energies

MDPI

Article

# New Interval-Valued Intuitionistic Fuzzy Behavioral MADM Method and Its Application in the Selection of Photovoltaic Cells

**Xiaolu Zhang**

The Collaborative Innovation Center, Jiangxi University of Finance and Economics, Nanchang 330013, China; xiaolu_jy@163.com; Tel.: +86-791-8381-3846

Academic Editor: Senthilarasu Sundaram
Received: 3 September 2016; Accepted: 10 October 2016; Published: 20 October 2016

**Abstract:** As one of the emerging renewable resources, the use of photovoltaic cells has become a promise for offering clean and plentiful energy. The selection of a best photovoltaic cell for a promoter plays a significant role in aspect of maximizing income, minimizing costs and conferring high maturity and reliability, which is a typical multiple attribute decision making (MADM) problem. Although many prominent MADM techniques have been developed, most of them are usually to select the optimal alternative under the hypothesis that the decision maker or expert is completely rational and the decision data are represented by crisp values. However, in the selecting processes of photovoltaic cells the decision maker is usually bounded rational and the ratings of alternatives are usually imprecise and vague. To address these kinds of complex and common issues, in this paper we develop a new interval-valued intuitionistic fuzzy behavioral MADM method. We employ interval-valued intuitionistic fuzzy numbers (IVIFNs) to express the imprecise ratings of alternatives; and we construct LINMAP-based nonlinear programming models to identify the reference points under IVIFNs contexts, which avoid the subjective randomness of selecting the reference points. Finally we develop a prospect theory-based ranking method to identify the optimal alternative, which takes fully into account the decision maker's behavioral characteristics such as reference dependence, diminishing sensitivity and loss aversion in the decision making process.

**Keywords:** multiple attribute decision making; interval-valued intuitionistic fuzzy information; photovoltaic cells; renewable energy

## 1. Introduction

With natural resource scarcity and environmental protection, the use of renewable energy has become a promise for offering clean and plentiful energy source [1]. Photovoltaic cell is one of the emerging renewable energy sources. For a promoter or inverter, the selection of a best photovoltaic cell plays a significant role in aspect of maximizing income, minimizing costs and conferring high maturity and reliability. The selection of photovoltaic cells usually needs to take into account multiple attributes, such as the manufacturing cost, the efficiency in energy conversion, the emissions of greenhouse gases, etc. This is a typical multiple attribute decision making (MADM) problem. Many prominent decision making techniques have been developed for solving MADM problems during the past decade years, such as Linear Programming Technique for Multidimensional Analysis of Preference (LINMAP) [2], the TOPSIS method [3], the VIKOR method [4,5], the ELECTRE method [6,7], the DEMATEL method [8], the WASPAS method [9], etc. During these well-known techniques, the LINMAP method is a practical and useful approach for determining the weights of attributes and the ideal solution in the decision making process, which has recently been extended to the decision environments of fuzzy numbers [10], the decision contexts of intuitionistic fuzzy numbers [11,12], the decision environments of hesitant fuzzy numbers [13,14], the decision environments of interval-valued intuitionistic fuzzy numbers (IVIFNs) [15], the decision contexts of grey [16], etc.

However, these aforementioned MADM techniques are derived from expected utility theory which is based on the strict assumption regarding complete rationality of the decision maker (DM), while many excellent papers involving behavioral experiments [17,18] have shown that the DM is usually bounded rational in real-life decision process. To well capture the DM's behavioral characteristics, prospect theory developed by Tversky and Kahneman [18] is integrated into the MADM process, which is called the behavioral MADM. Gomes and Lima [19] developed a discrete MADM approach based on prospect theory, which is called the TODIM approach and has been applied in the selection of the destination of natural gas [20], the evaluation of residential properties [21], oil spill response [22], etc. On the other hand, many fuzzy behavioral MADM techniques have recently been developed, for example, Liu et al. [23] developed a fuzzy MADM method based on prospect theory for solving risk decision problems with interval probability in which the attribute values take the form of uncertain linguistic variables. Krohling and de Souza [24] developed a fuzzy extension of TODIM for handling the fuzzy behavioral MADM problems. Krohling et al. [25] also developed an intuitionistic fuzzy TODIM. Liu et al. [26] introduced a risk decision method based on cumulative prospect theory to solve emergency response problems. In the case of considering aspiration-levels of attributes, Fan et al. [27] presented a method based on prospect theory to solve the MADM problem where attribute values are denoted by crisp numbers and interval numbers. Zhang and Xu [28] developed a hesitant fuzzy TODIM method for solving behavioral MADM problems with hesitant fuzzy information.

The most characteristic of fuzzy behavioral MADM is that it can deal simultaneously with risk and uncertainty in MADM problems. However, they suffer from some limitations: (1) in several practical behavioral MADM the reference point is completely unknown, while most of the aforementioned techniques under the hypothesis that the reference point is completely known in advance fail to deal with such a kind of problems; and (2) these existing prospect theory-based decision making methods fail to deal with the IVIFNs decision data in the decision making process. To overcome the above limitations, we attempt to develop a new interval-valued intuitionistic fuzzy behavioral MADM method to address the behavioral MADM problems under IVIFNs context in which the reference point is completely unknown in advance. We also explore how to solve the selection case of photovoltaic cells by using the proposed method.

The remainder of this paper is organized as follows: Section 2 reviews basic concepts related to IVIFNs. In Section 3, an optimal model is first constructed to determine the reference point, and a prospect theory-based ranking method is developed to identify the best alternative under IVIFNs environment. In Section 4, the proposed method is employed to assist the promoter to select the optimal photovoltaic cells. Section 5 presents the concluding remarks of this paper.

## 2. Preliminaries

Let $X$ be nonempty set, an interval-valued intuitionistic fuzzy set (IVIFS) $\widetilde{I}$ in $X$ is expressed as [29]: $\widetilde{I} = \{< x, \widetilde{I}(x) > \big| x \in X\}$, where $\widetilde{I}(x) = ([\widetilde{\mu}^L_{\widetilde{I}}(x), \widetilde{\mu}^U_{\widetilde{I}}(x)], [\widetilde{\nu}^L_{\widetilde{I}}(x), \widetilde{\nu}^U_{\widetilde{I}}(x)])$, $[\widetilde{\mu}^L_{\widetilde{I}}(x), \widetilde{\mu}^U_{\widetilde{I}}(x)] \subseteq [0,1]$ and $[\widetilde{\nu}^L_{\widetilde{I}}(x), \widetilde{\nu}^U_{\widetilde{I}}(x)] \subseteq [0,1]$ are intervals, respectively, $0 \leq \widetilde{\mu}^L_{\widetilde{I}}(x) \leq \widetilde{\mu}^U_{\widetilde{I}}(x) \leq 1$, $0 \leq \widetilde{\nu}^L_{\widetilde{I}}(x) \leq \widetilde{\nu}^U_{\widetilde{I}}(x) \leq 1$ and $\widetilde{\mu}^U_{\widetilde{I}}(x) + \widetilde{\nu}^U_{\widetilde{I}}(x) \leq 1$. Usually, the $\widetilde{I}(x)$ is called an IVIFN [30] and is denoted by $\widetilde{A} = ([\widetilde{\mu}^L, \widetilde{\mu}^U], [\widetilde{\nu}^L, \widetilde{\nu}^U])$ for convenience, where $[\widetilde{\mu}^L, \widetilde{\mu}^U] \subseteq [0,1]$, $[\widetilde{\nu}^L, \widetilde{\nu}^U] \subseteq [0,1]$ and $\widetilde{\mu}^U + \widetilde{\nu}^U \leq 1$.

In the real-life decision process, the DM usually uses the IVIFNs instead of IVIFSs to express the ratings for the alternatives on attributes (Case 1) or the pair-wised comparison assessment information over alternatives (Case 2). For example, Case 1: let $A_1$ be an alternative, and let $C_1$ be an attribute which the alternative $A_1$ satisfies, the rating of the alternative $A_1$ with respect to the attribute $C_1$ is represented by IVIFN as $C_1(A_1) = ([0.5, 0.6], [0.2, 0.3])$, which can express the meaning that the alternative $A_1$ is an excellent alternative for the DM on the attribute $C_1$ with a chance between 50% and 60%, and simultaneously $A_1$ is not an excellent choice with a chance between 20% and 30% [31]; and Case 2:

given two alternatives $A_1$ and $A_2$, if the DM prefers $A_1$ to $A_2$ with IVIFN preference information $([0.7, 0.8], [0.1, 0.2])$, which means that the degree to which the DM thinks the alternative $A_1$ is superior $A_2$ is the interval [70%, 80%] and $A_1$ is inferior to $A_2$ with a chance between 10% and 20%.

**Definition 2.1.** *[32]. For two IVIFNs* $\widetilde{A}_j = ([\widetilde{\mu}_j^L, \widetilde{\mu}_j^U], [\widetilde{v}_j^L, \widetilde{v}_j^U])$ $(j = 1, 2)$*, let* $s(\widetilde{A}_j) = \frac{1}{2}(\widetilde{\mu}_j^L - \widetilde{v}_j^L + \widetilde{\mu}_j^U - \widetilde{v}_j^U)$, $h(\widetilde{A}_j) = \frac{1}{2}(\widetilde{\mu}_j^L + \widetilde{\mu}_j^U + \widetilde{v}_j^L + \widetilde{v}_j^U)$, $T(\widetilde{A}_j) = \widetilde{\mu}_j^U + \widetilde{v}_j^L - \widetilde{\mu}_j^L - \widetilde{v}_j^U$ *and* $G(\widetilde{A}_j) = \widetilde{\mu}_j^U + \widetilde{v}_j^U - \widetilde{\mu}_j^L - \widetilde{v}_j^L$ $(j = 1, 2)$ *be the score function and the accuracy function, the membership uncertainty index, and the hesitation uncertainty index of* $\widetilde{A}_j$ $(j = 1, 2)$*, respectively, then we have:*

*(1)* *if* $s(\widetilde{A}_1) < s(\widetilde{A}_2)$*, then* $\widetilde{A}_1 \prec \widetilde{A}_2$*;*

*(2)* *if* $s(\widetilde{A}_1) = s(\widetilde{A}_2)$*, then* $\begin{cases} h(\widetilde{A}_1) < h(\widetilde{A}_2) \Rightarrow \widetilde{A}_1 \prec \widetilde{A}_2 \\ h(\widetilde{A}_1) = h(\widetilde{A}_2) \Rightarrow \begin{cases} T(\widetilde{A}_1) > T(\widetilde{A}_2) \Rightarrow \widetilde{A}_1 \prec \widetilde{A}_2 \\ T(\widetilde{A}_1) = T(\widetilde{A}_2) \Rightarrow \begin{cases} G(\widetilde{A}_1) > G(\widetilde{A}_2) \Rightarrow \widetilde{A}_1 \prec \widetilde{A}_2 \\ G(\widetilde{A}_1) = G(\widetilde{A}_2) \Rightarrow \widetilde{A}_1 \sim \widetilde{A}_2 \end{cases} \end{cases} \end{cases}$.

**Definition 2.2.** *[30]. Let* $\widetilde{A}_j = \left([\widetilde{\mu}_j^L, \widetilde{\mu}_j^U], [\widetilde{v}_j^L, \widetilde{v}_j^U]\right)$ $(j = 1, 2, \cdots, n)$ *be a collection of IVIFNs, the interval-valued intuitionistic fuzzy weighted averaging (IVIFWA) operator is a mapping* $H^n \to H$ *such that*

$$IVIFWA\left(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_n\right) = \left([1 - \prod_{j=1}^{n}(1 - \widetilde{\mu}_j^L)^{w_j}, 1 - \prod_{j=1}^{n}(1 - \widetilde{\mu}_j^U)^{w_j}], [\prod_{j=1}^{n}(\widetilde{v}_j^L)^{w_j}, \prod_{j=1}^{n}(\widetilde{v}_j^U)^{w_j}]\right) \quad (1)$$

*where* $\boldsymbol{w} = (w_1, w_2, \cdots, w_n)^T$ *is the weight vector of* $\widetilde{A}_j$ $(j = 1, 2, \cdots, n)$*.*

**Definition 2.3.** *[33]. Let* $\widetilde{A}_j = \left([\widetilde{\mu}_j^L, \widetilde{\mu}_j^U], [\widetilde{v}_j^L, \widetilde{v}_j^U]\right)$ $(j = 1, 2)$ *be two IVIFNs, the Euclidean distance between* $\widetilde{A}_1$ *and* $\widetilde{A}_2$ *is defined as follows:*

$$d\left(\widetilde{A}_1, \widetilde{A}_2\right) = \sqrt{\frac{1}{4}\left((\widetilde{\mu}_1^L - \widetilde{\mu}_2^L)^2 + (\widetilde{\mu}_1^U - \widetilde{\mu}_2^U)^2 + (\widetilde{v}_1^L - \widetilde{v}_2^L)^2 + (\widetilde{v}_1^U - \widetilde{v}_2^U)^2 + (\widetilde{\pi}_1^L - \widetilde{\pi}_2^L)^2 + (\widetilde{\pi}_1^U - \widetilde{\pi}_2^U)^2\right)} \quad (2)$$

*where* $\widetilde{\pi}_1^L = 1 - \widetilde{\mu}_1^U - \widetilde{v}_1^U$, $\widetilde{\pi}_1^U = 1 - \widetilde{\mu}_1^L - \widetilde{v}_1^L$, $\widetilde{\pi}_2^L = 1 - \widetilde{\mu}_2^U - \widetilde{v}_2^U$ *and* $\widetilde{\pi}_2^U = 1 - \widetilde{\mu}_2^L - \widetilde{v}_2^L$*.*

## 3. The Developed Decision Making Approach

Consider an MADM problem with IVIFNs which consists of a set of alternatives $\boldsymbol{A} = \{A_1, A_2, \cdots, A_m\}$ and a set of attributes $\boldsymbol{C} = \{C_1, C_2, \cdots, C_n\}$. Let $\widetilde{A}_{ij}$ be the rating of the alternative $A_i$ with respect to the attribute $C_j$, and the decision matrix is denoted by $\Re = (\widetilde{A}_{ij})_{m \times n}$ where all $\widetilde{A}_{ij}$ are represented by IVIFNs. The weight vector of attributes is denoted by $\boldsymbol{w} = \{w_1, w_2, \cdots, w_n\}$. To select the best alternative which is the most satisfactory with respect to attributes $\boldsymbol{C} = \{C_1, C_2, \cdots, C_n\}$ from the potential alternatives set $\boldsymbol{A} = \{A_1, A_2, \cdots, A_m\}$ according to the decision information $\Re = (\widetilde{A}_{ij})_{m \times n}$, we next construct the LINMAP-based nonlinear programming models to determine the reference points under IVIFNs context, and develop a prospect theory-based ranking method with IVIFNs data to identify the best alternative in case of considering the DM's behavioral characteristics.

### *3.1. LINMAP-Based Nonlinear Programming Models to Derive the Reference Point*

The LINMAP method developed by Srinivasan and Shocker [2] proves to be a practical and useful approach for determining the ideal solution based on the given decision information. In this method the derived ideal solution is neither the best point nor the worst point but an optimal point because it is obtained based on that the inconsistency index (poorness of fit) between the derived ranking order of each pair alternatives and the preorder given by the DM should be minimized and must be no bigger than the consistency index (goodness of fit). Obviously, the ideal solution is much in accord with the aspiration of the DM and is very appropriate to be as the reference point in the decision process in case of considering the DM's psychological behavior. Therefore, drawing on the idea of the LINMAP approach, in the next section we construct a nonlinear programming approach to obtain the reference point on the basis of the given decision information for solving the IVIF behavioral MCDM problem.

The LINMAP approach requires the DM to provide the incomplete preference relations on pair-wise comparisons of alternatives. We assume that the DM expresses the comparison preference information between the alternatives $A_\xi$ and $A_\zeta$ by using IVIFNs according to his/her subjective experiences and judgments. The comparison information is given by a set of ordered pairs $\widetilde{\Omega} = \{(\xi,\zeta)|A_\xi \succeq_{\widetilde{R}(\xi,\zeta)} A_\zeta, for\, \xi,\zeta = 1,2,\ldots,m\}$ where the $\widetilde{R}(\xi,\zeta)$ is an IVIFN denoted by $\widetilde{R}(\xi,\zeta) = ([\widetilde{u}^L_{\widetilde{R}(\xi,\zeta)}, \widetilde{u}^U_{\widetilde{R}(\xi,\zeta)}], [\widetilde{v}^L_{\widetilde{R}(\xi,\zeta)}, \widetilde{v}^U_{\widetilde{R}(\xi,\zeta)}])$, which indicates the degree that the DM prefers the alternative $A_\xi$ to $A_\zeta$.

**Remark 3.1.** *It is noted that if the comparison preference information is complete, then* $|\widetilde{\Omega}| = \frac{1}{2}m(m-1)$*, where* $|\widetilde{\Omega}|$ *is the cardinality of* $\widetilde{\Omega}$*; while if it is incomplete, then* $|\widetilde{\Omega}| < \frac{1}{2}m(m-1)$*. This study allows the pairwise comparison preference information given by the DM to be incomplete and/or intransitive.*

3.1.1. Definitions of Consistency and Inconsistency Indices under IVIFNs Context

Given a pair of alternatives $(\xi,\zeta) \in \widetilde{\Omega}$, the distance between each of the alternatives $A_i (i = \xi,\zeta)$ and the reference point $A^*$ can be calculated as follows:

$$\begin{aligned} D_i &= \textstyle\sum_{j=1}^n w_j d(\widetilde{A}_{ij}, \widetilde{A}^*_j)^2 \\ &= \frac{1}{4}\sum_{j=1}^n w_j \begin{pmatrix} \left(\widetilde{\mu}^L_{ij} - (\widetilde{\mu}^L_j)^*\right)^2 + \left(\widetilde{\mu}^U_{ij} - (\widetilde{\mu}^U_j)^*\right)^2 + \left(\widetilde{v}^L_{ij} - (\widetilde{v}^L_j)^*\right)^2 \\ + \left(\widetilde{v}^U_{ij} - (\widetilde{v}^U_j)^*\right)^2 + \left(\widetilde{\pi}^L_{ij} - (\widetilde{\pi}^L_j)^*\right)^2 + \left(\widetilde{\pi}^U_{ij} - (\widetilde{\pi}^U_j)^*\right)^2 \end{pmatrix} \end{aligned} \quad (3)$$

where the reference point vector is denoted by $A^* = (\widetilde{A}^*_1, \widetilde{A}^*_2, \cdots, \widetilde{A}^*_n)$, and $\widetilde{A}^*_j$ is the reference point with the attribute $C_j$ by expressing as an IVIFN $\widetilde{A}^*_j = ([(\widetilde{u}^L_j)^*, (\widetilde{u}^U_j)^*], [(\widetilde{v}^L_j)^*, (\widetilde{v}^U_j)^*])$.

Let $g_{\xi\zeta} = D_\zeta - D_\xi$, then

$$g_{\xi\zeta} = \frac{1}{4}\sum_{j=1}^n w_j \begin{pmatrix} \left(\widetilde{\mu}^L_{\zeta j} - (\widetilde{\mu}^L_j)^*\right)^2 + \left(\widetilde{\mu}^U_{\zeta j} - (\widetilde{\mu}^U_j)^*\right)^2 + \left(\widetilde{v}^L_{\zeta j} - (\widetilde{v}^L_j)^*\right)^2 + \left(\widetilde{v}^U_{\zeta j} - (\widetilde{v}^U_j)^*\right)^2 \\ + \left(\widetilde{\pi}^L_{\zeta j} - (\widetilde{\pi}^L_j)^*\right)^2 + \left(\widetilde{\pi}^U_{\zeta j} - (\widetilde{\pi}^U_j)^*\right)^2 - \left(\widetilde{\mu}^L_{\xi j} - (\widetilde{\mu}^L_j)^*\right)^2 - \left(\widetilde{\mu}^U_{\xi j} - (\widetilde{\mu}^U_j)^*\right)^2 \\ - \left(\widetilde{v}^L_{\xi j} - (\widetilde{v}^L_j)^*\right)^2 - \left(\widetilde{v}^U_{\xi j} - (\widetilde{v}^U_j)^*\right)^2 - \left(\widetilde{\pi}^L_{\xi j} - (\widetilde{\pi}^L_j)^*\right)^2 - \left(\widetilde{\pi}^U_{\xi j} - (\widetilde{\pi}^U_j)^*\right)^2 \end{pmatrix} \quad (4)$$

For convenience of description, we stipulate:

$$\begin{aligned} &\eta_{\zeta\xi j} = \frac{1}{4}\begin{pmatrix} (\widetilde{\mu}^L_{\zeta j})^2 + (\widetilde{\mu}^U_{\zeta j})^2 + (\widetilde{v}^L_{\zeta j})^2 + (\widetilde{v}^U_{\zeta j})^2 + (\widetilde{\pi}^L_{\zeta j})^2 + (\widetilde{\pi}^U_{\zeta j})^2 - 2\widetilde{\pi}^L_{\zeta j} - 2\widetilde{\pi}^U_{\zeta j} - \\ (\widetilde{\mu}^L_{\xi j})^2 - (\widetilde{\mu}^U_{\xi j})^2 - (\widetilde{v}^L_{\xi j})^2 - (\widetilde{v}^U_{\xi j})^2 - (\widetilde{\pi}^L_{\xi j})^2 - (\widetilde{\pi}^U_{\xi j})^2 + 2\widetilde{\pi}^L_{\xi j} + 2\widetilde{\pi}^U_{\xi j} \end{pmatrix} \\ &\hat{\mu}^L_j = w_j(\widetilde{\mu}^L_j)^*, \hat{\mu}^U_j = w_j(\widetilde{\mu}^U_j)^*, \hat{v}^L_j = w_j(\widetilde{v}^L_j)^*, \hat{v}^U_j = w_j(\widetilde{v}^U_j)^*, \alpha^L_{\zeta\xi j} = \frac{1}{2}(-\widetilde{\mu}^L_{\zeta j} + \widetilde{\pi}^U_{\zeta j} + \widetilde{\mu}^L_{\xi j} - \widetilde{\pi}^U_{\xi j}) \\ &\alpha^U_{\zeta\xi j} = \frac{1}{2}(-\widetilde{\mu}^U_{\zeta j} + \widetilde{\pi}^L_{\zeta j} + \widetilde{\mu}^U_{\xi j} - \widetilde{\pi}^L_{\xi j}), \beta^L_{\zeta\xi j} = \frac{1}{2}(-\widetilde{v}^L_{\zeta j} + \widetilde{\pi}^U_{\zeta j} + \widetilde{v}^L_{\xi j} - \widetilde{\pi}^U_{\xi j}), \\ &\beta^U_{\zeta\xi j} = \frac{1}{2}(-\widetilde{v}^U_{\zeta j} + \widetilde{\pi}^L_{\zeta j} + \widetilde{v}^U_{\xi j} - \widetilde{\pi}^L_{\xi j}) \end{aligned} \quad (5)$$

Therefore, $g_{\xi\zeta}$ can be written as follows:

$$g_{\xi\zeta} = \sum_{j=1}^n w_j \eta_{\zeta\xi j} + \sum_{j=1}^n \hat{\mu}^L_j \alpha^L_{\zeta\xi j} + \sum_{j=1}^n \hat{\mu}^U_j \alpha^U_{\zeta\xi j} + \sum_{j=1}^n \hat{v}^L_j \beta^L_{\zeta\xi j} + \sum_{j=1}^n \hat{v}^U_j \beta^U_{\zeta\xi j} \quad (6)$$

According to prospect theory, the DM usually prefers to the alternative that is closer to the reference point.

**Definition 3.1.** *For each pair of alternatives* $(\xi,\zeta) \in \widetilde{\Omega}$*, an inconsistency index* $(D_\zeta - D_\xi)^-$ *is defined to measure the degree of inconsistency between the ranking orders of the alternatives* $A_\xi$ *and* $A_\zeta$ *in which one ranking order is determined by* $D_\xi$ *and* $D_\zeta$*, and the other ranking order is obtained by the comparison preference relation* $(\xi,\zeta) \in \widetilde{\Omega}$ *given by the DM in advance as below:*

$$(D_\zeta - D_\xi)^- = \begin{cases} 0 & (D_\zeta \geq D_\xi) \\ \widetilde{R}(\xi,\zeta) \times (D_\xi - D_\zeta) & (D_\zeta < D_\xi) \end{cases} \tag{7}$$

According to the basic operation of IVIFNs, it is easy to see that the inconsistency index $(D_\zeta - D_\xi)^-$ is an IVIFN. Thus, $(D_\zeta - D_\xi)^-$ is also called the interval-valued intuitionistic fuzzy inconsistency index. Then, this inconsistency index $(D_\zeta - D_\xi)^-$ can be rewritten as:

$$(D_\zeta - D_\xi)^- = \widetilde{R}(\xi,\zeta) \max\{0, (D_\xi - D_\zeta)\} \tag{8}$$

The comprehensive inconsistency index is defined as:

$$\widetilde{B} = \sum_{(\xi,\zeta)\in\widetilde{\Omega}} (D_\zeta - D_\xi)^- = \sum_{(\xi,\zeta)\in\widetilde{\Omega}} \widetilde{R}(\xi,\zeta) \max\{0, (D_\xi - D_\zeta)\} \tag{9}$$

In a similar way, the consistent index can be introduced as follows:

**Definition 3.2.** *For each pair of alternatives* $(\xi,\zeta) \in \widetilde{\Omega}$*, the consistency index* $(D_\zeta - D_\xi)^+$ *is defined as:*

$$(D_\zeta - D_\xi)^+ = \begin{cases} \widetilde{R}(\xi,\zeta) \times (D_\zeta - D_\xi) & (D_\zeta \geq D_\xi) \\ 0 & (D_\zeta < D_\xi) \end{cases} \tag{10}$$

*which can measure the degree of consistency between the ranking orders of the alternatives* $A_\xi$ *and* $A_\zeta$ *in which one ranking order is determined by* $D_\xi$ *and* $D_\zeta$*, and the other ranking order is obtained by the preference relation* $(\xi,\zeta) \in \widetilde{\Omega}$ *given by the DM in advance.*

Obviously, the consistency index $(D_\zeta - D_\xi)^+$ in Equation (10) is also an IVIFN, which is called the interval-valued intuitionistic fuzzy consistency index and can also be rewritten as:

$$(D_\zeta - D_\xi)^+ = \widetilde{R}(\xi,\zeta) \max\{0, (D_\zeta - D_\xi)\} \tag{11}$$

Then, the comprehensive consistency index can be obtained by the following equation:

$$\widetilde{G} = \sum_{(\xi,\zeta)\in\widetilde{\Omega}} (D_\zeta - D_\xi)^+ = \sum_{(\xi,\zeta)\in\widetilde{\Omega}} \widetilde{R}(\xi,\zeta) \max\{0, (D_\zeta - D_\xi)\} \tag{12}$$

Using Equations (9) and (12), it can be easily derived that

$$\begin{aligned} \widetilde{G} - \widetilde{B} &= \sum_{(\xi,\zeta)\in\widetilde{\Omega}} \left((D_\zeta - D_\xi)^+ - (D_\zeta - D_\xi)^-\right) \\ &= \sum_{(\xi,\zeta)\in\widetilde{\Omega}} \left(\widetilde{R}(\xi,\zeta) \times (D_\zeta - D_\xi)\right) \\ &= \sum_{(\xi,\zeta)\in\widetilde{\Omega}} \left(\widetilde{R}(\xi,\zeta) \times g_{\xi\zeta}\right) \end{aligned} \tag{13}$$

### 3.1.2. Construction of the Nonlinear Programming Model

To determine the reference point $A^*$, we construct the following optimal model that intends to minimize $\widetilde{B}$ under the condition that $\widetilde{G}$ is no smaller than $\widetilde{B}$ by an IVIFN $\widetilde{\varepsilon}$:

$$
\begin{aligned}
&\min \left\{ \widetilde{B} \right\} \\
&s.t. \begin{cases} \widetilde{G} - \widetilde{B} \geq \widetilde{\varepsilon} \\ 0 \leq (\widetilde{\mu}_j^L)^* \leq (\widetilde{\mu}_j^U)^*, \ j \in \{1, 2, \cdots, n\} \\ 0 \leq (\widetilde{v}_j^L)^* \leq (\widetilde{v}_j^U)^*, \ j \in \{1, 2, \cdots, n\} \\ (\widetilde{\mu}_j^U)^* + (\widetilde{v}_j^U)^* \leq 1, \ j \in \{1, 2, \cdots, n\} \end{cases}
\end{aligned}
\tag{MOD-1}
$$

where $\widetilde{\varepsilon} = \left([u_{\widetilde{\varepsilon}}^L, u_{\widetilde{\varepsilon}}^U], [v_{\widetilde{\varepsilon}}^L, v_{\widetilde{\varepsilon}}^U]\right)$ which is an IVIFN given by the DM in advance represents the DM's lowest acceptable level towards the difference of $\widetilde{G} - \widetilde{B}$.

**Remark 3.2.** *According to the LINMAP method, the mode (MOD-1) is to determine the ideal solution based on the given decision information. This derived ideal solution is neither the best point nor the worst point but an optimal point because it is obtained based on that the inconsistency index between the derived ranking order of each pair alternatives and the preorder given by the DM should be minimized and must be no larger than the consistency index. Obviously, the ideal solution is much in accord with the aspiration of the DM and is very appropriate to be as the reference point in the decision process in case of considering the DM's psychological behavior. Apparently, it is reasonable to derive the reference points by using the mode (MOD-1).*

By Equations (6), (9) and (13), the optimal model (MOD-1) is equivalent to the following model:

$$
\begin{aligned}
&\min \left\{ \sum_{(\xi,\zeta)\in\widetilde{\Omega}} \widetilde{R}(\xi,\zeta) \max\left\{0, \left(D_\xi - D_\zeta\right)\right\} \right\} \\
&s.t. \begin{cases} \sum\limits_{(\xi,\zeta)\in\widetilde{\Omega}} \left(\widetilde{R}(\xi,\zeta) \times g_{\xi\zeta}\right) \geq \widetilde{\varepsilon} \\ 0 \leq \hat{u}_j^L \leq \hat{u}_j^U, 0 \leq \hat{v}_j^L \leq \hat{v}_j^U, 0 \leq \hat{u}_j^U + \hat{v}_j^U \leq w_j, \ j \in \{1, 2, \cdots, n\} \end{cases}
\end{aligned}
\tag{MOD-2}
$$

Furthermore, let $\lambda_{\xi\zeta} = \max\left\{0, \left(D_\xi - D_\zeta\right)\right\}$, then for each pairwise of alternatives $(\xi, \zeta) \in \widetilde{\Omega}$, it is obtained $\lambda_{\xi\zeta} \geq D_\xi - D_\zeta$, namely, $\lambda_{\xi\zeta} + D_\zeta - D_\xi \geq 0$ and $\lambda_{\xi\zeta} \geq 0$. Thus, the optimal model (MOD-2) can be converted into the following optimal model:

$$
\begin{aligned}
&\min \sum_{(\xi,\zeta)\in\widetilde{\Omega}} \lambda_{\xi\zeta} \widetilde{R}(\xi,\zeta) \\
&s.t. \begin{cases} \sum\limits_{(\xi,\zeta)\in\widetilde{\Omega}} \left(\widetilde{R}(\xi,\zeta) \times g_{\xi\zeta}\right) \geq \widetilde{\varepsilon} \\ g_{\xi\zeta} + \lambda_{\xi\zeta} \geq 0, \ \lambda_{\xi\zeta} \geq 0, \quad (\xi,\zeta) \in \widetilde{\Omega} \\ 0 \leq \hat{u}_j^L \leq \hat{u}_j^U, 0 \leq \hat{v}_j^L \leq \hat{v}_j^U, 0 \leq \hat{u}_j^U + \hat{v}_j^U \leq w_j, \ j \in \{1, 2, \cdots, n\} \end{cases}
\end{aligned}
\tag{MOD-3}
$$

It should be noted that both the objective function and the constraints of the model (MOD-3) contain the IVIFNs which cannot be solved by the existing programming methods. In the following, we discuss how to solve the model (MOD-3).

### 3.1.3. Obtain the Reference Points by Solving the Optimal Model

According to the interval-valued intuitionistic fuzzy weighted averaging operator introduced in Section 2, the objective function of the model (MOD-3) is obtained as:

$$
\sum_{(\xi,\zeta)\in\widetilde{\Omega}} \left(\lambda_{\xi\zeta} \widetilde{R}(\xi,\zeta)\right) = \begin{pmatrix} [1 - \prod\limits_{(\xi,\zeta)\in\widetilde{\Omega}} (1 - u_{\widetilde{R}(\xi,\zeta)}^L)^{\lambda_{\xi\zeta}}, 1 - \prod\limits_{(\xi,\zeta)\in\widetilde{\Omega}} (1 - u_{\widetilde{R}(\xi,\zeta)}^U)^{\lambda_{\xi\zeta}}], \\ [\prod\limits_{(\xi,\zeta)\in\widetilde{\Omega}} (v_{\widetilde{R}(\xi,\zeta)}^L)^{\lambda_{\xi\zeta}}, \prod\limits_{(\xi,\zeta)\in\widetilde{\Omega}} (v_{\widetilde{R}(\xi,\zeta)}^U)^{\lambda_{\xi\zeta}}] \end{pmatrix}
\tag{14}
$$

and the left of the first constraint the model (MOD-3) is obtained as below:

$$\left([1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}(1-u^{L}_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}},1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}(1-u^{U}_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}}],[\prod_{(\xi,\zeta)\in\widetilde{\Omega}}(v^{L}_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}},\prod_{(\xi,\zeta)\in\widetilde{\Omega}}(v^{U}_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}}]\right) \tag{15}$$

Apparently, both Equations (14) and (15) are IVIFNs. According to the definition of IVIFNs, the model (MOD-3) can be transformed to the following bi-objective interval programming model as:

$$\begin{aligned}
&\min\{[1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^{L}_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}},1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^{U}_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}}]\}\\
&\max\{[\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(v^{L}_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}},\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(v^{U}_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}}]\}\\
&s.t.\begin{cases}\begin{pmatrix}[1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}(1-u^{L}_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}},1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}(1-u^{U}_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}}],\\ [\prod_{(\xi,\zeta)\in\widetilde{\Omega}}(v^{L}_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}},\prod_{(\xi,\zeta)\in\widetilde{\Omega}}(v^{U}_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}}]\end{pmatrix}\geq([u^{L}_{\widetilde{\varepsilon}},u^{U}_{\widetilde{\varepsilon}}],[v^{L}_{\widetilde{\varepsilon}},v^{U}_{\widetilde{\varepsilon}}])\\
g_{\xi\zeta}+\lambda_{\xi\zeta}\geq 0,\ \lambda_{\xi\zeta}\geq 0,\quad(\xi,\zeta)\in\widetilde{\Omega}\\
0\leq\hat{u}^{L}_{j}\leq\hat{u}^{U}_{j},0\leq\hat{v}^{L}_{j}\leq\hat{v}^{U}_{j},0\leq\hat{u}^{U}_{j}+\hat{v}^{U}_{j}\leq w_j,\ j\in\{1,2,\cdots,n\}\end{cases}
\end{aligned} \tag{MOD-4}$$

**Theorem 3.1.** *The model (MOD-4) is equivalent to the following bi-objective interval programming model (MOD-5):*

$$\begin{aligned}
&max\left\{[\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\lambda_{\xi\zeta}ln(1-u^{U}_{\widetilde{R}(\xi,\zeta)}),\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\lambda_{\xi\zeta}ln(1-u^{L}_{\widetilde{R}(\xi,\zeta)})]\right\}\\
&max\left\{[\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\lambda_{\xi\zeta}ln(v^{L}_{\widetilde{R}(\xi,\zeta)}),\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\lambda_{\xi\zeta}ln(v^{U}_{\widetilde{R}(\xi,\zeta)})]\right\}\\
&s.t.\begin{cases}\sum_{(\xi,\zeta)\in\widetilde{\Omega}}(g_{\xi\zeta}ln(1-u^{L}_{\widetilde{R}(\xi,\zeta)}))\leq ln(1-u^{L}_{\widetilde{\varepsilon}}),\\
\sum_{(\xi,\zeta)\in\widetilde{\Omega}}(g_{\xi\zeta}ln(1-u^{U}_{\widetilde{R}(\xi,\zeta)}))\leq ln(1-u^{U}_{\widetilde{\varepsilon}})\\
\sum_{(\xi,\zeta)\in\widetilde{\Omega}}(g_{\xi\zeta}ln(v^{L}_{\widetilde{R}(\xi,\zeta)}))\leq ln(v^{L}_{\widetilde{\varepsilon}}),\\
\sum_{(\xi,\zeta)\in\widetilde{\Omega}}(g_{\xi\zeta}ln(v^{U}_{\widetilde{R}(\xi,\zeta)}))\leq ln(v^{U}_{\widetilde{\varepsilon}})\\
\lambda_{\xi\zeta}+g_{\xi\zeta}\geq 0,\quad\lambda_{\xi\zeta}\geq 0\quad\left((\xi,\zeta)\in\widetilde{\Omega}\right)\\
0\leq\hat{u}^{L}_{j}\leq\hat{u}^{U}_{j},\ 0\leq\hat{v}^{L}_{j}\leq\hat{v}^{U}_{j},0\leq\ \hat{u}^{U}_{j}+\hat{v}^{U}_{j}\leq w_j,\ j\in\{1,2,\cdots,n\}\end{cases}
\end{aligned} \tag{MOD-5}$$

The proof of Theorem 3.1 is provided in Appendix A.

**Lemma 3.1.** *[34] The maximization optimal problem with interval-objective function as*

$$\begin{cases}max & \widetilde{a}\\ s.t. & \widetilde{a}\in\Omega\end{cases}$$

*is equivalent to the bi-objective mathematical programming problem as:*

$$\begin{cases}max & \{a^L,m(\widetilde{a})\}\\ s.t. & \widetilde{a}\in\Omega\end{cases}$$

*where* $\widetilde{a}=[a^L,a^U]$ *is an interval number, the* $m(\widetilde{a})=\left(a^L+a^U\right)/2$ *is the midpoint of* $\widetilde{a}$*, and* $\Omega$ *is a set constraint condition in which the variable* $\widetilde{a}$ *should satisfy.*

According to Lemma 3.1., the model (MOD-5) can be further transformed into the multi-objective programming model as follows:

$$\begin{array}{l}
\max\{\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(1-u^{U}_{\tilde{R}(\xi,\zeta)})\} \\
\max\{\frac{1}{2}(\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(1-u^{U}_{\tilde{R}(\xi,\zeta)})+\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(1-u^{L}_{\tilde{R}(\xi,\zeta)}))\} \\
\max\{\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(v^{L}_{\tilde{R}(\xi,\zeta)})\} \\
\max\{\frac{1}{2}(\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(v^{L}_{\tilde{R}(\xi,\zeta)})+\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(v^{U}_{\tilde{R}(\xi,\zeta)}))\} \\
s.t.\left\{\begin{array}{l}
\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}(g_{\xi\zeta}\ln(1-u^{L}_{\tilde{R}(\xi,\zeta)}))\le\ln(1-u^{L}_{\tilde{\varepsilon}}),\ \sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}(g_{\xi\zeta}\ln(1-u^{U}_{\tilde{R}(\xi,\zeta)}))\le\ln(1-u^{U}_{\tilde{\varepsilon}}) \\
\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}(g_{\xi\zeta}\ln(v^{L}_{\tilde{R}(\xi,\zeta)}))\le\ln(v^{L}_{\tilde{\varepsilon}}),\ \sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}(g_{\xi\zeta}\ln(v^{U}_{\tilde{R}(\xi,\zeta)}))\le\ln(v^{U}_{\tilde{\varepsilon}}) \\
\lambda_{\xi\zeta}+g_{\xi\zeta}\ge0,\quad\lambda_{\xi\zeta}\ge0\quad\left((\xi,\zeta)\in\tilde{\Omega}\right) \\
0\le\hat{u}^{L}_{j}\le\hat{u}^{U}_{j},\ 0\le\hat{v}^{L}_{j}\le\hat{v}^{U}_{j},0\le\ \hat{u}^{U}_{j}+\hat{v}^{U}_{j}\le w_{j},\ j\in\{1,2,\cdots,n\}
\end{array}\right.
\end{array}\quad\text{(MOD-6)}$$

Using the weighted average approach, the objective function of model (MOD-6) can be defined as the following form:

$$\max\left\{\begin{array}{l}
\varpi_1\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(1-u^{U}_{\tilde{R}(\xi,\zeta)})+\frac{\varpi_2}{2}(\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(1-u^{U}_{\tilde{R}(\xi,\zeta)})+\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(1-u^{L}_{\tilde{R}(\xi,\zeta)})) \\
+\varpi_3\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(v^{L}_{\tilde{R}(\xi,\zeta)})+\frac{\varpi_4}{2}(\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(v^{L}_{\tilde{R}(\xi,\zeta)})+\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(v^{U}_{\tilde{R}(\xi,\zeta)}))
\end{array}\right\}$$

Namely,

$$\max\left\{\begin{array}{l}
(\varpi_1+\frac{\varpi_2}{2})\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(1-u^{U}_{\tilde{R}(\xi,\zeta)})+\frac{\varpi_2}{2}\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(1-u^{L}_{\tilde{R}(\xi,\zeta)})) \\
+(\varpi_3+\frac{\varpi_4}{2})\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(v^{L}_{\tilde{R}(\xi,\zeta)})+\frac{\varpi_4}{2}\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(v^{U}_{\tilde{R}(\xi,\zeta)}))
\end{array}\right\}$$

where $\varpi_f$ $(f\in\{1,2,3,4\})$ is the importance weight of the single object and $\varpi_1+\varpi_2+\varpi_3+\varpi_4=1$.

**Remark 3.3.** *The importance weight $\varpi_f$ $(f\in\{1,2,3,4\})$ of the single object is usually provided by the DM in advance. On the basis of the definition of IVIFNs, the values of $\varpi_1,\varpi_2$ should be bigger than $\varpi_3,\varpi_4$ because the membership degree in IVIFNs is usually more important than the non-membership degree. Without loss of generality, this study assumes that $\varpi_1=\varpi_2=0.3$ and $\varpi_3=\varpi_4=0.2$.*

Thus, the model (MOD-6) can be transformed into the single objective programming model as follows:

$$\begin{array}{l}
\max\left\{\begin{array}{l}
(\varpi_1+\frac{\varpi_2}{2})\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(1-u^{U}_{\tilde{R}(\xi,\zeta)})+\frac{\varpi_2}{2}\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(1-u^{L}_{\tilde{R}(\xi,\zeta)})) \\
+(\varpi_3+\frac{\varpi_4}{2})\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(v^{L}_{\tilde{R}(\xi,\zeta)})+\frac{\varpi_4}{2}\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}\lambda_{\xi\zeta}\ln(v^{U}_{\tilde{R}(\xi,\zeta)}))
\end{array}\right\} \\
s.t.\left\{\begin{array}{l}
\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}(g_{\xi\zeta}\ln(1-u^{L}_{\tilde{R}(\xi,\zeta)}))\le\ln(1-u^{L}_{\tilde{\varepsilon}}),\ \sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}(g_{\xi\zeta}\ln(1-u^{U}_{\tilde{R}(\xi,\zeta)}))\le\ln(1-u^{U}_{\tilde{\varepsilon}}) \\
\sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}(g_{\xi\zeta}\ln(v^{L}_{\tilde{R}(\xi,\zeta)}))\le\ln(v^{L}_{\tilde{\varepsilon}}),\ \sum\limits_{(\xi,\zeta)\in\tilde{\Omega}}(g_{\xi\zeta}\ln(v^{U}_{\tilde{R}(\xi,\zeta)}))\le\ln(v^{U}_{\tilde{\varepsilon}}) \\
\lambda_{\xi\zeta}+g_{\xi\zeta}\ge0,\quad\lambda_{\xi\zeta}\ge0\quad\left((\xi,\zeta)\in\tilde{\Omega}\right) \\
0\le\hat{u}^{L}_{j}\le\hat{u}^{U}_{j},\ 0\le\hat{v}^{L}_{j}\le\hat{v}^{U}_{j},0\le\ \hat{u}^{U}_{j}+\hat{v}^{U}_{j}\le w_{j},\ j\in\{1,2,\cdots,n\}
\end{array}\right.
\end{array}\quad\text{(MOD-7)}$$

Apparently, the model (MOD-7) can be easily solved by using the MATLAB 7.4.0 or LINGO 11.0 software, and its non-inferior solutions, namely the $\hat{u}^{L}_{j},\hat{u}^{U}_{j},\hat{v}^{L}_{j},\hat{v}^{U}_{j}(j=1,2,\cdots,n)$ can be obtained.

Furthermore, based on Equation (5) the reference point $\widetilde{A}_j^* = \left([(\widetilde{u}_j^L)^*, (\widetilde{u}_j^U)^*], [(\widetilde{v}_j^L)^*, (\widetilde{v}_j^U)^*]\right)$ can be obtained by using the following equations:

$$(\widetilde{u}_j^L)^* = \hat{u}_j^L / w_j,\ (\widetilde{u}_j^U)^* = \hat{u}_j^U / w_j,\ (\widetilde{v}_j^L)^* = \hat{v}_j^L / w_j,\ (\widetilde{v}_j^U)^* = \hat{v}_j^U / w_j\ (j \in \{1, 2, \cdots, n\}) \tag{16}$$

*3.2. Prospect Theory-Based Ranking Method for Identifying the Optimal Alternative*

Based on the derived reference points in Section 3.1, we present a prospect theory-based ranking method for identifying the optimal alternative under IVIFNs environment in case of considering the DM's psychological behavior.

Using the derived reference point vector $A^*$, we first need to calculate the gain and loss values under IVIFNs environment. We employ the ranking method of IVIFNs presented in Definition (2.1) to identify the relative "loss" and the "gain", and utilize the distance measure presented in Definition (2.3) to calculate the prospect value of the alternative $A_i$ $(i \in \{1, 2, \cdots, m\})$ on the attribute $C_j$ $(j \in \{1, 2, \cdots, n\})$. Then, the prospect value of the alternative $A_i$ $(i \in \{1, 2, \cdots, m\})$ with respect to the attribute $C_j$ $(j \in \{1, 2, \cdots, n\})$ is defined as below:

$$\mathcal{P}_{ij} = \begin{cases} \left(d(\widetilde{A}_{ij}, \widetilde{A}_j^*)\right)^{\alpha}, & if\ \widetilde{A}_{ij} \succeq \widetilde{A}_j^* \\ -\theta\left(d(\widetilde{A}_{ij}, \widetilde{A}_j^*)\right)^{\beta}, & if\ \widetilde{A}_{ij} \prec \widetilde{A}_j^* \end{cases} \tag{17}$$

where the ranking order between $\widetilde{A}_{ij}$ and $\widetilde{A}_j^*$ can be usually determined by Definition (2.1) and $d(\widetilde{A}_{ij}, \widetilde{A}_j^*)$ represents the interval-valued intuitionistic fuzzy Euclidean distance between the alternative $A_i$ and the preference point $\widetilde{A}_j^*$ with the attribute $C_j$.

It is apparent to see from Equation (17) that:

(1) If $\widetilde{A}_{ij} \succeq \widetilde{A}_j^*$, then the prospect value of the alternative $A_i$ with the attribute $C_j$ is regard as a "gain" result by comparing the reference point $\widetilde{A}_j^*$ with the attribute value $\widetilde{A}_{ij}$. According to the idea of prospect theory, the DM exhibits risk-averse tendency for gains and the DM's gains can be regard as $(d(\widetilde{A}_{ij}, \widetilde{A}_j^*))^{\alpha}(\alpha \in [0, 1])$, where the parameter $\alpha$ is the estimable coefficient representing the risk aversion of the DM with respect to gains.

(2) If $\widetilde{A}_{ij} \prec \widetilde{A}_j^*$, then the prospect value of the alternative $A_i$ with the attribute $C_j$ is regard as a "loss" result by comparing the reference point $\widetilde{A}_j^*$ with the attribute value $\widetilde{A}_{ij}$. Usually, the DM exhibits risk-seeking tendency for losses and the DM's losses can be regard as $-\theta(d(\widetilde{A}_{ij}, \widetilde{A}_j^*))^{\beta}(\beta \in [0, 1])$, where the parameter $\beta$ is the estimable coefficient representing the risk seeking of the DM with respect to losses and the parameter $\theta$ represents a characteristic of being steeper for losses than for gains. Moreover, the DM is more sensitive to losses than to equal gains. Therefore, the prospect function is steeper in the loss domain than in the gain domain, and the parameter $\theta$ should be bigger than 1.

**Remark 3.4.** *It is noted that the different values of three parameters $\alpha$, $\beta$, $\theta$ in the prospect function usually reflect the different psychological behavior of the DM concerned with reference dependence, diminishing sensitivity and loss aversion. Kahneman and Tversky [18] suggested $\alpha = \beta = 0.88$ and $\theta = 2.25$ which are also employed in this study.*

Based on the obtained prospect values $\mathcal{P}_{ij}(i \in \{1, 2, \cdots, m\}, j \in \{1, 2, \cdots, n\})$, we can get the collective prospect value of each alternative by using the following formula:

$$\mathcal{P}_i = \sum_{j \in \{k: |\widetilde{A}_{ik} \succ \widetilde{A}_j^*, k = 1, 2, \ldots, n\}} w_j \left(d(\widetilde{A}_{ij}, \widetilde{A}_j^*)\right)^{\alpha} - \sum_{j \in \{k: |\widetilde{A}_{ik} \prec \widetilde{A}_j^*, k = 1, 2, \ldots, n\}} w_j \theta \left(d(\widetilde{A}_{ij}, \widetilde{A}_j^*)\right)^{\beta} \tag{18}$$

Clearly, the bigger the collective prospect value $\mathcal{P}_i (i \in \{1, 2, \cdots, m)$ is, the better the alternative $A_i$ will be. Therefore, based on the collective prospect value of each alternative we can determine the ranking order of alternatives and the best alternative with the biggest collective prospect value is selected.

### *3.3. The Decision Process of the Proposed Approach*

The decision process of the proposed method can be roughly divided into three phases.

Phase I: Identify the potential alternatives as well as evaluation attributes, and provide the ratings of alternatives with each attribute represented by IVIFNs and the pair-wise comparison information of alternatives denoted by IVIFNs.

Phase II: Determine the reference points by solving the LINMAP-based nonlinear programming models.

Phase III: Sort the alternatives and choose the best alternative using the prospect theory-based ranking approach.

Detailed activities for these three phases are summarized in Figure 1, which reflects two main contributions of the proposed method: (1) we construct the LINMAP-based nonlinear programming models to identify the reference point under IVIFNs contexts; and (2) we develop a prospect theory-based ranking method with IVIFNs data to identify the optimal alternative.

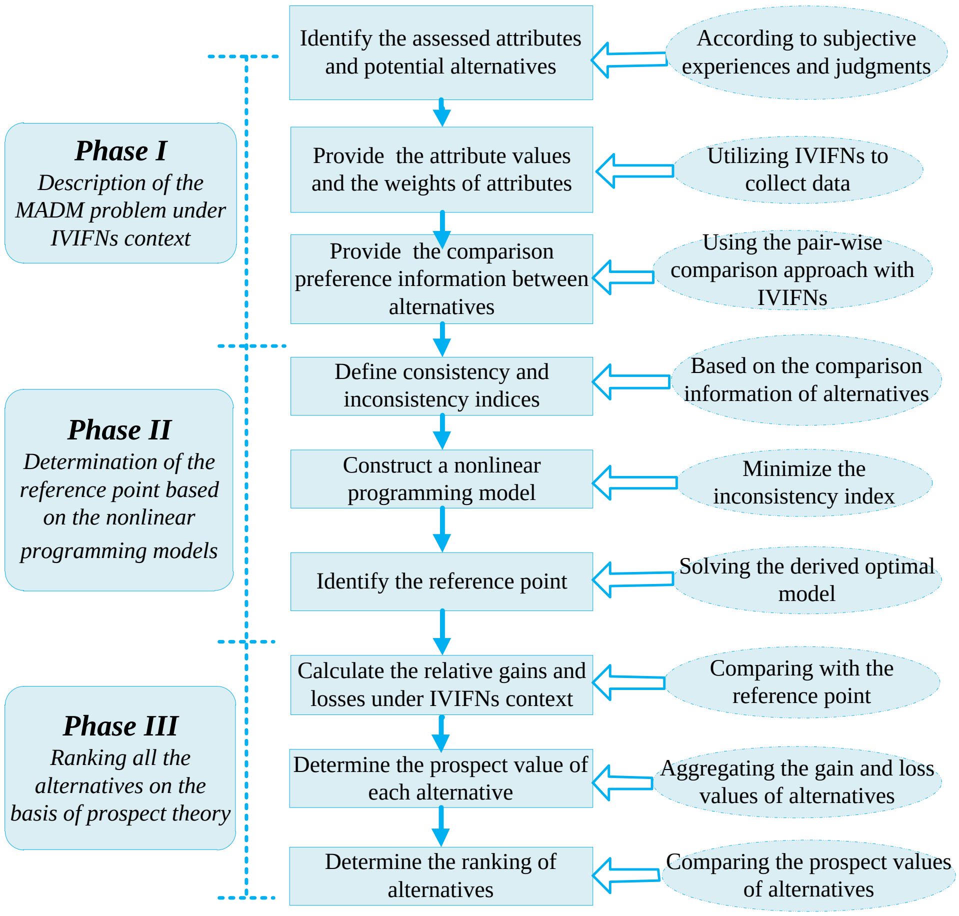


**Figure 1.** Schematic diagram of the proposed approach.

The steps of the proposed method can be summarized in Table 1.

**Table 1.** The steps of the proposed method.

| Decision Phases | Main Steps |
|---|---|
| Description of the MADM problems under IVIFNs context (Phase I) | Step 1: Identify the evaluation attributes and the feasible alternatives. |
| | Step 2: Provide the ratings of alternatives with attributes by using IVIFNs and the weights of attributes. |
| | Step 3: Give the incomplete pairwise comparison preference information between alternatives by using IVIFNs. |
| Determination of the reference point based on nonlinear programming models (Phase II) | Step 4: Calculate the consistency and inconsistency indices by using Equations (9) and (12), respectively. |
| | Step 5: Construct the nonlinear programming model by using the model (MOD-3). |
| | Step 6: Get the reference point vector $A^*$ through solving the model (MOD-3) which is converted into the model (MOD-7). |
| Ranking all the alternatives based on prospect theory (Phase III) | Step 7: Calculate the prospect values of the alternatives $A_i\,(i = 1, 2, \cdots, m)$ with respect to attributes by using Equation (17). |
| | Step 8: Calculate the collective prospect values of alternatives by using Equation (18). |
| | Step 9: Rank these alternatives and the best alternative with the biggest collective prospect value is selected. |

**Remark 3.5.** *In several existing literature reports related to behavioral MADM, the reference point is usually given by the DM in advance [24,26,27], or the following methods are used for the DM to select reference point [23]: (1) zero point; (2) mean value; (3) the medium value; (4) the worst point; and (5) the optimal point. But the developed method does not require knowing the reference point in advance, which can avoid the subjective randomness of selecting the reference point by the DM in the real-life decision process.*

## 4. Case Study

Now, we explore how to solve the selection case of photovoltaic cells by using the proposed method. The comparison analysis of the computational results is conducted to show the superiority of the proposed technique.

### *4.1. Description*

To protect environment and save the non-renewable energy sources, the use of renewable energy plays a significant role in aspect of the production of electric power. Photovoltaic cell is one of the emerging renewable energy sources, which possesses the desirable advantages of the simplicity and the modularity of the energy conversion system. For a promoter or inverter, the selection of a best photovoltaic cell plays a significant role in aspect of maximizing income, minimizing costs and conferring high maturity and reliability. In this study, we assume that there are five potential photovoltaic cells to be selected (please see [35] for more details): Photovoltaic cells with crystalline silicon ($A_1$), Photovoltaic cells with inorganic thin layer ($A_2$), Photovoltaic cells with inorganic thin layer ($A_3$), Photovoltaic cells with advanced, low cost, thin layers ($A_4$), and Photovoltaic cells with advanced III–V thin layer with tracking systems for solar concentration ($A_5$). After analyzing the potential photovoltaic cells, the attributes considered for the assessment of the selection problem are the following [35]: Manufacturing cost ($C_1$), Efficiency in energy conversion ($C_2$), Market share ($C_3$), and Emissions of greenhouse gases generated during the manufacturing process ($C_4$). We assume the weights of attributes are obtained in advance as $\boldsymbol{w} = (w_1, w_2, w_3, w_4)^{\mathrm{T}} = (0.25, 0.2, 0.15, 0.4)^{\mathrm{T}}$. The rating of each alternative under each attribute can be considered as an IVIFN and the results evaluated by the DM are contained in an IVIF decision matrix shown in Table 2. Each cell of the decision matrix $\Re$ denotes the IVIFN assessment of an alternative with respect to an attribute.

**Table 2.** Interval-valued intuitionistic fuzzy decision matrix $\Re$ .

| Attributes \ Alternatives | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $A_1$ | ([0.5, 0.6],[0.2, 0.3]) | ([0.3, 0.4],[0.4, 0.6]) | ([0.4, 0.5],[0.3, 0.5]) | ([0.3, 0.5],[0.4, 0.5]) |
| $A_2$ | ([0.3, 0.5],[0.4, 0.5]) | ([0.1, 0.3],[0.2, 0.4]) | ([0.7, 0.8],[0.1, 0.2]) | ([0.1, 0.2],[0.7, 0.8]) |
| $A_3$ | ([0.6, 0.7],[0.2, 0.3]) | ([0.3, 0.4],[0.4, 0.5]) | ([0.5, 0.8],[0.1, 0.2]) | ([0.1, 0.2],[0.5, 0.8]) |
| $A_4$ | ([0.5, 0.7],[0.1, 0.2]) | ([0.2, 0.4],[0.5, 0.6]) | ([0.4, 0.6],[0.2, 0.3]) | ([0.2, 0.3],[0.4, 0.6]) |
| $A_5$ | ([0.1, 0.4],[0.3, 0.5]) | ([0.7, 0.8],[0.1, 0.2]) | ([0.5, 0.6],[0.2, 0.3]) | ([0.2, 0.3],[0.5, 0.6]) |

According to the DM's subjective experiences and judgments, the incomplete pair-wise comparison information of alternatives is given by using IVIFNs as follows:

$$\widetilde{\Omega} = \left\{ \begin{array}{l} \langle (A_1, A_2), ([0.5, 0.6], [0.2, 0.4]) \rangle, \langle (A_3, A_1), ([0.65, 0.7], [0.1, 0.3]) \rangle, \\ \langle (A_4, A_3), ([0.85, 0.9], [0.05, 0.1]) \rangle, \langle (A_4, A_5), ([0.75, 0.8], [0.1, 0.2]) \rangle, \\ \langle (A_5, A_2), ([0.75, 0.8], [0.1, 0.2]) \rangle \end{array} \right\}$$

where the comparison information $\langle (A_1, A_2), ([0.5, 0.6], [0.2, 0.4]) \rangle$ in $\widetilde{\Omega}$ means that the degree to which the DM thinks the alternative $A_1$ is superior to $A_2$ is the interval [50%, 60%] and $A_1$ is inferior to $A_2$ with a chance between 20% and 40%; and the others have the similar meanings.

### *4.2. Illustration of the Proposed Approach*

In the following, the proposed method is used to aid the promoter to select the best suitable photovoltaic cells. Based on the procedure established in Section 3.3, we first need to derive the reference points.

According to the model (MOD-3) and let $\widetilde{\varepsilon} = ([0.01, 0.1], [0.8, 0.9])$, we construct the optimal model (MOD-8).

$$\min \left\{ \begin{array}{l} ([0.5, 0.6], [0.2, 0.4])\, \lambda_{12} + ([0.65, 0.7], [0.1, 0.3])\, \lambda_{31} + ([0.85, 0.9], [0.05, 0.1])\, \lambda_{43} \\ + ([0.75, 0.8], [0.1, 0.2])\, \lambda_{45} + ([0.75, 0.8], [0.1, 0.2])\, \lambda_{52} \end{array} \right\}$$
$$s.t. \left\{ \begin{array}{l} ([0.5, 0.6], [0.2, 0.4])\, g_{12} + ([0.65, 0.7], [0.1, 0.3])\, g_{31} + ([0.85, 0.9], [0.05, 0.1])\, g_{43} + \\ ([0.75, 0.8], [0.1, 0.2])\, g_{45} + ([0.75, 0.8], [0.1, 0.2])\, g_{52} \geq ([0.01, 0.1], [0.8, 0.9]) \\ g_{12} + \lambda_{12} \geq 0, g_{31} + \lambda_{31} \geq 0, g_{43} + \lambda_{43} \geq 0, g_{45} + \lambda_{45} \geq 0, g_{52} + \lambda_{52} \geq 0, \\ \lambda_{12} \geq 0, \lambda_{31} \geq 0, \lambda_{43} \geq 0, \lambda_{45} \geq 0, \lambda_{52} \geq 0 \\ g_{12} = 0.1\hat{u}_1^L + 0.3\hat{u}_1^U - 0.2\hat{v}_1^L + 0.05\hat{v}_1^U + 0.2\hat{u}_2^U - 0.15\hat{v}_2^L + 0.15\hat{v}_2^U - 0.1\hat{u}_3^L + 0.3\hat{u}_3^U \\ +0.05\hat{v}_3^L - 0.2\hat{v}_3^U - 0.15\hat{u}_4^L + 0.25\hat{u}_4^U + 0.15\hat{v}_4^L - 0.15\hat{v}_4^U + 0.0233 \\ g_{31} = 0.1\hat{u}_1^L - 0.15\hat{v}_1^U + 0.1\hat{u}_2^L - 0.05\hat{u}_2^U + 0.15\hat{v}_2^L - 0.15\hat{v}_2^U + 0.05\hat{u}_3^L - 0.15\hat{v}_3^L \\ +0.05\hat{u}_4^L - 0.1\hat{u}_4^U - 0.15\hat{v}_4^L + 0.15\hat{v}_4^U - 0.0295 \\ g_{43} = 0.05\hat{v}_1^U - 0.05\hat{u}_1^U - 0.05\hat{v}_1^L - 0.15\hat{u}_1^L - 0.05\hat{u}_2^L + 0.05\hat{u}_2^U - 0.15\hat{v}_2^L - 0.15\hat{u}_3^L \\ +0.05\hat{u}_3^U + 0.05\hat{v}_3^L - 0.05\hat{v}_3^U - 0.1\hat{u}_4^L + 0.1\hat{u}_4^U - 0.15\hat{v}_4^U + 0.0902 \\ g_{45} = 0.3\hat{u}_1^L - 0.3\hat{u}_1^U - 0.1\hat{v}_1^L - 0.05\hat{v}_1^U + 0.15\hat{u}_2^L - 0.2\hat{u}_2^U + 0.15\hat{u}_3^U \\ -0.05\hat{v}_3^L - 0.1\hat{v}_3^U - 0.15\hat{u}_4^L + 0.2\hat{u}_4^U + 0.0262 \\ g_{52} = 0.55\hat{u}_1^U - 0.25\hat{u}_1^L - 0.15\hat{v}_1^L - 0.1\hat{u}_2^L + 0.4\hat{u}_2^U - 0.15\hat{v}_2^L - 0.2\hat{u}_3^L \\ +0.2\hat{u}_3^U - 0.15\hat{v}_3^U - 0.05\hat{u}_4^L + 0.05\hat{u}_4^U - 0.15\hat{v}_4^U + 0.0577 \\ 0 \leq \hat{u}_j^L \leq \hat{u}_j^U, 0 \leq \hat{v}_j^L \leq \hat{v}_j^U, \hat{u}_j^U + \hat{v}_j^U \leq w_j,\ j \in \{1, 2, 3, 4\} \end{array} \right. \quad \text{(MOD-8)}$$

In the sense of the model (MOD-7) developed in Section 3.2, the optimal model (MOD-8) is equivalent to the following linear programming model:

$$\max\left\{\begin{array}{l}(0.15*\ln 0.5+0.45*\ln 0.4+0.3*\ln 0.2+0.1*\ln 0.4)*\lambda_{12}\\+(0.15*\ln 0.35+0.45*\ln 0.3+0.3*\ln 0.1+0.1*\ln 0.3)*\lambda_{31}\\+(0.15*\ln 0.35+0.45*\ln 0.2+0.3*\ln 0.15+0.1*\ln 0.2)*\lambda_{43}\\+(0.15*\ln 0.45+0.45*\ln 0.4+0.3*\ln 0.3+0.1*\ln 0.4)*\lambda_{45}\\+(0.15*\ln 0.3+0.45*\ln 0.2+0.3*\ln 0.1+0.1*\ln 0.2)*\lambda_{52}\end{array}\right\}$$

$$s.t.\left\{\begin{array}{l}\ln 0.5*g_{12}+\ln 0.35*g_{31}+\ln 0.35*g_{43}+\ln 0.45*g_{45}+\ln 0.3*g_{52}\le\ln 0.99\\\ln 0.4*g_{12}+\ln 0.3*g_{31}+\ln 0.2*g_{43}+\ln 0.4*g_{45}+\ln 0.2*g_{52}\le\ln 0.9\\\ln 0.2*g_{12}+\ln 0.1*g_{31}+\ln 0.15*g_{43}+\ln 0.3*g_{45}+\ln 0.1*g_{52}\le\ln 0.8\\\ln 0.4*g_{12}+\ln 0.3*g_{31}+\ln 0.2*g_{43}+\ln 0.4*g_{45}+\ln 0.2*g_{52}\le\ln 0.9\\g_{12}+\lambda_{12}\ge 0,\ g_{31}+\lambda_{31}\ge 0,\ g_{43}+\lambda_{43}\ge 0,\ g_{45}+\lambda_{45}\ge 0,\\g_{52}+\lambda_{52}\ge 0,\ \lambda_{12}\ge 0,\ \lambda_{31}\ge 0,\ \lambda_{43}\ge 0,\ \lambda_{45}\ge 0,\ \lambda_{52}\ge 0\\g_{12}=0.1\hat{u}_1^L+0.3\hat{u}_1^U-0.2\hat{v}_1^L+0.05\hat{v}_1^U+0.2\hat{u}_2^U-0.15\hat{v}_2^L+0.15\hat{v}_2^U-0.1\hat{u}_3^L\\+0.3\hat{u}_3^U+0.05\hat{v}_3^L-0.2\hat{v}_3^U-0.15\hat{u}_4^L+0.25\hat{u}_4^U+0.15\hat{v}_4^L-0.15\hat{v}_4^U+0.0233\\g_{31}=0.1\hat{u}_1^L-0.15\hat{v}_1^U+0.1\hat{u}_2^L-0.05\hat{u}_2^U+0.15\hat{v}_2^L-0.15\hat{v}_2^U+0.05\hat{u}_3^L\\-0.15\hat{v}_3^L+0.05\hat{u}_4^L-0.1\hat{u}_4^U-0.15\hat{v}_4^L+0.15\hat{v}_4^U-0.0295\\g_{43}=0.05\hat{v}_1^U-0.05\hat{u}_1^U-0.05\hat{v}_1^L-0.15\hat{u}_1^L-0.05\hat{u}_2^L+0.05\hat{u}_2^U-0.15\hat{v}_2^L-\\0.15\hat{u}_3^L+0.05\hat{u}_3^U+0.05\hat{v}_3^L-0.05\hat{v}_3^U-0.1\hat{u}_4^L+0.1\hat{u}_4^U-0.15\hat{v}_4^U+0.0902\\g_{45}=0.3\hat{u}_1^L-0.3\hat{u}_1^U-0.1\hat{v}_1^L-0.05\hat{v}_1^U+0.15\hat{u}_2^L-0.2\hat{u}_2^U+0.15\hat{u}_3^U\\-0.05\hat{v}_3^L-0.1\hat{v}_3^U-0.15\hat{u}_4^L+0.2\hat{u}_4^U+0.0262\\g_{52}=0.55\hat{u}_1^U-0.25\hat{u}_1^L-0.15\hat{v}_1^L-0.1\hat{u}_2^L+0.4\hat{u}_2^U-0.15\hat{v}_2^L-0.2\hat{u}_3^L+\\0.2\hat{u}_3^U-0.15\hat{v}_3^U-0.05\hat{u}_4^L+0.05\hat{u}_4^U-0.15\hat{v}_4^U+0.0577\\0\le\hat{u}_j^L\le\hat{u}_j^U,0\le\hat{v}_j^L\le\hat{v}_j^U,\hat{u}_j^U+\hat{v}_j^U\le w_j,\ j\in\{1,2,3,4\}\end{array}\right.\quad\text{(MOD-9)}$$

Let $\varpi_1=\varpi_2=0.3$, $\varpi_3=\varpi_4=0.2$, the model (MOD-9) is solved by using the LINGO software and the following results are obtained as:

$$\begin{array}{c}\hat{u}_1^L=0,\ \hat{u}_1^U=0,\ \hat{v}_1^L=0,\ \hat{v}_1^U=0.25,\ \hat{u}_2^L=0,\\\hat{u}_2^U=0.1067273,\ \hat{v}_2^L=0.09327273,\ \hat{v}_2^U=0.09327273,\\\hat{u}_3^L=0,\ \hat{u}_3^U=0.1393964,\ \hat{v}_3^L=0.0,\ \hat{v}_3^U=0,\ \hat{u}_4^L=0,\\\hat{u}_4^U=0.1641527,\ \hat{v}_4^L=0,\ \hat{v}_1^U=0.2358473.\end{array}$$

By Equation (16) and the weights of attributes provided by the DM in advance, the reference points can be calculated as follows:

$$\begin{array}{c}\widetilde{A}_1^*=([0.0,0.0],[0.0,1.0]),\ \widetilde{A}_2^*=([0.0,0.5336],[0.4664,0.4664]),\\\widetilde{A}_3^*=([0.0,0.9293],[0.0,0.0]),\ \widetilde{A}_4^*=([0.0,0.4104],[0.0,0.5896]).\end{array}$$

Then, using the Equation (17) the prospect value of each alternative $A_i$ $(i=1,2,3,4,5)$ with respect with each attribute $C_j$ $(j=1,2,3,4)$ can be calculated, i.e., $\mathcal{P}_{ij}$ $(i=1,2,3,4,5,\ j=1,2,3,4)$ is obtained and shown in Table 3.

**Table 3.** The prospect values of alternatives with attributes.

| Alternatives / Attributes | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $A_1$ | −1.5199 | −0.5814 | 0.5843 | 0.4805 |
| $A_2$ | −1.3439 | −0.6732 | 0.5891 | 0.5949 |
| $A_3$ | −1.6658 | −0.5720 | 0.4591 | 0.4669 |
| $A_4$ | −1.5678 | −0.5006 | 0.4814 | 0.4283 |
| $A_5$ | −1.0318 | 0.5130 | 0.5384 | 0.4932 |

Based on the data in Table 3, using the Equation (18) the collective prospect value of each alternative $\mathcal{P}(A_i)$ $(i = 1,2,3,4,5)$ can be obtained as follows:

$$\mathcal{P}(A_1) = -0.2164,\ \mathcal{P}(A_2) = -0.1443,\ \mathcal{P}(A_3) = -0.2752,$$
$$\mathcal{P}(A_4) = -0.2485,\ \mathcal{P}(A_5) = 0.1227.$$

By comparing the collective prospect values of alternatives, the ranking order of the five candidate alternatives is determined as: $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$. Obviously, the best alternative is $A_5$. That is to say, Photovoltaic cells with advanced III–V thin layer with tracking systems for solar concentration are the best choice for the promoter.

### *4.3. Discussion and Comparative Analysis*

It is noted that in our proposed method the reference points are completely unknown in advance. In some real-world decision process, the reference points may be known in advance [23,26,27], such as the following three common situations: (Case 1) the reference points are the positive ideal solutions; (Case 2) the reference points are the negative ideal solutions; and (Case 3) the reference points are the mean values of all alternatives with each attribute.

The positive ideal solutions for each attribute are assumed as below:

$$\widetilde{A}_1^+ = ([1.0, 1.0], [0.0, 0.0]), \quad \widetilde{A}_2^+ = ([1.0, 1.0], [0.0, 0.0]),$$
$$\widetilde{A}_3^+ = ([1.0, 1.0], [0.0, 0.0]), \quad \widetilde{A}_4^+ = ([1.0, 1.0], [0.0, 0.0]);$$

and the negative ideal solutions for each attribute as follows:

$$\widetilde{A}_1^- = ([0.0, 0.0], [1.0, 1.0]), \quad \widetilde{A}_2^- = ([0.0, 0.0], [1.0, 1.0]),$$
$$\widetilde{A}_3^- = ([0.0, 0.0], [1.0, 1.0]), \quad \widetilde{A}_4^- = ([0.0, 0.0], [1.0, 1.0]).$$

According to the interval-valued intuitionistic fuzzy averaging operator (i.e., Equation (2)), the mean values for each attribute can be computed as follows:

$$\widetilde{A}_1^M = ([0.4247, 0.5957], [0.2169, 0.3393]),\ \widetilde{A}_2^M = ([0.3618, 0.5033], [0.2579, 0.4282]),$$
$$\widetilde{A}_3^M = ([0.5144, 0.6830], [0.1644, 0.2825]),\ \widetilde{A}_4^M = ([0.1853, 0.3097], [0.4891, 0.6491]).$$

For given different reference points, the collective prospect values and the ranking order of all alternatives are obtained by using Equations (17) and (18) and are shown in Table 4.

**Table 4.** The prospect values and rankings of alternatives based on different reference points.

| Different Reference Points | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | The Ranking Orders of Alternatives |
|---|---|---|---|---|---|---|
| Case 1 | −0.2181 | −0.2992 | −0.1032 | −0.1881 | 0.0035 | $A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$ |
| Case 2 | −0.3178 | −0.3820 | −0.3744 | −0.3497 | 0.1105 | $A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$ |
| Case 3 | −0.015 | −0.141 | −0.0801 | −0.1253 | 0.0596 | $A_1 \succ A_5 \succ A_3 \succ A_4 \succ A_2$ |
| The proposed method (Case 4) | 0.0035 | 0.1105 | 0.5384 | 0.5384 | 0.4932 | $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$ |

We put the results of the rankings of alternatives based on different reference points into Figure 2. It is easy to see from Figure 2 that the results of the ranking orders of alternatives obtained by different reference points are usually different. If the reference points are the mean values, the best alternative is $A_1$; if the reference points are the positive ideal solutions or the negative ideal solutions, the best alternative is $A_5$; while if the reference points are completely unknown beforehand, our proposed method constructs the nonlinear programming models to determine the reference points and the best alternative is $A_5$. Usually, in the real-life decision process under the hypothesis that the DM is not completely rational, the different reference points will result in different decision results. Our proposed method does not require knowing the reference points in advance and avoids the subjective randomness of selecting the reference points in the decision analysis by the DM.

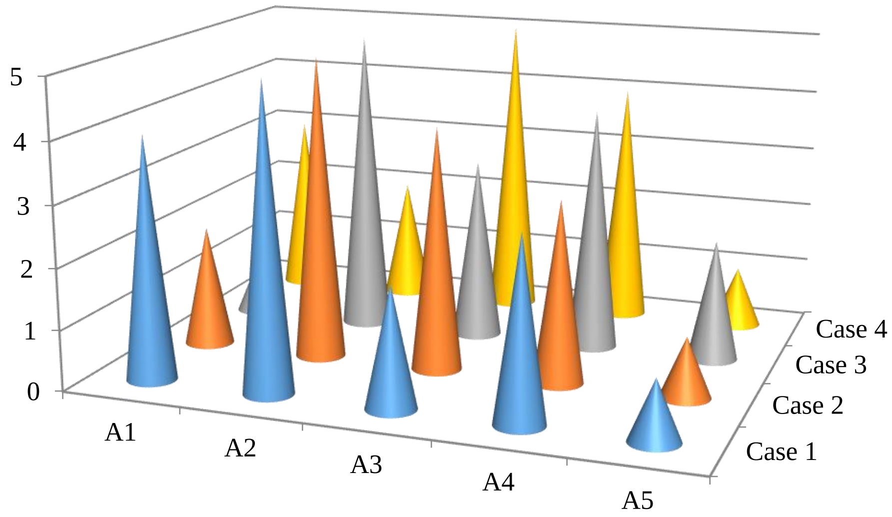


**Figure 2.** Comparison results of the ranking orders of alternatives based on different reference points.

On the other hand, we also note that in our proposed method the DM is bounded rational. However, some widely used decision methods in existing literatures, such as the interval-valued intuitionistic fuzzy TOPSIS method [36] and the interval-valued intuitionistic fuzzy LINMAP method [37] are assumed that the DM is completely rational in the decision making process and do not take into account the DM's psychological behavior in the decision making process. We here conduct a comparison analysis with these two methods.

Utilizing the decision method [36], we calculate the distances between each alternative and the positive ideal solutions as well as the negative ideal solutions, respectively. The calculation results are obtained as follows:

$$d_1^+ = 0.2879,\ d_2^+ = 0.4493,\ d_3^+ = 0.3431,\ d_4^+ = 0.3396,\ d_5^+ = 0.2843,$$
$$d_1^- = 0.2967,\ d_2^- = 0.3068,\ d_3^- = 0.3181,\ d_4^- = 0.3159,\ d_5^- = 0.4043.$$

Then, the closeness index of each alternative is calculated as

$$ci_1 = 0.5075,\ ci_2 = 0.4058,\ ci_3 = 0.4811,\ ci_4 = 0.4819,\ ci_5 = 0.5871$$

and the closeness index-based ranking of alternatives is also obtained as $A_2 \prec A_3 \prec A_4 \prec A_1 \prec A_5$.

Using the decision method [37], we calculate the distances between alternatives and the ideal solutions as below:

$$d_1^* = 0.2051,\ d_2^* = 0.2249,\ d_3^* = 0.2052,\ d_4^* = 0.1799,\ d_5^* = 0.2061$$

and the ranking of the five candidate alternatives is obtained as $A_4 \succ A_1 \succ A_3 \succ A_5 \succ A_2$.

We also depict the results of the ranking of alternatives obtained by the method [37], the method [36] and our proposed method into Figure 3. We clearly know from Figure 3 that the ranking orders of alternatives obtained by these three different approaches are remarkably different. Using the method in [36] and the method in [37], the best alternatives are $A_5$ and $A_4$, respectively, while, using our proposed approach, the best alternative is $A_5$. The main reason for these differences are that both the method [36] and the method [37] are assumed that the DM is fully rational in the decision process, and they fail to consider the DM's psychological behavior, while our proposed method can deal with the situation that the DM's bounded rational and can take fully into account the DM's behavioral characteristics in the ranking process of decision making.

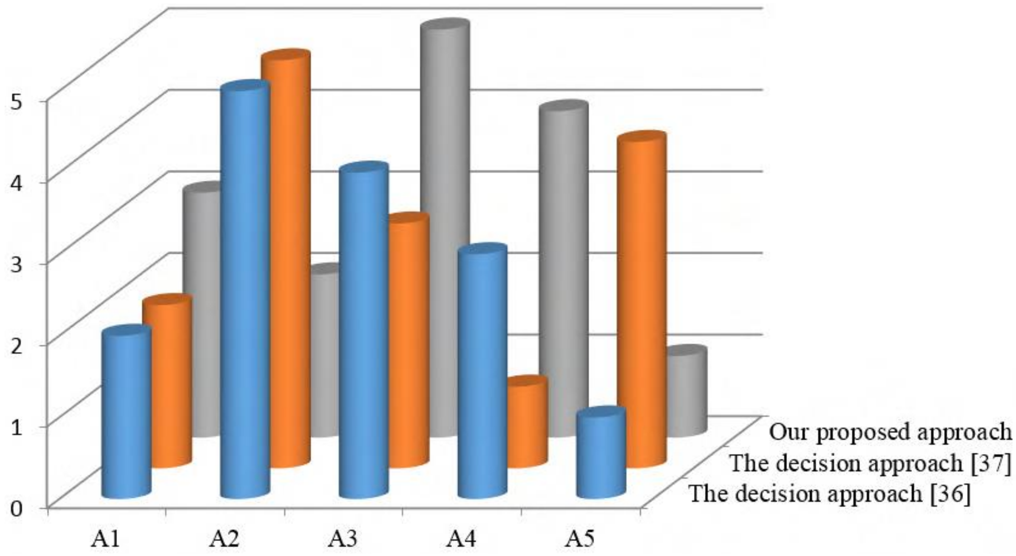


**Figure 3.** Comparison results of the ranking orders of alternatives by different methods.

## 5. Conclusions

In this paper, we have developed an LINMAP-based interval-valued intuitionistic fuzzy behavioral MADM method to help the promoter solve the selection problem of photovoltaic cells. The main characteristic of our developed approach is that it cannot only effectively capture the DM's psychological behavior by using the prospect function, but also sufficiently consider the uncertainty and ambiguity inherent in the human decision process by utilizing the IVIFNs. More importantly, our developed technique which constructs several nonlinear programming models based on LINMAP method to identify the reference points can avoid to the subjective randomness of selecting the reference points in the real-life decision process.

In addition, it is noted that in the developed technique the weights of attributes are assumed to be completely known in advance. But many real-life decision situations in which the weights of attributes are completely unknown or partially known in advance need to be taken into account in future. Meanwhile, in further research it would be interesting to extend the developed method to address the MCDM problems under Pythagorean fuzzy environments [38,39] or heterogeneous fuzzy contexts [40], etc.

**Acknowledgments:** The author is very grateful to the anonymous reviewers and the editor for their insightful and constructive comments and suggestions that have led to an improved version of this paper. The work was supported by the National Natural Science Foundation of China (Nos. 71661010, 71263020 and 71363020), the Major Program of the National Social Science Foundation of China (No. 15ZDC021), and the Natural Science Foundation of Jiangxi Province of China (Nos. 20161BAB211020 and 20142BAB201009).

**Author Contributions:** All work from this manuscript is finished by Xiaolu Zhang.

**Conflicts of Interest:** The author declares no conflict of interest.

## Appendix A

**Lemma A.1.** *[41] Let* $\widetilde{a} = [a^L, a^U]$ *and* $\widetilde{b} = [b^L, b^U]$ *be two intervals, a natural quasi-ordering on intervals is defined as follows:*

$$\widetilde{a} \leq \widetilde{b} \text{ if and only if } a^L \leq b^L \text{ and } a^U \leq b^U$$

*The Proof of Theorem 3.1*

**Proof:** For $0 \leq u^L_{\widetilde{R}(\xi,\zeta)} \leq 1$ and $0 \leq u^U_{\widetilde{R}(\xi,\zeta)} \leq 1$, where "⇔" means "is equivalent to", we have:

$$\begin{gathered}\min\left\{[1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^L_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}},1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^U_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}}]\right\}\\ \Leftrightarrow \max\left\{[\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^L_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}},\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^U_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}}]\right\}\end{gathered}$$

and, furthermore,

$$\begin{gathered}\max\left\{[\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^L_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}},\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^U_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}}]\right\}\\ \Leftrightarrow \max\left\{[\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\lambda_{\xi\zeta}\ln\left(1-u^U_{\widetilde{R}(\xi,\zeta)}\right),\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\lambda_{\xi\zeta}\ln\left(1-u^L_{\widetilde{R}(\xi,\zeta)}\right)]\right\}\end{gathered}$$

Consequently, we can obtain:

$$\begin{gathered}\min\left\{[1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^L_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}},1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^U_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}}]\right\}\\ \Leftrightarrow \max\left\{[\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\lambda_{\xi\zeta}\ln\left(1-u^U_{\widetilde{R}(\xi,\zeta)}\right),\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\lambda_{\xi\zeta}\ln\left(1-u^L_{\widetilde{R}(\xi,\zeta)}\right)]\right\}\end{gathered}$$

Similarly, for $0 < v^L_{\widetilde{R}(\xi,\zeta)} \leq 1$ and $0 < v^U_{\widetilde{R}(\xi,\zeta)} \leq 1$, we also obtain

$$\begin{gathered}\max\left\{[\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(v^L_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}},\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(v^U_{\widetilde{R}(\xi,\zeta)}\right)^{\lambda_{\xi\zeta}}]\right\}\\ \Leftrightarrow \max\left\{[\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(\lambda_{\xi\zeta}\ln(v^L_{\widetilde{R}(\xi,\zeta)})\right),\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(\lambda_{\xi\zeta}\ln(v^U_{\widetilde{R}(\xi,\zeta)})\right)]\right\}\end{gathered}$$

According to Lemma A.1, the first constraint condition of model (MOD-4) may be transformed into the following inequalities:

$$\begin{gathered}1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}(1-u^L_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}} \geq u^L_{\widetilde{\varepsilon}}, 1-\prod_{(\xi,\zeta)\in\widetilde{\Omega}}(1-u^U_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}} \geq u^U_{\widetilde{\varepsilon}},\\ \prod_{(\xi,\zeta)\in\widetilde{\Omega}}(v^L_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}} \leq v^L_{\widetilde{\varepsilon}}, \prod_{(\xi,\zeta)\in\widetilde{\Omega}}(v^U_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}} \leq v^U_{\widetilde{\varepsilon}}\end{gathered}$$

Namely,

$$\begin{gathered}\prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^L_{\widetilde{R}(\xi,\zeta)}\right)^{g_{\xi\zeta}} \leq 1-u^L_{\widetilde{\varepsilon}}, \prod_{(\xi,\zeta)\in\widetilde{\Omega}}\left(1-u^U_{\widetilde{R}(\xi,\zeta)}\right)^{g_{\xi\zeta}} \leq 1-u^U_{\widetilde{\varepsilon}},\\ \prod_{(\xi,\zeta)\in\widetilde{\Omega}}(v^L_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}} \leq v^L_{\widetilde{\varepsilon}}, \prod_{(\xi,\zeta)\in\widetilde{\Omega}}(v^U_{\widetilde{R}(\xi,\zeta)})^{g_{\xi\zeta}} \leq v^U_{\widetilde{\varepsilon}}\end{gathered}$$

which are equivalent to the following inequalities:

$$\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(g_{\xi\zeta}\ln\left(1-u^{L}_{\widetilde{R}(\xi,\zeta)}\right)\right)\leq\ln\left(1-u^{L}_{\widetilde{h}}\right),\ \sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(g_{\xi\zeta}\ln\left(1-u^{U}_{\widetilde{R}(\xi,\zeta)}\right)\right)\leq\ln\left(1-u^{U}_{\widetilde{h}}\right),$$
$$\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(g_{\xi\zeta}\ln\left(v^{L}_{\widetilde{R}(\xi,\zeta)}\right)\right)\leq\ln\left(v^{L}_{\widetilde{h}}\right),\ \sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(g_{\xi\zeta}\ln\left(v^{U}_{(\xi,\zeta)}\right)\right)\leq\ln\left(v^{U}_{\widetilde{h}}\right)$$

Then, the model (MOD-4) is transformed into the bi-objective interval programming model as below:

$$\max\left\{\left[\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\lambda_{\xi\zeta}\ln\left(1-u^{U}_{\widetilde{R}(\xi,\zeta)}\right),\ \sum_{(\xi,\zeta)\in\widetilde{\Omega}}\lambda_{\xi\zeta}\ln\left(1-u^{L}_{\widetilde{R}(\xi,\zeta)}\right)\right]\right\}$$
$$\max\left\{\left[\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(\lambda_{\xi\zeta}\ln(v^{L}_{\widetilde{R}(\xi,\zeta)})\right),\ \sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(\lambda_{\xi\zeta}\ln(v^{U}_{\widetilde{R}(\xi,\zeta)})\right)\right]\right\}$$
$$s.t.\begin{cases}\sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(g_{\xi\zeta}\ln\left(1-u^{L}_{\widetilde{R}(\xi,\zeta)}\right)\right)\leq\ln\left(1-u^{L}_{\widetilde{h}}\right),\ \sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(g_{\xi\zeta}\ln\left(1-u^{U}_{\widetilde{R}(\xi,\zeta)}\right)\right)\leq\ln\left(1-u^{U}_{\widetilde{h}}\right)\\ \sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(g_{\xi\zeta}\ln\left(v^{L}_{\widetilde{R}(\xi,\zeta)}\right)\right)\leq\ln\left(v^{L}_{\widetilde{h}}\right),\ \sum_{(\xi,\zeta)\in\widetilde{\Omega}}\left(g_{\xi\zeta}\ln\left(v^{U}_{(\xi,\zeta)}\right)\right)\leq\ln\left(v^{U}_{\widetilde{h}}\right)\\ \lambda_{\xi\zeta}+g_{\xi\zeta}\geq 0,\quad \lambda_{\xi\zeta}\geq 0\quad\left((\xi,\zeta)\in\widetilde{\Omega}\right)\\ \hat{u}^{L}_{j}\leq\hat{u}^{U}_{j},\ \hat{v}^{L}_{j}\leq\hat{v}^{U}_{j},\ \hat{u}^{U}_{j}+\hat{v}^{U}_{j}\leq w_j,\hat{u}^{L}_{j}\geq 0,\ \hat{u}^{U}_{j}\geq 0,\ \hat{v}^{L}_{j}\geq 0,\ \hat{v}^{U}_{j}\geq 0,\ j\in\{1,2,\cdots,n\}\end{cases}$$

Thus, the proof of Theorem 3.1 is completed.

## References


1. Lee, A.H.; Chen, H.H.; Kang, H.-Y. A model to analyze strategic products for photovoltaic silicon thin-film solar cell power industry. *Renew. Sustain. Energy Rev.* **2011**, *15*, 1271–1283. [CrossRef]
2. Srinivasan, V.; Shocker, A.D. Linear programming techniques for multidimensional analysis of preferences. *Psychometrika* **1973**, *38*, 337–369. [CrossRef]
3. Hwang, C.L.; Yoon, K.S. *Multiple Attibute Decision Methods and Applications*; Springer: Berlin/Heidelberg, Germany, 1981.
4. Opricovic, S.; Tzeng, G.H. Multicriteria planning of post-earthquake sustainable reconstruction. *Comput. Aided Civil Infrastruct. Eng.* **2002**, *17*, 211–220. [CrossRef]
5. Park, J.H.; Cho, H.J.; Kwun, Y.C. Extension of the VIKOR method for group decision making with interval-valued intuitionistic fuzzy information. *Fuzzy Optim. Decis. Mak.* **2011**, *10*, 233–253. [CrossRef]
6. Roy, B. Classement et choix en présence de points de vue multiples. *RAIRO Oper. Res. Rech. Opér.* **1968**, *2*, 57–75.
7. Hashemi, S.S.; Hajiagha, S.H.R.; Zavadskas, E.K.; Mahdiraji, H.A. Multicriteria group decision making with ELECTRE III method based on interval-valued intuitionistic fuzzy information. *Appl. Math. Model.* **2016**, *40*, 1554–1564. [CrossRef]
8. Govindan, K.; Khodaverdi, R.; Vafadarnikjoo, A. Intuitionistic fuzzy based DEMATEL method for developing green practices and performances in a green supply chain. *Expert Syst. Appl.* **2015**, *42*, 7207–7220. [CrossRef]
9. Zavadskas, E.K.; Antucheviciene, J.; Hajiagha, S.H.R.; Hashemi, S.S. Extension of weighted aggregated sum product assessment with interval-valued intuitionistic fuzzy numbers (WASPAS-IVIF). *Appl. Soft Comput.* **2014**, *24*, 1013–1021. [CrossRef]
10. Xia, H.C.; Li, D.F.; Zhou, J.Y.; Wang, J.M. Fuzzy LINMAP method for multiattribute decision making under fuzzy environments. *J. Comput. Syst. Sci.* **2006**, *72*, 741–759. [CrossRef]
11. Wan, S.P.; Li, D.F. Atanassov's intuitionistic fuzzy programming method for heterogeneous multiattribute group decision making with Atanassov's intuitionistic fuzzy truth degrees. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 300–312. [CrossRef]
12. Li, D.F. Extension of the LINMAP for multiattribute decision making under Atanassov’s intuitionistic fuzzy environment. *Fuzzy Optim. Decis. Mak.* **2008**, *7*, 17–34. [CrossRef]

13. Zhang, X.; Xu, Z. Interval programming method for hesitant fuzzy multi-attribute group decision making with incomplete preference over alternatives. *Comput. Ind. Eng.* **2014**, *75*, 217–229. [CrossRef]
14. Zhang, X.; Xu, Z.; Xing, X. Hesitant fuzzy programming technique for multidimensional analysis of hesitant fuzzy preferences. *OR Spectrum* **2016**, *38*, 789–817. [CrossRef]
15. Chen, T.Y. An interval-valued intuitionistic fuzzy LINMAP method with inclusion comparison possibilities and hybrid averaging operations for multiple criteria group decision making. *Knowl. Based Syst.* **2013**, *45*, 134–146. [CrossRef]
16. Hajiagha, S.H.R.; Hashemi, S.S.; Zavadskas, E.K.; Akrami, H. Extensions of LINMAP model for multi criteria decision making with grey numbers. *Technol. Econ. Dev. Econ.* **2012**, *18*, 636–650. [CrossRef]
17. Camerer, C. Bounded rationality in individual decision making. *Exp. Econ.* **1998**, *1*, 163–183. [CrossRef]
18. Kahneman, D.; Tversky, A. Prospect theory: An analysis of decision under risk. *Econometrica* **1979**, *47*, 263–291. [CrossRef]
19. Gomes, L.F.A.M.; Lima, M.M.P.P. TODIM: Basics and application to multicriteria ranking of projects with environmental impacts. *Found. Comput. Decis. Sci.* **1992**, *16*, 113–127.
20. Gomes, L.F.A.M.; Rangel, L.A.D.; Maranhão, F.J.C. Multicriteria analysis of natural gas destination in Brazil: An application of the TODIM method. *Math. Comput. Model.* **2009**, *50*, 92–100. [CrossRef]
21. Gomes, L.F.A.M.; Machado, M.A.S.; Rangel, L.A.D. Behavioral multi-criteria decision analysis: The TODIM method with criteria interactions. *Ann. Oper. Res.* **2013**, *211*, 531–548. [CrossRef]
22. Passos, A.C.; Teixeira, M.G.; Garcia, K.C.; Cardoso, A.M.; Gomes, L.F.A.M. Using the TODIM-FSE method as a decision-making support methodology for oil spill response. *Comput. Oper. Res.* **2014**, *42*, 40–48. [CrossRef]
23. Liu, P.D.; Jin, F.; Zhang, X.; Su, Y.; Wang, M.H. Research on the multi-attribute decision-making under risk with interval probability based on prospect theory and the uncertain linguistic variables. *Knowl. Based Syst.* **2011**, *24*, 554–561. [CrossRef]
24. Krohling, R.A.; de Souza, T.T.M. Combining prospect theory and fuzzy numbers to multi-criteria decision making. *Expert Syst. Appl.* **2012**, *39*, 11487–11493. [CrossRef]
25. Krohling, R.A.; Pacheco, A.G.C.; Siviero, A.L.T. IF-TODIM: An intuitionistic fuzzy TODIM to multi-criteria decision making. *Knowl. Based Syst.* **2013**, *53*, 142–146. [CrossRef]
26. Liu, Y.; Fan, Z.-P.; Zhang, Y. Risk decision analysis in emergency response: A method based on cumulative prospect theory. *Comput. Oper. Res.* **2014**, *42*, 75–82. [CrossRef]
27. Fan, Z.-P.; Zhang, X.; Chen, F.-D.; Liu, Y. Multiple attribute decision making considering aspiration-levels: A method based on prospect theory. *Comput. Ind. Eng.* **2013**, *65*, 341–350. [CrossRef]
28. Zhang, X.L.; Xu, Z.S. The TODIM analysis approach based on novel measured functions under hesitant fuzzy environment. *Knowl. Based Syst.* **2014**, *61*, 48–58. [CrossRef]
29. Atanassov, K.; Gargov, G. Interval valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349. [CrossRef]
30. Xu, Z.S. Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making. *Control Decis.* **2007**, *2*, 215–219.
31. Wang, Z.J.; Li, K.W.; Xu, J.H. A mathematical programming approach to multi-attribute decision making with interval-valued intuitionistic fuzzy assessment information. *Expert Syst. Appl.* **2011**, *38*, 12462–12469. [CrossRef]
32. Wang, Z.; Li, K.W.; Wang, W. An approach to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights. *Inform. Sci.* **2009**, *179*, 3026–3040. [CrossRef]
33. Xu, Z.S.; Chen, J. An overview of distance and similarity measures of intuitionistic fuzzy sets. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.* **2008**, *16*, 529–555. [CrossRef]
34. Ishibuchi, H.; Tanaka, H. Multiobjective programming in optimization of the interval objective function. *Eur. J. Oper. Res.* **1990**, *48*, 219–225. [CrossRef]
35. García-Cascales, M.S.; Lamata, M.T.; Sánchez-Lozano, J.M. Evaluation of photovoltaic cells in a multi-criteria decision making process. *Ann. Oper. Res.* **2012**, *199*, 373–391. [CrossRef]
36. Park, J.H.; Park, I.Y.; Kwun, Y.C.; Tan, X. Extension of the TOPSIS method for decision making problems under interval-valued intuitionistic fuzzy environment. *Appl. Math. Model.* **2011**, *35*, 2544–2556. [CrossRef]
37. Wang, Z.-J.; Li, K.W. An interval-valued intuitionistic fuzzy multiattribute group decision making framework with incomplete preference over alternatives. *Expert Syst. Appl.* **2012**, *39*, 13509–13516. [CrossRef]

38. Zhang, X.; Xu, Z. Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *Int. J. Intell. Syst.* **2014**, *29*, 1061–1078. [CrossRef]
39. Zhang, X. Multicriteria Pythagorean fuzzy decision analysis: a hierarchical QUALIFLEX approach with the closeness index-based ranking methods. *Inform. Sci.* **2016**, *330*, 104–124. [CrossRef]
40. Zhang, X.; Xu, Z.; Wang, H. Heterogeneous multiple criteria group decision making with incomplete weight information: A deviation modeling approach. *Inf. Fusion* **2015**, *25*, 49–62. [CrossRef]
41. Tong, S.C. Interval number and fuzzy number linear programmings. *Fuzzy Sets Syst.* **1994**, *66*, 301–306.