Referring to (4), we obtain for $n \geq 0$ the following formula:

$$u_n(s, m) = C(s, m)R_{n+1}(s) + \sum_{i=0}^n R_{n-i}(s)\phi_i(s, m), \tag{31}$$

where now (compare (7))

$$\begin{aligned} R_0(s) &= 0, \quad R_1(s) = a_0^{-1}(s), \\ R_{n+1}(s) &= R_1(s) \left(R_n(s) - \sum_{i=0}^n a_{i+1}(s)R_{n-i}(s) \right), \end{aligned} \tag{32}$$

where $n \geq 1$.

Taking $n = 0$ in (31), we obtain

$$u_0(s, m) = C(s, m)R_1(s). \tag{33}$$

Next, substituting $n = 1$ in (31) and using (29) and (33), we obtain

$$\begin{aligned} u_1(s, m) &= C(s, m)R_2(s) + R_1(s)\phi_0(s, m) = C(s, m)R_2(s) + R_1(s) \left(a_1(s)R_1(s)C(s, m) \right. \\ &\quad \left. - u_1(s, m) \sum_{i=1}^{\infty} a_i(s) - \beta_K(s, m) \right), \end{aligned} \tag{34}$$

and hence, we derive

$$u_1(s, m) = \zeta(s) \left[C(s, m) \left(R_2(s) + a_1(s)R_1^2(s) \right) - R_1(s)\beta_K(s, m) \right], \tag{35}$$

where

$$\zeta(s) \stackrel{def}{=} \frac{1}{1 + R_1(s) \sum_{i=1}^{\infty} a_i(s)} = \frac{f(\lambda + s)}{f(s)}. \tag{36}$$

Due to the fact that, having (33) and (35), the functionals $\phi_n(s, m)$, $n \geq 0$, can be expressed from (29) explicitly if only the representation for $C(s, m)$ is known, the main goal at this stage is to find the formula for $C(s, m)$.

Utilizing the representations (29) and (31), we can rewrite (30) in the following form:

$$\begin{aligned}
 u_K(s, m) &= \sum_{i=1}^{K-1} \sum_{r=0}^{K-i-1} \gamma_{i,r}(s) u_{K-i-r}(s) + \delta(s) u_0(s, m) + \eta(s, m) \\
 &= \sum_{i=1}^{K-1} \sum_{r=0}^{K-i-1} \gamma_{i,r}(s) \left[C(s, m) R_{K-i-r+1}(s) + \sum_{j=0}^{K-i-r} R_{K-i-r-j}(s) \phi_j(s, m) \right] \\
 &\quad + \delta(s) R_1(s) C(s, m) + \eta(s, m) = \sum_{i=1}^{K-1} \sum_{r=0}^{K-i-1} \gamma_{i,r}(s) \left\{ C(s, m) R_{K-i-r+1}(s) \right. \\
 &\quad + \sum_{j=0}^{K-i-r} R_{K-i-r-j}(s) \left[a_{j+1}(s) R_1(s) C(s, m) - \zeta(s) \left[C(s, m) (R_2(s) \right. \right. \\
 &\quad \left. \left. + a_1(s) R_1^2(s)) - R_1(s) \beta_K(s, m) \right] \sum_{z=j+1}^{\infty} a_z(s) - \beta_{K-j}(s, m) \right] \left. \right\} \\
 &\quad + \delta(s) R_1(s) C(s, m) + \eta(s, m) = \Theta_1(s) C(s, m) + \Omega_1(s, m), \tag{37}
 \end{aligned}$$

where we denote

$$\begin{aligned}
 \Theta_1(s) &\stackrel{def}{=} \sum_{i=1}^{K-1} \sum_{r=0}^{K-i-1} \gamma_{i,r}(s) \left\{ R_{K-i-r+1}(s) + \sum_{j=0}^{K-i-r} R_{K-i-r-j}(s) \right. \\
 &\quad \times \left[a_{j+1}(s) R_1(s) - \zeta(s) (R_2(s) + a_1(s) R_1^2(s)) \sum_{z=j+1}^{\infty} a_z(s) \right] \left. \right\} \\
 &\quad + \delta(s) R_1(s) \tag{38}
 \end{aligned}$$

and

$$\begin{aligned}
 \Omega_1(s, m) &\stackrel{def}{=} \sum_{i=1}^{K-1} \sum_{r=0}^{K-i-1} \gamma_{i,r}(s) \sum_{j=0}^{K-i-r} R_{K-i-r-j}(s) \\
 &\quad \times \left[\zeta(s) R_1(s) \beta_K(s, m) \sum_{z=j+1}^{\infty} a_z(s) - \beta_{K-j}(s, m) \right] + \eta(s, m). \tag{39}
 \end{aligned}$$

Similarly, let us substitute $n = K$ into (31) and use the formulae (29), (33) and (35). As a consequence, we obtain

$$\begin{aligned}
 u_K(s, m) &= C(s, m) R_{K+1}(s) + \sum_{i=0}^K R_{K-i}(s) \left\{ a_{i+1}(s) R_1(s) C(s, m) - \zeta(s) \right. \\
 &\quad \times \left[C(s, m) (R_2(s) + a_1(s) R_1^2(s)) - R_1(s) \beta_K(s, m) \right] \sum_{j=i+1}^{\infty} a_j(s) - \beta_{K-i}(s, m) \left. \right\} \\
 &= C(s, m) \left\{ R_{K+1}(s) + \sum_{i=0}^K R_{K-i}(s) \left[a_{i+1}(s) R_1(s) \right. \right. \\
 &\quad \left. \left. - \zeta(s) (R_2(s) + a_1(s) R_1^2(s)) \sum_{j=i+1}^{\infty} a_j(s) \right] \right\} + \sum_{i=0}^K R_{K-i}(s) \\
 &\quad \times \left(\zeta(s) R_1(s) \beta_K(s, m) \sum_{j=i+1}^{\infty} a_j(s) - \beta_{K-i}(s, m) \right) = \Theta_2(s) C(s, m) + \Omega_2(s, m), \tag{40}
 \end{aligned}$$

where

$$\Theta_2(s) \stackrel{def}{=} R_{K+1}(s) + \sum_{i=0}^K R_{K-i}(s) \left[a_{i+1}(s)R_1(s) - \zeta(s)(R_2(s) + a_1(s)R_1^2(s)) \sum_{j=i+1}^{\infty} a_j(s) \right] \tag{41}$$

and

$$\Omega_2(s, m) \stackrel{def}{=} \sum_{i=0}^K R_{K-i}(s) \left(\zeta(s)R_1(s)\beta_K(s, m) \sum_{j=i+1}^{\infty} a_j(s) - \beta_{K-i}(s, m) \right). \tag{42}$$

By equating the right sides of the Equations (37) and (40), we eliminate $C(s, m)$ as follows:

$$C(s, m) = \frac{\Omega_2(s, m) - \Omega_1(s, m)}{\Theta_1(s) - \Theta_2(s)}. \tag{43}$$

Now, the formulae (27), (29), (31), (33), (35) and (43) lead to the following main theorem:

Theorem 1. *The representation for the LT of the conditional transient queue-size distribution in the $M^X/G/1/K$ -type model with batched arrivals and generally distributed setup times is the following:*

$$q_n(s, m) = \int_0^{\infty} e^{-st} \mathbf{P}\{X(t) = m \mid X(0) = n\} dt = \frac{\Omega_2(s, m) - \Omega_1(s, m)}{\Theta_1(s) - \Theta_2(s)} \times \left\{ R_{K-n+1}(s) + \sum_{i=0}^{K-n} R_{K-n-i}(s) \left[a_{i+1}(s)R_1(s) - \zeta(s)(R_2(s) + a_1(s)R_1^2(s)) \sum_{j=i+1}^{\infty} a_j(s) \right] \right\} + \sum_{i=0}^{K-n} R_{K-n-i}(s) \left(\zeta(s)R_1(s)\beta_K(s, m) \sum_{j=i+1}^{\infty} a_j(s) - \beta_{K-i}(s, m) \right), \tag{44}$$

where the formulae for $\beta_i(s, m)$, $a_i(s)$, $R_i(s)$, $\zeta(s)$, $\Theta_1(s)$, $\Omega_1(s, m)$, $\Theta_2(s)$ and $\Omega_2(s, m)$ are given in (19), (26), (32), (36), (38), (39), (41) and (42), respectively.

As a corollary from Theorem 1, let us note that from the formula (44) the stationary queue-size distribution

$$\pi_m \stackrel{def}{=} \mathbf{P}\{X(\infty) = m\} = \lim_{t \rightarrow \infty} \mathbf{P}\{X(t) = m\}, \tag{45}$$

where $m = 0, \dots, K$, can be found by using the Tauberian theorem.

Indeed, we have

$$\pi_m = \lim_{s \downarrow 0} s \cdot q_n(s, m) = \lim_{s \downarrow 0} s \cdot \int_0^{\infty} e^{-st} \mathbf{P}\{X(t) = m \mid X(0) = n\} dt, \tag{46}$$

where n can be chosen arbitrarily from $\{0, \dots, K\}$ as the stationary queue-size distribution does not depend on the initial number of jobs present in the system.

In practice, for larger values of the constant K , instead of evaluating successive terms of the functional sequence $(R_k(s))$ via recursive formulae (32), we can apply the following approach. Observe that

$$\sum_{k=0}^{\infty} \mu^k R_k(s) = \frac{1}{M(\mu, s) - 1}, \tag{47}$$

where

$$M(\mu, s) \stackrel{\text{def}}{=} \sum_{k=-1}^{\infty} \mu^k a_{k+1}(s), \quad |\mu| < 1. \quad (48)$$

In consequence, the representation (47) can be utilized to find successive terms of the sequence $(R_k(s))$. Using Maclaurin's expansion, we obtain

$$R_k(s) = \frac{1}{k!} \frac{\partial}{\partial \mu} \left(\frac{1}{M(\mu, s) - 1} \right) \Big|_{\mu=0}. \quad (49)$$

5. Numerical Examples

As an illustrating example, let us consider the stream of packets of average sizes 250 B arriving into the node of a wireless network. Assume that the incoming process is modeled via a compound Poisson process with intensity 360 Kbps and let $K = 6$.

Let us study two different batch size distributions:

- P_1 : $p_1 = 0.8$, $p_2 = 0.2$, $p_k = 0$, $k > 2$,
- P_2 : $p_1 = p_2 = 0.5$, $p_k = 0$, $k > 2$,

which give arrival rate parameters $\lambda_1 = 150$ and $\lambda_2 = 120$, respectively. In relation to batch distributions, in practice, P_1 can be used as a model of the situation in which packets of size 250 B make 80 percent of incoming traffic while packets of size 500 are "responsible" for 20 percent only. Similarly, in the case of P_2 , the incoming flow consists of packets of sizes 250 B and 500 B which appear with equal frequencies.

Furthermore, let packets be transmitted with speeds 480 Kbps and 360 Kbps, respectively, according to exponential service time distribution. That corresponds to parameters $\mu_1 = 240$, $\mu_2 = 180$ of the service time distribution, respectively. Having given arrival and serving rates, we obtain the utilization factor ρ of the considered system equal to 0.75 or 1, respectively. Additionally, let us implement an energy saving mechanism: the first arrival after an idle period restarts the server and, simultaneously, initializes a setup time with exponential distribution with mean 2 ms or 20 ms.

The probabilities $\mathbf{P}\{X(t) = m | X(0) = 0\}$ for factors $\rho = 0.75$, $\rho = 1$, batch size distributions P_1, P_2 and mean setup times equal 2 ms and 20 ms are presented in Table 1 (for the stationary state, $t \rightarrow \infty$) and in Figures 1–8 (for the transient case), respectively.

Table 1. Stationary distributions π_m for utilization factors $\rho = 0.75$, $\rho = 1$, batch size distributions P_1, P_2 and mean setup times equal 2 and 20 ms.

m	$\pi_m = \mathbf{P}\{X(\infty) = m X(0) = 0\}$			
	$\rho = 0.75$		$\rho = 1$	
	P_1	P_2	P_1	P_2
Mean setup time equals 2 ms				
1	0.193688	0.157534	0.134491	0.118967
3	0.136140	0.130957	0.146078	0.141245
5	0.085300	0.089495	0.147105	0.144444
Mean setup time equals 20 ms				
1	0.115951	0.106959	0.088952	0.085723
3	0.134825	0.135847	0.136159	0.136760
5	0.121891	0.128069	0.164976	0.166223

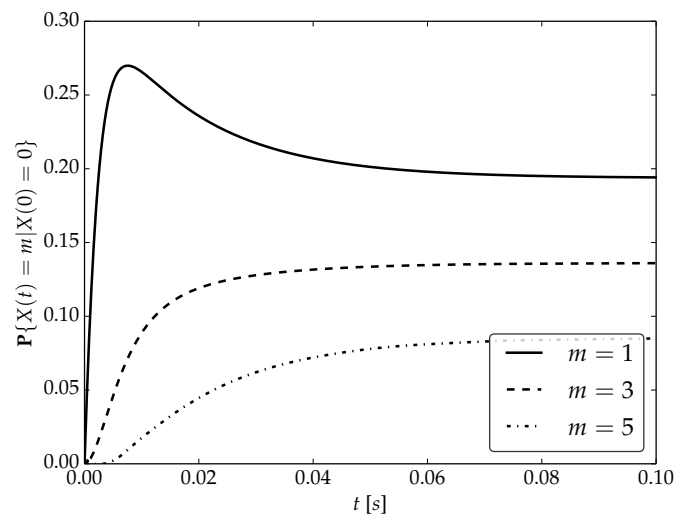


Figure 1. Transient probabilities $P\{X(t) = m | X(0) = 0\}$ for P_1 , $\rho = 0.75$ and mean setup time equal 2 ms.

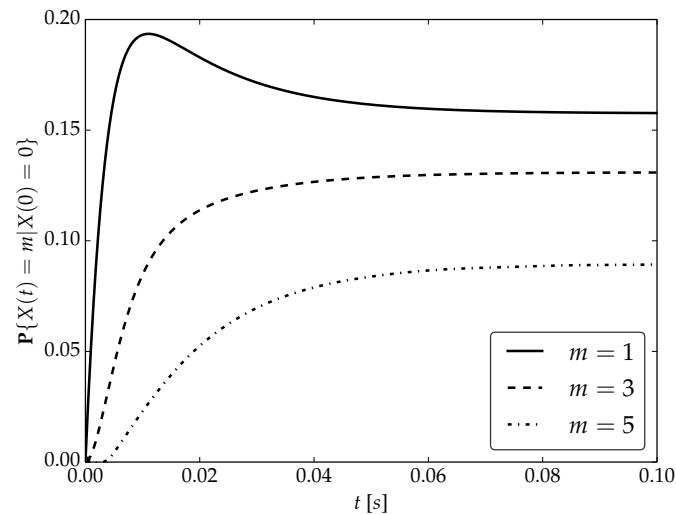


Figure 2. Transient probabilities $P\{X(t) = m | X(0) = 0\}$ for P_2 , $\rho = 0.75$ and mean setup time equal 2 ms.

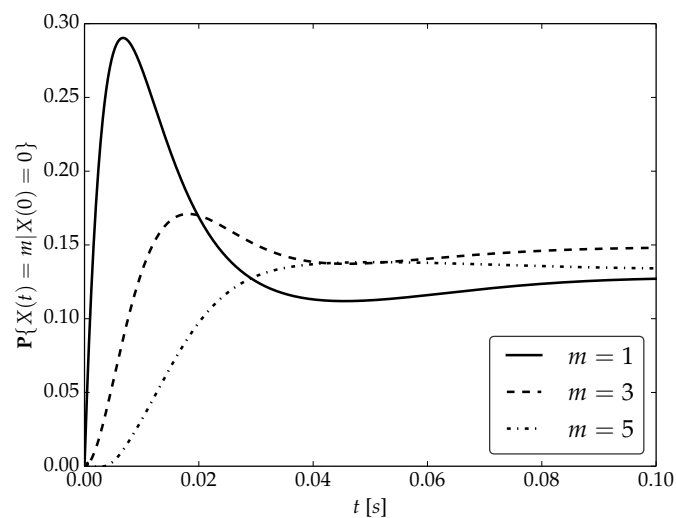


Figure 3. Transient probabilities $P\{X(t) = m | X(0) = 0\}$ for P_1 , $\rho = 0.75$ and mean setup time equal 20 ms.

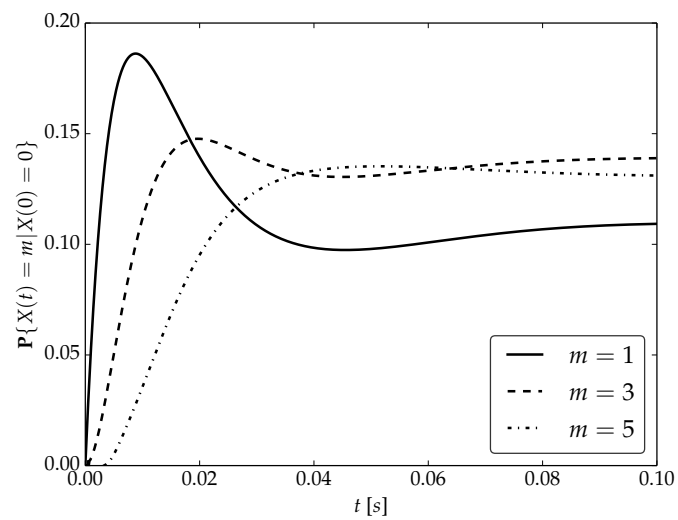


Figure 4. Transient probabilities $P\{X(t) = m | X(0) = 0\}$ for $P_2, \rho = 0.75$ and mean setup time equal 20 ms.

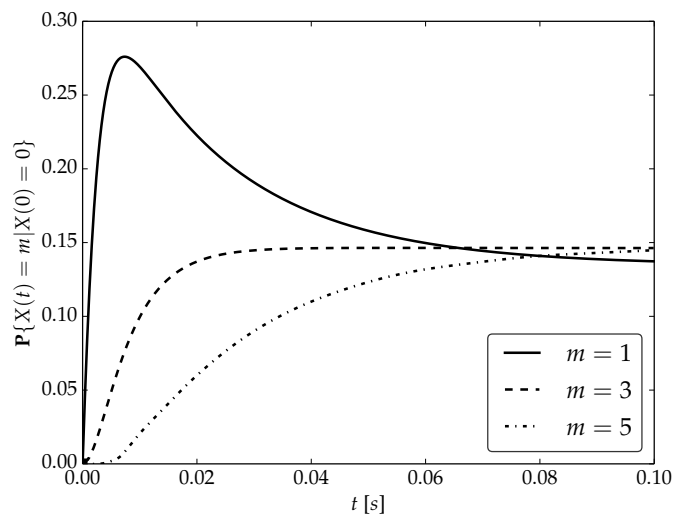


Figure 5. Transient probabilities $P\{X(t) = m | X(0) = 0\}$ for $P_1, \rho = 1$ and mean setup time equal 2 ms.

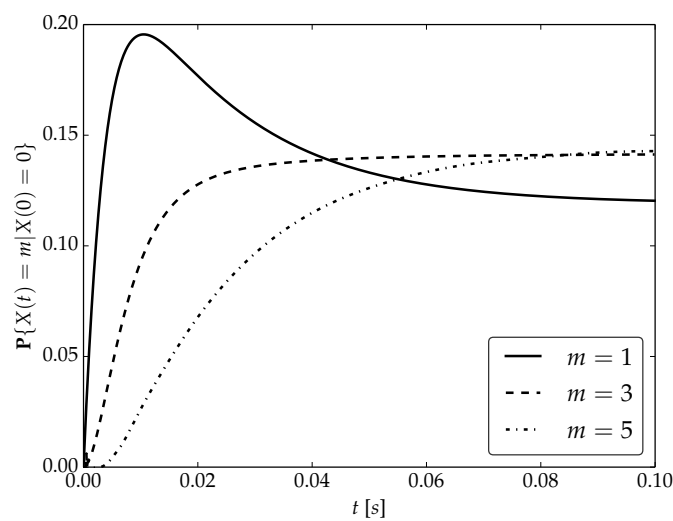


Figure 6. Transient probabilities $P\{X(t) = m | X(0) = 0\}$ for $P_2, \rho = 1$ and mean setup time equal 2 ms.

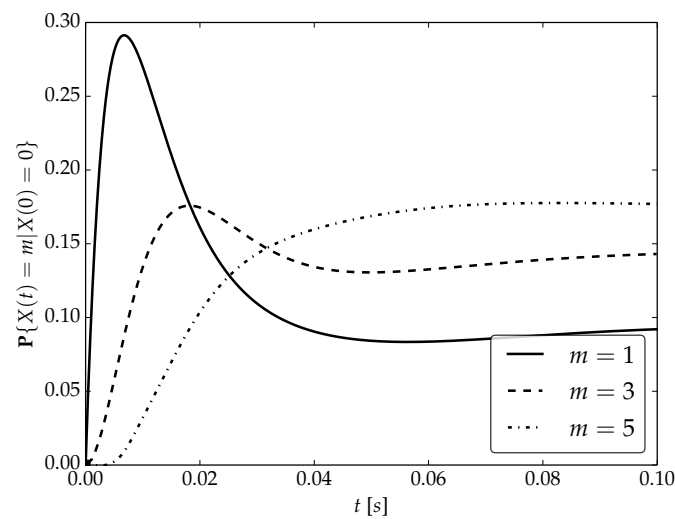


Figure 7. Transient probabilities $P\{X(t) = m | X(0) = 0\}$ for P_1 , $\rho = 1$ and mean setup time equal 20 ms.

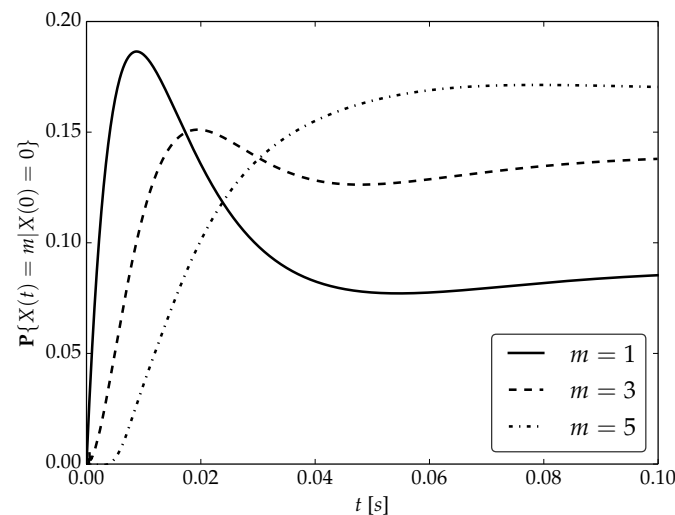


Figure 8. Transient probabilities $P\{X(t) = m | X(0) = 0\}$ for P_2 , $\rho = 1$ and mean setup time equal 20 ms.

As one can observe, in Figures 1–8, essential dependence of the transient queue-size distribution on the utilization factor (occupation rate) is visible. In fact, the greater the value of ρ , the longer the process of “stabilization” of the characteristic about the stationary value. Moreover, the dependence on the type of probability distribution of the batch size is shown. Finally, the impact of the mean setup time duration on the convergence rate of the transient characteristic to the stationary one can be noticed: the longer the setup time duration, the lower the corresponding convergence rate.

6. Conclusions

In the paper, a queuing system in which entering messages occur in groups according to a compound Poisson process and are served individually with arbitrary service time distribution is studied. It is assumed that a setup time with a general-type distribution function precedes the first processing in a new busy period. The investigated model has potential applications in the analysis of the transmission process, e.g., in nodes of WSNs, in which an energy saving mechanism is implemented. A system of integral equations for conditional non-steady queue-size distribution is found and next solved explicitly in terms of LTs. From the representation of the solution, the stationary queue-size distribution can

be derived simply by using the Tauberian theorem. Illustrative numerical computations are presented.

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