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# On Classical Solutions for A Kuramoto–Sinelshchikov–Velarde-Type Equation

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**Abstract:** The Kuramoto–Sinelshchikov–Velarde equation describes the evolution of a phase turbulence in reaction-diffusion systems or the evolution of the plane flame propagation, taking into account the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front. In this paper, we prove the well-posedness of the classical solutions for the Cauchy problem.

**Keywords:** existence; uniqueness; stability; Kuramoto–Sinelshchikov–Velarde-type equation; Cauchy problem

**MSC:** 35G25; 35K55

## 1. Introduction

In this paper, we investigate the well-posedness of the following Cauchy problem:

$$\begin{cases} \partial_t u + \kappa(\partial_x u)^2 + q(\partial_x u)^3 + r(\partial_x u)^4 + \delta\partial_x^3 u \\ \qquad +\beta^2\partial_x^4 u + \mu\partial_x^2 u_\varepsilon + \gamma u\partial_x^2 u = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

with

$$\gamma = 0, \quad \beta \neq 0, \quad \text{or} \tag{2}$$
$$q = r = 0, \quad \beta \neq 0, \quad \kappa = 2\gamma. \tag{3}$$

Under Assumption (2), we assume on the initial datum

$$u_0 \in H^2(\mathbb{R}). \tag{4}$$

Instead, under Assumption (3), we assume (4) or

$$u_0 \in H^3(\mathbb{R}). \tag{5}$$

Observe that if $q = r = \delta = \mu = \beta = \gamma = 0$, Equation (1) reads

$$\partial_t u + \kappa(\partial_x u)^2 + \delta\partial_x^3 u = 0. \tag{6}$$

Using the variable (see [1]),

$$v = \partial_x u \tag{7}$$

Equation (6) is equivalent to the Korteweg-de Vries equation [2]

$$\partial_t v + \kappa \partial_x v^2 + \delta \partial_x^3 v = 0, \tag{8}$$

that has a very wide range of applications, such as magnetic fluid waves, ion sound waves, and longitudinal astigmatic waves.

From a mathematical point of view, in [3–5], the Cauchy problem for (8) is studied, while in [6], the author reviewed the travelling wave solutions for (8). Moreover, in [7–9], the convergence of the solution of (8) to the unique entropy one of the Burgers equation is proven.

Taking $\kappa = r = \delta = \mu = \beta = \gamma = 0$ and using the variable (7), (1) becomes

$$\partial_t v + q \partial_x v^3 + \delta \partial_x^3 v = 0, \tag{9}$$

which is known as the modified Korteweg-de Vries equation.

[10–15] show that (9) is a non-slowly varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons. In [3,5], the Cauchy problem for (9) is studied, while in [9,16], the convergence of the solution of (9) to the unique entropy solution of the following scalar conservation law

$$\partial_t v + q \partial_x v^3 = 0. \tag{10}$$

Assuming $\kappa = 1$ and $q = r = \gamma = 0$, (1) reads

$$\partial_t u + (\partial_x u)^2 + \delta \partial_x^3 u + \beta^2 \partial_x^4 u + \mu \partial_x^2 u = 0. \tag{11}$$

Equation (11) arises in interesting physical situations, for example as a model for long waves on a viscous fluid owing down an inclined plane [17] and to derive drift waves in a plasma [18]. Equation (11) was derived also independently by Kuramoto [19–21] as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky [22] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

Equation (11) also describes incipient instabilities in a variety of physical and chemical systems [23–25]. Moreover, (11), which is also known as the Benney–Lin equation [26,27], was derived by Kuramoto in the study of phase turbulence in the Belousov–Zhabotinsky reaction [28].

The dynamical properties and the existence of exact solutions for (11) have been investigated in [29–34]. In [35–37], the control problem for (11) with periodic boundary conditions, and on a bounded interval are studied, respectively. In [38], the problem of global exponential stabilization of (11) with periodic boundary conditions is analyzed. In [39], it is proposed a generalization of optimal control theory for (11), while in [40] the problem of global boundary control of (11) is considered. In [41], the existence of solitonic solutions for (11) is proven. In [1,42], the well-posedness of the Cauchy problem for (11) is proven, using the energy space technique and the fixed-point method, respectively. In particular, in [1], the well-posedness is proven, under the assumption

$$\delta = 0, \qquad \mu = \beta^2 = 1. \tag{12}$$

Observe that thanks to (7), Equation (11) is equivalent to the following one

$$\partial_t v + \partial_x v^2 + \delta \partial_x^3 v + \beta^2 \partial_x^4 v = 0. \tag{13}$$

Consequently, following [8,9,43], in [44], it is proven that when $\delta$, $\beta^2$ go to zero, the solution of (13) converges to the unique entropy one of the Burgers equation.

Taking $q = r = 0$, (1) is known as the Kuramoto-Velarde (KV) equation [45,46], which describes slow space-time variations of disturbances at interfaces, diffusion-reaction fronts and plasma instability fronts.

From a mathematical point of view, in [47] the exact solutions for the KV equation are studied, while in [48], the initial boundary problem is analyzed. In [49], the well-posedness of the Cauchy problem for the KV equation is proven in the energy spaces.

The main result of this paper is the following theorem.

**Theorem 1.** *Let $T > 0$ be given. The following statements hold.*

*(i) If (2) and (4) hold then there exists a solution $u$ of (1), such that*

$$u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^2(\mathbb{R})). \tag{14}$$

*(ii) If (3) and (4) hold then there exists a solution $u$ of (1) satisfying (14).*

*(iii) If (3) and (5) hold then there exists a solution $u$ of (1), such that*

$$u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^3(\mathbb{R})). \tag{15}$$

*(iv) If (2) and (4) hold then $u$ is unique.*

*(v) If (3) and (5) hold then $u$ is unique.*

*(vi) If (2) and (4) hold and $u_1$ and $u_2$ are two solutions of (1), we have that*

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R})} \le e^{C(T)t}\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})}, \tag{16}$$

*for some suitable $C(T) > 0$, and every $0 \le t \le T$.*

*(vii) If (3) and (5) hold then we have (16).*

Theorem 1 improves the existing literature (see [1]) because it gives the well-posedness of (1) under Assumption (2), without additional assumption on the constants. Under Assumptions (3) and (4), Theorem 1 gives only the existence of the solution, while the uniqueness is guaranteed by Assumption (5). The argument of Theorem 1 relies on deriving suitable a priori estimates together with an application of the Cauchy–Kovalevskaya Theorem [50]. We conjecture that our argument can be applied also to the the initial boundary value problem and to multidimensional version of the problem.

The paper is organized as follows. In Section 2, we prove Theorem 1, under Assumption (2) and (4). In Section 3, Theorem 1 is proven, under Assumption (3) and (4) or (5). We state the conclusions in Section 4.

## 2. Proof of Theorem 1, under the Assumptions (2) and (4)

In this section, we prove Theorem 1, under the assumptions (2) and (4). Thanks to (2), (1) reads

$$\begin{cases} \partial_t u + \kappa(\partial_x u)^2 + q(\partial_x u)^3 + r(\partial_x u)^4 + \delta\partial_x^3 u + \beta^2\partial_x^4 u + \mu\partial_x^2 u_\varepsilon, & t>0, \quad x\in\mathbb{R},\\ u(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases} \tag{17}$$

Let us prove some a priori estimates on $u$. We denote with $C_0$ the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on $T$.

We prove the following lemma.

**Lemma 1.** *Fix $T > 0$. Then, we have that*

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + e^{\frac{\mu^2 t}{\beta^2}}\beta^2\int_0^t e^{-\frac{\mu^2 s}{\beta^2}}\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{18}$$

*for every* $0 \leq t \leq T$.

**Proof.** Let $0 \leq t \leq T$. Multiplying (17) by $-2\partial_x^2 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} =& -2\int_{\mathbb{R}}\partial_x^2 u\partial_t u dx\\ =&2\kappa\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u dx + 2q\int_{\mathbb{R}}(\partial_x u)^3 dx + 2r\int_{\mathbb{R}}(\partial_x u)^4\partial_x^2 u dx\\ &+ 2\alpha\int_{\mathbb{R}}\partial_x^3 u\partial_x^2 u dx + 2\beta^2\int_{\mathbb{R}}\partial_x^2 u\partial_x^4 u dx + 2\mu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ =& -2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\mu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, we have that

$$\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 2\mu\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \tag{19}$$

Observe that

$$2\mu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 2\mu\int_{\mathbb{R}}\partial_x^2 u\partial_x^2 u dx = -2\mu\int_{\mathbb{R}}\partial_x u\partial_x^3 u dx \leq 2|\mu|\int_{\mathbb{R}}|\partial_x u||\partial_x^3 u| dx.$$

Consequently, by the Young inequality,

$$\begin{aligned}2\mu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq& 2\int_{\mathbb{R}}\left|\frac{\mu\partial_x u}{\beta}\right|\left|\beta\partial_x^3 u\right| dx\\ \leq&\frac{\mu^2}{\beta^2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (19) that

$$\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq \frac{\mu^2}{\beta^2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

The Gronwall Lemma and (4) give

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 e^{\frac{\mu^2 t}{\beta^2}}\int_0^t e^{-\frac{\mu^2 s}{\beta^2}}\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq \beta^2 e^{\frac{\mu^2 t}{\beta^2}}\|u_0\|^2_{H^2(\mathbb{R})} \leq C(T),$$

that is (18). □

**Lemma 2.** *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T), \tag{20}$$

*for every $0 \leq t \leq T$. In particular,*

$$\|u\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T). \tag{21}$$

*Moreover,*

$$\int_0^t \|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R})} ds \leq C(T), \tag{22}$$

*for every $0 \leq t \leq T$.*

The proof of the previous lemma is based on the regularity of the functions $u$ and the following result.

**Lemma 3.** *For each $t \geq 0$, we have that*

$$\int_{\mathbb{R}} |u||\partial_x u|^3 dx \leq 2\sqrt{\|u(t,\cdot)\|_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^2 u(t,\cdot)\|_{L^2(\mathbb{R})}}\,\|\partial_x u(t,\cdot)\|^3_{L^2(\mathbb{R})}, \tag{23}$$

$$\int_{\mathbb{R}} |u||\partial_x u|^4 dx \leq 2\sqrt{2}\sqrt{\|u(t,\cdot)\|_{L^2(\mathbb{R})}}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\sqrt{\|\partial_x u(t,\cdot)\|^7_{L^2(\mathbb{R})}}. \tag{24}$$

**Proof.** We begin by proving (24). Thanks to the regularity of the function $u$ and the Hölder inequality,

$$u^2(t,x) = 2\int_{-\infty}^{x} u\partial_x u dy \leq 2\int_{\mathbb{R}} |u||\partial_x u| dx \leq 2\|u(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})}.$$

Therefore,

$$\|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \leq 2\|u(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})}. \tag{25}$$

Again, by the regularity of the function $u$ and the Hölder inequality,

$$(\partial_x u(t,x))^2 = 2\int_{-\infty}^{x} \partial_x u\partial_x^2 u dy \leq 2\int_{\mathbb{R}} |\partial_x u||\partial_x^2 u| dx \leq 2\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}.$$

Hence,

$$\|\partial_x u(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \leq 2\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}. \tag{26}$$

Consequently, by (25) and (26),

$$\begin{aligned}\int_{\mathbb{R}} |u||\partial_x u|^3 dx \leq& \|u(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \leq& 2\sqrt{\|u(t,\cdot)\|_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^2 u(t,\cdot)\|_{L^2(\mathbb{R})}}\,\|\partial_x u(t,\cdot)\|^3_{L^2(\mathbb{R})},\end{aligned}$$

that is (23).

Finally, we prove (24). By (25) and (26),

$$\begin{aligned}\int_{\mathbb{R}} |u||\partial_x u|^4 dx \leq& \|u(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \leq& 2\sqrt{2}\sqrt{\|u(t,\cdot)\|_{L^2(\mathbb{R})}}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\sqrt{\|\partial_x u(t,\cdot)\|^7_{L^2(\mathbb{R})}},\end{aligned}$$

which gives (24). □

**Proof of Lemma 2.** Let $0 \leq t \leq T$. Multiplying (17) by $2u$, integrating on $\mathbb{R}$, we have that

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} u\partial_t u dx\\ =& -2\kappa\int_{\mathbb{R}} u(\partial_x u)^2 dx - 2q\int_{\mathbb{R}} u(\partial_x u)^3 dx - 2r\int_{\mathbb{R}} u(\partial_x u)^4 dx\\ &- 2\delta\int_{\mathbb{R}} u\partial_x^3 u dx - 2\beta\int_{\mathbb{R}} u\partial_x^4 u dx - 2\mu\int_{\mathbb{R}} u\partial_x^2 u dx\\ =& -2\kappa\int_{\mathbb{R}} u(\partial_x u)^2 dx - 2q\int_{\mathbb{R}} u(\partial_x u)^3 dx - 2r\int_{\mathbb{R}} u(\partial_x u)^4 dx\\ &+ 2\delta\int_{\mathbb{R}} \partial_x u\partial_x^2 u dx + 2\beta^2\int_{\mathbb{R}} \partial_x u\partial_x^3 u dx + 2\mu\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ =& -2\kappa\int_{\mathbb{R}} u(\partial_x u)^2 dx - 2q\int_{\mathbb{R}} u(\partial_x u)^3 dx - 2r\int_{\mathbb{R}} u(\partial_x u)^4 dx\end{aligned}$$

$$- 2\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\mu \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad = -2\kappa \int_{\mathbb{R}} u(\partial_x u)^2 dx - 2q \int_{\mathbb{R}} u(\partial_x u)^3 dx\\
&\qquad\quad - 2r \int_{\mathbb{R}} u(\partial_x u)^4 dx + 2\mu \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{27}$$

Due to (18), (23), (25) and the Young inequality,

$$\begin{aligned}
2|\kappa| \int_{\mathbb{R}} |u|(\partial_x u)^2 dx \leq& 2|\kappa| \left\|u(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\leq& C_0 \left\|u(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2}\left\|u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} + C_0\\
\leq& \left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})} \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0\\
\leq& C_0 \left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})} + C_0\\
\leq& C_0 \left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0,\\
2|q| \int_{\mathbb{R}} |u||\partial_x u|^3 dx \leq& 4|q|\sqrt{\left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})}}\sqrt{\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}} \left\|\partial_x u(t,\cdot)\right\|^3_{L^2(\mathbb{R})}\\
\leq& C_0\sqrt{\left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})}}\sqrt{\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}}\\
\leq& C_0 \left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})} + \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
=& C_0 \left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})} + \frac{|\beta|}{|\beta|}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
\leq& C_0 \left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0 + \frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{\beta^2}\\
\leq& C_0 \left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0,\\
2|r| \int_{\mathbb{R}} |u|(\partial_x u)^4 dx \leq& 4\sqrt{2}|r|\sqrt{\left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})}}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\sqrt{\left\|\partial_x u(t,\cdot)\right\|^7_{L^2(\mathbb{R})}}\\
\leq& C_0\sqrt{\left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})}}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
=& \frac{C_0}{|\beta|}\sqrt{\left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})}}|\beta|\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
\leq& C_0 \left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\leq& C_0 \left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

It follows from (18) and (27) that

$$\begin{aligned}
&\frac{d}{dt}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad \leq C_0 \left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0 + 2|\mu| \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad \leq C_0 \left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0.
\end{aligned}$$

The Gronwall Lemma and (4) give

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ \le & e^{C_0 t}\|u_0\|^2_{H^2(\mathbb{R})} + C_0 e^{C_0 t}\int_0^t e^{-C_0 s} ds \le C(T),\end{aligned}$$

which gives (20).

Equation (21) follows from (18), (20) and (25).

Finally, we prove (22). We begin by observing that ([51] Lemma 2.3) says that

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \le 6\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (18) and (20),

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \le C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (20), we have that

$$\begin{aligned}\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^4_{L^4(\mathbb{R})} ds \le & C_0\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ \le & C_0 e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T),\end{aligned}$$

which gives (22). □

**Lemma 4.** *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{28}$$

*In particular, we have that*

$$\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{29}$$

*for every $0 \le t \le T$.*

**Proof.** Let $0 \le t \le T$. Multiplying (17) by $2\partial_x^4 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}}\partial_x^4 u\partial_t u dx\\ =& -2\kappa\int_{\mathbb{R}}(\partial_x u)^2\partial_x^4 u dx - 2q\int_{\mathbb{R}}(\partial_x u)^3\partial_x^4 u dx - 2r\int_{\mathbb{R}}(\partial_x u)^4\partial_x^4 u dx\\ & - 2\delta\int_{\mathbb{R}}\partial_x^3 u\partial_x^4 u dx - 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\mu\int_{\mathbb{R}}\partial_x^2 u dx\partial_x^4 u dx\\ =& -2\kappa\int_{\mathbb{R}}(\partial_x u)^2\partial_x^4 u dx - 2q\int_{\mathbb{R}}(\partial_x u)^3\partial_x^4 u dx + 8r\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\partial_x^2 u\partial_x^3 u dx\\ & - 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\mu\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &= -2\kappa\int_{\mathbb{R}}(\partial_x u)^2\partial_x^4 u dx - 2q\int_{\mathbb{R}}(\partial_x u)^3\partial_x^4 u dx\\ &\quad + 8r\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\partial_x^2 u\partial_x^3 u dx + 2\mu\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{30}$$

Due to the Young inequality,

$$\begin{aligned}2|\kappa|\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^4|u dx =& 2\int_{\mathbb{R}}\left|\frac{\kappa(\partial_x u)^2}{\beta}\right|\left|\beta\partial_x^4 u\right| dx\\ \leq& \frac{\kappa^2}{\beta^2}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ 2|q|\int_{\mathbb{R}}|\partial_x u|^3|\partial_x^4 u|dx =& \int_{\mathbb{R}}\left|\frac{2q(\partial_x u)^2}{\beta}\right|\left|\beta\partial_x^4 u\right| dx\\ \leq& \frac{2q^2}{\beta^2}\int_{\mathbb{R}}(\partial_x u)^6 dx + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq& \frac{2q^2}{\beta^2}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ 8|r|\int_{\mathbb{R}}|\partial_x u_\varepsilon|^3|\partial_x^2 u||\partial_x^3 u|dx \leq& 8\|\partial_x u\|^3_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|r\partial_x^2 u||\partial_x^3 u|dx\\ \leq& 4r^2\|\partial_x u\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &+ 4\|\partial_x u\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, by (30),

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\leq \frac{\kappa^2}{\beta^2}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{2q^2}{\beta^2}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})}\\ &\quad + 4r^2\|\partial_x u\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad + 4\|\partial_x u\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (4), (18), (20) and an integration on $(0,t)$ that

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\leq \|u_0\|^2_{H^2(\mathbb{R})} + \frac{\kappa^2}{\beta^2}\int_0^t\|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R})} ds\\ &\quad + \frac{2q^2}{\beta^2}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R})} ds\\ &\quad + 4r^2\|\partial_x u\|^3_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\quad + 4\|\partial_x u\|^3_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\end{aligned} \tag{31}$$

$$
\begin{aligned}
&\leq C(T)\left(1+\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\\
&\quad+4r^2\|\partial_x u\|^3_{L^\infty((0,T)\times\mathbb{R})}e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad+4\|\partial_x u\|^3_{L^\infty((0,T)\times\mathbb{R})}e^{\mu^2 t}\beta^2\int_0^t e^{-\mu^2 s}\beta^2\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\leq C(T)\left(1+\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}+\|\partial_x u\|^3_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned}
$$

Due to the Young inequality,

$$
\|\partial_x u\|^3_{L^\infty((0,T)\times\mathbb{R})}\leq\frac{D_1}{2}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}+\frac{1}{2D_1}\|\partial_x u\|^4_{L^\infty((0,T)\times\mathbb{R})},
$$

where $D_1$ is a positive constant, which will be specified later. It follows from (31) that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\leq C(T)\left(1+(1+D_1)\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}+\frac{1}{D_1}\|\partial_x u\|^4_{L^\infty((0,T)\times\mathbb{R})}\right)
\end{aligned}
\tag{32}
$$

We prove (28). Thanks to (18), (26) and (32),

$$
\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\leq C(T)\sqrt{\left(1+(1+D_1)\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}+\frac{1}{D_1}\|\partial_x u\|^4_{L^\infty((0,T)\times\mathbb{R})}\right)}.
$$

Hence,

$$
\left(1-\frac{C(T)}{D_1}\right)\|\partial_x u\|^4_{L^\infty((0,T)\times\mathbb{R})}-C(T)(1+D_1)\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}-C(T)\leq 0.
$$

Choosing

$$
D_1=2C(T),
\tag{33}
$$

we have that

$$
\frac{1}{2}\|\partial_x u\|^4_{L^\infty((0,T)\times\mathbb{R})}-C(T)\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}-C(T)\leq 0,
$$

which gives (28).

Finally, (29) follows from (28), (32) and (33). □

**Lemma 5.** *Fix $T>0$. There exists a constant $C(T)>0$, such that*

$$
\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\int_0^t\|\partial_t u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\leq C(T),
\tag{34}
$$

*for every $0\leq t\leq T$.*

**Proof.** Let $0\leq t\leq T$. Multiplying (17) by $2\partial_t u$, an integration on $\mathbb{R}$ gives

$$
\begin{aligned}
&\beta^2\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad=-2\kappa\int_{\mathbb{R}}(\partial_x u)^2\partial_t u dx-2q\int_{\mathbb{R}}(\partial_x u)^3\partial_t u dx\\
&\qquad-2r\int_{\mathbb{R}}(\partial_x u)^4\partial_t u dx-2\delta\int_{\mathbb{R}}\partial_x^3 u\partial_t u dx-2\mu\int_{\mathbb{R}}\partial_x^2 u\partial_t u dx.
\end{aligned}
\tag{35}
$$

Due to the Young inequality,

$$
\begin{aligned}
2|\kappa| \int_{\mathbb{R}} (\partial_x u)^2 |\partial_t u| dx =& 2 \int_{\mathbb{R}} \left| \frac{\kappa (\partial_x u)^2}{\sqrt{D_2}} \right| \left| \sqrt{D_2} \partial_t u \right| dx \\
&\leq \frac{\kappa^2}{D_2} \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
2|q| \int_{\mathbb{R}} |\partial_x u|^3 |\partial_t u| dx =& 2 \int_{\mathbb{R}} \left| \frac{q (\partial_x u)^3}{\sqrt{D_2}} \right| \left| \sqrt{D_2} \partial_t u \right| dx \\
\leq& \frac{q^2}{D_2} \int_{\mathbb{R}} (\partial_x u)^6 dx + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
\leq& \frac{q^2}{D_2} \|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})} \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
\leq& \frac{C(T)}{D_2} \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
2|r| \int_{\mathbb{R}} (\partial_x u)^4 |\partial_t u| dx =& 2 \int_{\mathbb{R}} \left| \frac{r (\partial_x u)^4}{\sqrt{D_2}} \right| \left| \sqrt{D_2} \partial_t u \right| dx \\
\leq& \frac{r^2}{D_2} \int_{\mathbb{R}} (\partial_x u)^8 dx + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
\leq& \frac{r^2}{D_2} \|\partial_x u\|^4_{L^\infty((0,T)\times\mathbb{R})} \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
\leq& \frac{C(T)}{D_2} \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
2|\delta| \int_{\mathbb{R}} |\partial_x^3 u| |\partial_t u| dx =& 2 \int_{\mathbb{R}} \left| \frac{\delta \partial_x^3 u}{D_2} \right| |D_2 \partial_t u| \, dx \\
\leq& \frac{\delta^2}{D_2} \left\| \partial_x^3 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
2|\mu| \int_{\mathbb{R}} |\partial_x^2 u| |\partial_t u| dx =& 2 \int_{\mathbb{R}} \left| \frac{\mu \partial_x^2 u}{D_2} \right| |D_2 \partial_t u| \, dx \\
\leq& \frac{\mu^2}{D_2} \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

where $D_2$ is a positive constant, which will be specified later. It follows from (35) that

$$
\begin{aligned}
&\beta^2 \frac{d}{dt} \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + (2 - 5D_2) \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad \leq \frac{C(T)}{D_2} \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{\delta^2}{D_2} \left\| \partial_x^3 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\mu^2}{D_2} \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Choosing $D_2 = \frac{1}{5}$, we have that

$$
\begin{aligned}
&\beta^2 \frac{d}{dt} \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad \leq C(T) \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + 5\delta^2 \left\| \partial_x^3 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + 5\mu^2 \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Integrating on $(0,t)$, by (4), (18), (20) and (22), we obtain

$$
\begin{aligned}
&\beta^2 \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \int_0^t \|\partial_t u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \\
&\qquad \leq C_0 + C(T) \int_0^t \|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R})} ds + 5\delta^2 \int_0^t \left\| \partial_x^3 u(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds
\end{aligned}
$$

$$
\begin{aligned}
&+5\mu^2\int_0^t\left\|\partial_x^2u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
\leq&C(T)+5\delta^2e^{\mu^2t}\beta^2\int_0^t e^{-\mu^2 s}\beta^2\left\|\partial_x^3u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&+5\mu^2e^{C_0t}\int_0^t e^{-C_0 s}\left\|\partial_x^2u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\leq C(T),
\end{aligned}
$$

which gives (34). □

Now, we prove Theorem 1.

**Proof of Theorem 1.** Fix $T>0$. Thanks to Lemmas 1, 2, 4 5 and the Cauchy–Kovalevskaya Theorem [50], we have that $u$ is solution of (1) and (14) holds.

We prove (16). Let $u_1$ and $u_2$ be two solutions of (1), which verify (14) that is

$$
\begin{cases}
\partial_t u_1+\kappa(\partial_x u_1)^2+q(\partial_x u_1)^3+r(\partial_x u_1)^4+\delta\partial_x^3u_1+\mu\partial_x^2u_1+\beta^2\partial_x^4u_1=0, & t>0,\ x\in\mathbb{R},\\
u_1(0,x)=u_{1,0}(x), & x\in\mathbb{R},
\end{cases}
$$

$$
\begin{cases}
\partial_t u_2+\kappa(\partial_x u_2)^2+q(\partial_x u_2)^3+r(\partial_x u_2)^4+\delta\partial_x^3u_2+\mu\partial_x^2u_2+\beta^2\partial_x^4u_2=0, & t>0,\ x\in\mathbb{R},\\
u_1(0,x)=u_{1,0}(x), & x\in\mathbb{R},
\end{cases}
$$

Then, the function

$$
\omega=u_1-u_2 \tag{36}
$$

is the solution of the following Cauchy problem:

$$
\begin{cases}
\partial_t\omega+\kappa\left[(\partial_x u_1)^2-(\partial_x u_2)^2\right]+q\left[(\partial_x u_1)^3-(\partial_x u_2)^3\right]\\
\qquad+r\left[(\partial_x u_1)^4-(\partial_x u_2)^4\right]+\delta\partial_x^3\omega\\
\qquad+\mu\partial_x^2\omega+\beta^2\partial_x^4\omega=0, & t>0,\ x\in\mathbb{R},\\
\omega(0,x)=u_{1,0}(x)-u_{2,0}(x), & x\in\mathbb{R}.
\end{cases} \tag{37}
$$

Observe that thanks to (36),

$$
\begin{aligned}
(\partial_x u_1)^2-(\partial_x u_2)^2=&(\partial_x u_1+\partial_x u_2)(\partial_x u_1-\partial_x u_2)=(\partial_x u_1+\partial_x u_2)\partial_x\omega,\\
(\partial_x u_1)^3-(\partial_x u_2)^3=&[(\partial_x u_1)^2+(\partial_x u_2)^2+\partial_x u_1\partial_x u_2](\partial_x u_1-\partial_x u_2)\\
=&[(\partial_x u_1)^2+(\partial_x u_2)^2+\partial_x u_1\partial_x u_2]\partial_x\omega,\\
(\partial_x u_1)^4-(\partial_x u_2)^4=&[(\partial_x u_1)^2+(\partial_x u_2)^2][(\partial_x u_1)^2-(\partial_x u_2)^2]\\
=&[(\partial_x u_1)^2+(\partial_x u_2)^2](\partial_x u_1+\partial_x u_2)(\partial_x u_1-\partial_x u_2)\\
=&[(\partial_x u_1)^2+(\partial_x u_2)^2](\partial_x u_1+\partial_x u_2)\partial_x\omega.
\end{aligned}
$$

Consequently, (37) is equivalent to the following equation:

$$
\begin{aligned}
&\partial_t\omega+\kappa(\partial_x u_1+\partial_x u_2)\partial_x\omega+q[(\partial_x u_1)^2+(\partial_x u_2)^2+\partial_x u_1\partial_x u_2]\partial_x\omega\\
&\quad r[(\partial_x u_1)^2+(\partial_x u_2)^2](\partial_x u_1+\partial_x u_2)\partial_x\omega+\delta\partial_x^3\omega+\mu\partial_x^2\omega+\beta^2\partial_x^4\omega=0.
\end{aligned} \tag{38}
$$

Since

$$
\begin{aligned}
2\delta\int_{\mathbb{R}}\omega\partial_x^3\omega=&-2\delta\int_{\mathbb{R}}\partial_x\omega\partial_x^2\omega dx=0,\\
2\mu\int_{\mathbb{R}}\omega\partial_x^2\omega=&-2\mu\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2\beta^2\int_{\mathbb{R}}\omega\partial_x^4\omega dx=&-2\beta^2\int_{\mathbb{R}}\partial_x\omega\partial_x^3\omega dx=2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned} \tag{39}
$$

multiplying (38) by $2\omega$, an integration on $\mathbb{R}$ gives,

$$\begin{aligned}\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ 2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &= 2\mu\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\kappa\int_{\mathbb{R}}(\partial_x u_1+\partial_x u_2)\omega\partial_x\omega dx\\ &\quad - 2q\int_{\mathbb{R}}[(\partial_x u_1)^2+(\partial_x u_2)^2+\partial_x u_1\partial_x u_2]\omega\partial_x\omega dx\\ &\quad - 2r\int_{\mathbb{R}}[(\partial_x u_1)^2+(\partial_x u_2)^2](\partial_x u_1+\partial_x u_2)\omega\partial_x\omega dx.\end{aligned} \tag{40}$$

Observe that since $u_1,\ u_2\in H^2(\mathbb{R})$, for every $0\le t\le T$, we have

$$\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})},\ \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})}\le C(T). \tag{41}$$

Thanks to (41), we have that

$$\begin{aligned}&(\partial_x u_1+\partial_x u_2)^2\le\left(\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})}+\|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})}\right)^2\le C(T),\\ &[(\partial_x u_1)^2+(\partial_x u_2)^2+\partial_x u_1\partial_x u_2]^2\\ &\quad\le\left(\|\partial_x u_1\|^2_{L^\infty((0,T)\times\mathbb{R})}+\|\partial_x u_2\|^2_{L^\infty((0,T)\times\mathbb{R})}+\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})}\right)^2\\ &\quad\le C(T),\\ &[(\partial_x u_1)^2+(\partial_x u_2)^2]^2\le\left(\|\partial_x u_1\|^2_{L^\infty((0,T)\times\mathbb{R})}+\|\partial_x u_2\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)^2\le C(T).\end{aligned}$$

Consequently, by the Young inequality,

$$\begin{aligned}2|\kappa|&\int_{\mathbb{R}}|(\partial_x u_1+\partial_x u_2)||\omega||\partial_x\omega|dx\\ &\le\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\kappa^2\int_{\mathbb{R}}(\partial_x u_1+\partial_x u_2)^2(\partial_x\omega)^2dx\\ &\le\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ 2|q|&\int_{\mathbb{R}}|(\partial_x u_1)^2+(\partial_x u_2)^2+\partial_x u_1\partial_x u_2||\omega||\partial_x\omega|dx\\ &\le\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+q^2\int_{\mathbb{R}}[(\partial_x u_1)^2+(\partial_x u_2)^2+\partial_x u_1\partial_x u_2]^2(\partial_x\omega)^2dx\\ &\le\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ 2|r|&\int_{\mathbb{R}}|(\partial_x u_1)^2+(\partial_x u_2)^2||\partial_x u_1+\partial_x u_2||\omega||\partial_x\omega|dx\\ &\le\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+r^2\int_{\mathbb{R}}((\partial_x u_1)^2+(\partial_x u_2)^2)^2(\partial_x u_1+\partial_x u_2)^2(\partial_x\omega)^2dx\\ &\le\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (40) that

$$\begin{aligned}\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}&+2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le 3\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{42}$$

Due to the Young inequality,

$$C(T)\|\partial_x\omega(t,\cdot)\|^2=-C(T)\int_{\mathbb{R}}\omega\partial_x^2\omega dx\le C(T)\int_{\mathbb{R}}|\omega||\partial_x^2\omega|dx=2\int_{\mathbb{R}}\left|\frac{C(T)\omega}{2\beta}\right|\left|\beta\partial_x^2\omega\right|dx$$

$$\leq C(T) \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Therefore, by (42),

$$\frac{d}{dt} \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C(T) \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

The Gronwall Lemma and (37) gives

$$\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 e^{C(T)t} \int_0^t e^{-C(T)s} \left\|\partial_x^2\omega(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq e^{C(T)t} \|\omega_0\|_{L^2(\mathbb{R})}. \tag{43}$$

Equation (16) follows from (36) and (43). □

## 3. Proof of Theorem 1, under the assumptions (3) and (4) or (5)

In this section, we prove Theorem 1, under the assumptions (3) and (4). Thanks to (3), (1) reads

$$\begin{cases} \partial_t u + 2\gamma(\partial_x u)^2 + \delta\partial_x^3 u + \beta^2\partial_x^4 u + \mu\partial_x^2 u_\varepsilon + \gamma u\partial_x^2 u = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{44}$$

Let us prove some a priori estimates on $u$. We denote with $C_0$ the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on $T$.

**Lemma 6.** *Fix $T > 0$ and assume (4) or (5). There exists a constant $C(T) > 0$, such that*

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 e^{\frac{\mu^2 t}{\beta^2}} \int_0^t e^{-\frac{\mu^2 s}{\beta^2}} \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T), \tag{45}$$

*for every $0 \leq t \leq T$.*

**Proof.** Let $0 \leq t \leq T$. Multiplying (44) by $2u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} u\partial_t u dx \\ =& -4\gamma \int_{\mathbb{R}} u(\partial_x u)^2 dx - 2\delta \int_{\mathbb{R}} u\partial_x^3 u dx - 2\beta^2 \int_{\mathbb{R}} u\partial_x^4 u dx \\ & - 2\mu \int_{\mathbb{R}} u\partial_x^2 u dx - 2\gamma \int_{\mathbb{R}} u^2\partial_x^2 u dx \\ =& -4\gamma \int_{\mathbb{R}} u(\partial_x u)^2 dx + 2\delta \int_{\mathbb{R}} \partial_x u\partial_x^2 u dx + 2\beta^2 \int_{\mathbb{R}} \partial_x u\partial_x^3 u dx \\ & - 2\mu \int_{\mathbb{R}} u\partial_x^2 u dx + 4\gamma \int_{\mathbb{R}} u(\partial_x^2 u)^2 dx \\ =& -2\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\mu \int_{\mathbb{R}} u\partial_x^2 u dx. \end{aligned}$$

Therefore,

$$\frac{d}{dt} \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = -2\mu \int_{\mathbb{R}} u\partial_x^2 u dx. \tag{46}$$

Due to the Young inequality,

$$\begin{aligned} |\mu| \int_{\mathbb{R}} |u||\partial_x^2 u| dx =& 2\int_{\mathbb{R}} \left|\frac{\mu u}{\beta}\right| \left|\beta\partial_x^2 u\right| dx \\ \leq& \frac{\mu^2}{\beta^2} \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Consequently by (46),

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le\frac{\mu^2}{\beta^2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

It follows from the Gronwall Lemma and (4) that

$$\begin{aligned}&\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2 e^{\frac{\mu^2 t}{\beta^2}}\int_0^t e^{-\frac{\mu^2 s}{\beta^2}}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ \le&\beta^2 C_0 e^{\frac{\mu^2 t}{\beta^2}}\le C(T),\end{aligned}$$

which gives (45). □

**Lemma 7.** *Fix $T>0$ and assume (4) or (5). There exists a positive constant $C(T)>0$, such that (21) holds. In particular,*

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T), \tag{47}$$

*for every $0\le t\le T$. Moreover, (22) holds, for every $0\le t\le T$.*

**Proof.** Let $0\le t\le T$. Multiplying (44) by $-2\partial_x^2 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}=&-2\int_{\mathbb{R}}u\partial_x^2 u dx\\ =&2\kappa\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u dx+2\delta\int_{\mathbb{R}}\partial_x^2 u\partial_x^3 u dx+2\beta^2\int_{\mathbb{R}}\partial_x^2 u\partial_x^4 u dx\\ &+2\mu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2\gamma\int_{\mathbb{R}}u(\partial_x^2 u)^2 dx\\ =&-2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\mu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2\gamma\int_{\mathbb{R}}u(\partial_x^2 u)^2 dx.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &=-2\gamma\int_{\mathbb{R}}u(\partial_x^2 u)^2 dx+2\mu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le 2|\gamma|\int_{\mathbb{R}}\int_{\mathbb{R}}|u|(\partial_x^2 u)^2 dx+2|\mu|\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le 2|\gamma|\,\|u\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2|\mu|\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le C_0\left(1+\|u\|_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (4), (45) and an integration on $(0,t)$ that

$$\begin{aligned}&\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\le C_0+C_0\left(1+\|u\|_{L^\infty((0,T)\times\mathbb{R})}\right)\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\le C_0+C_0\left(1+\|u\|_{L^\infty((0,T)\times\mathbb{R})}\right)e^{\frac{\mu^2 t}{\beta^2}}\int_0^t e^{-\frac{\mu^2 s}{\beta^2}}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\le C(T)\left(1+\|u\|_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned} \tag{48}$$

We prove (21). Thanks to (25), (45) and (48),

$$\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \le C(T)\sqrt{\left(1+\|u\|_{L^\infty((0,T)\times\mathbb{R})}\right)}.$$

Thus, we obtain that

$$\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0.$$

Arguing as in [52, Lemma 2.4] or [53, Lemma 2.3], we have (21).

Finally, (47) follows from (21) and (48), while thanks to (45) and (47), arguing as in Lemma 2, we have (22). □

**Lemma 8.** *Fix $T > 0$ and assume (4) or (5). There exists a positive constant $C(T) > 0$, such that*

$$\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \|\partial_t u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{49}$$

*for every $0 \le t \le T$. Moreover, (28) holds.*

**Proof.** Let $0 \le t \le T$. Multiplying (44) by $2\partial_t u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\beta^2 \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R})} \\
&= -2\gamma\int_{\mathbb{R}}(\partial_x u)^2\partial_t u dx - 2\delta\int_{\mathbb{R}}\partial_x^3 u\partial_t u dx \\
&\quad - 2\mu\int_{\mathbb{R}}\partial_x^2 u\partial_t u dx - 2\gamma\int_{\mathbb{R}} u\partial_x^2 u\partial_t u dx.
\end{aligned} \tag{50}$$

Due to (21) and the Young inequality,

$$\begin{aligned}
2|\gamma|\int_{\mathbb{R}}(\partial_x u)^2|\partial_t u| dx =& 2\int_{\mathbb{R}}\left|\frac{\gamma(\partial_x u)^2}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_t u\right| dx \\
\le& \frac{\gamma^2}{D_3}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
2|\delta|\int_{\mathbb{R}}|\partial_x^3 u||\partial_t u| dx =& 2\int_{\mathbb{R}}\left|\frac{\delta\partial_x^3 u}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_t u\right| dx \\
\le& \frac{\delta^2}{D_3}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
2|\mu|\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u| dx =& 2\int_{\mathbb{R}}\left|\frac{\mu\partial_x^2 u}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_t u\right| dx \\
\le& \frac{\mu^2}{D_3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
2|\gamma|\int_{\mathbb{R}}|u\partial_x^2 u||\partial_t u| dx =& 2\int_{\mathbb{R}}\left|\frac{\gamma u\partial_x^2 u}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_t u\right| dx \\
\le& \frac{\gamma^2}{D_3}\int_{\mathbb{R}} u^2(\partial_x^2 u)^2 dx + D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
\le& \frac{\gamma^2}{D_3}\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
\le& \frac{C(T)}{D_3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where $D_3$ is a positive constant, which will be specified later. Consequently, by (50),

$$\begin{aligned}&\beta^2 \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\left(1-2D_3\right)\left\|\partial_t u(t,\cdot)\right\|_{L^2(\mathbb{R})}\\ &\quad\le \frac{\gamma^2}{D_3}\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{\delta^2}{D_3}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{C(T)}{D_3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Taking $D_3 = \frac{1}{2}$, we have

$$\begin{aligned}&\beta^2 \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_t u(t,\cdot)\right\|_{L^2(\mathbb{R})}\\ &\quad\le 2\gamma^2\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + 2\delta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (4), (45), (47) that

$$\begin{aligned}&\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\quad\le \|u_0\|^2_{H^2(\mathbb{R})} + 2\gamma^2\int_0^t \left\|\partial_x u(s,\cdot)\right\|^4_{L^4(\mathbb{R})} ds\\ &\qquad + 2\delta^2\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + C(T)\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T),\end{aligned}$$

which gives (49). □

**Lemma 9.** *Fix $T > 0$ and assume (5). There exists a positive constant $C(T) > 0$, such that*

$$\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\int_{\mathbb{R}}\left\|\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{51}$$

*for every $0 \le t \le T$. In particular, we have that*

$$\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{52}$$

**Proof.** Let $0 \le t \le T$. Multiplying (44) by $-2\partial_x^6 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& -2\int_{\mathbb{R}}\partial_x^6 u\partial_t u dx\\ =&2\gamma\int_{\mathbb{R}}(\partial_x u)^2\partial_x^6 u dx + 2\delta\int_{\mathbb{R}}\partial_x^3 u\partial_x^6 u dx + 2\beta^2\int_{\mathbb{R}}\partial_x^4 u\partial_x^6 u dx\\ &+ 2\mu\int_{\mathbb{R}}\partial_x^2 u\partial_x^6 u dx + 2\gamma\int_{\mathbb{R}} u\partial_x^2 u\partial_x^6 u dx\\ =& -4\gamma\int_{\mathbb{R}}\partial_x u\partial_x^2 u\partial_x^5 u dx - 2\delta\int_{\mathbb{R}}\partial_x^4 u\partial_x^5 u dx - 2\beta^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &- 2\mu\int_{\mathbb{R}}\partial_x^3 u\partial_x^5 u dx - 2\gamma\int_{\mathbb{R}}\partial_x u\partial_x^2 u\partial_x^5 u dx - 2\gamma\int_{\mathbb{R}} u\partial_x^3 u\partial_x^5 u dx\\ =& -6\gamma\int_{\mathbb{R}}\partial_x u\partial_x^2 u\partial_x^5 u dx - 2\beta^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\mu\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &- 2\gamma\int_{\mathbb{R}} u\partial_x^3 u\partial_x^5 u dx.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}&\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\quad=-6\gamma\int_{\mathbb{R}}\partial_x u\partial_x^2 u\partial_x^5 u dx+2\mu\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2\gamma\int_{\mathbb{R}}u\partial_x^3 u\partial_x^5 u dx.\end{aligned}\tag{53}$$

Due to (21), (28), (49) and the Young inequality,

$$\begin{aligned}6|\gamma|\int_{\mathbb{R}}|\partial_x u\partial_x^2 u||\partial_x^5 u|dx=&6\int_{\mathbb{R}}\left|\frac{\gamma\partial_x u\partial_x^2 u}{\beta\sqrt{D_4}}\right|\left|\beta\sqrt{D_4}\partial_x^5 u\right|dx\\\leq&\frac{3\gamma^2}{D_4}\int_{\mathbb{R}}(\partial_x u)^2(\partial_x^2 u)^2 dx+3\beta^2 D_3\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\\leq&\frac{3\gamma^2}{\beta^2 D_4}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+3\beta^2 D_4\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\\leq&\frac{C(T)}{D_4}+3\beta^2 D_4\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\2|\gamma|\int_{\mathbb{R}}|u\partial_x^3 u||\partial_x^5 u|dx=&2\int_{\mathbb{R}}\left|\frac{\gamma u\partial_x^3 u}{\beta\sqrt{D_4}}\right|\left|\beta\sqrt{D_4}\partial_x^5 u dx\right|dx\\\leq&\frac{\gamma^2}{\beta^2 D_4}\int_{\mathbb{R}}u^2(\partial_x^3 u)^2 dx+\beta^2 D_4\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\\leq&\frac{\gamma^2}{\beta^2 D_4}\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2 D_4\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\\leq&\frac{C(T)}{D_4}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2 D_4\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

where $D_4$ is a positive constant, which will be specified later. Therefore, by (53),

$$\begin{aligned}&\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2(2-4D_4)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\quad\leq\frac{C(T)}{D_4}+\frac{C(T)}{D_4}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\mu\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{54}$$

Observe that

$$2\mu\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=2\mu\int_{\mathbb{R}}\partial_x^4 u\partial_x^4 u dx=-2\mu\int_{\mathbb{R}}\partial_x^3 u\partial_x^5 u dx.$$

Thus, by the Young inequality,

$$\begin{aligned}2|\mu|\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\leq&2|\mu|\int_{\mathbb{R}}\partial_x^3 u\partial_x^5 u dx\\=&2\int_{\mathbb{R}}\left|\frac{\mu\partial_x^3 u}{\beta\sqrt{D_4}}\right|\left|\beta\sqrt{D_4}\partial_x^4 u\right|dx\\\leq&\frac{\mu^2}{\beta^2 D_4}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2 D_4\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (54) that

$$\begin{aligned}&\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2(2-5D_4)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&\quad\leq\frac{C(T)}{D_4}+\frac{C(T)}{D_4}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Choosing $D_4 = \frac{1}{5}$, we have that

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq C(T) + C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Integrating on $(0,t)$, by (5) and (47), we obtain that

$$\begin{aligned}&\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\int_0^t\left\|\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\leq \|u_0\|^2_{H^3(\mathbb{R})} + C(T)t + C(T)\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \leq C(T),\end{aligned}$$

which gives (51).

Finally, we prove (52). Due to (49), (51) and the Hölder inequality,

$$\begin{aligned}(\partial_x^2 u(t,x))^2 =&2\int_{-\infty}^x \partial_x^2 u\partial_x^3 u dy \leq \int_{\mathbb{R}}|\partial_x^2 u||\partial_x^3 u|dx\\ \leq&\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq C(T).\end{aligned}$$

Hence,

$$\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \leq C(T),$$

which gives (52). □

Now, we prove Theorem 1.

**Proof of Theorem 1.** Let $T > 0$. Assuming (3) and (4), thanks to Lemmas 6, 7, 8 and the Cauchy–Kovalevskaya Theorem [50], we have that $u$ is solution of (1) and (14) holds.

Instead, assuming (3) and (5), thanks to Lemmas 6, 7, 8, 9 and the Cauchy–Kovalevskaya Theorem [50], we have that $u$ is solution of (1) and (15) holds.

We prove (16), under Assumptions (3) and (5). Let $u_1$ and $u_2$ be two solutions of (1), which verify (15) that is

$$\begin{cases}\partial_t u_1 + 2\gamma(\partial_x u_1)^2 + \delta\partial_x^3 u_1 + \mu\partial_x^2 u_1 + \beta^2\partial_x^4 u_1 + \gamma u_1\partial_x^2 u_1 = 0, & t>0,\ x\in\mathbb{R},\\ u_1(0,x) = u_{1,0}(x), & x\in\mathbb{R},\end{cases}$$
$$\begin{cases}\partial_t u_2 + 2\gamma(\partial_x u_2)^2 + \delta\partial_x^3 u_2 + \mu\partial_x^2 u_2 + \beta^2\partial_x^4 u_2 + \gamma u_2\partial_x^2 u_2 = 0, & t>0,\ x\in\mathbb{R},\\ u_1(0,x) = u_{1,0}(x), & x\in\mathbb{R},\end{cases}$$

Then, the function $\omega$, defined in (36), is the solution of the following Cauchy problem:

$$\begin{cases}\partial_t\omega + 2\gamma\left[(\partial_x u_1)^2 - (\partial_x u_2)^2\right] + \delta\partial_x^3\omega + \mu\partial_x^2\omega\\ \qquad +\beta^2\partial_x^4\omega + \gamma u_1\partial_x^2 u_1 - \gamma u_2\partial_x^2 u_2 = 0, & t>0,\ x\in\mathbb{R},\\ \omega(0,x) = u_{1,0}(x) - u_{2,0}(x), & x\in\mathbb{R}.\end{cases} \tag{55}$$

Observe that thanks to (36),

$$\begin{aligned}&u_1\partial_x^2 u_1 - u_2\partial_x^2 u_2 = u_1\partial_x^2 u_1 - u_2\partial_x^2 u_1 + u_2\partial_x^2 u_1 - u_2\partial_x^2 u_2 = \omega\partial_x^2 u_1 + u_2\partial_x^2\omega,\\ &(\partial_x u_1)^2 - (\partial_x u_2)^2 = (\partial_x u_1 + \partial_x u_2)\partial_x\omega.\end{aligned}$$

Consequently, (55) is equivalent to the following equation:

$$\partial_t\omega + 2\gamma(\partial_x u_1 + \partial_x u_2)\partial_x\omega + \gamma\omega\partial_x^2 u_1 + \gamma u_2\partial_x^2\omega + \delta\partial_x^3\omega + \mu\partial_x^2\omega + \beta^2\partial_x^4\omega = 0. \tag{56}$$

Therefore, multiplying (56), by (39) and an integration on $\mathbb{R}$ of, we have that

$$\begin{aligned}\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ 2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &= 2\mu\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\gamma\int_{\mathbb{R}}(\partial_x u_1 + \partial_x u_2)\omega\partial_x\omega dx\\ &\quad - 2\gamma\int_{\mathbb{R}}\partial_x^2 u_1\omega^2 dx - 2\gamma\int_{\mathbb{R}} u_2\omega\partial_x^2\omega dx.\end{aligned} \tag{57}$$

Since $u_1$, $u_2 \in H^3(\mathbb{R})$, for every $0 \le t \le T$, we have

$$\begin{aligned}&\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T),\\ &\left\|\partial_x^2 u_1\right\|_{L^\infty((0,T)\times\mathbb{R})}, \|u_2\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T).\end{aligned} \tag{58}$$

Due to (58) and the Young inequality,

$$\begin{aligned}2|\gamma| &\int_{\mathbb{R}}(\partial_x u_1 + \partial_x u_2)\omega\partial_x\omega dx\\ &\le \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \gamma^2\int_{\mathbb{R}}(\partial_x u_1 + \partial_x u_2)^2(\partial_x\omega)^2 dx\\ &\le \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \gamma^2\left(\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})} + \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})}\right)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\le \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ 2|\gamma| &\int_{\mathbb{R}}|\partial_x^2 u_1|\omega^2 dx\\ &\le 2|\gamma|\left\|\partial_x^2 u_1\right\|_{L^\infty((0,T)\times\mathbb{R})}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ 2|\gamma| &\int_{\mathbb{R}}|u_2\omega||\partial_x^2\omega| dx\\ &= 2\int_{\mathbb{R}}\left|\frac{\gamma u_2\omega}{\beta^2}\right|\left|\beta\partial_x^2\omega\right| dx\\ &\le \frac{\gamma^2}{\beta^2}\|u_2\|_{L^\infty((0,T)\times\mathbb{R})}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (57) that

$$\begin{aligned}\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ \beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, arguing as in Section 2, we have (16). □

## 4. Conclusions

In this paper, we prove the well-posedness of the classical solutions for the Cauchy problem of the Kuramoto–Sinelshchikov–Velarde equation, that describes the evolution of a phase turbulence in reaction-diffusion systems or the evolution of the plane flame propagation, taking in account the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front. Our result requires very general assumptions on the coefficients and the argument is based on energy estimates and the Cauchy–Kovalevskaya Theorem.

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