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Mathematical Modeling and Numerical Approximation of Heat Conduction in Three-Phase-Lag Solid

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Abstract: In this article, we propose a mathematical model for one-dimensional heat conduction in a three-layered solid considering that an interfacial condition is present for the temperature and heat flux conditions between the layers. The numerical approach is developed by constructing a finite difference scheme to solve the initial boundary–interface problem. The numerical scheme is designed by considering the accuracy of the model on the inner part of each layer, then extending to the interfaces and boundaries by incorporating the continuous interfacial conditions. The finite difference scheme is unconditionally stable, convergent, and easy to implement since it consists of the solution of two algebraic systems. We provide three numerical examples to confirm that our numerical approximation is consistent with the analytical solution and the physical phenomenon.

Keywords: heat conduction; finite difference method; unconditional numerical method; second-order finite difference scheme

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1. Introduction

In recent years, and mainly driven by concerns regarding climate change, there is a growing concern to investigate issues related to the global environment, where one of the areas is that related to energy [1–3]. More specifically, regarding energy, there are various investigations focused on its distinct aspects that range from improving energy efficiency [4,5], encouraging the energy refill of clean energy devices for transportation and heating [6,7], promoting the construction of buildings that guarantee low energy consumption [8–10], and the proposal of global policies that regulate the practices and guide the industrial activities that reduce CO_2 emissions [11,12]. In this context, engineering applications based on multilayered materials have been encouraged. For example, in the case of the construction of modern buildings in geographical areas where extreme temperatures are reached, it is widespread to find the inclusion of thermopane windows [10], for the conservation of foods, thin films are used [13,14], and photovoltaic panels are considered relevant in the industries oriented to green energy production. In all these cases, heat conduction research is a relevant topic [15–17].

Traditionally, heat conduction is modeled mathematically by considering the well-known Fourier law of heat conduction that relates the temperature gradient and the heat flux linearly by using the thermal conductivity as the proportionality constant, which implies an immediate change in the heat flux as a consequence of the temperature gradient, which is not the case in several physical environments and materials [18,19]. However, it is widely known that this kind of behavior of heat conduction is strongly restrictive, requiring, for instance, a homogeneous and isotropic thermally conducting medium, and, even in this case, the Fourier law is only local in time [20]. Several authors have developed improvements or corrections of this basic constitutive relation, which are currently grouped under the term of non-Fourier heat conduction [18].

Nowadays, there has been great progress in the development of advanced mathematical models as well as physical applications; see, for instance, [10,21–33] and the references therein. In the case of mathematical analysis, the works are focused on topics like the local existence of solutions, the global existence of solutions, the well-posedness of the mathematical models, the asymptotic behavior of the solutions, energy decay, the numerical solutions, and the convergence of the numerical methods. Meanwhile, from the physical study, we possess articles studying different materials, the thermal conductivity in different mediums, the study of diffusion phenomena in different problems, including atmospheric ones like climate and weather, and the alimentary industry. In addition, the study of the phenomenon of heat has influenced the development of other theories, like probability theory, financial mathematics, and hydrodynamics.

In this paper, we are interested in the heat transfer in a three-layered solid of the form provided in Figure 1 (see also Table 1 for the notation). These kinds of devices appear in several engineering applications: the aerodynamic heating of spacecraft structures [34], multilayered porous-medium microheat exchangers [35], superconducting cables of the cylindrical type [36], airfoil thrust bearings [37], and thermal protection systems for aircraft [38]. Other applications are reviewed below in Section 2.

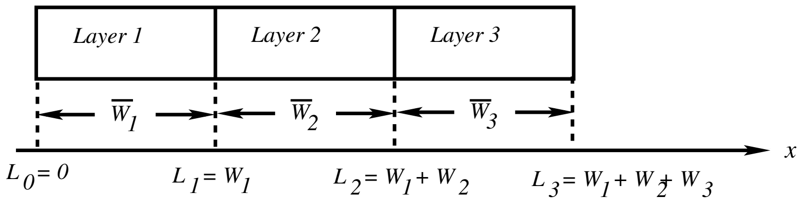

Figure 1. Schematic form of the three-layered solid.

Our main contributions are the following: (i) the construction of a mathematical model based on non-Fourier heat conduction law, (ii) the numerical approximation, and (iii) the numerical simulations. Concerning (i), in Section 2, firstly, we discuss some types of non-Fourier heat conduction; then, we introduce the mathematical model for the conduction equation on the interior of each layer by considering the dual-phase-lagging [39] equation, assuming the continuous behavior of the temperature and flux through the interfaces and the flux conditions on the boundaries. The details for (ii) are developed in Section 3 and are summarized as follows. We approximate the dual-phase-lagging mathematical model by applying the finite difference method. The methodology for discretization consists of three steps: approximate the equation on the interior of each layer, discretize the interfaces appropriately, and approximate the boundary conditions. We prove that the numerical discretization consists of the solution of two linear systems of algebraic equations. Then, the scheme is easy to implement. Then, we invite the arduous computations provided in [40,41], where the authors considered a change in the variable, a semi-discrete finite difference scheme for the new variable, a fully finite difference scheme for the new variable, and then approximated the original variable. Moreover, we show that the scheme is unconditionally stable. Concerning (iii), we present three numerical simulations focused on the comparison with analytical solutions and the influence of some physical constants that appear in the definition of the boundary conditions.

Table 1. Notations for the geometrical and physical parameters associated with the three-layered solid. Here, we consider the notation $\ell \in \{1,2,3\}$.

Notation	**Definition**
Geometrical	
$\overline{W}_\ell$	width of ℓth-layer
$L_0 = 0$	left boundary
$L_1 = \overline{W}_1$	interface 1
$L_2 = \overline{W}_1 + \overline{W}_2$	interface 2
$L_3 = L = \overline{W}_1 + \overline{W}_2 + \overline{W}_3$	right boundary
$\mathcal{I}_\ell =]L_{\ell-1}, L_\ell[$	interval denoting the ℓth-layer
$\mathcal{I}^{lay} = \cup_{\ell=1}^{3} \mathcal{I}_\ell,\ \ \overline{\mathcal{I}^{lay}} = [L_0, L_3]$	space domain
$[0, T]$	time domain
$Q_T^{lay} = \cup_{\ell=1}^{3} Q_{\ell,T},\ \ Q_{\ell,T} = \mathcal{I}_\ell \times [0, T]$	space–time domain
Physical	
C^ℓ	the heat capacitance of ℓth-layer
τ_q^ℓ	heat flux phase lags of ℓth-layer
τ_T^ℓ	temperature gradient phase lags of ℓth-layer
k_ℓ	thermal conductivity of ℓth-layer
α_1, α_2	some proportionality constants
$\overline{K}_1, \overline{K}_2$	Knudsen numbers
$f_\ell(x,t)$	heat source function of ℓth-layer
$\psi_1(x)$	initial distribution of the temperature
$\psi_2(x)$	initial distribution of the temporal derivative of temperature
$\varphi_1(t)$	temperature flux at the left boundary of the solid
$\varphi_2(t)$	temperature flux at the right boundary of the solid

2. A Mathematical Model for Heat Conduction in a Three-Phase-Lag Solid

One of the lines of research developed in the last few years is related to considering generalized and complex constitutive laws for the flux of heat. In order to put this in context, we recall three of those laws: Fourier, Cattaneo, and dual-phase lagging. The prototypical model for heat conduction in a region $\Omega \subset \mathbb{R}^d$ ($d = 1,2,3$) is provided by the heat equation as follows:

$$\frac{\partial T}{\partial t}(x,t) = \alpha \Delta T(x,t) + S(x,t), \tag{1}$$

where t is the time, $x \in \Omega$ is the space position, T is the temperature, $\alpha > 0$ is the thermal diffusivity of the medium, and S is the volumetric heat generation. In a broader sense, (1) is based on two facts: (i) the balance of temperature in a region, which is provided in differential form as follows

$$C\frac{\partial T}{\partial t}(x,t) = -\mathrm{div}(\mathbf{q}(x,t)) + \overline{Q}(x,t), \tag{2}$$

where C is the heat capacity of the material, $\mathbf{q}$ is the heat flux, and $\overline{Q}$ is the volumetric heat generation; and (ii) the constitutive relation proposed by Joseph Fourier

$$\mathbf{q}(x,t) = -k\nabla T(x,t), \tag{3}$$

where k is a positive constant t that measures the thermal conductivity of the material. We notice that we can deduce (1) by replacing (3) in (2) with $\alpha = k/C$ and $S = \overline{Q}/C$. The Fourier model for heat conduction (1) has several disadvantages, which are presented well in [42]. Particularly, for instance, (1) is not adequate for modelling the heat transport at very high frequencies and short wavelengths. To eliminate this inconsistency, Cattaneo proposed an improved constitutive relation of the following type

$$\tau \frac{\partial \mathbf{q}}{\partial t}(x,t) = -\Big(\mathbf{q}(x,t) + k\nabla T(x,t)\Big), \tag{4}$$

where τ is positive constant denoting a relaxation parameter [43,44]. Then, replacing (4) in (2), we deduce that

$$\tau \frac{\partial^2 T}{\partial t^2}(x,t) + \frac{\partial T}{\partial t}(x,t) = \frac{k}{C}\Delta T(x,t) + \frac{1}{C}\left(\overline{Q}(x,t) + \tau \frac{\partial \overline{Q}}{\partial t}(x,t)\right). \tag{5}$$

Notice that, when $\tau \to 0$ in model (5), we recover (1). Another consistent extension of Fourier law was derived by [39] (see also [30]), by assuming that

$$\mathbf{q}(x, t+\tau_q) = -k\nabla T(x, t+\tau_T), \tag{6}$$

with τ_T and τ_q, the phase lags of the temperature gradient and the heat flux, respectively. In the case of one-dimensional domain $(d = 1)$, a Taylor expansion in (6) implies that

$$q(x,t) + \tau_q \frac{\partial q}{\partial x}(x,t) = -k\left[\frac{\partial T}{\partial x}(x,t) + \tau_T \frac{\partial^2 T}{\partial t \partial x}(x,t)\right]. \tag{7}$$

From (2) and (7), we obtain

$$C\left(\frac{\partial T}{\partial t} + \tau_q \frac{\partial^2 T}{\partial t \partial x}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \tau_T \frac{\partial^3 T}{\partial t \partial^2 x}\right) + \overline{Q}(x,t) + \tau_q \frac{\partial \overline{Q}}{\partial x}(x,t), \tag{8}$$

which is known as the heat conduction equation under the dual-phase-lagging effect or briefly as dual-phase-lagging model. We observe that (8) is reduced to (1) when $(\tau_q, \tau_T) \to (0, 0)$.

Let us consider the solid of Figure 1, where the geometric and physical information of each layer $\ell = 1, 2, 3$, provided in Table 1. We assume that the heat conduction on each layer is modeled by a dual-phase-lagging Equation (8), i.e.,

$$C_\ell\left(\frac{\partial u}{\partial t} + \tau_q^{(\ell)} \frac{\partial^2 u}{\partial t^2}\right) = k_\ell\left(\frac{\partial^2 u}{\partial x^2} + \tau_T^{(\ell)} \frac{\partial^3 u}{\partial t \partial x^2}\right) + f_\ell(x,t),\ (x,t) \in Q_{\ell,T}, \quad \ell = 1,2,3;$$

and on the interfaces we assume the temperature and the heat flux are continuous. In order to formulate the equation on Q_T^{lay} and precise interface conditions, we consider $C, k, \tau_q, \tau_T : \overline{\mathcal{I}^{lay}} \to \mathbb{R}$ for the piecewise constant functions and $f : Q_T \to \mathbb{R}$ the continuous function of the form

$$\left.\begin{array}{ll} C(x) = \sum_{\ell=1}^{3} C_\ell \mathbb{1}_{[L_{\ell-1}, L_\ell[}(x), & k(x) = \sum_{\ell=1}^{3} k_\ell \mathbb{1}_{[L_{\ell-1}, L_\ell[}(x), \\ \tau_q(x) = \sum_{\ell=1}^{3} \tau_q^{(\ell)} \mathbb{1}_{[L_{\ell-1}, L_\ell[}(x), & \tau_T(x) = \sum_{\ell=1}^{3} \tau_T^{(\ell)} \mathbb{1}_{[L_{\ell-1}, L_\ell[}(x), \\ f(x,t) = \sum_{\ell=1}^{3} f_\ell(x,t) \mathbb{1}_{[L_{\ell-1}, L_\ell[}(x), & \end{array}\right\} \tag{9}$$

with $\mathbb{1}_A$ the indicator function defined as $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise. Hence, the mathematical model is provided by the initial interface–boundary value problem

$$C(x)\left(\frac{\partial u}{\partial t} + \tau_q(x) \frac{\partial^2 u}{\partial t^2}\right) = k(x)\left(\frac{\partial^2 u}{\partial x^2} + \tau_T(x) \frac{\partial^3 u}{\partial t \partial x^2}\right) + f, \qquad (x,t) \in Q_T^{lay}, \tag{10}$$

$$u(x,0) = \psi_1(x), \quad \frac{\partial u}{\partial t}(x,0) = \psi_2(x), \qquad x \in \overline{\mathcal{I}^{lay}}, \tag{11}$$

$$\left(-\alpha_1 \overline{K}_1 \frac{\partial u}{\partial x} + u\right)(L_0, t) = \varphi_1(t), \qquad t \in [0, T], \tag{12}$$

$$\left(\alpha_2\overline{K}_2\frac{\partial u}{\partial x}+u\right)(L_3,t)=\varphi_2(t), \qquad t\in[0,T], \tag{13}$$

$$[\![u(x,t)]\!]=0, \qquad (x,t)\in I_T^{int}, \tag{14}$$

$$\left[\!\!\left[\left[k(x)\left(\frac{\partial u}{\partial x}+\tau_T(x)\frac{\partial^2 u}{\partial x\partial t}\right)\right](x,t)\right]\!\!\right]=0, \qquad (x,t)\in I_T^{int}, \tag{15}$$

where $I_T^{int}:=\{L_1,L_2\}\times[0,T]$; α_1 and α_2 are some coefficients; $\overline{K}_1$ and $\overline{K}_2$ are the Knudsen numbers; ψ_1 and ψ_2 are the initial conditions; φ_1 and φ_2 are two provided functions modelling the boundary conditions; and the bracket $[\![\cdot]\!]$ is defined by $[\![G(x,t)]\!]=G(x-0,t)-G(x+0,t)$. The relationship between K_n and k is provided by $K_n^2CL_c^2=3k\tau_q$ with L_c a characteristic length, boundary conditions (12) and (13) are a consequence of assuming a temperature-jump condition, and the model is not in dimensionless form; see [40,41,45] for details.

3. Finite Difference Terminology and Preliminary Results

3.1. Discretization of the Domain and Notation

Let us consider the notation in (1). To discretize the space and time domains, we select $m_1,m_2,m_3,N\in\mathbb{N}$, and consider that the ℓ-th layer $\mathcal{I}_\ell$ and the time interval $[0,T]$ are divided into m_ℓ and N parts of sizes Δx_ℓ and Δt, respectively. Then, we define the following notation and terminology

$$\left.\begin{array}{l}
M_0=0 \text{ and } M_\ell=\sum_{i=1}^{\ell}m_i \text{ for } \ell\in\{1,2,3\},\quad \Delta W_\ell=(L_\ell-L_{\ell-1})/m_\ell,\\
x_j=L_{\ell-1}+(j-M_{\ell-1})\Delta W_\ell \text{ for } (\ell,j)\in\{1,2,3\}\times\{M_{\ell-1},\ldots,M_\ell\},\\
\Delta x_j=\Delta W_\ell \text{ for } j\in\{M_{\ell-1},\ldots,M_\ell\},\quad \mathcal{I}_{\Delta x}^{int}=\{x_j\,:\,j=M_1,M_2\},\\
\mathcal{I}_{\ell,\Delta x}=\{x_j\,:\,j=M_{\ell-1}+1,\ldots,M_\ell-1\},\quad \mathcal{I}_{\Delta x}^{lay}=\cup_{\ell=1}^{d}\mathcal{I}_{\ell,\Delta x},\\
\partial\mathcal{I}_{\Delta x}^{lay}=\{x_j\,:\,j=M_0,M_3\},\quad \overline{\mathcal{I}}_{\Delta x}=\mathcal{I}_{\Delta x}^{lay}\cup\mathcal{I}_{\Delta x}^{int}\cup\partial\mathcal{I}_{\Delta x}^{lay},\\
\Delta t=T/N,\quad t_n=n\Delta t \text{ for } n=0,\ldots,N,\quad \mathcal{T}_{\Delta t}=\{t_n\,:\,n=0,\ldots,N\},\\
Q_{\Delta x,\Delta t}=\overline{\mathcal{I}}_{\Delta x}\times\mathcal{T}_{\Delta t}, \mathbb{I}=\{M_0,\ldots,M_3\}, \mathbb{I}^{int}=\{M_1,M_2\},\\
\partial\mathbb{I}=\{M_0,M_3\}, \mathbb{I}^{lay}=\mathbb{I}-(\mathbb{I}^{int}\cup\partial\mathbb{I}).
\end{array}\right\} \tag{16}$$

Moreover, we can define the grid function space by

$$\mathcal{U}_{\Delta x,\Delta t}=\left\{\mathbb{U}=(\mathbf{u}^0,\ldots,\mathbf{u}^N)\in\mathbb{R}^{M_d+1}\times\mathbb{R}^{N+1}\quad:\quad \mathbf{u}^n=(u_0^n,\ldots,u_{M_3}^n)\right\}.$$

Here, the notation u_j^n is defined by $u_j^n=u(x_j,t_n)$ for $(j,n)\in\mathbb{I}\times\{0,\ldots,N\}$.

3.2. Finite Difference Notation for Discretization

In this section, we consider some notations used to discretize the system, (10)–(15). We remark that the finite difference approximation constructed is based on the following lemma.

Lemma 1 ([46,47])**.** *Let $[a,b]\subset\mathbb{R}$ be an interval partitioned in m sub-intervals $[z_{i-1},z_i]$ of the same size $h=(b-a)/m$ with $z_i=a+ih$ for $i=0,\ldots,m$. If $g\in C^4([a,b])$, and then there is ξ_i such that the following approximation of second order derivative at z_i*

$$g''(z_i)=\begin{cases}
\dfrac{2}{h}\left[\dfrac{g(z_{i+1})-g(z_i)}{h}-g'(z_i)\right]-\dfrac{h}{3}g'''(\xi_i),\ \xi_i\in]z_i,z_{i+1}[, & i=0,\\
\dfrac{1}{h^2}[g(z_{i+1})-2g(z_i)+g(z_{i-1})]-\dfrac{h^2}{12}g^{(4)}(\xi_i),\ \xi_i\in]z_{i-1},z_{i+1}[, & i=1,\ldots,m-1,\\
\dfrac{2}{h}\left[g'(z_i)-\dfrac{g(z_i)-g(z_{i-1})}{h}\right]+\dfrac{h}{3}g'''(\xi_i),\ \xi_i\in]z_{i-1},z_i[, & i=m,
\end{cases} \tag{17}$$

is satisfied. Moreover, if $g\in C^4([z_i,z_{i+1}])$ for each $i=1,\ldots,m-1$, then the following relation

$$\frac{1}{2}\left[g'(z_i)+g'(z_{i+1})\right] = \frac{g(z_{i+1})-g(z_i)}{h} + \frac{h^2}{8}\int_0^1\left[g'''\left(t_{i+1/2}+\frac{h}{s}\right)+g'''\left(t_{i+1/2}-\frac{h}{s}\right)\right](1-s^2)ds, \quad (18)$$

is satisfied.

Proposition 1. *Consider notation (16) and* $\mathbb{L}$ *defined as follows*

$$\mathbb{L}u(x,t) = C(x)\left(\frac{\partial u}{\partial t}(x,t) + \tau_q(x)\frac{\partial^2 u}{\partial t^2}(x,t)\right), \quad (19)$$

for $(x,t) \in \overline{Q_T^{lay}}$. *If* $u \in C^4(\overline{Q_T^{lay}})$, *then* $(\mathbb{L}_\Delta^1 u)_j^{n+1/2}$ *the average of* $\mathbb{L}u(x_j,t_n)$ *and* $\mathbb{L}u(x_j,t_{n+1})$ *is approximated by*

$$(\mathbb{L}_\Delta^1 u)_j^{n+1/2} = C(x_j)\left(\frac{u_j^{n+1}-u_j^n}{\Delta t} + \frac{\tau_q(x_j)}{\Delta t}\left[\frac{\partial u}{\partial t}(x_j,t_{n+1}) - \frac{\partial u}{\partial t}(x_j,t_n)\right]\right), \quad (20)$$

for all $j \in \mathbb{I}$ *and* $n \in \{1,\ldots,N-1\}$. *Moreover, the approximation of* $\mathbb{L}u(x_j,t_1)$ *is provided by*

$$(\mathbb{L}_\Delta^1 u)_j^{1/2} = C(x_j)\left(\frac{u_j^1-u_j^0}{\Delta t} + \frac{2\tau_q(x_j)}{\Delta t}\left[\frac{u_j^1-u_j^0}{\Delta t} - \frac{\partial u}{\partial t}(x_j,0)\right]\right), \quad (21)$$

for all $j \in \mathbb{I}$ *and* $n = 0$.

Proof. The proof is constructive and based on the application of Lemma 1. We observe that

$$\begin{aligned}(\mathbb{L}_\Delta^1 u)_j^{n+1/2} &= \frac{1}{2}\Big(\mathbb{L}u(x_j,t_n) + \mathbb{L}u(x_j,t_{n+1})\Big)\\ &= C(x_j)\left(\frac{1}{2}\left[\frac{\partial u}{\partial t}(x_j,t_n) + \frac{\partial u}{\partial t}(x_j,t_{n+1})\right] + \tau_q(x_j)\frac{\partial}{\partial t}\left(\frac{1}{2}\left[\frac{\partial u}{\partial t}(x_j,t_n) + \frac{\partial u}{\partial t}(x_j,t_{n+1})\right]\right)\right)\\ &= C(x_j)\left(\frac{u_j^{n+1}-u_j^n}{\Delta t} + \frac{\tau_q(x_j)}{\Delta t}\left[\frac{\partial u}{\partial t}(x_j,t_{n+1}) - \frac{\partial u}{\partial t}(x_j,t_n)\right]\right), \quad (22)\end{aligned}$$

for all $j \in \mathbb{I}$ and $n \in \{1,\ldots,N-1\}$. Notice that $\Big(\partial_t u(x_j,t_n) + \partial_t u(x_j,t_{n+1})\Big)/2 = \Big(u(x_j,t_{n+1}) - u(x_j,t_n)\Big)/\Delta t$ is deduced by (18).

For $n = 0$, using (17) with $i = 0$, we follow that

$$\begin{aligned}(\mathbb{L}_\Delta^1 u)_j^{1/2} = \mathbb{L}(x_j,t_1) &= C(x_j)\left(\frac{\partial u}{\partial t}(x_j,t_1) + \tau_q(x_j)\frac{\partial^2 u}{\partial t^2}(x_j,t_1)\right)\\ &= C(x_j)\left(\frac{u_j^1-u_j^0}{\Delta t} + \tau_q(x_j)\frac{2}{\Delta t}\left[\frac{u_j^1-u_j^0}{\Delta t} - \frac{\partial u}{\partial t}(x_j,0)\right]\right), \quad (23)\end{aligned}$$

for all $j \in \mathbb{I}$.

From (22) and (23), we obtain (20), (24), and (21), respectively. □

Proposition 2. *Consider the notation and assumptions of Proposition 1. Then,* $\mathbb{L}_\Delta^2 u_j^n$ *defined as the average of* $(\mathbb{L}_\Delta^1 u)_j^{n-1/2}$ *and* $(\mathbb{L}_\Delta^1 u)_j^{n+1/2}$ *is approximated by*

$$(\mathbb{L}^2_\Delta u)^n_j = C(x_j)\left(\frac{u^{n+1}_j - u^{n-1}_j}{2\Delta t} + \frac{\tau_q(x_j)}{(\Delta t)^2}\left(u^{n+1}_j - 2u^n_j + u^{n-1}_j\right)\right), \tag{24}$$

for all $j \in \mathbb{I}$ and $n \in \{2, \ldots, N-1\}$.

Proof. Regarding the application of relation (18), we have that

$$\begin{aligned}
&\frac{1}{2}\left\{\left[\frac{\partial u}{\partial t}(x_j, t_{n+1}) - \frac{\partial u}{\partial t}(x_j, t_n)\right] + \left[\frac{\partial u}{\partial t}(x_j, t_n) - \frac{\partial u}{\partial t}(x_j, t_{n-1})\right]\right\} \\
&\quad = \frac{1}{2}\left[\frac{\partial u}{\partial t}(x_j, t_{n+1}) + \frac{\partial u}{\partial t}(x_j, t_n)\right] - \left[\frac{\partial u}{\partial t}(x_j, t_n) + \frac{\partial u}{\partial t}(x_j, t_{n-1})\right] \\
&\quad = \frac{u^{n+1}_j - u^n_j}{\Delta t} - \frac{u^n_j - u^{n-1}_j}{\Delta t} = \frac{u^{n+1}_j - 2u^n_j + u^{n-1}_j}{\Delta t}.
\end{aligned}$$

From (22), we deduce the following relation

$$\begin{aligned}
(\mathbb{L}^2_\Delta u)^n_j &= \frac{1}{2}\left[(\mathbb{L}^1_\Delta u)^{n+1/2}_j + (\mathbb{L}^1_\Delta u)^{n-1/2}_j\right] \\
&= C(x_j)\left(\frac{1}{2}\left[\frac{u^{n+1}_j - u^n_j}{\Delta t} + \frac{u^n_j - u^{n-1}_j}{\Delta t}\right] + \frac{\tau_q(x_j)}{\Delta t}\left[\frac{1}{2}\left\{\left[\frac{\partial u}{\partial t}(x_j, t_{n+1}) - \frac{\partial u}{\partial t}(x_j, t_n)\right] + \left[\frac{\partial u}{\partial t}(x_j, t_n) - \frac{\partial u}{\partial t}(x_j, t_{n-1})\right]\right\}\right]\right) \\
&= C(x_j)\left(\frac{u^{n+1}_j - u^{n-1}_j}{2\Delta t} + \frac{\tau_q(x_j)}{(\Delta t)^2}\left(u^{n+1}_j - 2u^n_j + u^{n-1}_j\right)\right),
\end{aligned}$$

for all $j \in \mathbb{I}$ and $n \in \{2, \ldots, N-1\}$, which clearly prove the result. □

Proposition 3. *Consider notation (16) and define $\mathbb{D}$, $(\mathbb{D}^1_{lay}u)^{n+1/2}_j$, and $(\mathbb{D}^1_{int,\pm}u)^{n+1/2}_j$ as follows*

$$\mathbb{D}u(x,t) = k(x)\left(\frac{\partial^2 u}{\partial x^2}(x,t) + \tau_T(x)\frac{\partial^3 u}{\partial t \partial x^2}(x,t)\right), \quad \text{for } (x,t) \in \overline{Q^{lay}_T}. \tag{25}$$

$$\begin{aligned}
(\mathbb{D}^1_{lay}u)^{n+1/2}_j = \frac{k(x_j)}{(\Delta x_j)^2}\Bigg\{&\left[u^1_{j+1} + \frac{\tau_T(x_j)}{\Delta t}(u^1_{j+1} - u^0_{j+1})\right] \\
&-2\left[u^1_j + \frac{\tau_T(x_j)}{\Delta t}(u^1_j - u^0_j)\right] + \left[u^1_{j-1} + \frac{\tau_T(x_j)}{\Delta t}(u^1_{j-1} - u^0_{j-1})\right]\Bigg\},
\end{aligned} \tag{26}$$

$$\begin{aligned}
(\mathbb{D}^1_{int,+}u)^{n+1/2}_j = \frac{2k(x_j)}{\Delta x_j}\Bigg\{&\frac{1}{\Delta x_j}\left(\left[u^1_{j+1} + \frac{\tau_T(x_j)}{\Delta t}(u^1_{j+1} - u^0_{j+1})\right] - \left[u^1_j + \frac{\tau_T(x_j)}{\Delta t}(u^1_j - u^0_j)\right]\right) \\
&-\left(\frac{\partial u}{\partial x} + \tau_T(x_j)\frac{\partial^2 u}{\partial t \partial x}\right)\left(x_j+, t^{1/2}\right)\Bigg\},
\end{aligned} \tag{27}$$

$$\begin{aligned}
(\mathbb{D}^1_{int,-}u)^{n+1/2}_j = \frac{2k(x_{j-1})}{\Delta x_{j-1}}\Bigg\{&\left(\frac{\partial u}{\partial x} + \tau_T(x_{j-1})\frac{\partial^2 u}{\partial t \partial x}\right)(x_j-, t^{1/2}) \\
&-\frac{1}{\Delta x_{j-1}}\left(\left[u^1_j + \frac{\tau_T(x_{j-1})}{\Delta t}(u^1_j - u^0_j)\right] - \left[u^1_{j-1} + \frac{\tau_T(x_{j-1})}{\Delta t}(u^1_{j-1} - u^0_{j-1})\right]\right)\Bigg\}.
\end{aligned} \tag{28}$$

If $u \in C^4(\overline{Q^{lay}_T})$, then $(\mathbb{D}^1_\Delta u)^{1/2}_j$, the approximation of $\mathbb{D}u(x_j, t_1)$, is provided by

$$(\mathbb{D}^1_\Delta u)^{1/2}_j = \begin{cases} (\mathbb{D}^1_{lay}u)^{1/2}_j, & j \in \mathbb{I}^{lay}, \\ (\mathbb{D}^1_{int,+}u)^{1/2}_j, & j \in \mathbb{I}^{int} \cup \{M_0\}, \\ (\mathbb{D}^1_{int,-}u)^{1/2}_j, & j \in \mathbb{I}^{int} \cup \{M_3\}, \end{cases} \tag{29}$$

for all $j \in \mathbb{I}$.

Proof. By application of Lemma 1, we deduce that

$$\frac{\partial^2 u}{\partial x^2}(x_j,t_1)=\begin{cases}\dfrac{u_{j+1}^1-2u_j^1+u_{j-1}^1}{(\Delta x_j)^2}, & j\in\mathbb{I}^{lay},\\ \dfrac{2}{\Delta x_j}\left[\left(\dfrac{u_{j+1}^1-u_j^1}{\Delta x_j}\right)-\dfrac{\partial u}{\partial x}(x_j+,t_1)\right], & j\in\mathbb{I}^{int}\cup\{M_0\},\\ \dfrac{2}{\Delta x_{j-1}}\left[\dfrac{\partial u}{\partial x}(x_j-,t_1)-\left(\dfrac{u_j^1-u_{j-1}^1}{\Delta x_{j-1}}\right)\right], & j\in\mathbb{I}^{int}\cup\{M_3\},\end{cases} \tag{30}$$

$$\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_1)=\begin{cases}\dfrac{1}{(\Delta x_j)^2}\left(\dfrac{\partial u}{\partial t}(x_{j+1},t_1)-2\dfrac{\partial u}{\partial t}(x_j,t_1)+\dfrac{\partial u}{\partial t}(x_{j-1},t_1)\right), & j\in\mathbb{I}^{lay},\\ \dfrac{2}{\Delta x_j}\left[\dfrac{1}{\Delta x_j}\left(\dfrac{\partial u}{\partial t}(x_{j+1},t_1)-\dfrac{\partial u}{\partial t}(x_j,t_1)\right)-\dfrac{\partial^2 u}{\partial t\partial x}(x_j+,t_1)\right], & j\in\mathbb{I}^{int}\cup\{M_0\},\\ \dfrac{2}{\Delta x_{j-1}}\left[\dfrac{\partial^2 u}{\partial t\partial x}(x_j-,t_1)-\dfrac{1}{\Delta x_{j-1}}\left(\dfrac{\partial u}{\partial t}(x_j,t_1)-\dfrac{\partial u}{\partial t}(x_{j-1},t_1)\right)\right], & j\in\mathbb{I}^{int}\cup\{M_3\}.\end{cases} \tag{31}$$

Considering the fact that $u_t(x_j,t_1)=(u_j^1-u_j^0)/\Delta t$ for all $j\in\mathbb{I}$, we notice that (31) is equivalent to

$$\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_1)=\begin{cases}\dfrac{1}{(\Delta x_j)^2}\left(\dfrac{1}{\Delta t}(u_{j+1}^1-u_{j+1}^0)-\dfrac{2}{\Delta t}(u_j^1-u_j^0)+\dfrac{1}{\Delta t}(u_{j-1}^1-u_{j-1}^0)\right), & j\in\mathbb{I}^{lay},\\ \dfrac{2}{\Delta x_j}\left[\dfrac{1}{\Delta x_j}\left(\dfrac{1}{\Delta t}(u_{j+1}^1-u_{j+1}^0)-\dfrac{1}{\Delta t}(u_j^1-u_j^0)\right)-\dfrac{\partial^2 u}{\partial t\partial x}(x_j+,t_1)\right], & j\in\mathbb{I}^{int}\cup\{M_0\},\\ \dfrac{2}{\Delta x_{j-1}}\left[\dfrac{\partial^2 u}{\partial t\partial x}(x_j-,t_1)-\dfrac{1}{\Delta x_{j-1}}\left(\dfrac{1}{\Delta t}(u_j^1-u_j^0)-\dfrac{1}{\Delta t}(u_{j-1}^1-u_{j-1}^0)\right)\right], & j\in\mathbb{I}^{int}\cup\{M_3\}.\end{cases}$$

Then, rearranging the terms, we obtain the result. □

Proposition 4. *Consider the notation and assumptions of Proposition 3 and define* Γ, $(\mathbb{D}_{lay}^1 u)_j^{n+1/2}$, *and* $(\mathbb{D}_{int,\pm}^1 u)_j^{n+1/2}$ *as follows*

$$\Gamma u(x_j\pm,t^{n+1/2})=\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}+\tau_T(x)\frac{\partial^2 u}{\partial t\partial x}\right)(x_j\pm,t_n)+\left(\frac{\partial u}{\partial x}+\tau_T(x)\frac{\partial^2 u}{\partial t\partial x}\right)(x_j\pm,t_{n+1})\right], \tag{32}$$

$$\begin{aligned}(\mathbb{D}_{lay}^1 u)_j^{n+1/2}=\frac{k(x_j)}{(\Delta x_j)^2}\Bigg\{&\frac{1}{2}(u_{j+1}^{n+1}+u_{j+1}^n)+\frac{\tau_T(x_j)}{\Delta t}(u_{j+1}^{n+1}-u_{j+1}^n)-2\left[\frac{1}{2}(u_j^{n+1}+u_j^n)+\frac{\tau_T(x_j)}{\Delta t}(u_j^{n+1}-u_j^n)\right]\\ &+\frac{1}{2}(u_{j-1}^{n+1}+u_{j-1}^n)+\frac{\tau_T(x_j)}{\Delta t}(u_{j+1}^{n-1}-u_{j-1}^n)\Bigg\},\end{aligned} \tag{33}$$

$$\begin{aligned}(\mathbb{D}_{int,+}^1 u)_j^{n+1/2}=\frac{2k(x_j)}{\Delta x_j}\Bigg\{&\frac{1}{\Delta x_j}\left(\frac{1}{2}(u_{j+1}^{n+1}+u_{j+1}^n)+\frac{\tau_T(x_j)}{\Delta t}(u_{j+1}^{n+1}-u_{j+1}^n)\right.\\ &\left.-\left[\frac{1}{2}(u_j^{n+1}+u_j^n)+\frac{\tau_T(x_j)}{\Delta t}(u_j^{n+1}-u_j^n)\right]\right)-\Gamma u\left(x_j+,t^{n+1/2}\right)\Bigg\}\end{aligned} \tag{34}$$

$$\begin{aligned}(\mathbb{D}_{int,-}^1 u)_j^{n+1/2}=\frac{2k(x_{j-1})}{\Delta x_{j-1}}\Bigg\{&\Gamma u(x_j-,t^{n+1/2})-\frac{1}{\Delta x_{j-1}}\left(\frac{1}{2}(u_j^{n+1}+u_j^n)+\frac{\tau_T(x_j)}{\Delta t}(u_j^{n+1}-u_j^n)\right.\\ &\left.-\left[\frac{1}{2}(u_{j-1}^{n+1}+u_{j-1}^n)+\frac{\tau_T(x_j)}{\Delta t}(u_{j+1}^{n-1}-u_{j-1}^n)\right]\right)\Bigg\}.\end{aligned} \tag{35}$$

If $u\in C^4(\overline{Q_T^{lay}})$, *then* $(\mathbb{D}_\Delta^1 u)_j^{n+1/2}$, *the average of* $\mathbb{D}u(x_j,t_n)$ *and* $\mathbb{D}u(x_j,t_{n+1})$, *is approximated by*

$$(\mathbb{D}^1_\Delta u)^{n+1/2}_j = \begin{cases} (\mathbb{D}^1_{lay}u)^{n+1/2}_j, & j \in \mathbb{I}^{lay}, \\ (\mathbb{D}^1_{int,+}u)^{n+1/2}_j, & j \in \mathbb{I}^{int} \cup \{M_0\}, \\ (\mathbb{D}^1_{int,-}u)^{n+1/2}_j, & j \in \mathbb{I}^{int} \cup \{M_3\}, \end{cases} \tag{36}$$

for all $j \in \mathbb{I}$ and $n \in \{2, \ldots, N-1\}$.

Proof. By application of Lemma 1, similarly to (30) and (31), we deduce that

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_n) = \begin{cases} \dfrac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{(\Delta x_j)^2}, & j \in \mathbb{I}^{lay}, \\ \dfrac{2}{\Delta x_j}\left[\left(\dfrac{u^n_{j+1} - u^n_j}{\Delta x_j}\right) - \dfrac{\partial u}{\partial x}(x_j+, t_n)\right], & j \in \mathbb{I}^{int} \cup \{M_0\}, \\ \dfrac{2}{\Delta x_{j-1}}\left[\dfrac{\partial u}{\partial x}(x_j-, t_n) - \left(\dfrac{u^n_j - u^n_{j-1}}{\Delta x_{j-1}}\right)\right], & j \in \mathbb{I}^{int} \cup \{M_3\}, \end{cases} \tag{37}$$

$$\frac{\partial^3 u}{\partial t \partial x^2}(x_j, t_n) = \begin{cases} \dfrac{1}{(\Delta x_j)^2}\left(\dfrac{\partial u}{\partial t}(x_{j+1}, t_n) - 2\dfrac{\partial u}{\partial t}(x_j, t_n) + \dfrac{\partial u}{\partial t}(x_{j-1}, t_n)\right), & j \in \mathbb{I}^{lay}, \\ \dfrac{2}{\Delta x_j}\left[\dfrac{1}{\Delta x_j}\left(\dfrac{\partial u}{\partial t}(x_{j+1}, t_n) - \dfrac{\partial u}{\partial t}(x_j, t_n)\right) - \dfrac{\partial^2 u}{\partial t \partial x}(x_j+, t_n)\right], & j \in \mathbb{I}^{int} \cup \{M_0\}, \\ \dfrac{2}{\Delta x_{j-1}}\left[\dfrac{\partial^2 u}{\partial t \partial x}(x_j-, t_n) - \dfrac{1}{\Delta x_{j-1}}\left(\dfrac{\partial u}{\partial t}(x_j, t_n) - \dfrac{\partial u}{\partial t}(x_{j-1}, t_n)\right)\right], & j \in \mathbb{I}^{int} \cup \{M_3\}. \end{cases} \tag{38}$$

To calculate $(\mathbb{D}^1_\Delta u)^{n+1/2}_j$, we consider three cases: $j \in \mathbb{I}^{lay}$, $j \in \mathbb{I}^{int}$, and $j \in \partial\mathbb{I}^{lay}$.

For $j \in \mathbb{I}^{lay}$, we deduce the following two relations

$$\begin{aligned} \frac{1}{2}\left\{\frac{\partial^2 u}{\partial x^2}(x_j, t_n) + \frac{\partial^2 u}{\partial x^2}(x_j, t_{n+1})\right\} &= \frac{1}{2}\left\{\frac{u^{n+1}_{j+1} - 2u^{n+1}_j + u^{n+1}_{j-1}}{(\Delta x_j)^2} + \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{(\Delta x_j)^2}\right\} \\ &= \frac{1}{(\Delta x_j)^2}\left[\frac{1}{2}(u^{n+1}_{j+1} + u^n_{j+1}) - (u^{n+1}_j + u^n_j) + \frac{1}{2}(u^{n+1}_{j-1} + u^n_{j-1})\right], \end{aligned}$$

$$\begin{aligned} &\frac{1}{2}\left\{\tau_T(x_j)\frac{\partial^3 u}{\partial t \partial x^2}(x_j, t_n) + \tau_T(x_j)\frac{\partial^3 u}{\partial t \partial x^2}(x_j, t_{n+1})\right\} \\ &= \frac{\tau_T(x_j)}{2}\left\{\left(\frac{\partial u}{\partial t}(x_{j+1}, t_n) - 2\frac{\partial u}{\partial t}(x_j, t_n) + \frac{\partial u}{\partial t}(x_{j-1}, t_n)\right) + \left(\frac{\partial u}{\partial t}(x_{j+1}, t_{n+1}) - 2\frac{\partial u}{\partial t}(x_j, t_{n+1}) + \frac{\partial u}{\partial t}(x_{j-1}, t_{n+1})\right)\right\} \\ &= \frac{\tau_T(x_j)}{2(\Delta x_j)^2}\left\{\frac{\partial u}{\partial t}(x_{j+1}, t_n) + \frac{\partial u}{\partial t}(x_{j+1}, t_{n+1}) - 2\left(\frac{\partial u}{\partial t}(x_j, t_n) + \frac{\partial u}{\partial t}(x_j, t_{n+1})\right) + \frac{\partial u}{\partial t}(x_{j-1}, t_n) + \frac{\partial u}{\partial t}(x_{j-1}, t_{n+1})\right\} \\ &= \frac{\tau_T(x_j)}{2(\Delta x_j)^2}\left\{\frac{2}{\Delta t}(u^{n+1}_{j+1} - u^n_{j+1}) - \frac{4}{\Delta t}(u^{n+1}_j - u^n_j) + \frac{2}{\Delta t}(u^{n+1}_{j-1} - u^n_{j-1})\right\} \\ &= \frac{1}{(\Delta x_j)^2}\left[\frac{\tau_T(x_j)}{\Delta t}(u^{n+1}_{j+1} - u^n_{j+1}) - 2\frac{\tau_T(x_j)}{\Delta t}(u^{n+1}_j - u^n_j) + \frac{\tau_T(x_j)}{\Delta t}(u^{n+1}_{j-1} - u^n_{j-1})\right]. \end{aligned}$$

By the definition of $\mathbb{D}$, we deduce that

$$\begin{aligned} (\mathbb{D}^1_{lay}u)^{n+1/2}_j &:= \frac{1}{2}\Big(\mathbb{D}(x_j, t_n) + \mathbb{D}(x_j, t_{n+1})\Big) \\ &= \frac{k(x_j)}{(\Delta x_j)^2}\Bigg(\left[\frac{1}{2}(u^{n+1}_{j+1} + u^n_{j+1}) - (u^{n+1}_j + u^n_j) + \frac{1}{2}(u^{n+1}_{j-1} + u^n_{j-1})\right] \\ &\quad + \left[\frac{\tau_T(x_j)}{\Delta t}(u^{n+1}_{j+1} - u^n_{j+1}) - 2\frac{\tau_T(x_j)}{\Delta t}(u^{n+1}_j - u^n_j) + \frac{\tau_T(x_j)}{\Delta t}(u^{n+1}_{j-1} - u^n_{j-1})\right]\Bigg), \end{aligned}$$

which implies (33).

For $j \in \mathbb{I}^{int}$, we deduce that

$$(\mathbb{D}^1_{int,+}u)_j^{n+1/2} := \frac{1}{2}\Big(\mathbb{D}(x_j+,t_n) + \mathbb{D}(x_j+,t_{n+1})\Big)$$
$$= \frac{2k(x_j)}{\Delta x_j}\left(\frac{1}{\Delta x_j}\left[\frac{1}{2}(u_{j+1}^{n+1}+u_{j+1}^n) - \frac{1}{2}(u_j^{n+1}+u_j^n) + \frac{\tau_T(x_j)}{\Delta t}(u_{j+1}^{n+1}-u_{j+1}^n) - \frac{\tau_T(x_j)}{\Delta t}(u_j^{n+1}-u_j^n)\right] - \Gamma u(x_j+,t_{n+1})\right),$$
$$(\mathbb{D}^1_{int,-}u)_j^{n+1/2} := \frac{1}{2}\Big(\mathbb{D}(x_j-,t_n) + \mathbb{D}(x_j-,t_{n+1})\Big)$$
$$= \frac{2k(x_{j-1})}{\Delta x_{j-1}}\left(\Gamma u(x_j-,t_{n+1}) - \frac{1}{\Delta x_{j-1}}\left[\frac{1}{2}(u_j^{n+1}+u_j^n) - \frac{1}{2}(u_{j-1}^{n+1}+u_{j-1}^n) + \frac{\tau_T(x_j)}{\Delta t}(u_j^{n+1}-u_j^n) - \frac{\tau_T(x_j)}{\Delta t}(u_{j-1}^{n+1}-u_{j-1}^n)\right]\right).$$

Hence, we obtain relations (34) and (35).

For $j \in \partial\mathbb{I}^{lay}$, we observe that the evaluation of $\mathbb{D}$ at $(x_j\pm,t_n)$ for $x_j \in \partial\mathcal{I}^{lay}_{\Delta x}$ can be developed using relations (34) and (35) with $j = M_0$ and $j = M_3$, respectively. □

Proposition 5. *Consider the assumptions and notation of Proposition 4. Then, $(\mathbb{D}^2_\Delta u)_j^n$, the average of $(\mathbb{D}^1_\Delta u)_j^{n+1/2}$ and $(\mathbb{D}^1_\Delta u)_j^{n-1/2}$, is approximated by*

$$(\mathbb{D}^2_\Delta u)_j^n = \frac{1}{2}\begin{cases} (\mathbb{D}^1_{lay}u)_j^{n+1/2} + (\mathbb{D}^1_{lay}u)_j^{n-1/2}, & j \in \mathbb{I}^{lay}, \\ (\mathbb{D}^1_{int,+}u)_j^{n+1/2} + (\mathbb{D}^1_{int,+}u)_j^{n-1/2}, & j \in \mathbb{I}^{int} \cup \{M_0\}, \\ (\mathbb{D}^1_{int,-}u)_j^{n+1/2} + (\mathbb{D}^1_{int,-}u)_j^{n-1/2}, & j \in \mathbb{I}^{int} \cup \{M_3\}, \end{cases} \tag{39}$$

for all $j \in \mathbb{I}$ and $n \in \{2,\ldots,N-1\}$.

Proof. The calculus of $(\mathbb{D}^2_\Delta u)_j^n$ provided in (39) is a straightforward application of definition $(\mathbb{D}^2_\Delta u)_j^n$ and the relation for $(\mathbb{D}^1_\Delta u)_j^n$ provided in (36). □

4. Numerical Approximation of the System (10)–(15)

4.1. Discretization of the System (10)–(15) for $n = 0$ and $n = 1$

We initialize the discretization of the system by evaluating the initial conditions, i.e., functions ψ_1 and ψ_2 provided in (11), on the mesh $\overline{\mathcal{I}}_{\Delta x}$:

$$\mathbf{u}^0 = (u^0_{M_0},\ldots,u^0_{M_3}),\ \boldsymbol{\delta}\mathbf{u}^0 = (\delta u^0_{M_0},\ldots,\delta u^0_{M_3}),\ u_j^0 = \psi_1(x_j),\ \delta u_j^0 = \psi_2(x_j),\ \ j \in \mathbb{I}. \tag{40}$$

Then, the discretization of the system (10)–(15) at $n = 0$ is provided by $\mathbf{u}^0$.

On the other hand, to provide a simplified presentation of the discretization at $n = 1$, we need to introduce an appropriate notation. In terms of the discretization of boundary conditions (12) and (13) and the physical parameters, we introduce the notation

$$\left.\begin{array}{l}
\boldsymbol{\varphi}_k = (\varphi_k^0,\ldots,\varphi_k^N), \quad \varphi_k^n = \varphi_k(t_n), \quad n \in \{0,1,\ldots,N\}, \quad k \in \{0,1\}; \\
\boldsymbol{\delta\varphi}_k = (\varphi_k'^0,\ldots,\varphi_k'^N), \quad \varphi_k'^n = \varphi_k'(t_n), \quad n \in \{0,1,\ldots,N\}, \quad k \in \{0,1\}; \\
\boldsymbol{\phi}_1 = \boldsymbol{\varphi}_1 + \tau_T(x_{M_0})\boldsymbol{\delta\varphi}_1; \quad \boldsymbol{\phi}_2 = \boldsymbol{\varphi}_2 + \tau_T(x_{M_3})\boldsymbol{\delta\varphi}_2; \\
\mathcal{L}_j^C = \dfrac{C(x_j)}{(\Delta t)^2}\Big(\Delta t + 2\tau_q(x_j)\Big), \quad \mathcal{L}_j^L = 2C(x_j)\dfrac{\tau_q(x_j)}{(\Delta t)^2}, \quad j \in \mathbb{I}; \\
\Psi_j^R = 1 + \dfrac{\tau_T(x_j)}{\Delta t}, \quad \Psi_j^L = -\dfrac{\tau_T(x_j)}{\Delta t}, \quad j \in \mathbb{I}; \\
\mu_j = \dfrac{k(x_j)}{(\Delta x_j)^2}, \quad \Psi_j^\pm = \left(\dfrac{1}{2} \pm \dfrac{\tau_T(x_j)}{\Delta t}\right), \quad j \in \mathbb{I}; \\
\mathcal{Z}_j^L = \dfrac{C(x_j)}{2(\Delta t)^2}\Big(-\Delta t + 2\tau_q(x_j)\Big),\ \mathcal{Z}_j^U = \dfrac{C(x_j)}{2(\Delta t)^2}\Big(\Delta t + 2\tau_q(x_j)\Big), \\
\mathcal{Z}_j^C = \mathcal{Z}_j^L + \mathcal{Z}_j^U, \quad j \in \mathbb{I}.
\end{array}\right\} \tag{41}$$

Lemma 2. *Consider notations (16) and (41) . Then, $\mathbf{u}^1$ is defined as the solution of the linear system*

$$\hat{\mathbb{A}}\mathbf{u}^1 = \hat{\mathbb{B}}\mathbf{u}^0 + \mathbf{s}^0, \tag{42}$$

where $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ are the tridiagonal matrices and $\mathbf{s}^0$ is the vector defined in Table 2.

Table 2. Entries of the tridiagonal matrices $\hat{\mathbb{A}}$, $\hat{\mathbb{B}}$ and the vector $\mathbf{s}^0$ defining the system (42). For $j = M_0$, see (61); for $j \in \mathbb{I}^{lay}$, see (49); for $j \in \mathbb{I}^{int}$, see (52), and, for $j = M_3$, see (62).

			under-diagonal $i = j+1$	diagonal $i = j$	upper-diagonal $i = j-1$
$\hat{a}_{i,j}$	$j = M_0$	(61)		$\mathcal{L}^C_{M_0} + 2\mu_{M_0}\left(1 + \dfrac{\Delta x_{M_0}}{\alpha_1 \overline{K}_1}\right)\Psi^R_{M_0}$	$-2\mu_{M_0}\Psi^R_{M_0}$
	$j \in \mathbb{I}^{lay}$	(49)	$-\mu_j\Psi^R_j$	$\mathcal{L}^C_j + 2\mu_j\Psi^R_j$	$-\mu_j\Psi^R_j$
	$j \in \mathbb{I}^{int}$	(52)	$-2\Delta x_{j-1}\mu_{j-1}\Psi^R_{j-1}$	$\Delta x_{j-1}\left(\mathcal{L}^C_j + 2\mu_{j-1}\Psi^R_j\right) + \Delta x_j\left(\mathcal{L}^C_j + 2\mu_j\Psi^R_j\right)$	$-2\Delta x_j\mu_j\Psi^R_j$
	$j = M_3$	(62)	$2\mu_{M_3-1}\Psi^R_{M_3-1}$	$\mathcal{L}^C_{M_3} + 2\mu_{M_3-1}\left(1 + \dfrac{\Delta x_{M_3-1}}{\alpha_2 \overline{K}_2}\right)\Psi^R_{M_3-1}$	
			under-diagonal $i = j+1$	diagonal $i = j$	upper-diagonal $i = j-1$
$\hat{b}_{i,j}$	$j = M_0$	(61)		$\mathcal{L}^C_{M_0} - 2\mu_{M_0}\left(1 + \dfrac{\Delta x_{M_0}}{\alpha_1 \overline{K}_1}\right)\Psi^L_{M_0}$	$2\mu_{M_0}\Psi^L_{M_0}$
	$j \in \mathbb{I}^{lay}$	(49)	$\mu_j\Psi^L_j$	$\mathcal{L}^C_j - 2\mu_j\Psi^L_j$	$\mu_j\Psi^L_j$
	$j \in \mathbb{I}^{int}$	(52)	$2\Delta x_{j-1}\mu_{j-1}\Psi^L_{j-1}$	$\Delta x_{j-1}\left(\mathcal{L}^C_j - 2\mu_{j-1}\Psi^L_{j-1}\right) + \Delta x_j\left(\mathcal{L}^C_j - 2\mu_j\Psi^L_j\right)$	$2\Delta x_j\mu_j\Psi^L_j$
	$j = M_3$	(62)	$2\mu_{M_3-1}\Psi^L_{M_3-1}$	$\mathcal{L}^C_{M_3} - 2\mu_{M_3-1}\left(1 + \dfrac{\Delta x_{M_3-1}}{\alpha_2 \overline{K}_2}\right)\Psi^L_{M_3-1}$	
s^0_j	$j = M_0$	(61)	$\mathcal{L}^U_{M_0}\Delta t\psi_2(x_{M_0}) + 2\mu_{M_0}\dfrac{\Delta x_{M_0}}{\alpha_1 \overline{K}_1}\phi^1_1 + f^1_{M_0}$		
	$j \in \mathbb{I}^{lay}$	(49)	$\mathcal{L}^U_j\Delta t\psi_2(x_j) + f^1_j$		
	$j \in \mathbb{I}^{int}$	(52)	$\left(\Delta x_{j-1} + \Delta x_j\right)\mathcal{L}^L_j\Delta t\psi_2(x_j) + \Delta x_{j-1}f^1_{j-1} + \Delta x_j f^1_j$		
	$j = M_3$	(62)	$\mathcal{L}^U_{M_3}\Delta t\psi_2(x_{M_3}) + 2\mu_{M_3-1}\dfrac{\Delta x_{M_3-1}}{\alpha_2 \overline{K}_2}\phi^1_2 + f^1_{M_3-1}$		

Proof. We observe that Equation (10) is equivalent to

$$\mathbb{L}u = \mathbb{D}u + f \text{ with } \mathbb{L} \text{ and } \mathbb{R} \text{ defined in (19) and (32).} \tag{43}$$

Then, the discretization of the system (10)–(15) will be developed by discretizing (43) by applying Propositions 1–5 and incorporating appropriately the initial, boundary, and interface conditions (11)–(15). Evaluating (43) at (x_j, t_1), from Propositions 1 and 4, we deduce that the approximation of (43) is provided by

$$(\mathbb{L}^1_\Delta u)^{1/2}_j = (\mathbb{D}^1_\Delta u)^{1/2}_j + f^1_j, \tag{44}$$

where $(\mathbb{L}^1_\Delta u)^{1/2}_j$ and $(\mathbb{D}^1_\Delta u)^{1/2}_j$ are defined in (21) and (29). Then, the rest of the proof will be focused on rewriting (44) in matrix form.

We rewrite $(\mathbb{L}^1_\Delta u)^{1/2}_j$ and $(\mathbb{D}^1_\Delta u)^{1/2}_j$ in notation (41). We begin by $(\mathbb{L}^1_\Delta u)^{1/2}_j$, from (21) and the initial condition (11), and we deduce that

$$\begin{aligned}
(\mathbb{L}^1_\Delta u)^{1/2}_j &= C(x_j)\left(\frac{u^1_j - u^0_j}{\Delta t} + \frac{2\tau_q(x_j)}{\Delta t}\left[\frac{u^1_j - u^0_j}{\Delta t} - \psi_2(x_j)\right]\right) \\
&= C(x_j)\left(1 + \frac{2\tau_q(x_j)}{\Delta t}\right)\left(\frac{u^1_j - u^0_j}{\Delta t}\right) - C(x_j)\frac{2\tau_q(x_j)}{\Delta t}\psi_2(x_j) \\
&= \mathcal{L}^C_j u^1_j - \mathcal{L}^C_j u^0_j - \mathcal{L}^L_j \Delta t\psi_2(x_j), \quad \text{for } j \in \mathbb{I}.
\end{aligned} \tag{45}$$

Now, to rewrite $(\mathbb{D}^1_\Delta u)^{1/2}_j$, we notice that

$$u^1_{j+1} + \frac{\tau_T(x_j)}{\Delta t}(u^1_{j+1} - u^0_{j+1}) = \Psi^R_j u^1_{j+1} + \Psi^L_j u^0_{j+1},$$
$$u^1_j + \frac{\tau_T(x_j)}{\Delta t}(u^1_j - u^0_j) = \Psi^R_j u^1_j + \Psi^L_j u^0_j,$$
$$u^1_{j-1} + \frac{\tau_T(x_j)}{\Delta t}(u^1_{j-1} - u^0_{j-1}) = \Psi^R_j u^1_{j-1} + \Psi^L_j u^0_{j-1}$$

which implies that $(\mathbb{D}^1_{lay} u)^{1/2}_j$, $(\mathbb{D}^1_{int,\pm} u)^{1/2}_j$ defined in (26)–(28) can be expressed as follows

$$(\mathbb{D}^1_{lay} u)^{1/2}_j = \mu_j\Big(\Psi^R_j u^1_{j-1} - 2\Psi^R_j u^1_j + \Psi^R_j u^1_{j+1} + \Psi^L_j u^0_{j-1} - 2\Psi^L_j u^0_j + \Psi^L_j u^0_{j+1}\Big), \quad j \in \mathbb{I}^{lay}, \tag{46}$$

$$(\mathbb{D}^1_{int,+} u)^{1/2}_j = 2\mu_j\Big(\Psi^R_j u^1_{j+1} - \Psi^R_j u^1_j + \Psi^L_j u^0_{j+1} - \Psi^L_j u^0_j\Big) - \frac{2\,k(x_j)}{\Delta x_j}\left(\frac{\partial u}{\partial x} + \tau_T(x_j)\frac{\partial^2 u}{\partial t \partial x}\right)(x_j+, t_1), \quad j \in \mathbb{I}^{int} \cup \{M_0\}, \tag{47}$$

$$(\mathbb{D}^1_{int,-} u)^{1/2}_j = \frac{2\,k(x_{j-1})}{\Delta x_{j-1}}\left(\frac{\partial u}{\partial x} + \tau_T(x_{j-1})\frac{\partial^2 u}{\partial t \partial x}\right)(x_j-, t_1) - 2\mu_{j-1}\Big(\Psi^R_{j-1} u^1_j - \Psi^R_{j-1} u^1_{j-1} + \Psi^L_{j-1} u^0_j - \Psi^L_{j-1} u^0_{j-1}\Big), \quad j \in \mathbb{I}^{int} \cup \{M_3\}. \tag{48}$$

Then, $(\mathbb{D}^1_\Delta u)^{1/2}_j$ is rewritten in notation (41) by (46)–(48).

In order to rewrite (44) in notation (41), we consider three cases: $j \in \mathbb{I}^{lay}$, $j \in \mathbb{I}^{int}$, and $j \in \partial\mathbb{I}$. In the first case, i.e., $j \in \mathbb{I}^{lay}$, from (44)–(46), we deduce that

$$-\mu_j \Psi^R_j u^1_{j-1} + \Big(\mathcal{L}^C_j + 2\mu_j \Psi^R_j\Big)u^1_j - \mu_j \Psi^R_j u^1_{j+1} = \mu_j \Psi^L_j u^0_{j-1} + (\mathcal{L}^L_j - 2\mu_j \Psi^L_j)u^0_j + \mu_j \Psi^L_j u^0_{j+1} + \mathcal{L}^L_j \Delta t \psi_2(x_j) + f^1_j, \quad j \in \mathbb{I}^{lay}. \tag{49}$$

For the second case, i.e., $j \in \mathbb{I}^{int}$, from (44), (45), (47), and (48), we obtain

$$\begin{aligned}
&\Delta x_j\Big[\mathcal{L}^C_j u^1_j - \mathcal{L}^C_j u^0_j - \Delta t \mathcal{L}^L_j \psi_2(x_j)\Big] + \Delta x_{j-1}\Big[\mathcal{L}^C_j u^1_j - \mathcal{L}^C_j u^0_j - \Delta t \mathcal{L}^L_j \psi_2(x_j)\Big] \\
&= \Delta x_j\Big[2\mu_j\Big(\Psi^R_j u^1_{j+1} - \Psi^R_j u^1_j + \Psi^L_j u^0_{j+1} - \Psi^L_j u^0_j\Big)\Big] - 2\,k(x_j)\left(\frac{\partial u}{\partial x} + \tau_T(x_j)\frac{\partial^2 u}{\partial t \partial x}\right)(x_j+, t_1) \\
&\quad - \Delta x_{j-1}\Big[2\mu_{j-1}\Big(\Psi^R_{j-1} u^1_j - \Psi^R_{j-1} u^1_{j-1} + \Psi^L_{j-1} u^0_j - \Psi^L_{j-1} u^0_{j-1}\Big)\Big] \\
&\quad + 2\,k(x_{j-1})\left(\frac{\partial u}{\partial x} + \tau_T(x_j)\frac{\partial^2 u}{\partial t \partial x}\right)(x_j-, t_1) + \Delta x_j f^1_j + \Delta x_{j-1} f^1_{j-1}, \quad \text{for } j \in \mathbb{I}^{int}.
\end{aligned} \tag{50}$$

We observe that boundary condition (15) implies the following identity

$$-2k(x_j)\left(\frac{\partial u}{\partial x} + \tau_T(x_j)\frac{\partial^2 u}{\partial t \partial x}\right)(x_j+, t_n) + 2k(x_{j-1})\left(\frac{\partial u}{\partial x} + \tau_T(x_j)\frac{\partial^2 u}{\partial t \partial x}\right)(x_j-, t_n) = 0, \quad j \in \mathbb{I}^{int},\ n \in \{0, \ldots, N\}. \tag{51}$$

Consequently, (50) is equivalent to

$$\begin{aligned}
&-2\Delta x_{j-1}\mu_{j-1}\Psi^R_{j-1} u^1_{j-1} + \Big[\Delta x_{j-1}\Big(\mathcal{L}^C_j + 2\mu_{j-1}\Psi^R_{j-1}\Big) + \Delta x_j\Big(\mathcal{L}^C_j + 2\mu_j \Psi^R_j\Big)\Big]u^1_j - 2\Delta x_j \mu_j \Psi^R_j u^1_{j+1} \\
&= 2\Delta x_{j-1}\mu_{j-1}\Psi^L_{j-1} u^0_{j-1} + \Big[\Delta x_{j-1}\Big(\mathcal{L}^C_j - 2\mu_{j-1}\Psi^L_{j-1}\Big) + \Delta x_j\Big(\mathcal{L}^C_j - 2\mu_j \Psi^L_j\Big)\Big]u^0_j + 2\Delta x_j \mu_j \Psi^L_j u^0_{j+1} \\
&\quad + \Big(\Delta x_{j-1} + \Delta x_j\Big)\mathcal{L}^L_j \Delta t \psi_2(x_j) + \Delta x_{j-1} f^1_{j-1} + \Delta x_j f^1_j, \quad j \in \mathbb{I}^{int}.
\end{aligned} \tag{52}$$

Before deducing the discretization for the third case, we obtain some relations for the boundary condition. Boundary conditions (12) and (13) imply that

$$\frac{\partial u}{\partial x}(L_0,t) = -\frac{1}{\alpha_1\overline{K}_1}\Big(\varphi_1(t) - u(L_0,t)\Big), \quad \frac{\partial^2 u}{\partial x\partial t}(L_0,t) = -\frac{1}{\alpha_1\overline{K}_1}\Big(\varphi_1'(t) - \frac{\partial u}{\partial t}(L_0,t)\Big), \tag{53}$$

$$\frac{\partial u}{\partial x}(L_3,t) = \frac{1}{\alpha_2\overline{K}_2}\Big(\varphi_2(t) - u(L_3,t)\Big), \quad \frac{\partial^2 u}{\partial x\partial t}(L_3,t) = \frac{1}{\alpha_2\overline{K}_2}\Big(\varphi_2'(t) - \frac{\partial u}{\partial t}(L_3,t)\Big), \tag{54}$$

and we can deduce the following relations

$$\left(\frac{\partial u}{\partial x} + \tau_T(L_0)\frac{\partial^2 u}{\partial t\partial x}\right)(L_0+,t) = \frac{1}{\alpha_1\overline{K}_1}\left(u + \tau_T(L_0)\frac{\partial u}{\partial t}(L_0,t) - \varphi_1(t) - \tau_T(L_0)\varphi_1'(t)\right), \tag{55}$$

$$\left(\frac{\partial u}{\partial x} + \tau_T(L_3)\frac{\partial^2 u}{\partial t\partial x}\right)(L_3-,t) = \frac{1}{\alpha_2\overline{K}_2}\left(\varphi_2(t) + \tau_T(L_3)\varphi_2'(t) - u - \tau_T(L_3)\frac{\partial u}{\partial t}(L_3,t)\right). \tag{56}$$

Then, using the fact that $\varphi_1^0 + \tau_T(M_0)\delta\varphi_1^0 = \phi_1^0$ and $\varphi_2^0 + \tau_T(M_3)\delta\varphi_2^0 = \phi_2^0$, the discretization for $j \in \partial\mathbb{I}$ is provided by

$$\left(\frac{\partial u}{\partial x} + \tau_T(L_0+)\frac{\partial^2 u}{\partial t\partial x}\right)(L_0+,t) = \frac{1}{\alpha_1\overline{K}_1}\Big(\Psi^R_{M_0}u^1_{M_0} + \Psi^L_{M_0}u^0_{M_0} - \phi_1^1\Big), \tag{57}$$

$$\left(\frac{\partial u}{\partial x} + \tau_T(L_3)\frac{\partial^2 u}{\partial t\partial x}\right)(L_3-,t) = \frac{1}{\alpha_2\overline{K}_2}\Big(\phi_2^1 - \Big[\Psi^R_{M_3-1}u^1_{M_3} + \Psi^L_{M_3-1}u^0_{M_3}\Big]\Big). \tag{58}$$

We notice that, replacing (57) and (58) in relations (47) and (48), we deduce that

$$\begin{aligned}
(\mathbb{D}^1_{int,+}u)^{1/2}_j &= 2\mu_j\Big(\Psi^R_j u^1_{j+1} - \Psi^R_j u^1_j + \Psi^L_j u^0_{j+1} - \Psi^L_j u^0_j\Big) - \frac{2\mu_j\Delta x_j}{\alpha_1\overline{K}_1}\Big(\Psi^R_j u^1_j + \Psi^L_j u^0_j - \phi_1^1\Big)\\
&= 2\mu_j\left(\Psi^R_j u^1_{j+1} - \left[1 + \frac{\Delta x_j}{\alpha_1\overline{K}_1}\right]\Psi^R_j u^1_j + \Psi^L_j u^0_{j+1} - \left[1 + \frac{\Delta x_j}{\alpha_1\overline{K}_1}\right]\Psi^L_j u^0_j\right) + \frac{2\mu_j\Delta x_j}{\alpha_1\overline{K}_1}\phi_1^1, \quad j = M_0,
\end{aligned} \tag{59}$$

$$\begin{aligned}
(\mathbb{D}^1_{int,-}u)^{1/2}_j &= \frac{2\mu_{j-1}\Delta x_{j-1}}{\alpha_2\overline{K}_2}\Big(\phi_2^1 - \Big[\Psi^R_{j-1}u^1_j + \Psi^L_{j-1}u^0_j\Big]\Big) - 2\mu_{j-1}\Big(\Psi^R_{j-1}u^1_j - \Psi^R_{j-1}u^1_{j-1} + \Psi^L_{j-1}u^0_j - \Psi^L_{j-1}u^0_{j-1}\Big)\\
&= -2\mu_{j-1}\left(\left[1 + \frac{\Delta x_{j-1}}{\alpha_2\overline{K}_2}\right]\Psi^R_{j-1}u^1_j - \Psi^R_{j-1}u^1_{j-1} + \left[1 + \frac{\Delta x_{j-1}}{\alpha_2\overline{K}_2}\right]\Psi^L_{j-1}u^0_j - \Psi^L_{j-1}u^0_{j-1}\right) + \frac{2\mu_{j-1}\Delta x_{j-1}}{\alpha_2\overline{K}_2}\phi_2^1, \quad j = M_3.
\end{aligned} \tag{60}$$

Then, from (45), (59), and (60), we deduce that (44), for $j \in \partial\mathbb{I}^{lay}$, is provided by the following expressions

$$\begin{aligned}
&\mathcal{L}^C_{M_0}u^1_{M_0} - \mathcal{L}^C_{M_0}u^0_{M_0} - \mathcal{L}^L_{M_0}\Delta t\psi_2(x_{M_0})\\
&\quad = 2\mu_{M_0}\left(\Psi^R_{M_0}u^1_{M_0+1} - \left[1 + \frac{\Delta x_{M_0}}{\alpha_1\overline{K}_1}\right]\Psi^R_{M_0}u^1_{M_0} + \Psi^L_{M_0}u^0_{M_0+1} - \left[1 + \frac{\Delta x_{M_0}}{\alpha_1\overline{K}_1}\right]\Psi^L_{M_0}u^0_{M_0}\right) + \frac{2\mu_{M_0}\Delta x_{M_0}}{\alpha_1\overline{K}_1}\phi_1^0 + f^1_{M_0},\\
&\mathcal{L}^C_{M_3}u^1_{M_3} - \mathcal{L}^C_{M_3}u^0_{M_3} - \mathcal{L}^L_{M_3}\Delta t\psi_2(x_{M_3})\\
&\quad = -2\mu_{M_3-1}\left(\left[1 + \frac{\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\right]\Psi^R_{M_3-1}u^1_{M_3} - \Psi^R_{M_3-1}u^1_{M_3-1} + \left[1 + \frac{\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\right]\Psi^L_{M_3-1}u^0_{M_3} - \Psi^L_{M_3-1}u^0_{M_3-1}\right)\\
&\qquad + \frac{2\mu_{M_3-1}\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\phi_2^0 + f^1_{M_3-1}.
\end{aligned}$$

Consequently, (59) and (60) can be rewritten as follows

$$\begin{aligned}
&\left[\mathcal{L}^C_{M_0} + 2\mu_{M_0}\left(1 + \frac{\Delta x_{M_0}}{\alpha_1\overline{K}_1}\right)\psi^R_{M_0}\right]u^1_{M_0} - 2\mu_{M_0}\psi^R_{M_0}u^1_{M_0+1}\\
&\qquad = \left[\mathcal{L}^C_{M_0} - 2\mu_{M_0}\left(1 + \frac{\Delta x_{M_0}}{\alpha_1\overline{K}_1}\right)\psi^L_{M_0}\right]u^0_{M_0} + 2\mu_{M_0}\psi^L_{M_0+1}u^0_{M_0+1}
\end{aligned}$$

$$+\mathcal{L}^L_{M_0}\Delta t\psi_2(x_{M_0}) + 2\mu_{M_0}\frac{\Delta x_{M_0}}{\alpha_1\overline{K}_1}\phi_1^1 + f^1_{M_0} \tag{61}$$

$$\begin{aligned} -2\mu_{M_3-1}\psi^R_{M_3-1}u^1_{M_3-1} &+ \left[\mathcal{L}^C_{M_3} + 2\mu_{M_3-1}\left(1+\frac{\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\right)\psi^R_{M_3-1}\right]u^1_{M_3} \\ &= 2\mu_{M_3-1}\psi^L_{M_3-1}u^0_{M_3-1} + \left[\mathcal{L}^C_{M_3} - 2\mu_{M_3-1}\left(1+\frac{\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\right)\psi^L_{M_3-1}\right]u^0_{M_3} \\ &+ \mathcal{L}^L_{M_3}\Delta t\psi_2(x_{M_3}) + 2\mu_{M_3-1}\frac{\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\phi_2^1 + f^1_{M_3-1}. \end{aligned} \tag{62}$$

In conclusion, $\mathbf{u}^1$ is defined from the system provided by relations (44), (50), (59), and (60) or equivalently by (49), (52), (61), and (62), which are clearly rewritten in a matrix form as the liner system (42). □

4.2. Discretization of System (10)–(15) for $n \geq 2$

Lemma 3. *Consider notations* (16) *and* (41)*. Then,* $\mathbf{u}^{n+1}$ *is defined as the solution of the linear system*

$$\mathbb{A}\mathbf{u}^{n+1} = \mathbb{B}\mathbf{u}^n + \mathbb{C}\mathbf{u}^{n-1} + \mathbf{s}^n, \tag{63}$$

where $\mathbb{A}, \mathbb{B}$, *and* $\mathbb{C}$ *are the tridiagonal matrices and* $\mathbf{s}^n$ *is the vector defined in Table 3.*

Table 3. Entries of the tridiagonal matrices $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and the vector $\mathbf{s}^n$ defining the system (42). For $j = M_0$, see (75); for $j \in \mathbb{I}^{lay}$, see (73); for $j \in \mathbb{I}^{int}$, see (74) and (76), and, for $j = M_3$, see (76).

			under-diagonal $i=j+1$	diagonal $i=j$	upper-diagonal $i=j-1$
$a_{i,j}$	$j=M_0$	(75)		$\mathcal{Z}^U_{M_0} + \mu_{M_0}\left(1+\frac{\Delta x_{M_0}}{\alpha_1\overline{K}_1}\right)\Psi^+_{M_0}$	$-\mu_{M_0}\Psi^+_{M_0}$
	$j\in\mathbb{I}^{lay}$	(73)	$-\frac{1}{2}\mu_j\Psi^+_j$	$\mathcal{Z}^U_j + \mu_j\Psi^+_j$	$-\frac{1}{2}\mu_j\Psi^+_j$
	$j\in\mathbb{I}^{int}$	(74)	$-\Delta x_{j-1}\mu_{j-1}\Psi^+_{j-1}$	$\Delta x_{j-1}\left(\mathcal{Z}^U_j + \mu_{j-1}\Psi^+_{j-1}\right) + \Delta x_j\left(\mathcal{Z}^U_j + \mu_j\Psi^+_j\right)$	$-\Delta x_j\mu_j\Psi^+_j$
	$j=M_3$	(76)	$-\mu_{M_3-1}\Psi^+_{M_3-1}$	$\mathcal{Z}^U_{M_3} + \mu_{M_3-1}\left(1+\frac{\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\right)\Psi^+_{M_3-1}$	
			under-diagonal $i=j+1$	diagonal $i=j$	upper-diagonal $i=j-1$
$b_{i,j}$	$j=M_0$	(75)		$\mathcal{Z}^C_{M_0} - \mu_{M_0}\left(1+\frac{\Delta x_{M_0}}{\alpha_1\overline{K}_1}\right)$	μ_{M_0}
	$j\in\mathbb{I}^{lay}$	(73)	$\frac{1}{2}\mu_j$	$\mathcal{Z}^C_j - \mu_j$	$\frac{1}{2}\mu_j$
	$j\in\mathbb{I}^{int}$	(74)	$\Delta x_{j-1}\mu_{j-1}$	$\Delta x_{j-1}\left(\mathcal{Z}^C_j - \mu_{j-1}\right) + \Delta x_j\left(\mathcal{Z}^C_j - \mu_j\right)$	$\Delta x_j\mu_j$
	$j=M_3$	(76)	μ_{M_3-1}	$\mathcal{Z}^C_{M_3} - \mu_{M_3-1}\left(1+\frac{\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\right)$	
			under-diagonal $i=j+1$	diagonal $i=j$	upper-diagonal $i=j-1$
$c_{i,j}$	$j=M_0$	(75)		$-\left[\mathcal{Z}^L_{M_0} + \mu_{M_0}\left(1-\frac{\Delta x_{M_0}}{\alpha_1\overline{K}_1}\right)\Psi^-_{M_0}\right]$	$\mu_{M_0}\Psi^-_{M_0}$
	$j\in\mathbb{I}^{lay}$	(73)	$\frac{1}{2}\mu_j\Psi^-_j$	$-\left[\mathcal{Z}^L_j + \mu_j\Psi^-_j\right]$	$\frac{1}{2}\mu_j\Psi^-_j$
	$j\in\mathbb{I}^{int}$	(74)	$\Delta x_{j-1}\mu_{j-1}\Psi^-_{j-1}$	$-\left[\Delta x_{j-1}\left(\mathcal{Z}^L_j + \mu_{j-1}\Psi^-_{j-1}\right) + \Delta x_j\left(\mathcal{Z}^L_j + \mu_j\Psi^-_j\right)\right]$	$\Delta x_j\mu_j\Psi^-_j$
	$j=M_3$	(76)	$\mu_{M_3}\Psi^-_{M_3-1}$	$-\left[\mathcal{Z}^L_{M_3} + \mu_{M_3-1}\left(1+\frac{\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\right)\Psi^-_{M_3}\right]$	

Table 3. *Cont.*

s_j^n	$j = M_0$	(75)	$\mu_{M_0}\dfrac{\Delta x_{M_0}}{2\alpha_1\overline{K}_1}\left(\phi_1^{n-1}+2\phi_1^n+\phi_1^{n+1}\right)+\dfrac{1}{4}\left(f_{M_0}^{n-1}+2f_{M_0}^n+f_{M_0}^{n+1}\right)$
	$j \in \mathbb{I}^{lay}$	(73)	$\dfrac{1}{4}(f_j^{n-1}+2f_j^n+f_j^{n+1})$
	$j \in \mathbb{I}^{int}$	(74)	$\dfrac{\Delta x_{j-1}}{4}(f_{j-1}^{n-1}+2f_{j-1}^n+f_{j-1}^{n+1})+\dfrac{\Delta x_j}{4}(f_j^{n-1}+2f_j^n+f_j^{n+1})$
	$j = M_3$	(76)	$\mu_{M_3-1}\dfrac{\Delta x_{M_3-1}}{2\alpha_2\overline{K}_2}\left(\phi_2^{n-1}+2\phi_2^n+\phi_2^{n+1}\right)+\dfrac{1}{4}(f_{M_3}^{n-1}+2f_{M_3}^n+f_{M_3}^{n+1})$

Proof. We consider that Equation (10) is rewritten as (43). Evaluating both sides of (43) at (x_j, t_n) and (x_j, t_{n+1}) and averaging the results, by Propositions 1 and 4, we obtain

$$(\mathbb{L}_\Delta^1 u)_j^{n+1/2} = (\mathbb{D}_\Delta^1 u)_j^{n+1/2} + \frac{1}{2}(f_j^n + f_j^{n+1}), \quad j \in \mathbb{I}, \quad n = 1, \dots, N-1, \tag{64}$$

where the expressions for $(\mathbb{L}_\Delta^1 u)_j^{n+1/2}$ and $(\mathbb{D}_\Delta^1 u)_j^{n+1/2}$ are provided by (20) and (36), respectively. Then, if we consider Equation (64) at $n-1/2$ and $n+1/2$ and average the results, we obtain

$$(\mathbb{L}_\Delta^2 u)_j^n = (\mathbb{D}_\Delta^2 u)_j^n + \frac{1}{4}(f_j^{n-1} + 2f_j^n + f_j^{n+1}), \quad j \in \mathbb{I}, \quad n = 2, \dots, N-1, \tag{65}$$

where $(\mathbb{L}_\Delta^2 u)_j^n$ and $(\mathbb{D}_\Delta^2 u)_j^n$ are provided by (24) and (39), respectively.

We rewrite $(\mathbb{L}_\Delta^2 u)_j^n$ in notation (41)

$$\begin{aligned}(\mathbb{L}_\Delta^2 u)_j^n &= C(x_j)\left(\frac{1}{2\Delta t} + \frac{\tau_q(x_j)}{(\Delta t)^2}\right)u_j^{n+1} - 2C(x_j)\frac{\tau_q(x_j)}{(\Delta t)^2}u_j^n + C(x_j)\left(-\frac{1}{2\Delta t} + \frac{\tau_q(x_j)}{(\Delta t)^2}\right)u_j^{n+1} \\ &= \mathcal{Z}_j^U u_j^{n+1} - \mathcal{Z}_j^C u_j^n + \mathcal{Z}_j^L u_j^{n-1}.\end{aligned} \tag{66}$$

Meanwhile, to rewritte $(\mathbb{D}_\Delta^2 u)_j^n$ in notation (41), we begin by rewriting (33)–(35) using notation (41), i.e.,

$$\begin{aligned}(\mathbb{D}_{lay}^1 u)_j^{n+1/2} &= \mu_j\Big(\Psi_j^+ u_{j-1}^{n+1} - 2\Psi_j^+ u_j^{n+1} + \Psi_j^+ u_{j+1}^{n+1} + \Psi_j^- u_{j-1}^n - 2\Psi_j^- u_j^n + \Psi_j^- u_{j+1}^n\Big), \\ (\mathbb{D}_{int,+}^1 u)_j^{n+1/2} &= 2\mu_j\Big(-\Psi_j^+ u_j^{n+1} + \Psi_j^+ u_{j+1}^{n+1} - \Psi_j^- u_j^n + \Psi_j^- u_{j+1}^n\Big) - \frac{2\,k(x_j)}{\Delta x_j}\Gamma u(x_j+, t_{n+1}), \\ (\mathbb{D}_{int,-}^1 u)_j^{n+1/2} &= 2\mu_{j-1}\Big(\Psi_{j-1}^+ u_{j-1}^{n+1} - \Psi_{j-1}^+ u_j^{n+1} + \Psi_{j-1}^- u_{j-1}^n - \Psi_{j-1}^- u_j^n\Big) + \frac{2\,k(x_{j-1})}{\Delta x_{j-1}}\Gamma u(x_j-, t_{n+1}),\end{aligned}$$

for $j \in \mathbb{I}^{lay}$, $j \in \mathbb{I}^{int} \cup \{M_0\}$, and $j \in \mathbb{I}^{int} \cup \{M_3\}$, respectively. Here, Γ is the notation provided in (32). Then, using relation (39) and the fact that $\Psi_j^- + \Psi_j^+ = 1$ for all $j \in \mathbb{I}$, we obtain that

$$\begin{aligned}(\mathbb{D}_{lay}^2 u)_j^n &= \frac{1}{2}\mu_j\Psi_j^+ u_{j-1}^{n+1} - \mu_j\Psi_j^+ u_j^{n+1} + \frac{1}{2}\mu_j\Psi_j^+ u_{j+1}^{n+1} + \frac{1}{2}\mu_j u_{j-1}^n - \mu_j u_j^n \\ &\quad + \frac{1}{2}\mu_j u_{j+1}^n + \frac{1}{2}\mu_j\Psi_j^- u_{j-1}^{n-1} - \mu_j\Psi_j^- u_j^{n-1} + \frac{1}{2}\mu_j\Psi_j^- u_{j+1}^{n-1}, \qquad j \in \mathbb{I}^{lay}; \quad (67) \\ (\mathbb{D}_{int,+}^2 u)_j^n &= -\mu_j\Psi_j^+ u_j^{n+1} + \mu_j\Psi_j^+ u_{j+1}^{n+1} - \mu_j u_j^n + \mu_j u_{j+1}^n - \mu_j\Psi_j^- u_j^{n-1} + 2\mu_j\Psi_j^- u_{j+1}^{n-1} \\ &\quad - \frac{k(x_j)}{\Delta x_j}\Big(\Gamma u(x_j+, t^{n+1/2}) + \Gamma u(x_j+, t^{n-1/2})\Big), \qquad j \in \mathbb{I}^{int} \cup \{M_0\}; \quad (68) \\ (\mathbb{D}_{int,-}^2 u)_j^n &= \mu_{j-1}\Psi_{j-1}^+ u_{j-1}^{n+1} - \mu_{j-1}\Psi_j^+ u_j^{n+1} + \mu_{j-1}u_{j-1}^n - \mu_{j-1}u_j^n + \mu_{j-1}\Psi_{j-1}^- u_{j-1}^{n-1} \\ &\quad - \mu_{j-1}\Psi_{j-1}^- u_j^{n-1} \qquad (69)\end{aligned}$$

$$+\frac{k(x_{j-1})}{\Delta x_{j-1}}\Big(\Gamma u(x_j-,t^{n+1/2})+\Gamma u(x_j+,t^{n-1/2})\Big),\qquad j\in\mathbb{I}^{int}\cup\{M_3\}.\quad(70)$$

Moreover, similarly to (57)–(58), at the boundaries, we obtain the following relations

$$\begin{aligned}\Gamma u(x_{M_0}+,t^{n+1/2})&=\frac{1}{\alpha_1\overline{K}_1}\left(\Psi^+_{M_0}u^{n+1}_{M_0}+\Psi^-_{M_0}u^n_{M_0}-\left[\frac{1}{2}\left(\varphi^n_1+\varphi^{n+1}_1\right)+\frac{\tau_T(M_0)}{2}\left(\delta\varphi^n_1+\delta\varphi^{n+1}_1\right)\right]\right)\\&=\frac{1}{\alpha_1\overline{K}_1}\Psi^+_{M_0}u^{n+1}_{M_0}+\frac{1}{\alpha_1\overline{K}_1}\Psi^-_{M_0}u^n_{M_0}-\frac{1}{2\alpha_1\overline{K}_1}\left(\phi^n_1+\phi^{n+1}_1\right),\\\Gamma u(x_{M_3}-,t^{n+1/2})&=\frac{1}{\alpha_2\overline{K}_2}\left(\left[\frac{1}{2}\left(\varphi^n_2+\varphi^{n+1}_2\right)+\frac{\tau_T(M_3)}{2}\left(\delta\varphi^n_2+\delta\varphi^{n+1}_2\right)\right]-\left[\Psi^+_{M_3-1}u^{n+1}_{M_3}+\Psi^-_{M_3-1}u^n_{M_3}\right]\right)\\&=\frac{1}{2\alpha_2\overline{K}_2}\left(\phi^n_2+\phi^{n+1}_2\right)-\frac{1}{\alpha_2\overline{K}_2}\Psi^+_{M_3-1}u^{n+1}_{M_3}-\frac{1}{\alpha_2\overline{K}_2}\Psi^-_{M_3-1}u^n_{M_3},\end{aligned}$$

which implies that

$$\begin{aligned}&\Gamma u(x_{M_0}+,t^{n+1/2})+\Gamma u(x_{M_0}+,t^{n-1/2})\\&=\frac{1}{\alpha_1\overline{K}_1}\Psi^+_{M_0}u^{n+1}_{M_0}+\frac{1}{\alpha_1\overline{K}_1}u^n_{M_0}+\frac{1}{\alpha_1\overline{K}_1}\Psi^-_{M_0}u^{n-1}_{M_0}-\frac{1}{2\alpha_1\overline{K}_1}\left(\phi^{n+1}_1+2\phi^n_1+\phi^{n-1}_1\right),\qquad(71)\\&\Gamma u(x_{M_3}+,t^{n+1/2})+\Gamma u(x_{M_3}+,t^{n-1/2})\\&=-\frac{1}{\alpha_2\overline{K}_2}\Psi^+_{M_3-1}u^{n+1}_{M_3}-\frac{1}{\alpha_2\overline{K}_2}u^n_{M_3-1}-\frac{1}{\alpha_2\overline{K}_2}\Psi^-_{M_3-1}u^{n-1}_{M_3}+\frac{1}{2\alpha_2\overline{K}_2}\left(\phi^{n+1}_2+2\phi^n_2+\phi^{n-1}_2\right).\qquad(72)\end{aligned}$$

We notice that we can discretize the boundary terms by replacing (71) in (68) with $j=M_0$ and (72) in (70) with $j=M_3$.

We consider the explicit expressions for (65) in notation (41) by studying three cases: (i) $j\in\mathbb{I}^{lay}$, (ii) $j\in\mathbb{I}^{int}$, and (iii) $j\in\partial\mathbb{I}$. In case (i), i.e., $j\in\mathbb{I}^{lay}$, Equation (65) can be rewritten by using (66) and (67), and after simplifying we obtain

$$\begin{aligned}&-\frac{1}{2}\mu_j\Psi^+_{j-1}u^{n+1}_{j-1}+\left[\mathcal{Z}^U_j+\mu_j\Psi^+_j\right]u^{n+1}_j-\frac{1}{2}\mu_j\Psi^+_{j+1}u^{n+1}_{j+1}\\&=\frac{1}{2}\mu_ju^n_{j-1}+\left[\mathcal{Z}^C_j-\mu_j\right]u^n_j+\frac{1}{2}\mu_ju^n_{j+1}+\frac{1}{2}\mu_j\Psi^-_{j-1}u^{n-1}_{j-1}-\left[\mathcal{Z}^L_j+\mu_j\Psi^-_j\right]u^{n-1}_j\\&\quad+\frac{1}{2}\mu_j\Psi^-_{j+1}u^{n-1}_{j+1}+\frac{1}{4}(f^{n-1}_j+2f^n_j+f^{n+1}_j),\quad\text{for } j\in\mathbb{I}^{lay}.\qquad(73)\end{aligned}$$

For $j\in\mathbb{I}^{int}$ (case (ii)), from (65) and (66), we deduce the following relation

$$\begin{aligned}&\Delta x_{j-1}\left[\mathcal{Z}^L_ju^{n-1}_j-\mathcal{Z}^C_ju^n_j+\mathcal{Z}^U_ju^{n+1}_j\right]+\Delta x_j\left[\mathcal{Z}^L_ju^{n-1}_j-\mathcal{Z}^C_ju^n_j+\mathcal{Z}^U_ju^{n+1}_j\right]\\&=\Delta x_{j-1}(\mathbb{D}^2_{int,-}u)^n_j+\Delta x_j(\mathbb{D}^2_{int,+}u)^n_j+\frac{\Delta x_{j-1}}{4}(f^{n-1}_{j-1}+2f^n_{j-1}+f^{n+1}_{j-1})\\&\quad+\frac{\Delta x_j}{4}(f^{n-1}_j+2f^n_j+f^{n+1}_j),\quad j\in\mathbb{I}^{int},\quad n=2,\dots,N-1.\end{aligned}$$

Then, using the relations in (68) and (70) and applying (51), we deduce the following discretization at the interfaces

$$\begin{aligned}&-\Delta x_{j-1}\mu_{j-1}\Psi^+_{j-1}u^{n+1}_{j-1}+\left[\Delta x_{j-1}\left(\mathcal{Z}^U_j+\mu_{j-1}\Psi^+_{j-1}\right)+\Delta x_j\left(\mathcal{Z}^U_j+\mu_j\Psi^+_j\right)\right]u^{n+1}_j-\Delta x_j\mu_j\Psi^+_ju^{n+1}_{j+1}\\&=\Delta x_{j-1}\mu_{j-1}u^n_{j-1}+\left[\Delta x_{j-1}\left(\mathcal{Z}^C_j-\mu_{j-1}\right)+\Delta x_j\left(\mathcal{Z}^C_j-\mu_j\right)\right]u^n_j+\Delta x_j\mu_ju^n_{j+1}\\&\quad+\Delta x_{j-1}\mu_{j-1}\Psi^-_{j-1}u^{n-1}_{j-1}-\left[\Delta x_{j-1}\left(\mathcal{Z}^L_j+\mu_{j-1}\Psi^-_{j-1}\right)+\Delta x_j\left(\mathcal{Z}^L_j+\mu_j\Psi^-_j\right)\right]u^{n-1}_j+\Delta x_j\mu_j\Psi^-_{j+1}u^{n-1}_{j+1}\\&\quad+\frac{\Delta x_{j-1}}{4}(f^{n-1}_{j-1}+2f^n_{j-1}+f^{n+1}_{j-1})+\frac{\Delta x_j}{4}(f^{n-1}_j+2f^n_j+f^{n+1}_j),\quad j\in\mathbb{I}^{int},\quad n=2,\dots,N-1.\qquad(74)\end{aligned}$$

Meanwhile, from (65), (68), and (71), we obtain

$$\left[\mathcal{Z}^U_{M_0}+\mu_{M_0}\left(1+\frac{\Delta x_{M_0}}{\alpha_1\overline{K}_1}\right)\Psi^+_{M_0}\right]u^{n+1}_{M_0}-\mu_{M_0}\Psi^+_{M_0+1}u^{n+1}_{M_0+1}$$
$$=\left[\mathcal{Z}^C_{M_0}-\mu_{M_0}\left(1+\frac{\Delta x_{M_0}}{\alpha_1\overline{K}_1}\right)\right]u^n_{M_0}+\mu_{M_0}u^n_{M_0+1}-\left[\mathcal{Z}^L_{M_0}+\mu_{M_0}\left(1+\frac{\Delta x_{M_0}}{\alpha_1\overline{K}_1}\right)\Psi^-_{M_0}\right]u^{n-1}_{M_0}+\mu_{M_0}\Psi^-_{M_0+1}u^{n-1}_{M_0+1}$$
$$+\mu_{M_0}\frac{\Delta x_{M_0}}{2\alpha_1\overline{K}_1}\left(\phi^{n-1}_1+2\phi^n_1+\phi^{n+1}_1\right)+\frac{1}{4}\left(f^{n-1}_{M_0}+2f^n_{M_0}+f^{n+1}_{M_0}\right), \tag{75}$$

and from (65), (70), and (72), we deduce that

$$-\mu_{M_3-1}\Psi^+_{M_3-1}u^{n+1}_{M_3-1}+\left[\mathcal{Z}^U_{M_3}+\mu_{M_3-1}\left(1+\frac{\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\right)\Psi^+_{M_3-1}\right]u^{n+1}_{M_3}$$
$$=\mu_{M_3-1}u^n_{M_3-1}+\left[\mathcal{Z}^C_{M_3}-\mu_{M_3-1}\left(1+\frac{\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\right)\right]u^n_{M_3}+\mu_{M_3}\Psi^-_{M_3-1}u^{n-1}_{M_3-1}$$
$$-\left[\mathcal{Z}^L_{M_3}+\mu_{M_3-1}\left(1+\frac{\Delta x_{M_3-1}}{\alpha_2\overline{K}_2}\right)\Psi^-_{M_3-1}\right]u^{n-1}_{M_3}$$
$$+\mu_{M_3-1}\frac{\Delta x_{M_3-1}}{2\alpha_2\overline{K}_2}\left(\phi^{n-1}_2+2\phi^n_2+\phi^{n+1}_2\right)+\frac{1}{4}(f^{n-1}_{M_3}+2f^n_{M_3}+f^{n+1}_{M_3}). \tag{76}$$

Then, for $j\in\partial\mathbb{I}$ the discretization of (10) is provided by (75) and (76).

Summarizing, the state $\mathbf{u}^{n+1}$ is defined from (73)–(76), which clearly rewritten in a matrix form as the liner system (63). □

4.3. The Numerical Scheme and Properties

The numerical scheme is defined fundamentally by the relations provided by (40), (42), and (63). However, to be precise, the guidelines for discretization of the system (10)–(15) can be summarized in the following five steps:

Step 1. Define the input data provided in Table 1. Require

the physical parameters	$T,\overline{W}_\ell,C^\ell,\tau^\ell_q,\tau^\ell_T,k_\ell,\alpha_1,\alpha_2,\overline{K}_1,\overline{K}_2,$
the functions	$f_\ell(x,t),\psi_1(x),\psi_2(x),\varphi_1(t),\varphi_2(t),$
mesh parameters	$m_\ell,N,$

where $\ell=1,2,3$. Moreover, calculate $L_0=0$, $L_1=\overline{W}_1$, $L_2=\overline{W}_1+\overline{W}_2$, and $L_3=L=\overline{W}_1+\overline{W}_2+\overline{W}_3$.

Step 2. Discretization of space and time. Using the relations provided in (16), calculate M_ℓ, ΔW_ℓ, x_j, Δx_j, Δt, and t_n. Here, $\ell\in\{1,2,3\}$, $j\in\{M_0,\ldots,M_3\}$, and $n\in\{0,\ldots,N\}$.

Step 3. Evaluation of functions of the mesh. Evaluate

- the functions ψ_1 and ψ_2 on x_j for $j\in\{M_0,\ldots,M_3\}$ and using relation (40) define the vectors $\mathbf{u}^0$ and $\delta\mathbf{u}^0$;
- the functions φ_1 and φ_2 on t_n for $n\in\{0,\ldots,N\}$ and define the vectors $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ using the relations provided in (41);
- the functions C, k, τ_q, and τ_T, on x_j, using the relations provided in (9), for $j\in\{M_0,\ldots,M_3\}$ and define the vectors $\boldsymbol{C}$, $\boldsymbol{k}$, $\boldsymbol{\tau_q}$, and $\boldsymbol{\tau_T}$;
- the function f on (x_j,t_n), using the relation provided in (9), for $j\in\{M_0,\ldots,M_3\}$ and define the matrix f^n_j for $(j,n)\in\{M_0,\ldots,M_3\}\times\{0,\ldots,N\}$.

Moreover calculate $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ by the relation provided in (40) and the preliminary symbols $\mathcal{L}^C_j$, $\mathcal{L}^L_j$, Ψ^L_j, Ψ^R_j, μ_j, $\Psi^\pm_j$, $\mathcal{Z}^L_j$, $\mathcal{Z}^U_j$, and $\mathcal{Z}^C_j$ for $j\in\{M_0,\ldots,M_3\}$ provided in (41).

Step 4. Calculate the matrices $\hat{\mathbb{A}}$, $\hat{\mathbb{B}}$ and the vector $\mathbf{s}^0$ using the relations summarized in Table 2. Calculate the matrices $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and the vector $\mathbf{s}^n$ using the relations summarized in Table 3.

Step 5. Discretization of the equations. The equations are discretized as follows:

- The initial condition $\mathbf{u}^0$ is calculated using (40) as specified in Step 3.
- Calculate $\mathbf{u}^1$ solving the linear system (42).
- Calculate $\mathbf{u}^{n+1}$ for $n = 1, \ldots, N-1$ solving the linear system (63).

Applying the similar ideas of Theorems 3 and 4 in [41], we deduce the following results:

Theorem 1. *Let $\{u_j^n \, : \, n = 0, \ldots, N, \; j = M_0, \ldots, M_d\}$ be the solution of the finite difference scheme (40), (42), and (63) and consider the notation*

$$E^n = \sum_{j=M_0}^{M_3} C(x_j)\tau_q(x_j)\left(\frac{u_j^{n+1} - u_j^n}{\Delta t}\right)^2 + \sum_{j=M_0}^{M_3} k(x_j)\left(\frac{u_{j+1}^n - u_j^n}{\Delta x_j}\right)^2 + \frac{k(x_{M_0})}{\alpha_1 \overline{K}_1}(u_{M_0}^n)^2 + \frac{k(x_{M_3})}{\alpha_2 \overline{K}_2}(u_{M_3}^n)^2$$

Then, the following estimate

$$E^{n+1} \leq E^0 + \frac{\Delta t}{2}\sum_{k=0}^{n}\sum_{j=M_0}^{M_3}\frac{1}{C(x_j)}\left(f_j^{n+1}\right)^2 \tag{77}$$

is satisfied for $n = 0, \ldots, N-1$.

Theorem 2. *The finite difference scheme (40) and (42) is unconditionally stable with respect to initial conditions and source terms.*

5. Numerical Examples

In this section, we consider three examples: an example for comparison of the numerical solution with the analytical solution and two examples with physical parameters for specific materials: gold and chromium.

5.1. Example 1

In this example, we consider the physical and geometric parameters provided in Table 4 with initial conditions

$$\psi_1(x) = \begin{cases} \sin(3\pi x/4), & x \in [0, 1/3[, \\ -\cos(3\pi(2x-1)/4) + \sqrt{2}, & x \in [1/3, 2/3[, \\ \cos(\pi(3x-1)/4), & x \in [2/3, 1], \end{cases} \qquad \psi_2(x) = -\frac{1}{2}\psi_1(x);$$

boundary conditions $\varphi_1(t) = \varphi_2(t) = -3\pi\exp(-t/2)/8$; and source term

$$f(x,t) = \begin{cases} 8^{-1}(-2+9\pi^2)\exp(-t/2)\sin(3\pi x/4), & x \in [0, 1/3[, \\ 4^{-1}(1+9\pi^2)\exp(-t/2)\cos(3\pi(2x-1)/4), & x \in [1/3, 2/3[, \\ 8^{-1}(-2+9\pi^2)\exp(-t/2)\cos(\pi(3x-1)/4), & x \in [2/3, 1]. \end{cases}$$

We observe that the analytical solution of the system (10)–(15) is provided by

$$u(x,t) = \begin{cases} \exp(-t/2)\sin(3\pi x/4), & x \in [0, 1/3[, \\ \exp(-t/2)\Big(-\cos(3\pi(2x-1)/4) + \sqrt{2}\Big), & x \in [1/3, 2/3[, \\ \exp(-t/2)\cos(\pi(3x-1)/4), & x \in [2/3, 1]. \end{cases}$$

In our simulations, we have considered $T = 1$, $\alpha_1 = \alpha_2 = 1/2$ and $\overline{K}_1 = \overline{K}_2 = 1$. The comparison of analytical solution and the numerical solution is provided in Figure 2a,b. In Figure 2a, we show the initial condition and in Figure 2b we show the profiles of the solution at $t = T$. Meanwhile, in Figure 2c,d, we show the analytical and the numerical solutions, respectively.

Table 4. Parameters for examples.

	Example 1			**Example 2**			**Example 3**		
	Layer 1	**Layer 2**	**Layer 3**	**Layer 1**	**Layer 2**	**Layer 3**	**Layer 1**	**Layer 2**	**Layer 3**
	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 1$	$\ell = 2$	$\ell = 3$
$\overline{W}_\ell$	1/3	1/3	1/3	1	1	1	1	1	1
C^ℓ	1	1	1	129	449	129	449	129	449
τ_q^ℓ	1	1	1	8.5	0.136	8.5	0.136	8.5	0.136
τ_T^ℓ	1	4	4/3	90	7.86	90	7.86	90	7.86
k^ℓ	4	1	6	317	94	317	317	94	317

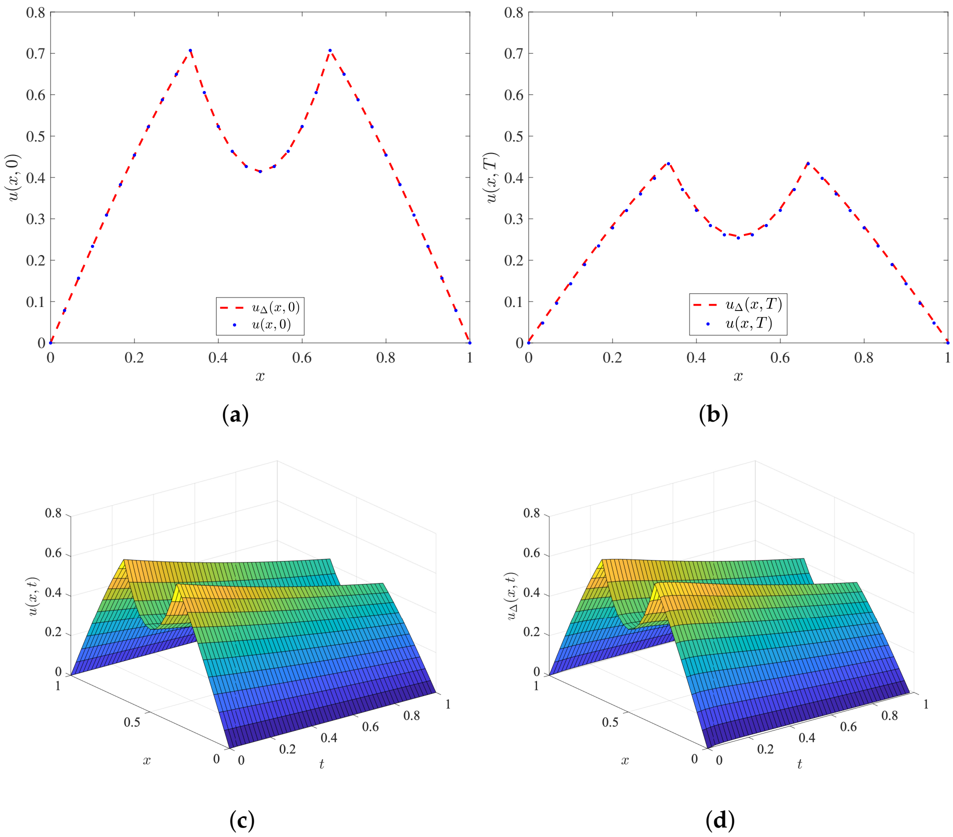

Figure 2. Numerical results for Example 1. Here, we consider $T = 1$, and, for (**a**–**d**), the discretization parameters are $m_1 = m_2 = m_3 = 10$ and $N = 50$. (**a**) Initial condition profile. (**b**) End time solution profile. (**c**) Analytical solution. (**d**) Numerical solution.

5.2. Example 2

We consider that layers 1 and 3 are of gold and layer 2 is of chromium. Using the physical parameters for gold and chromium provided in [40], we define the numerical values provided in Table 4. The source function is provided by

$$f(x,t) = \begin{cases} Q(x,t) + \tau_q^1 \partial_t Q(x,t), & x \in [0,1[, \\ Q(x,t) + \tau_q^2 \partial_t Q(x,t), & x \in [1/3,2/3[, \\ Q(x,t) + \tau_q^3 \partial_t Q(x,t), & x \in [2/3,1], \end{cases} \quad (78)$$

where

$$Q(x,t) = 5.8919e - 04 \exp\left[-\frac{x}{15.3} - 2.77\left(\frac{t-100}{10000}\right)^2\right].$$

The initial condition and boundary conditions are provided by $\psi_1(x) = 300$, $\psi_2(x) = 0$, and $\varphi_1(t) = \varphi_2(t) = 300$. The numerical simulation occurs with $\Delta x = 1.0e - 02$; $\Delta t = 5.0e - 05$, $\alpha_1 = \alpha_2 = 1/2$; $\overline{K}_1 = \overline{K}_2 = 0.1, 0.2, 0.4$; and $T = 0.2, 0.25, 0.5$. The results are shown in Figure 3.

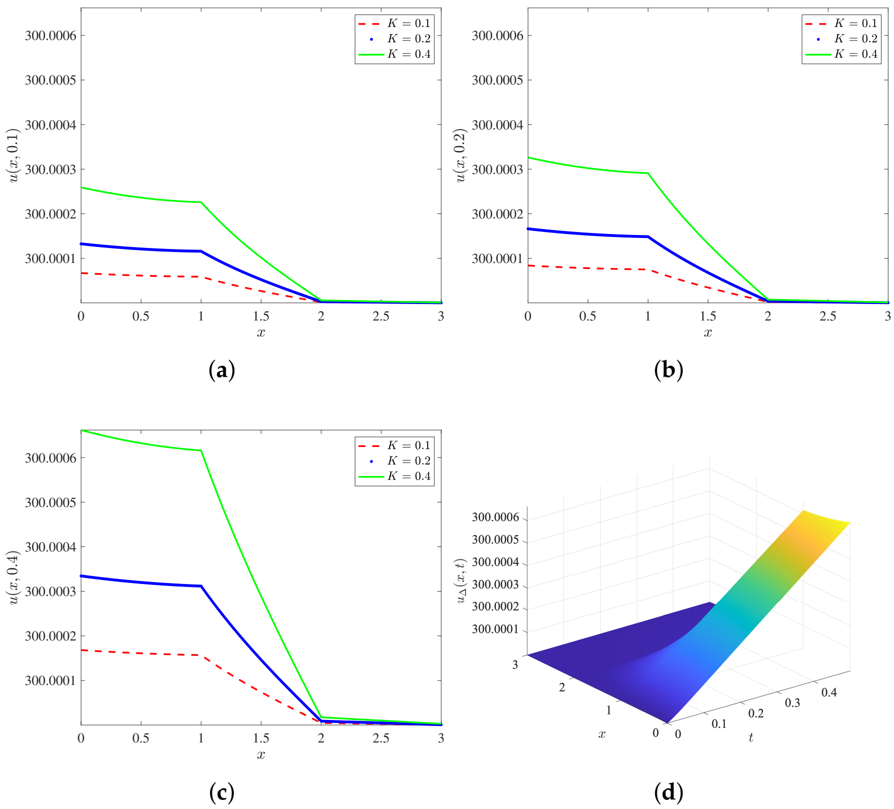

Figure 3. Numerical results for Example 2. (**a**), (**b**), and (**c**) show the comparison of profiles with T = 0.1, 0.2, and 0.4, respectively. (**d**) The numerical solution in domain $[0,3] \times [0,0.5]$ with $K = 0.4$.

5.3. Example 3

In this example, we consider the physical parameters provided in Table 4, obtained from [40] and corresponding to a three-layer structure formed from chromium–gold–chromium. The source term is defined by the relation provided in (78). We assume that $\psi_1(x) = 300$, $\psi_2(x) = 0$, and $\varphi_1(t) = \varphi_2(t) = 300$. The numerical simulation occurs with $\Delta x = 1.0e - 02$; $\Delta t = 5.0e - 05$, $\alpha_1 = \alpha_2 = 1/2$; $\overline{K}_1 = \overline{K}_2 = 0.1, 0.2, 0.4$; and $T = 0.2, 0.25, 0.5$. The results are shown in Figure 4.

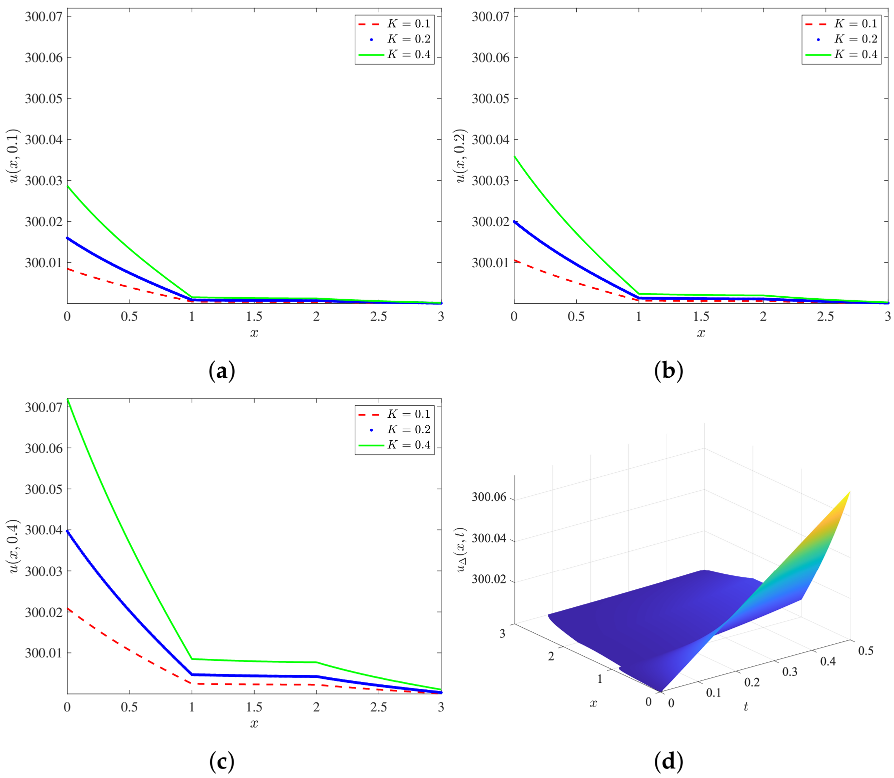

Figure 4. Numerical results for Example 2. (**a**–**c**) show the comparison of profiles with T = 0.1, 0.2, and 0.4, respectively. (**d**) The numerical solution in domain $[0, 3] \times [0, 0.5]$ with $K = 0.4$.

6. Conclusions

We introduced a mathematical model of the heat transfer in three-layered solids by considering the non-Fourier law $\mathbf{q}(x, t + \tau_q) = -k\nabla T(x, t + \tau_T)$, where τ_T and τ_q are the phase lags of the temperature gradient and the heat flux, respectively. We considered two interphase conditions: continuous temperature and continuous flux. Likewise, we introduced an appropriate notation for coefficient discretization, and, using the finite difference methodology, we deduced a numerical scheme consisting of the solution of two kinds of linear systems. Then, we validated our numerical approximation using analytical data and also constructed two examples with physical data where the solids are gold and chromium.

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