Supplementary Information

Beyond Newton: a new root-finding fixed-point iteration for nonlinear equations

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1 Inverse of x

We consider a function

$$r(x) = \frac{1}{x} - H, \tag{S1}$$

which is discontinuous at $x = 0$. The convergence behavior using three methods are shown (Figs. S1–S3), where discontinuity leads to non-smooth results.

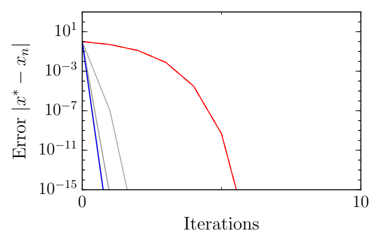

Figure S1: Convergence of Eq. (S1) using standard Newton (red), Extended Newton (gray), and Halley's (blue) methods for $H = 0.5$, $x_0 = 1$, and $c \in (1, 50)$

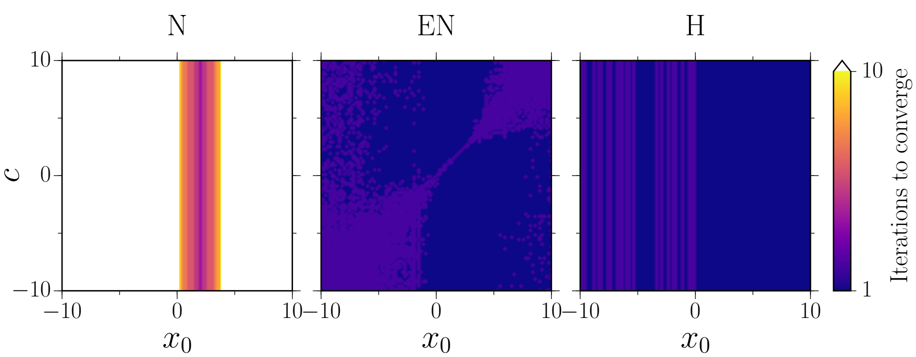

Figure S2: Iterations to converge for Eq. (S1) using (from left) standard Newton, Extended Newton, and Halley's methods for $H = 0.5$ and varying x_0 and c

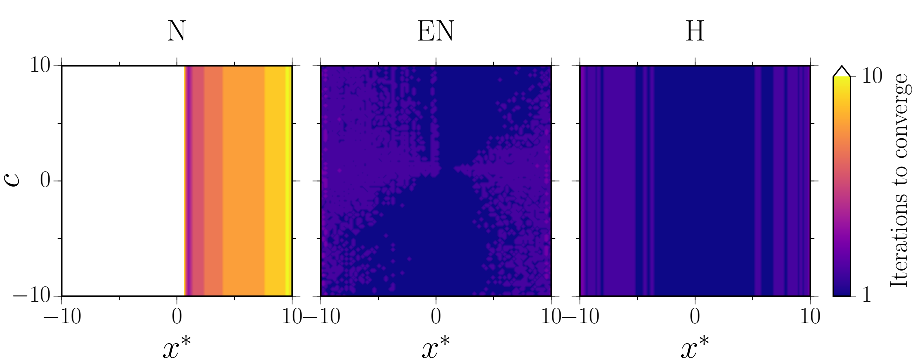

Figure S3: Iterations to converge for Eq. (S1) using (from left) standard Newton, Extended Newton, and Halley's methods for $x_0 = 1$ and varying x^* and c

2 Nonlinear compression

We consider a function from nonlinear elasticity, which is only defined in $\mathbb{R}^+$:

$$r(x) = \begin{cases} x^2 - \frac{1}{x} + H & \text{if } x > 0 \\ \text{Not defined} & \text{if } x \leq 0 \end{cases} . \tag{S2}$$

If the current guess x_n becomes negative, the iterations are considered to be non-converged. The resulting convergence behavior is shown (Fig. S4–S6)

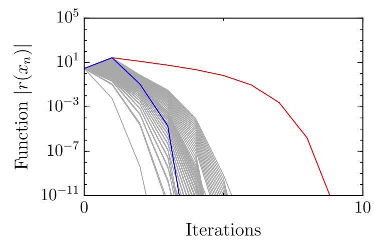

Figure S4: Convergence of Eq. (S2) using standard Newton (red), Extended Newton (gray), and Halley's (blue) methods for $H = 2.9$, $x_0 = 1$, and $c \in (0.01, 1)$

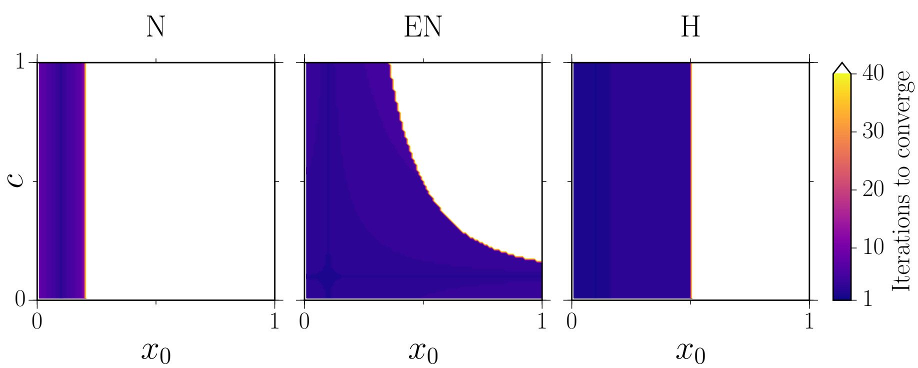

Figure S5: Iterations to converge for Eq. (S2) using (from left) standard Newton, Extended Newton, and Halley's methods for $H = 10$ and varying x_0 and c

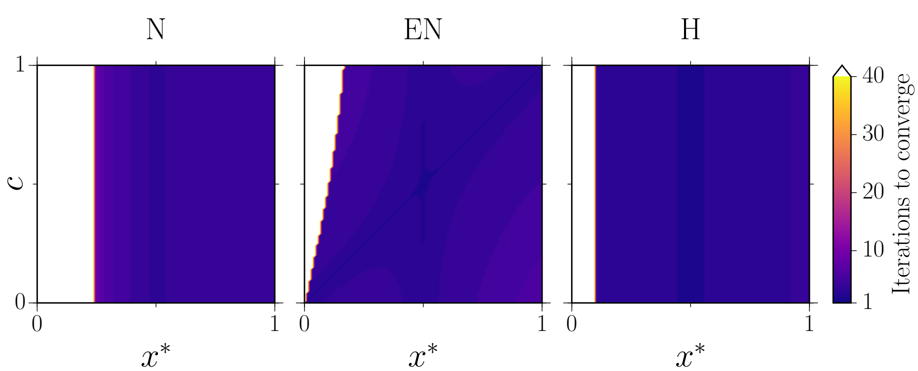

Figure S6: Iterations to converge for Eq. (S2) using (from left) standard Newton, Extended Newton, and Halley's methods for $x_0 = 0.5$ and varying x^* and c

3 Transcendental equation

We consider the following transcendental equation with restricted domain

$$r(x) = \begin{cases} \tan(x) - x & \text{if } x \in [\pi, 3\pi/2] \\ \text{Not defined} & \text{if } x < \pi \text{ or } x > 3\pi/2 \end{cases} . \tag{S3}$$

If the current guess goes outside the domain ($x_n < \pi$ or $x_n > 3\pi/2$), the iterations are considered to be non-converged. Resulting convergence behavior is shown (Fig. S7).

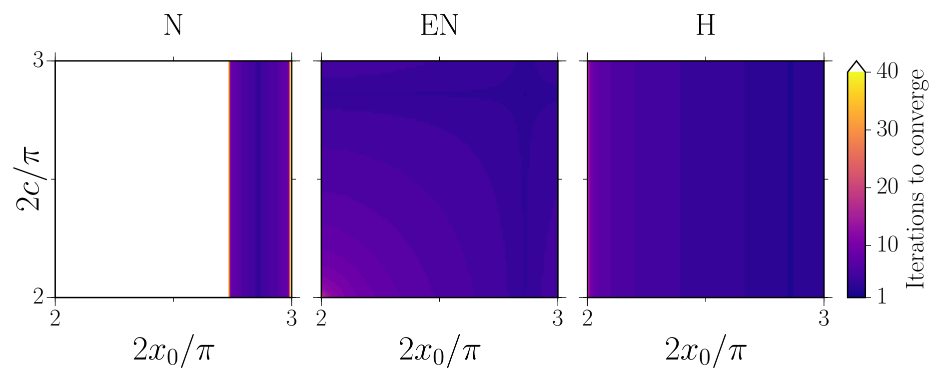

Figure S7: Iterations to converge for Eq. (S3) using (from left) standard Newton, Extended Newton, and Halley's methods for varying x_0 and c

4 Complex Plane

Applying Newton's method to a complex function $r(z) = z^3 - 1$ gives us a Newton fractal, and the Extended Newton and Halley's methods modify that fractal (Fig. S8). The new schemes not only find the root in fewer iterations, even the dimension of the fractal is reduced (Fig. S9) and the basins of attraction appear more connected (generally speaking) than before. Choosing c in Extended Newton breaks the three-fold symmetry of this system, and the solution close to c is heavily favored. Halley's method did not have such a bias and reduced the sensitivity equally for all the three roots.

4.1 Dimension of Newton Fractal

Furthermore, for the above Newton fractal (Fig. S8), we calculate the dimension of the boundary between basins of attraction (i.e. Julia set) using box counting method (Fig. S9).

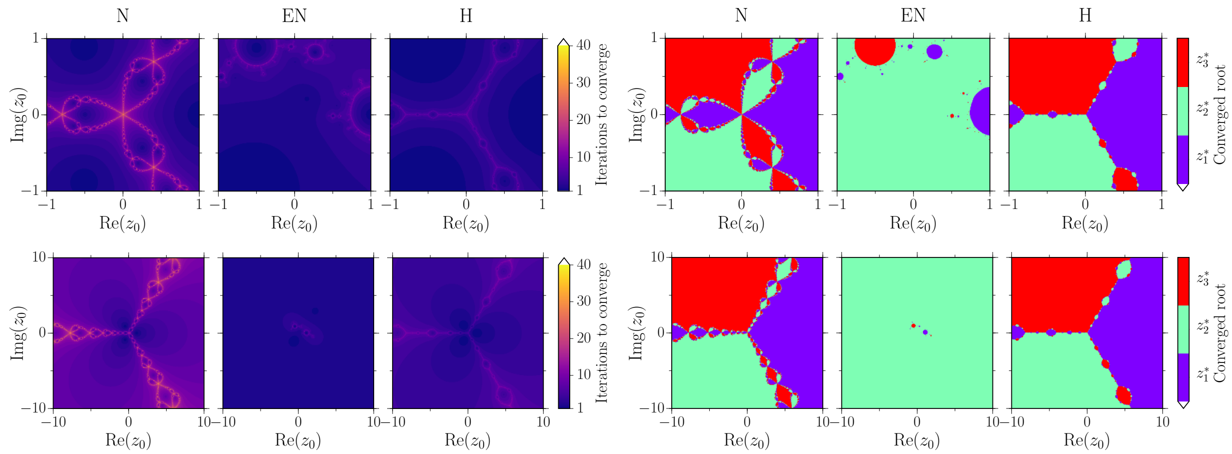

Figure S8: Newton fractal for $r(z) = z^3 - 1$ in complex plane, (top two rows) colored by number of iterations and (bottom two rows) colored by the final root: the standard Newton's, Extended Newton, and Halley's methods (from left to right). $c = -0.65 - 0.65i$ is used for EN, and $z_1^* = 1 + 0i$, $z_2^* = -0.5 - \sqrt{0.75}i$ and $z_3^* = -0.5 + \sqrt{0.75}i$.

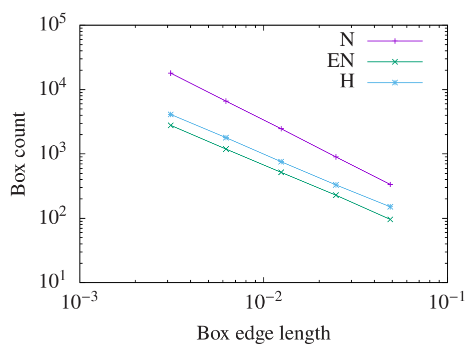

Figure S9: Dimension d of the boundary between basins of attraction for $r(z) = z^3 - 1$ calculated by box counting: $d \approx 1.45$ for standard Newton's method, $d \approx 1.22$ for Extended Newton method with $c = -0.65 - 0.65i$, and $d \approx 1.20$ for Halley's method