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Unbiased Fuzzy Estimators in Fuzzy Hypothesis Testing

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Abstract: In this paper, we develop fuzzy, possibilistic hypothesis tests for testing crisp hypotheses for a distribution parameter from crisp data. In these tests, fuzzy statistics are used, which are produced by the possibility distribution of the estimated parameter, constructed by the known from crisp statistics confidence intervals. The results of these tests are in much better agreement with crisp statistics than the ones produced by the respective tests of a popular book on fuzzy statistics, which uses fuzzy critical values. We also present an error that we found in the implementation of the unbiased fuzzy estimator of the variance in this book, due to a poor interpretation of its mathematical content, which leads to disagreement of some fuzzy hypotheses tests with their respective crisp ones. Implementing correctly this estimator, we produce test statistics that achieve results in hypotheses tests that are in much better agreement with the results of the respective crisp ones.

Keywords: fuzzy hypothesis tests; unbiased fuzzy estimator of the variance; possibility of rejection or acceptance

1. Introduction

Testing statistical hypotheses is a main branch of inferential statistics (see [1]). A statistical hypothesis is an assertion about a parameter of the probability distribution of a random variable.

After the introduction of the notion of fuzzy sets by Zadeh [2], many approaches have been proposed for fuzzy hypothesis testing, using fuzzy set theory (see [3]).

The problem of testing hypotheses with fuzzy data was analyzed for the first time in [4] and in the sequel in [5,6], where the author extended both Neyman–Pearson and Bayes theories to this framework (in [5]) and worked on the same problem in the context of fuzzy decision problems (in [6]).

Fuzzy hypothesis testing with crisp data is presented in [7] in the sense of Neyman–Pearson, using the extension principle and γ-cuts, where fuzzy critical regions are intro-duced, leading to a fuzzy conclusion, as well as in [8,9], where the author provides new definitions for the probability of type I and type II errors and presented the best test for the one-parameter exponential family. The problem of testing fuzzy hypotheses when the observations are crisp is also studied in [10], where the authors give new definitions for the probability of type I and type II errors and prove a version of the Neyman–Pearson Lemma. They also study the problem of testing hypotheses from a Bayesian point of view for which the observations are ordinary and the hypotheses are fuzzy (see [11]).

In [12], testing hypotheses about the mean of a fuzzy random variable is introduced. This approach is applicable to practical situations where either the observed data or the hypotheses are fuzzy (formalized in linguistic terms). These fuzzy tests result in a fuzzy decision for the acceptance or rejection of the null hypotheses with a degree of confidence between 0 and 1. They are generalizations of the classical tests, so that they are reduced to the classical tests if both the data and the parameters are crisp. Fuzzy hypothesis tests are also developed in [13–15] for cases in which the available data are fuzzy and in [16], where the authors propose fuzzy hypothesis testing for a proportion with crisp data as the exact generalized one-tailed hypergeometric test with fuzzy hypotheses. Fuzzy hypothesis

testing in the framework of the randomized and non-randomized hypergeometric test for a proportion is presented in [17].

In [18], the fuzzy *p*-value concept is used for testing hypotheses with fuzzy data and in [19] it is generalized on the basis of Zadeh's probability measure of fuzzy events [2] for testing fuzzy hypotheses with crisp data. In [20], a fuzzy *p*-value is obtained, using fuzzy test statistics constructed by fuzzy estimators for cases in which both the data and the hypotheses are crisp.

Two different ways of making inference from set-valued data are presented in [21]. A detailed review on possibilistic interpretations of statistical tests (where hypotheses and/or data are fuzzy) and statistical decisions is presented in [22,23] and a comprehensive review with regard to statistical properties of different approaches for calculating fuzzy *p*-values, in [24].

In [25], the authors demonstrate how to accomplish a fuzzy test with fuzzy data and fuzzy formulated hypotheses and discuss the defuzzification of fuzzy test decisions by means of the signed distance method. In [26], the author reviews and compares the R packages "FPV" and "Fuzzy.p.value" for hypothesis testing in fuzzy environments by using the fuzzy *p*-value for decision making.

In [27], the authors systematically review the literature, identifying papers proposing advances in fuzzy hypothesis testing and its applications. Then, they look at each contained paper through the lens of the key research questions.

In [28,29], fuzzy hypotheses testing is developed for cases in which both the data and the hypotheses are crisp. This approach uses fuzzy critical values and fuzzy test statistics constructed by fuzzy estimators produced by a set of confidence intervals. So, the null hypothesis H_0 is rejected or not at a certain significance level, comparing a fuzzy statistic $\widetilde{U}$ constructed by a proper fuzzy estimator with fuzzy critical values $\widetilde{CV}_i$, created using probabilistic concepts. In [30], this approach is generalized using fuzzy statistics produced by non asymptotic fuzzy estimators (see [31]) and a degree of rejection or acceptance of the null hypothesis.

As it is proved in [32], the possibility distribution (see [2]) induced by confidence intervals around the mode is identical to the one obtained by the maximal specificity probability–possibility transformation. Based on this principle, we develop in this paper possibilistic fuzzy statistical tests of crisp hypotheses, which lead to a possibility of rejection or acceptance of the null hypothesis for cases in which both the hypothesis and the data are crisp, whereas the other fuzzy tests (for example, those of [12]) give crisp results when applied in such cases.

This paper is organized as follows. The next section is concerned with classical hypotheses testing. In Section 3, we present the concept of fuzzy estimators and, also, find out that in the examples of [29,33–35], the unbiased fuzzy estimator of the variance $\widetilde{\sigma_u^2}$ introduced in [29] is not implemented as described there and propose its correct implementation. In Section 4, we develop possibilistic hypothesis tests for cases in which both the data and the hypotheses are crisp. In Sections 5–7, we present fuzzy hypothesis tests about the mean and the variance, and compare their results as well as the results of the respective tests developed in [29,30] with those of crisp statistics (Examples 2 and 3).

2. Classical Hypothesis Testing

In classical crisp statistics, the problem of testing a hypothesis for a parameter θ of the distribution of a random variable X is deciding whether to reject or accept the null hypothesis $H_0 : \theta = \theta_0$ at a significance level γ against the alternative hypothesis H_1 from a random sample of observations of X, using a test statistic U, which is evaluated for the sample observations, resulting in a value u. The space of possible values of U is decomposed into a rejection region and its complement, the acceptance region [1]. Depending on the alternative hypothesis H_1, the rejection region has one of the following forms:

(a) $U \geq u_{c1}$, if the alternative hypothesis is $H_1 : \theta > \theta_0$ (one sided test from the right), where the following is true:

$$P(U \geq u_{c1}) = \gamma \qquad (1)$$

(b) $U \leq u_{c2}$, if the alternative hypothesis is $H_1 : \theta < \theta_0$ (one sided test from the left), where the following is true:

$$P(U \leq u_{c2}) = \gamma \qquad (2)$$

(c) $U \geq u_{c3}$ or $U \leq u_{c4}$, if the alternative hypothesis is $H_1 : \theta \neq \theta_0$ (two sided test), where the following is true:

$$P(U \geq u_{c3}) = \frac{\gamma}{2} \quad \text{and} \quad P(U \leq u_{c4}) = \frac{\gamma}{2} \qquad (3)$$

and u_{ci} are the critical values of the test, which are determined by the distribution of U. So, H_0 is rejected if the value u of the test statistic U is in the rejection region and not rejected if u is in the acceptance region.

If the value u of the test statistic of a hypothesis test is close to its critical value, then the crisp test is unstable since a very small change in the sample may lead from rejection to no rejection of H_0 or vice-versa, as shown in Example 1. Using a degree of rejection or acceptance of H_0 (see [30]) the above problem is eliminated.

3. Fuzzy Estimation

The estimation of a parameter of the probability density function (or probability function of a discrete random variable) of a distribution is one of the main purposes of inferential statistics.

Let X be a random variable with probability density function (or probability function for a discrete random variable) $f(x; \theta)$, where θ is nan unknown parameter, which has to be estimated from a random sample of observations $X_1, X_2, \ldots, X_n$ of X. For the estimation of θ, a statistic U is used (estimator of θ), which is a function of $X_1, X_2, \ldots, X_n$. For a specific sample where $x_i, i = 1, 2, \ldots, n$ are the observed values of X_i, a point estimator $\theta_0 = u(x_1, x_2, \ldots, x_n)$ for θ is obtained, which is not of high interest since the probability of this being the required value of θ is zero. A way to estimate θ in the crisp statistics is to find a $(1 - \beta)100\%$ confidence interval for θ in which the value of θ can be found with probability $1 - \beta$. We use β here since α, usually employed for confidence intervals, is reserved for α-cuts of fuzzy numbers. Usually, $\beta = 0.005$ or 0.01 or 0.02 or 0.05 or 0.1 for 99.5%, 99%, 98%, 95% or 90% confidence intervals. Starting arbitrarily with $\beta = 0.01$ (we could begin with 0.001 or 0.005, etc.) and using $[\theta_0, \theta_0]$ as the 0% confidence interval, we have the following confidence intervals:

$$[\theta_l(\beta), \theta_r(\beta)], \quad 0.01 \leq \beta \leq 1$$

Among other methods, a fuzzy estimation method is proposed in [29], according to which by placing all the above confidence intervals one on the top of the other, a triangular shaped fuzzy number $\widetilde{\theta}$ is constructed, the $\alpha-$cuts of which are the confidence intervals:

$$\mu_{\widetilde{\theta}_B}[\alpha] = [\theta_l(\alpha), \theta_r(\alpha)], \quad 0.01 \leq \alpha \leq 1$$

To finish the "bottom" of $\widetilde{\theta}$ in order to make it a complete fuzzy number, one needs to drop its graph straight down to complete its $\alpha-$cuts (see Figure 1). So, a fuzzy estimator $\widetilde{\theta}$ of θ is produced from a given sample, the possibility distribution of which are the confidence intervals of θ

$$\mu_{\widetilde{\theta}_B}[\alpha] = \begin{cases} [\theta_l(\alpha), \theta_r(\alpha)], & 0.01 \leq \alpha \leq 1 \\ [\theta_l(0.01), \theta_r(0.01)], & 0 \leq \alpha < 0.01 \end{cases} \qquad (4)$$

The fuzzy estimator $\widetilde{\theta}$ produced in this way contains much more information than just a single interval estimate.
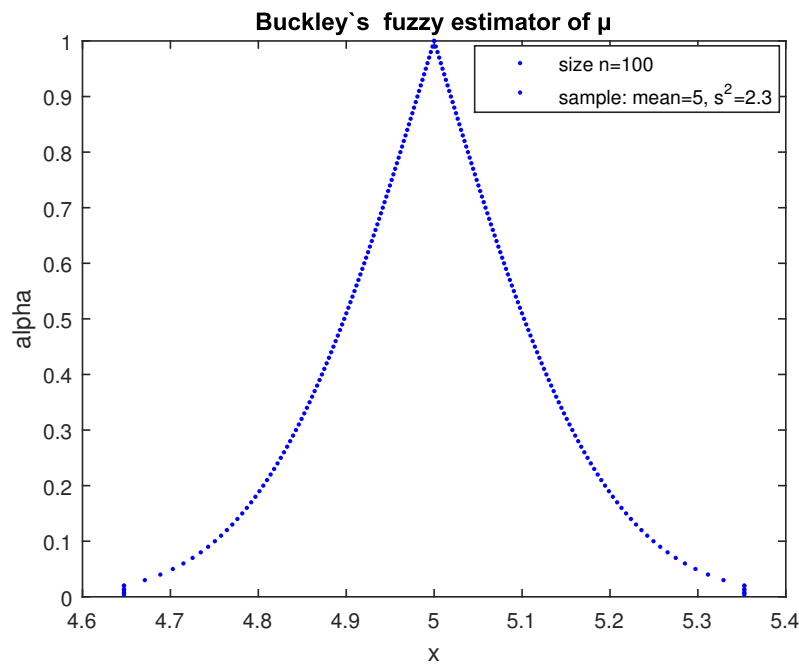
Figure 1. Buckley's fuzzy estimator of the mean μ of a random variable X, which follows normal distribution with variance $\sigma^2 = 2.3$, derived from a sample of $n = 100$ observations with sample mean $\overline{x} = 5$.

For the construction of test statistics used in fuzzy hypothesis testing about an unknown parameter θ, we use the fuzzy estimator $\widetilde{\theta}$, the α-cuts of the possibility distribution of which are the confidence intervals of θ

$$\mu_{\widehat{\theta}}[\alpha] = [\theta_l(\alpha), \, \theta_r(\alpha)], \quad 0 \leq \alpha \leq 1 \tag{5}$$

since, as proved in [32], this possibility distribution is identical to the one obtained by the maximal specificity probability–possibility transformation. In order to easily handle the possibility representation, especially for further computations, the possibility distribution is restricted to the interval that corresponds to the largest 99% confidence interval for the considered probability distribution (see [32]).

3.1. Estimation of the Mean of a Normal Variable with Known Variance

If the random variable X follows normal distribution with known variance σ, then the $(1 - \beta)100\%$ confidence interval of the mean μ of X derived from a random sample of observations of X of size n and sample mean $\overline{x}$ is [1].

$$\left[\overline{x} - z_{\frac{\beta}{2}} \frac{\sigma}{\sqrt{n}}, \; \overline{x} + z_{\frac{\beta}{2}} \frac{\sigma}{\sqrt{n}} \right] \tag{6}$$

where $z_{\frac{\beta}{2}} = \Phi^{-1}\left(1 - \dfrac{\beta}{2}\right)$ and Φ^{-1}, the inverse distribution function of the standard normal distribution. So according to (6), the α-cuts of the possibility distribution of the fuzzy estimator $\widetilde{\mu}$ of mean are as follows (see [29,30]):

$$\mu_{\widetilde{\mu}}[\alpha] = \left[\overline{x} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \; \overline{x} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right], \quad 0 \leq \alpha \leq 1 \tag{7}$$

where

$$z_{\alpha} = \Phi^{-1}(1 - \alpha) \tag{8}$$

Implementing (7), we get the fuzzy estimator $\widetilde{\mu}$ of Figure 2.
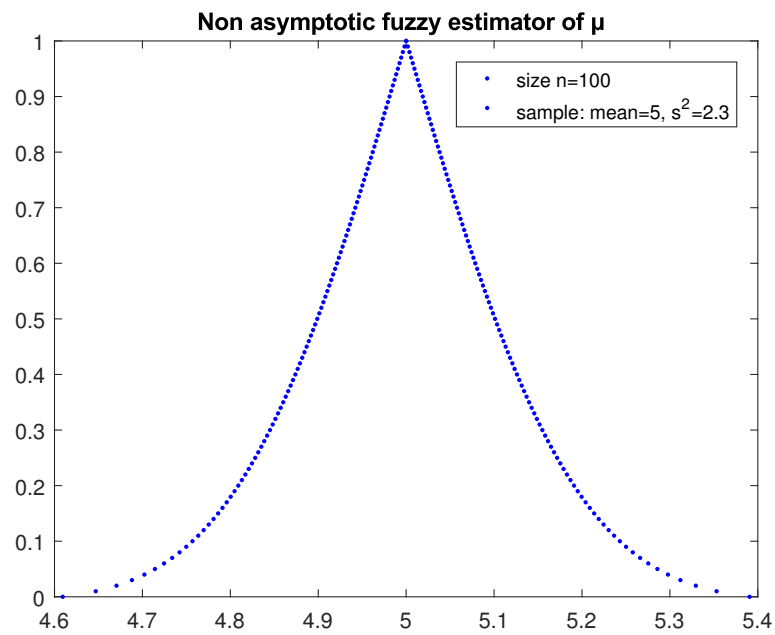
Figure 2. The possibility distribution of the fuzzy estimator $\widetilde{\mu}$ of the mean μ of a random variable X, which follows normal distribution with variance $\sigma^2 = 2.3$ derived from a sample of $n = 100$ observations with sample mean $\overline{x} = 5$.

3.2. Estimation of the Mean of a Random Variable from a Large Sample (with Unknown Variance)

If the random variable X follows any distribution, then, according to the central limit theorem (see [1]), the $(1 - \beta)100\%$ confidence interval of the mean μ of X derived from a random sample of observations of X of large size n ($n > 30$) with sample mean and variance $\overline{x}$ and s^2 is given by (6), substituting σ with s. So according to (5), the α-cuts of the possibility distribution of the fuzzy estimator $\widetilde{\mu}$ of the mean are as follows:

$$\mu_{\widetilde{\mu}}[\alpha] = \left[\overline{x} - z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}, \ \overline{x} + z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \right], \quad 0 \leq \alpha \leq 1 \tag{9}$$

where z_α is given by (8).

3.3. Estimation of the Variance of a Normal Variable

If the random variable X follows normal distribution, then the $(1 - \beta)100\%$ confidence interval of the variance σ^2 of X derived from a random sample of observations of X of size n and sample variance s^2 is as follows [1]:

$$\left[\frac{(n-1)s^2}{\chi^2_{\frac{\beta}{2};n-1}}, \ \frac{(n-1)s^2}{\chi^2_{1-\frac{\beta}{2};n-1}} \right] \tag{10}$$

where (F_{n-1}^{-1} the inverse distribution function of the χ^2_{n-1} distribution)

$$\begin{aligned} \chi^2_{1-\alpha;n-1} &= F_{n-1}^{-1}(\alpha) \\ \chi^2_{\alpha;n-1} &= F_{n-1}^{-1}(1-\alpha) \end{aligned} \tag{11}$$

So according to (5), the α-cuts of the possibility distribution of the fuzzy estimator of the variance σ^2 are as follows:

$$\mu_{\widetilde{\sigma_b^2}}[\alpha] = \left[\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2};n-1}}, \ \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2};n-1}} \right], \quad \alpha \in [0,1] \tag{12}$$

Therefore, the α-cuts of the fuzzy estimator of the standard deviation σ of X are as follows:

$$\mu_{\widetilde{\sigma_b}}[\alpha] = \left[s\sqrt{\frac{(n-1)}{\chi^2_{\frac{\alpha}{2};n-1}}}, \ s\sqrt{\frac{(n-1)}{\chi^2_{1-\frac{\alpha}{2};n-1}}} \right], \quad \alpha \in [0,1] \tag{13}$$

It can be shown that the above fuzzy estimator $\widetilde{\sigma_b^2}$ is biased because its core is not at s^2. A fuzzy estimator of a parameter is defined as biased if its core is not at the crisp point estimator of the parameter.

In [29], an unbiased fuzzy estimator $\widetilde{\sigma_u^2}$ of the variance σ^2 of a normal random variable is obtained from a sample of size n and variance s^2 by putting one above the other for the following confidence intervals:

$$\left[\frac{(n-1)s^2}{L(\lambda)}, \ \frac{(n-1)s^2}{R(\lambda)} \right], \quad 0 \le \lambda \le 1 \tag{14}$$

where

$$L(\lambda) = (1-\lambda)\chi^2_{0.005;n-1} + (n-1)\lambda \tag{15}$$
$$R(\lambda) = (1-\lambda)\chi^2_{1-0.005;n-1} + (n-1)\lambda \tag{16}$$

and (F_{n-1}^{-1} the inverse distribution function of the χ^2_{n-1} distribution)

$$\chi^2_{1-0.005;n-1} = F_{n-1}^{-1}(0.005) \quad \text{and} \quad \chi^2_{0.005;n-1} = F_{n-1}^{-1}(0.995)$$

For a value of λ, the corresponding value of α is as follows [29]:

$$\alpha = \phi(\lambda) = \int_0^{R(\lambda)} \chi^2_{n-1}(x)dx + 1 - \int_0^{L(\lambda)} \chi^2_{n-1}(x)dx \tag{17}$$

where $\chi^2_{n-1}(x)$ the probability density function of the χ^2_{n-1} distribution. The range of this one to one and onto function is $[0.01, 1]$. So, the $\alpha-$cuts of the correct unbiased fuzzy estimator $\widetilde{\sigma_u^2}$ are as follows:

$$\mu_{\widetilde{\sigma_u^2}}[\alpha] = \begin{cases} \left[\dfrac{(n-1)s^2}{L(\lambda)}, \ \dfrac{(n-1)s^2}{R(\lambda)} \right], & \lambda = \phi^{-1}(\alpha) \quad 0.01 \le \alpha \le 1 \\[3mm] \left[\dfrac{(n-1)s^2}{L(0.01)}, \ \dfrac{(n-1)s^2}{R(0.01)} \right], & 0 \le \alpha < 0.01 \end{cases} \tag{18}$$

where the function $\alpha = \phi(\lambda)$ and its inverse $\lambda = \phi^{-1}(\alpha)$ (it gives λ as a function of α) are implemented numerically.

In the implementation of $\widetilde{\sigma_u^2}$ of [29], the λ-cuts defined by (14) (instead of its α-cuts of (18)) are used for the construction of $\mu_{\widetilde{\sigma_u^2}}$ and the statistics generated by it.

As illustrated in Figure 3, the shape of the correct implementation (18) of $\widetilde{\sigma_u^2}$ is significantly different than the shape of its implementation used in [29] and quite similar to the shape of the biased estimator $\widetilde{\sigma_b^2}$.
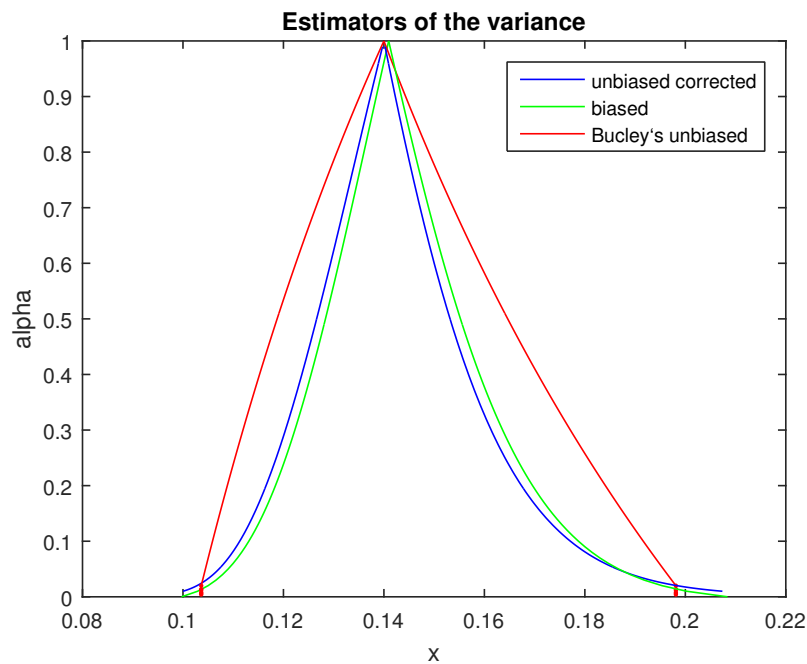
Figure 3. Non-asymptotic fuzzy estimators of the variance σ^2: correct unbiased (blue line), biased (green line), unbiased used in [29] (red line).

4. Possibilistic Statistical Tests of Crisp Hypotheses

We denote by 1 the rejection and by 0 the acceptance of a hypothesis H_0 for a parameter ϑ of the distribution of a random variable X from a sample of n observations $X_1, X_2, \ldots, X_n$ of X. Hence, a possibilistic test of a crisp hypothesis H_0 with alternative H_1 is a decision rule, defined as two functions: $\varphi_R(X_1, X_2, \ldots, X_n) : R^n \to [0, 1]$, which gives the degree in which the observed value u of the test statistic U belongs to the rejection region (see Section 1) (the possibility of the proposition "u is in the rejection region"), and $\varphi_A(X_1, X_2, \ldots, X_n) : R^n \to [0, 1]$, which gives the degree in which the observed value u of the test statistic U belongs to the acceptance region (see Section 1) (the possibility of the proposition "u is in the acceptance region").

Using the fuzzy estimator $\widetilde{\vartheta}$ of ϑ (see Section 2) and interval arithmetics, we derive the α-cuts of the possibility distribution $\mu_{\widetilde{U}}$ of the respective fuzzy test statistic $\widetilde{U}$. So, the possibilities of rejection or acceptance of H_0 are as follows:

(a) For *one sided test from the right* (alternative hypothesis $H_1 : \theta > \theta_0$), the possibility of rejection of H_0 is equal to the Necessity of Strict Dominance index (see [23,36]).

$$\varphi_R(X_1, X_2, \ldots, X_n) = \text{Ness}(\widetilde{U} > \widetilde{CV}_1) = 1 - \sup_{x,y; x \leq y} \min\{\mu_{\widetilde{U}}(x), \mu_{\widetilde{CV}_1}(y)\} \quad (19)$$

which represents the necessity that the fuzzy set $\widetilde{U}$ strictly dominates the fuzzy set of the right critical value $\widetilde{CV}_1$, the membership function of which is as follows:

$$\mu_{\widetilde{CV}_1}(u) = \begin{cases} 1, & u = u_{c1} \\ 0, & u \neq u_{c1} \end{cases} \quad (20)$$

where u_{c1} is the critical value of the one sided from the right, defined in (1). So if the core of $\widetilde{U}$ is at the right of u_{c1}, (19) gives the following:

$$\varphi_R(X_1, X_2, \ldots, X_n) = 1 - \mu_{\widetilde{U}}^L(u_{c1}) = 1 - y_1 \quad (21)$$

where $\mu_{\widetilde{U}}^L$ is the left part of the possibility distribution $\mu_{\widetilde{U}}$ and y_1 is the ordinate of the point of intersection of $\mu_{\widetilde{U}}^L$ with the vertical at u_{c1}.

The possibility of acceptance of H_0 is as follows:

$$\varphi_A(X_1, X_2, \ldots, X_n) = \text{Ness}(\widetilde{U} < \widetilde{CV}_1) = 1 - \sup_{x,y;y\leq x} \min\{\mu_{\widetilde{U}}(x), \mu_{\widetilde{CV}_1}(y)\} \qquad (22)$$

which represents the necessity that the fuzzy set $\widetilde{CV}_1$ strictly dominates the fuzzy set of $\widetilde{U}$. So if the core of $\widetilde{U}$ is at the left of u_{c1}, (22) gives the following:

$$\varphi_A(X_1, X_2, \ldots, X_n) = 1 - \mu_{\widetilde{U}}^R(u_{c1}) = 1 - y_1 \qquad (23)$$

where $\mu_{\widetilde{U}}^R$ is the right part of the possibility distribution $\mu_{\widetilde{U}}$ and y_1 is the ordinate of the point of intersection of $\mu_{\widetilde{U}}^R$ with the vertical at u_{c1}.

(b) For , *the one sided test from the left* (alternative hypothesis $H_1 : \theta < \theta_0$), the possibility of rejection of H_0 is equal to the Necessity of Strict Dominance index of $\widetilde{CV}_2$ to $\widetilde{U}$, where the membership function of the critical value $\widetilde{CV}_2$ is as follows (u_{c2} the critical value of the one sided from the left defined in (2)):

$$\mu_{\widetilde{CV}_2}(u) = \begin{cases} 1, & u = u_{c2} \\ 0, & u \neq u_{c2} \end{cases} \qquad (24)$$

$$\varphi_R(X_1, X_2, \ldots, X_n) = \text{Ness}(\widetilde{U} < \widetilde{CV}_2) = 1 - \sup_{x,y;y\leq x} \min\{\mu_{\widetilde{U}}(x), \mu_{\widetilde{CV}_2}(y)\} \qquad (25)$$

So if the core of $\widetilde{U}$ is at the left of u_{c2}, (25) gives the following:

$$\varphi_R(X_1, X_2, \ldots, X_n) = 1 - \mu_{\widetilde{U}}^R(u_{c2}) = 1 - y_2 \qquad (26)$$

where y_2 is the ordinate of the point of intersection of the right part $\mu_{\widetilde{U}}^R$ of the possibility distribution of $\widetilde{U}$ with the vertical at u_{c2}.

The possibility of acceptance of H_0 is as follows:

$$\varphi_A(X_1, X_2, \ldots, X_n) = \text{Ness}(\widetilde{U} > \widetilde{CV}_2) = 1 - \sup_{x,y;x\leq y} \min\{\mu_{\widetilde{U}}(x), \mu_{\widetilde{CV}_2}(y)\} \qquad (27)$$

So if the core of $\widetilde{U}$ is at the right of u_{c2}, (27) gives the following:

$$\varphi_A(X_1, X_2, \ldots, X_n) = 1 - \mu_{\widetilde{U}}^L(u_{c2}) = 1 - y_2 \qquad (28)$$

where y_2 is the ordinate of the point of intersection of the left part $\mu_{\widetilde{U}}^L$ of the possibility distribution of $\widetilde{U}$ with the vertical at u_{c2}.

(c) For the *two-sided test* (alternative hypothesis $H_1 : \theta \neq \theta_0$), the possibility of rejection of H_0 is equal to the following:

$$\begin{aligned} \varphi_R(X_1, X_2, \ldots, X_n) &= \text{Ness}(\widetilde{U} > \widetilde{CV}_1 \ or \ \widetilde{U} < \widetilde{CV}_2) \\ &= \max\left\{\widetilde{U} > \widetilde{CV}_1, \widetilde{U} < \widetilde{CV}_2\right\} \end{aligned} \qquad (29)$$

where (u_{c3} and u_{c4} are the critical values of the two-sided defined in (3)).

$$\mu_{\widetilde{CV}_1}(u) = \begin{cases} 1, & u = u_{c3} \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \mu_{\widetilde{CV}_2}(u) = \begin{cases} 1, & u = u_{c4} \\ 0, & \text{elsewhere} \end{cases} \qquad (30)$$

So if the core of $\widetilde{U}$ is at the right of u_{c3}, (33) gives the following:

$$\varphi_R(X_1, X_2, \ldots, X_n) = 1 - y_1 \qquad (31)$$

where $y_1 = \mu^L_{\widetilde{U}}(u_{c3})$ and if it is at the left of u_{c4}, it gives the following:

$$\varphi_R(X_1, X_2, \ldots, X_n) = 1 - y_2 \tag{32}$$

where $y_2 = \mu^R_{\widetilde{U}}(u_{c4})$.

The possibility of acceptance of H_0 is as follows:

$$\varphi_A(X_1, X_2, \ldots, X_n) = \text{Ness}(\widetilde{U} < \widetilde{CV}_1 \text{ and } \widetilde{U} > \widetilde{CV}_2) \tag{33}$$

So if the core of $\widetilde{U}$ is between u_{c3} and u_{c4}, (34) gives the following:

$$\varphi_A(X_1, X_2, \ldots, X_n) = \min\{1 - y_1, 1 - y_2\} \tag{34}$$

where $y_1 = \mu^R_{\widetilde{U}}(u_{c3})$ and $y_2 = \mu^L_{\widetilde{U}}(u_{c4})$ (the ordinates of the point of intersection of the right part $\mu^R_{\widetilde{U}}$ of the possibility distribution of $\widetilde{U}$ with the vertical at u_{c3} and of its left part $\mu^L_{\widetilde{U}}$ with the vertical at u_{c4}).

If the possibility of the rejection or acceptance of H_0 is very low (for example lower than 0.2), then we *cannot make a decision* on rejecting or accepting H_0.

5. Tests on the Mean of a Normal Distribution with Known Variance or of Any Distribution from a Large Sample

We test at the significance level γ the null hypothesis $H_0 : \mu = \mu_0$ for the mean μ of a random variable X, which follows the normal distribution with known variance σ, using a random sample of observations of X of size n.

In the crisp case, we test H_0, using the statistic (see [1]).

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \tag{35}$$

where $\overline{X}$ is the statistic of the sample mean. It is known that under the null hypothesis ($\mu = \mu_0$) Z follows the standard normal distribution $N(0,1)$, so H_0 is rejected from a given sample:
(a) For the *one sided test from the right*, if

$$z_0 \geq z_\gamma, \quad \text{where} \quad z_\gamma = \Phi^{-1}(1 - \gamma),$$

(b) For the *one sided test from the left*, if $z_0 \leq -z_\gamma = \Phi^{-1}(\gamma)$,
(c) For the *two sided* test, if $z_0 \leq -z_{\gamma/2}$ or $z_0 \geq z_{\gamma/2}$, where

$$z_0 = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} \tag{36}$$

the value of the statistic (35) for the given sample,

$$z_{\gamma/2} = \Phi^{-1}\left(1 - \frac{\gamma}{2}\right) \tag{37}$$

and Φ^{-1} the inverse distribution function of the standard normal distribution. Meanwhile, H_0 is not rejected if $-z_{\gamma/2} < z_0 < z_{\gamma/2}$.

In the fuzzy case, the test of H_0 is based on the fuzzy statistic

$$\widetilde{Z} = \frac{\widetilde{\mu} - \mu_0}{\sigma/\sqrt{n}}, \tag{38}$$

which is generated by substituting $\overline{X}$ in (35) with the fuzzy estimator $\widetilde{\mu}$ of the mean value for the given sample, the $\alpha-$cuts of the membership function of which are given by (7).

From (38), (7) and interval arithmetics follow that the $\alpha-$cuts of the possibility distribution of the fuzzy statistic $\widetilde{Z}$ are as follows:

$$\mu_{\widetilde{Z}}[\alpha] = [Z - z_\alpha, \ Z + z_\alpha], \quad \alpha \in [0, 1]$$

Hence, the $\alpha-$cuts of the possibility distribution of $\widetilde{Z}$ for the given sample are as follows:

$$\mu_{\widetilde{Z}_0}[\alpha] = [z_0 - z_\alpha, \ z_0 + z_\alpha], \quad \alpha \in [0, 1] \tag{39}$$

where z_0 the crisp value of the statistic Z for this sample, which is given by (36).

The membership functions of the critical values $\widetilde{CV}_1$ and $\widetilde{CV}_2$ of the one-sided tests are as follows:

$$\mu_{\widetilde{CV}_1}(z) = \begin{cases} 1, & z = \Phi^{-1}(1 - \gamma) \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \mu_{\widetilde{CV}_2}(z) = \begin{cases} 1, & z = \Phi^{-1}(\gamma) \\ 0, & \text{elsewhere} \end{cases} \tag{40}$$

and for the two-sided test,

$$\mu_{\widetilde{CV}_1}(z) = \begin{cases} 1, & z = \Phi^{-1}\left(1 - \frac{\gamma}{2}\right) \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \mu_{\widetilde{CV}_2}(z) = \begin{cases} 1, & z = \Phi^{-1}\left(\frac{\gamma}{2}\right) \\ 0, & \text{elsewhere} \end{cases} \tag{41}$$

Having the $\alpha-$cuts of the possibility distribution of the fuzzy statistic $\widetilde{Z}_0$ of (39) and the critical values $\widetilde{CV}_1$ and $\widetilde{CV}_2$, we can evaluate the possibility of the rejection or acceptance of H_0 from (21), (23), (26), (28) for $\widetilde{U} = \widetilde{Z}_0$ and (40) for one sided tests or from (31), (32), (34) and (41) for the two-sided test.

If the sample is large and X follows any distribution, then in the crisp case, we test H_0 using the statistic

$$Z = \frac{\overline{X} - \mu_0}{s / \sqrt{n}}$$

where $\overline{X}$ is the statistic of the sample mean and s, the sample standard deviation [1]. It is known that under the null hypothesis ($\mu = \mu_0$), Z follows the standard normal distribution $N(0, 1)$ according to the central limit theorem [1], so H_0 is rejected or not by one sided tests, using (21), (23), (26), (28) for $\widetilde{U} = \widetilde{Z}_0$ given in (39) for the following sample value:

$$z_0 = \frac{\overline{x} - \mu_0}{s / \sqrt{n}} \tag{42}$$

of the statistic Z or (31), (32), (34) for the two-sided test.

Example 1. *We test the null hypothesis H_0: $\mu = 20.2$ at significance level $\gamma = 0.05$ with alternative H_1: $\mu \neq 20.2$ (two-sided test) for the mean value μ of the temperature X, using the two large random samples of 50 observations each of the Appendix A (monthly values for selected Greek weather stations) [37].*

The mean and the standard deviation of the first sample are $\overline{x}_1 = 17.61$ and $s_1 = 8.96$, so in the crisp test, we evaluate the value of the statistic Z for the first sample from (42):

$$z_0 = \frac{17.61 - 20.2}{8.96 / \sqrt{50}} = -2.044$$

Since

$$z_0 = -2.044 < -z_{0.05/2} = -z_{0.025} = -1.96$$

H_0 is rejected by this crisp test.

For the second sample the mean and variance are $\overline{x}_2 = 17.95$ and $s_2 = 8.51$, so the value of the statistic Z in this case is the following:

$$z_0 = \frac{17.95 - 20.2}{8.51/\sqrt{50}} = -1.87$$

Therefore, since the following is true:

$$z_0 = -1.87 > -z_{0.05/2} = -z_{0.025} = -1.96$$

H_0 is not rejected by the crisp test.

Applying the above described two-sided fuzzy test of H_0, implementing the possibility distribution (39) of $\widetilde{Z}_0$ for the value z_0 of (42) and the critical values (41) for the first sample, we get the results of Figure 4, where the point of intersection of $\mu_{\widetilde{Z}_0}$ and $\mu_{\widetilde{CV}_2}$ has $y_0 = 0.94$. So according to (32), the possibility of rejection of H_0 is $1 - 0.94 = 0.06$. Hence, we cannot make a decision on whether to reject or not H_0 from this sample.

Implementing (39) and (41) for the second sample, we take Figure 5, where the point of intersection of $\mu_{\widetilde{Z}_0}$ and $\mu_{\widetilde{CV}_2}$ has $y_0 = 0.95$, so according to (34), the possibility of acceptance of H_0 is $1 - 0.95 = 0.05$. Therefore, we cannot make a decision on whether to reject or not H_0 from this sample.
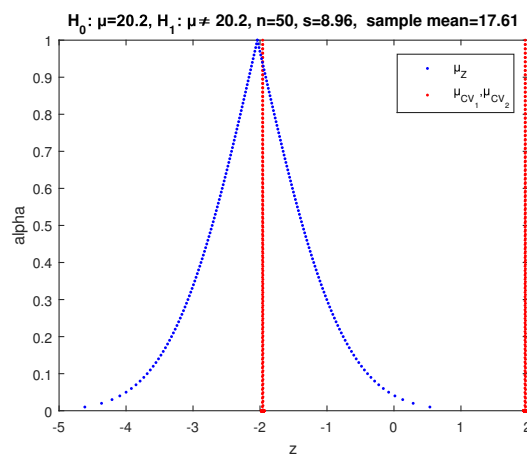

Figure 4. The possibility distribution of the fuzzy statistic Z for the test of Example 1 from a sample with $\overline{x}_1 = 17.6$.
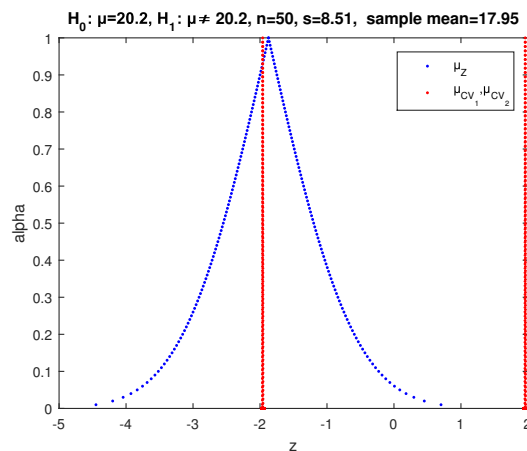

Figure 5. The possibility distribution of the fuzzy statistic Z for the test of Example 1 from a sample with $\overline{x}_2 = 17.95$.

Example 2. *We test at significance level* 0.05 *the null hypothesis* H_0: $\mu = 11$ *with alternative* H_1: $\mu \neq 11$ *(two-sided test) for the mean* μ *of a random variable X, which follows normal distribution with variance* $\sigma^2 = 10.24$, *using a sample of* $n = 25$ *observations with sample mean* $\bar{x} = 11.05$.

The value of the test statistic (35) is found by (36) to be as follows:

$$z_0 = \frac{11.05 - 11}{\sqrt{10.24}/\sqrt{25}} = 0.08$$

so the crisp *p*-value of the test is as follows:

$$p = 2(1 - \Phi(0.08)) = 0.94$$

Therefore, in this case, the crisp test gives acceptance of H_0 with a very high *p*-value (near to the maximum value, 1).

We apply the above two-sided fuzzy test of H_0 implementing (39) and (41). So, we obtain the results of Figure 6, where the core of $\widetilde{Z}$ is between the cores of $\widetilde{CV}_1$ and the point of intersection of $\mu_{\widetilde{Z}}$, and $\mu_{\widetilde{CV}_1}$ has $y_1 = 0.04$ and of $\mu_{\widetilde{Z}}$ and $\mu_{\widetilde{CV}_2}$ has $y_2 = 0.06$. Hence, according to (34), the possibility of acceptance of H_0 is as follows:

$$\varphi_A = \min(1 - y_1, 1 - y_2) = 1 - 0.06 = 0.94,$$

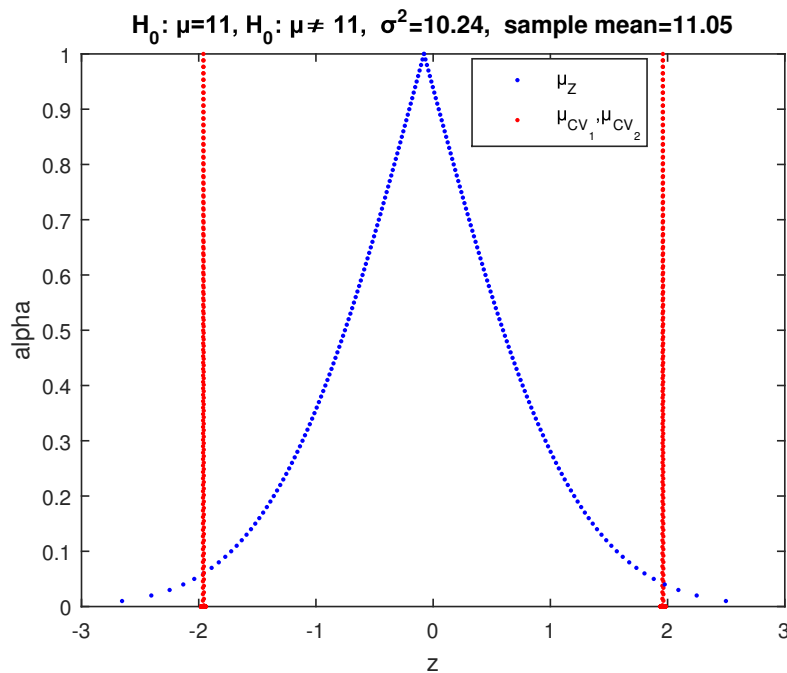so H_0 is accepted by this test with possibility $d = 0.94$.

Figure 6. The possibility distribution of the fuzzy statistic $\widetilde{Z}$ for the two-sided test of Example 2.

Applying the respective test of [29,30], we get the results of Figure 7, where we see that the possibility of acceptance of H_0 is much lower than 1:
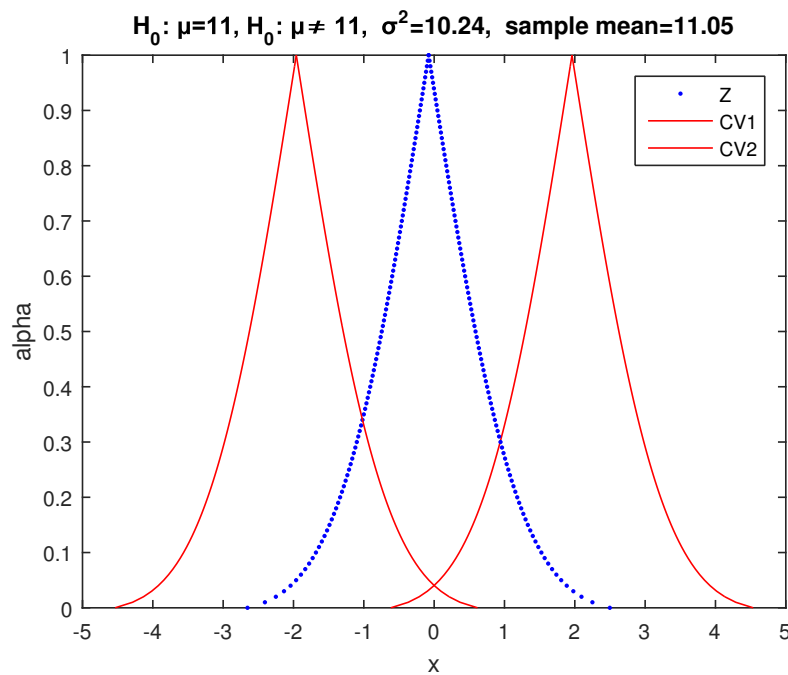
$$\varphi_A = 1 - y_1 = 1 - 0.31 = 0.69$$

Figure 7. Fuzzy statistic $\widetilde{Z}$ and critical values $\widetilde{CV}_i$ for the respective test of Example 2 in [30].

6. Hypotheses Tests for the Variance of a Normal Distribution

We test at significance level γ the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ for the variance σ^2 of a normal random variable X, using a random sample of observations of X of size n and variance s^2.

In the crisp case, we test H_0, using the test statistic

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} \tag{43}$$

where S^2 is the sample variance, which follows χ^2_{n-1} distribution with $n-1$ degrees of freedom under the null hypothesis [1]. H_0 is rejected from a given random sample (omitting in χ^2 the implied $n_1 - 1$ degrees of freedom):

(a) For *one-sided test from the left* (alternative $H_1 : \sigma^2 < \sigma_0^2$), if

$$\chi_0^2 < \chi^2_{1-\gamma}, \quad \text{where} \quad \chi^2_{1-\gamma} = F^{-1}(1-(1-\gamma)) = F^{-1}(\gamma) \tag{44}$$

and F^{-1} is the inverse distribution function of the χ^2_{n-1} distribution.

(b) For *one-sided test from the right* (alternative $H_1 : \sigma^2 > \sigma_0^2$), if

$$\chi_0^2 > \chi_\gamma^2, \quad \text{where} \quad \chi_\gamma^2 = F^{-1}(1-\gamma) \tag{45}$$

(c) For *two-sided* test (alternative $H_1 : \sigma^2 \neq \sigma_0^2$), if

$$\chi_0^2 < \chi^2_{1-\gamma/2} \quad \text{or} \quad \chi_0^2 > \chi^2_{\gamma/2}, \tag{46}$$

where

$$\chi^2_{1-\gamma/2} = F^{-1}(\gamma/2) \quad \text{and} \quad \chi^2_{\gamma/2} = F^{-1}(1-\gamma/2)$$

and

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2} \tag{47}$$

the crisp value of the statistic (43). Otherwise, H_0 is not rejected.

In the fuzzy case for the test of H_0, we use the fuzzy statistic $\widetilde{\chi^2}$ generated by substituting S^2 in (43) with a fuzzy estimator $\widetilde{s^2}$ of the following variance:

$$\widetilde{\chi^2} = \frac{(n-1)\widetilde{s^2}}{\sigma_0^2} \qquad (48)$$

Using the α-cuts of the unbiased fuzzy estimator given in (14) and interval arithmetics, we obtain from (48) the α-cuts of the possibility distribution of the following test statistic:

$$\mu_{\widetilde{\chi^2}}[\alpha] = \left[\frac{(n-1)\chi_0^2}{L(\lambda)}, \ \frac{(n-1)\chi_0^2}{R(\lambda)} \right], \quad \lambda = \phi^{-1}(\alpha) \qquad (49)$$

As illustrated in Figures 8 and 9, the shape of this test statistic is significantly different than the shape of the respective test statistic produced by the implementation of $\mu_{\widetilde{\sigma_u^2}}$ used in [29], which is as expected since the shape of the two implementations (14) and (18) of $\mu_{\widetilde{\sigma_u^2}}$ are different.
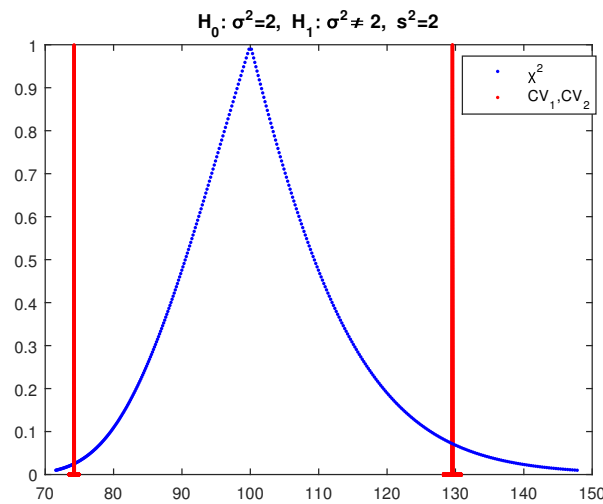

Figure 8. The possibility distribution of the fuzzy statistic $\widetilde{\chi^2}[\alpha]$ for the test of H_0 of Example 3.
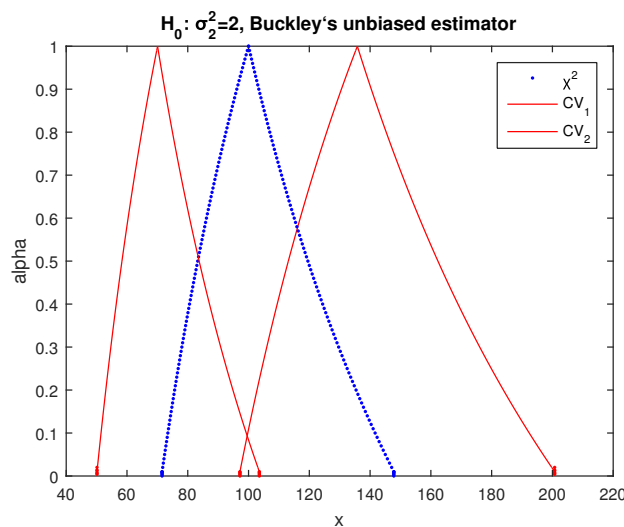

Figure 9. The possibility distribution of the fuzzy statistic $\widetilde{\chi^2}[\alpha]$ and the fuzzy critical values for the respective test of H_0 of Example 3 in [29].

The membership functions of the critical values $\widetilde{CV}_1$ and $\widetilde{CV}_2$ of one-sided tests are as follows:

$$\mu_{\widetilde{CV}_1}(x) = \begin{cases} 1, & x = F^{-1}(1-\gamma) \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \mu_{\widetilde{CV}_2}(x) = \begin{cases} 1, & x = F^{-1}(\gamma) \\ 0, & \text{elsewhere} \end{cases} \quad (50)$$

and for the two-sided test

$$\mu_{\widetilde{CV}_1}(x) = \begin{cases} 1, & x = F^{-1}\left(1 - \dfrac{\gamma}{2}\right) \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \mu_{\widetilde{CV}_2}(x) = \begin{cases} 1, & x = F^{-1}\left(\dfrac{\gamma}{2}\right) \\ 0, & \text{elsewhere} \end{cases} \quad (51)$$

where F^{-1} the inverse distribution function of the χ^2 distribution with $n-1$ degrees of freedom.

Having the $\alpha-$cuts of the possibility distribution of the fuzzy statistic χ_0^2 and the critical values $\widetilde{CV}_1$ and $\widetilde{CV}_2$, we can evaluate the possibility of rejection or acceptance of H_0 from (21), (23), (26), (28) for $\widetilde{U} = \widetilde{\chi^2}$ and (50) for one sided tests or from (31), (32), (34) and (51) for two sided test.

Example 3. *We test at significance level* 0.01 *the null hypothesis* $H_0 : \sigma^2 = 2$ *with alternative* $H_1 : \sigma^2 \neq 2$ *for the variance of a normal random variable, using a random sample of* 101 *observations with sample variance* $s^2 = 2$.

The crisp test in this case gives acceptance of H_0 with the largest possible difference between the test statistic and the critical values since the p-value is 1 (the test statistic is exactly in the middle of the no-rejection region), so it is the best case for the acceptance of H_0.

Applying the above fuzzy test, we get the results of Figure 8, where the core of $\widetilde{\chi^2}$ is between the cores of $\widetilde{CV}_1$ and $\widetilde{CV}_2$ and the point of intersection of $\mu_{\widetilde{\chi^2}}$, and $\mu_{\widetilde{CV}_1}$ has $y_1 = 0.06$ and of $\mu_{\widetilde{\chi^2}}$ and $\mu_{\widetilde{CV}_2}$, $y_2 = 0.03$, so according to (34) the possibility of acceptance of H_0 is as follows:

$$\varphi_A = \text{Ness}\left(\widetilde{\chi^2} < \widetilde{CV}_1, \widetilde{\chi^2} > \widetilde{CV}_2\right) = \min(1 - 0.06, 1 - 0.03) = 0.94.$$
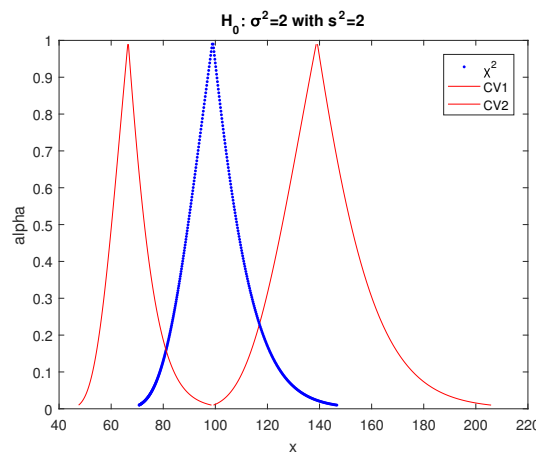
The respective fuzzy test of H_0 of [29] gives the results of Figure 9, where the possibility of acceptance of H_0 is as follows:

$$\varphi_A = \text{Ness}\left(\widetilde{\chi^2} < \widetilde{CV}_1, \widetilde{\chi^2} > \widetilde{CV}_2\right) = \min(1 - y_1, 1 - y_2) = 0.44.$$

and the test of H_0 of [30] gives the results of Figure 10, where the possibility of acceptance of H_0 is as follows:

$$\varphi_A = \text{Ness}\left(\widetilde{\chi^2} < \widetilde{CV}_1, \widetilde{\chi^2} > \widetilde{CV}_2\right) = \min(1 - 0.17, 1 - 0.23) = 0.77.$$

Figure 10. The possibility distribution of the fuzzy statistic $\widetilde{\chi^2}[\alpha]$ and critical values for the respective test of H_0 of Example 3 in [30].

The results of the fuzzy hypothesis tests proposed in this paper are in much better agreement with the results of the respective crisp tests than the results of the respective tests of [29]. This happens because of the following:

(1)   The former are based on the correct implementation of $\widetilde{\sigma_u^2}$, according to the theory, whereas the latter contain the error of using λ-cuts instead of α-cuts.
(2)   The tests of [29] use fuzzy critical values, which are created using probabilistic concepts.

The above are illustrated by the characteristic case of the hypothesis test of Example 3 in which the crisp value of the test statistic is exactly in the middle of the acceptance region (p-value $=$ 1), which is the best case of acceptance. As shown in this example, the test, which uses the implementation of $\widetilde{\sigma_u^2}$ of [29], gives a significantly lower possibility of acceptance of H_0 (0.44) than the test which uses our implementation of (0.94). So, the latter is in much better agreement with the crisp statistical tests than the former.

The possibilistic tests developed in this paper are in better agreement with the results of the respective crisp tests than the results of the tests of [30], which also use the same fuzzy critical values as [29]. This is illustrated in Examples 2 and 3 in which the crisp value of the test statistic is exactly in the middle of the acceptance region. As shown in these examples, the tests of [30] give a significantly lower degree of acceptance of H_0 (0.69 for Example 2 and 0.77 for Example 3) than the respective tests of this paper (0.94 for both examples). So, the latter are in better agreement with the crisp statistical test than the former. This happens in all relevant examples (statistic in the middle of the acceptance region).

Example 4. *We test the null hypothesis $H_0 : \sigma^2 = 47$ at significance level $\gamma = 0.01$ with alternative $H_1 : \mu \neq 47$ (two-sided test) for the variance σ^2 of the temperature X, using the two large random samples of 50 observations each of the Appendix A (monthly values for selected Greek weather stations) [37].*

The crisp value of the test statistic (43) for the first sample is found by (47) to be as follows (the sample variance is 80.28):

$$\chi_0^2 = \frac{(50-1) \cdot 80.28}{47} = 83.7$$

The critical values of this crisp test are (F^{-1}, the inverse distribution function of the χ_{50-1}^2 distribution).

$$\chi_{1-0.01/2}^2 = F^{-1}\left(\frac{0.01}{2}\right) = 27.25 \quad \text{and} \quad \chi_{0.01/2}^2 = F^{-1}\left(1 - \frac{0.01}{2}\right) = 78.23$$

So since $\chi_0^2 > \chi_{0.01/2}^2$, H_0 is rejected by the crisp test for this sample.

For the second sample (variance $s^2 = 72.42$), the value of the test statistic (43) is found by (47) to be as follows:

$$\chi_0^2 = \frac{(50 - 1) \cdot 72.42}{47} = 75.5$$

So since $\chi_{1-0.01/2}^2 < \chi_0^2 < \chi_{0.01/2}^2$, H_0 is not rejected by the crisp test for this sample.

Applying the above fuzzy test for the first sample, we get the results of Figure 11, where the core of $\widetilde{\chi^2}$ is at the right of the core of $\widetilde{CV}_1$ and their point of intersection has $y_2 = 0.78$. So according to (31), the possibility of rejection of H_0 is as follows:

$$\varphi_R = \text{Ness}\left(\widetilde{\chi^2} > \widetilde{CV}_1\right) = 1 - 0.78 = 0.22.$$
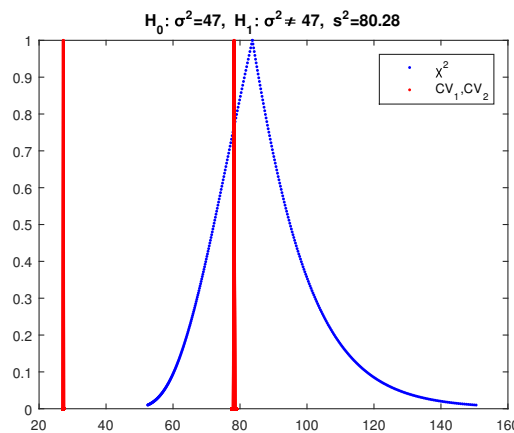

Figure 11. The fuzzy statistic $\widetilde{\chi}^2$ for the test of Example 4 for a sample with variance $s^2 = 82.28$.

For the second sample, we get the results of Figure 12, where the core of $\widetilde{\chi^2}$ is between the cores of $\widetilde{CV}_1$ and $\widetilde{CV}_2$ and the point of intersection of $\widetilde{\chi^2}$ and $\widetilde{CV}_1$ has $y_1 = 0.93$ and of $\widetilde{\chi^2}$ and $\widetilde{CV}_2$, $y_2 = 0$. So according to (34), the possibility of acceptance of H_0 is as follows:

$$\varphi_A = \text{Ness}\left(\widetilde{\chi^2} < \widetilde{CV}_1, \ \widetilde{\chi^2} > \widetilde{CV}_2\right) = \min(1 - 0.83, 1 - 0) = 0.17.$$

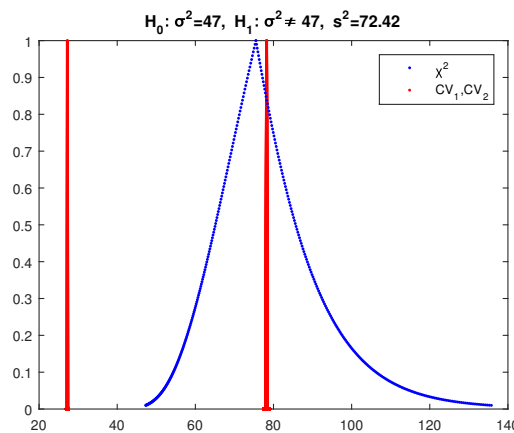Therefore, we cannot make a decision on accepting H_0 from this test.

Figure 12. The possibility distribution of the fuzzy statistic $\widetilde{\chi}^2$ for the test of Example 4 for a sample with variance $s^2 = 72.42$.

7. Hypothesis Tests for the Mean of a Normal Random Variable with Unknown Variance

We test at significance level γ the null hypothesis $H_0 : \mu = \mu_0$ for the mean value μ of a random variable X, which follows normal distribution with unknown variance, using a random sample of observations of X of size n with sample mean and variance $\overline{x}$ and s^2.

In this case, the test statistic is

$$T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} \tag{52}$$

where $\overline{X}$ and S are the statistics of the sample mean and the standard deviation.

It is known [1] that under the null hypothesis ($\mu = \mu_0$), T follows t distribution with $n - 1$ degrees of freedom. So in the crisp case, H_0 is rejected from a given random sample:
(a) For the *one sided test from the right* (alternative $H_1 : \mu > \mu_0$), if $t_0 > t_{\gamma;n-1}$, where

$$t_0 = \frac{\overline{x} - \mu_0}{s/\sqrt{n}} \tag{53}$$

the value of the statistic (52),

$$t_{\gamma;n-1} = F_{n-1}^{-1}(1 - \gamma) \tag{54}$$

the critical value of the test and F_{n-1}^{-1} is the inverse distribution function of the t distribution with $n - 1$ degrees of freedom.
(b) For the *one sided test from the left* (alternative $H_1 : \mu < \mu_0$), if $t_0 < -t_{\gamma;n-1}$.
(c) For the *two-sided* test (alternative $H_1 : \mu \neq \mu_0$), if $t_0 < -t_{\gamma/2;n-1}$ or $t_0 > t_{\gamma/2;n-1}$ while, if $-t_{\gamma/2;n-1} < t_0 < t_{\gamma/2;n-1}$, then H_0 is not rejected.

In the fuzzy case for the test of H_0 we use the following fuzzy statistic:

$$\overline{T} = \frac{\overline{\mu} - \mu_0}{\overline{s}/\sqrt{n}} \tag{55}$$

which is generated by substituting $\overline{X}$ and S in (52) with the fuzzy estimators $\widetilde{\mu}$ and $\widetilde{s}$ of the mean and standard deviation. The $\alpha-$cuts of $\widetilde{s}$ are given by (13) and (as described in [29,30]) the $\alpha-$cuts of the possibility distribution of $\widetilde{\mu}$ are as follows:

$$\mu_{\widetilde{\mu}}[\alpha] = \left[\overline{x} - t_{\alpha;n-1}\frac{s}{\sqrt{n}}, \ \overline{x} + t_{\alpha;n-1}\frac{s}{\sqrt{n}} \right], \quad \alpha \in [0,1] \tag{56}$$

From (53), (55), (56), (13) and fuzzy number arithmetics follows that the $\alpha-$cuts of the possibility distribution of the fuzzy statistic $\widetilde{T}$ are as follows:

$$\mu_{\widetilde{T}}[\alpha] = \left[\frac{\overline{x} - \mu_0 - t_{\alpha;n-1}\frac{s}{\sqrt{n}}}{\frac{1}{\sqrt{n}}s\sqrt{\frac{(n-1)}{\chi^2_{1-\alpha;n-1}}}}, \ \frac{\overline{x} - \mu_0 + t_{\alpha;n-1}\frac{s}{\sqrt{n}}}{\frac{1}{\sqrt{n}}s\sqrt{\frac{(n-1)}{\chi^2_{\alpha;n-1}}}} \right]$$

$$= \left[\sqrt{\frac{\chi^2_{1-\alpha;n-1}}{n-1}}(t_0 - t_{\alpha;n-1}), \ \sqrt{\frac{\chi^2_{\alpha;n-1}}{n-1}}(t_0 + t_{\alpha;n-1}) \right] \tag{57}$$

where $t_\alpha = F_{n-1}^{-1}(1 - \alpha)$.

The membership functions of the critical values $\widetilde{CV}_1$ and $\widetilde{CV}_2$ of the one-sided tests are as follows:

$$\mu_{\widetilde{CV}_1}(x) = \begin{cases} 1, & x = F_{n-1}^{-1}(1 - \gamma) \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \mu_{\widetilde{CV}_2}(x) = \begin{cases} 1, & x = F_{n-1}^{-1}(\gamma) \\ 0, & \text{elsewhere} \end{cases} \tag{58}$$

and for the two-sided test,

$$\mu_{\widetilde{CV_1}}(x) = \begin{cases} 1, & x = F_{n-1}^{-1}\left(1 - \dfrac{\gamma}{2}\right) \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \mu_{\widetilde{CV_2}}(x) = \begin{cases} 1, & x = F_{n-1}^{-1}\left(\dfrac{\gamma}{2}\right) \\ 0, & \text{elsewhere} \end{cases} \tag{59}$$

where F^{-1} is the inverse distribution function of the t distribution with $n - 1$ degrees of freedom.

Having the $\alpha-$cuts (57) of the possibility distribution of the fuzzy statistic T and the critical values $\widetilde{CV}_1$ and $\widetilde{CV}_2$, we can evaluate the possibility of rejection or acceptance of H_0 from (21), (23), (26), (28) for $\widetilde{U} = \widetilde{T}$ and (58) for one-sided tests or from (31), (32), (34) and (59) for the two-sided test.

Example 5. *We test the null hypothesis* $H_0 : \mu = 11$ *at significance level* 0.02 *with alternative* $H_1 : \mu \neq 11$ *(two sided test) for the mean* μ *of a normal random variable X, using a sample of* $n = 25$ *observations with sample mean and variance* $\overline{x} = 10,9$ *and* $s^2 = 7.57$.

In the crisp case, the value of the statistic (52) is evaluated by (53)

$$t_0 = \frac{10,9 - 11}{\sqrt{3,57}/\sqrt{25}} = 0,26$$

so the $p-$value is $p = 2(1 - F_{24}(0,26)) = 0.8$, which is much higher than the significance level $\gamma = 0.02$. Hence, H_0 is accepted by the crisp test with high p-value (the test statistic is almost at the center of the acceptance region).

We apply the above two-sided fuzzy test of H_0, implementing (57) and (59). So, we obtain the results of Figure 13, where the core of $\widetilde{T}$ is between the cores of $\widetilde{CV}_1$ and $\widetilde{CV}_2$ and the points of intersection of $\mu_{\widetilde{T}}$ and $\mu_{\widetilde{CV}_1}$ and of $\mu_{\widetilde{T}}$ and $\mu_{\widetilde{CV}_2}$ have $y_1 = 0.07$ and $y_2 = 0.05$. Hence, according to (34), the possibility of acceptance of H_0 is

$$\varphi_A = \min(1 - y_1, 1 - y_2) = 0.93,$$

so H_0 is accepted by this test with possibility $d = 0.93$.

Applying the respective test of [29,30], we get the results of Figure 14, where we see that the possibility of acceptance of H_0 is as follows:
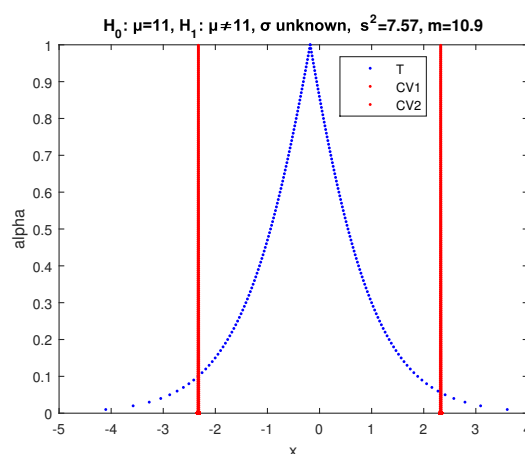
$$\varphi_A = \min(1 - y_1, 1 - y_2) = 0.69$$

Figure 13. The possibility distribution of the fuzzy statistic $\overline{T}$ for the test of H_0 of Example 5.
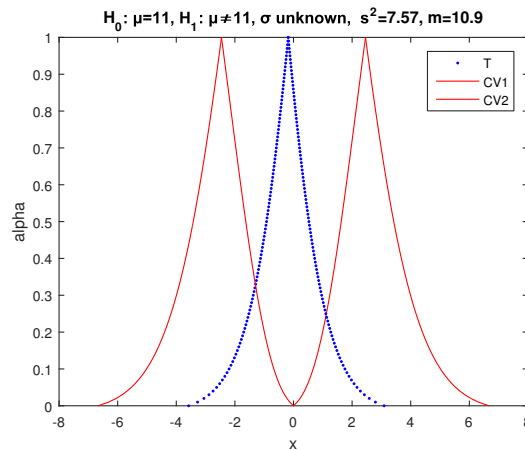
Figure 14. The possibility distribution of the fuzzy statistic $\overline{T}$ and the fuzzy critical values for the respective test of [15] of H_0 of Example 5.

Example 6. *We test the null hypothesis $H_0 : \mu = 6$ at significance level 0.05 with alternative $H_1 : \mu \neq 6$ (two sided test) for the mean μ of a normal random variable X, using a sample of $n = 25$ observations with sample mean and variance $\overline{x} = 4.5$ and $s^2 = 3.66$.*

In the crisp case, the value of the statistic (52) is evaluated by (53)

$$t_0 = \frac{4.5 - 6}{\sqrt{3,66} / \sqrt{25}} = -3.92$$

so the $p-$value is $p = 2F_{24}(-3.92)) = 6.5 \times 10^{-4}$, which is almost zero. Hence, H_0 is rejected by the crisp test.

We apply the above test implementing (57) and (59). So, we take the results of Figure 15, where the core of $\overline{T}$ is at the left of the core of $\overline{CV}_2$ and the point of intersection of $\overline{T}$ and $\overline{CV}_2$ has $y_2 = 0.03$. Hence, according to (32), the possibility of rejection of H_0 is $1 - 0.0.3 = 0.97$.
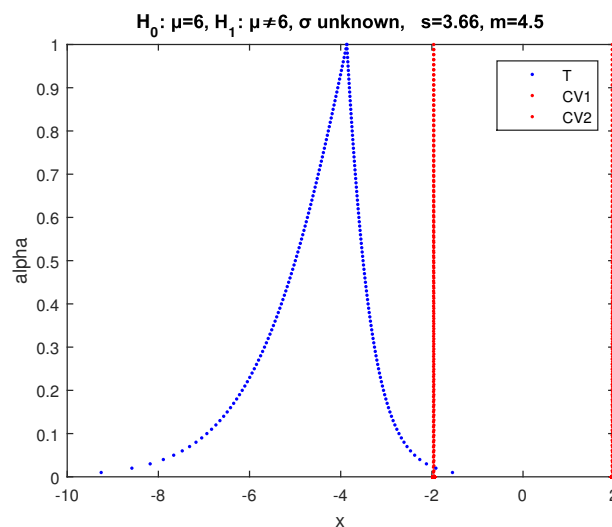

Figure 15. The fuzzy statistic $\overline{T}$ for the test of Example 6.

Applying the respective test of [30], we get the results of Figure 16, where we see that the possibility of acceptance of H_0 is as follows:

$$\varphi_A = \min(1 - y_1, 1 - y_2) = 1 - 0.33 = 0.67$$

A similar result with even lower possibility of acceptance is obtained by the respective test of [29].
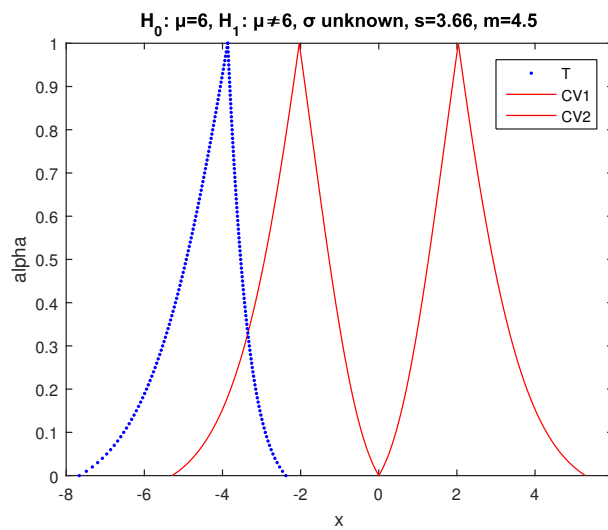

Figure 16. The fuzzy statistic $\overline{T}$ and the fuzzy critical values for the test of H_0 of [30] for Example 6.

8. Conclusions

If the value of the test statistic of a hypothesis is close to a critical value of the test, then the crisp hypothesis test is unstable since a very small change in the sample may lead from rejection to no rejection of H_0 or vice-versa, as shown in Examples 1 and 4. Our possibilistic approach, which uses fuzzy estimators for the construction of the test statistic and a possibility of rejection or acceptance of the null hypothesis H_0, gives better results than the crisp test since it gives us the possibility of rejection or acceptance of H_0, as shown in Examples 1 and 4, where we get a very low possibility of rejection or acceptance of the null hypothesis, which means "no decision".

We can also conclude the following:

(1) The use of the fuzzy critical values of [29,30] in hypothesis tests leads to wrong results, since they are not in good agreement with the crisp tests, which have been known for nearly one hundred years (Examples 2, 3 and 5). Meanwhile, the possibilistic fuzzy hypothesis tests developed in this paper give results which are in much better agreement with the crisp tests.

(2) The use of the implementation of $\widetilde{\sigma}_u^2$ of [29] in the hypothesis tests leads, also, to wrong results since they are not in good agreement with the crisp tests (Example 3). Meanwhile, the fuzzy hypothesis tests, which are based on test statistics produced by the correct implementation of $\widetilde{\sigma}_u^2$ (taking into consideration the relation $a(\lambda)$ between the significance level a and the parameter λ), give results which are in better agreement with the crisp tests.

We believe that with the help of all the above, new horizons can be opened in the fuzzy hypothesis testing of several topics, such as the following:

(a) Proportions and difference of proportions, using fuzzy test statistics constructed by fuzzy estimators produced by confidence intervals with the help of the central limit theorem, correcting the results of the respective tests of [29,30].

(b) The ratio of variances of two normal random variables, using test statistics constructed by the correct implementation of $\widetilde{\sigma}_u^2$, correcting the respective results produced by tests which use the wrong implementation of $\widetilde{\sigma}_u^2$ of [29] or the biased estimator $\widetilde{\sigma}_b^2$ of [30].

(c) The regression coefficient β and the predicted value $y_p(x_0)$ of Y for a given x_0 in a linear regression model $Y = a + \beta x$, using fuzzy test statistics constructed by the fuzzy estimators $\widetilde{\beta}$ and $\widehat{y}_p(x_0)$ produced by the known classical statistics confidence

intervals, $[\widehat{\beta} - t_{a/2;n-1}s_{\widehat{\beta}}, \; \widehat{\beta} - t_{a/2;n-1}s_{\widehat{\beta}}]$ of β ($\widehat{\beta}$ and $s_{\widehat{\beta}}$ the sample regression coefficient and its standard deviation) and $[\widehat{y} - t_{a/2;n-1}s_{\widehat{y}}, \; \widehat{y} - t_{a/2;n-1}s_{\widehat{y}}]$ of y_p ($\widehat{y}$ and $s_{\widehat{y}}$ the predicted value of Y for $x = x_0$ from the given sample and its standard deviation).

Appendix A

Samples of temperature taken from Hellenic national meteorological service [37].

Sample 1

12.14	21.48	20.82	20.96	22.66	20.61	21.5	27.86	27.33	25.75
27.8	29.32	28.52	28.76	28.31	28.1	27.99	27.43	28.8	28.17
29.02	24.55	22.29	23.35	23.37	23.96	5.96	16.54	17.25	17.98
18.78	13.59	10.85	10.82	13.42	15.19	8	5.56	8.66	6.5
10.21	4.87	5.66	3.2	6.2	2.97	4.4	10.19	6.2	6.9

Sample 2

26.4	27.7	28.7	29.63	29.44	29.89	28.15	29.15	29.05	29.34
29.08	29.94	25.66	26.85	24.55	26.09	25.16	22.33	20.11	19.8
20.41	21.97	18.44	17.01	14.54	16.72	18.34	10.88	7.63	12.99
10.35	12.54	6.6	6.61	0.83	6.68	4.43	6.59	11.52	10.18
8.82	8.68	9.1	12.16	12.87	12.43	12.32	15.3	17.47	15.86

References

1. Hogg, R.V.; Tanis E.A. *Probability and Statistical Inference*, 6th ed.; Prentice Hall: Upper Saddle River, NJ, USA, 2001.
2. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
3. Taheri, S.M. Trends in fuzzy statistics. *Aust. J. Stat.* **2003**, *32*, 239–257. [CrossRef]
4. Tanaka, H.; Okuda, T.; Asai, K. Fuzzy information and decision in a statistical model. In *Advances in Fuzzy Set Theory and Applications*; Gupta, M.M., Ragade, R. K., Yager R. R. , Eds; North-Holland: Amsterdam, The Netherlands, 1979; pp. 303–320
5. Casals, M.R.; Gil, M.A.; Gil, P. On the use of Zadeh's probabilistic definition for testing statistical hypotheses from fuzzy information. *Fuzzy Sets Syst.* **1986**, *20*, 175–190. [CrossRef]
6. Casals, M.R. Bayesian testing of fuzzy parametric hypotheses from fuzzy information. *Oper. Res.* **1993**, *27*, 189–199. [CrossRef]
7. Watanabe, N.; Imaizumi, T. A fuzzy statistical test of fuzzy hypotheses. *Fuzzy Sets Syst.* **1993**, *53*, 167–178. [CrossRef]
8. Arnold, B.F. An approach to fuzzy hypothesis testing. *Metrika* **1996**, *44*, 119–126. [CrossRef]
9. Arnold, B.F. Testing fuzzy hypotheses with crisp data. *Fuzzy Sets Syst.* **1998**, *94*, 323–333. [CrossRef]
10. Taheri, S.M.; Behboodian, J. Neyman-Pearson Lemma for fuzzy hypotheses testing. *Metrika* **1999**, *49*, 3–17. [CrossRef]
11. Taheri, S.M.; Behboodian, J. ABayesian approach to fuzzy hypotheses testing. *Fuzzy Sets Syst.* **2001**, *123*, 39–48. [CrossRef]
12. Chachi, J.; Taheri, S.M. Optimal statistical tests based on fuzzy random variables. *Iran. J. Fuzzy Syst.* **2018**, *15*, 27–45.
13. Akbari, M.G.; Hesamian, G. Testing statistical hypotheses for intuitionistic fuzzy data. *Soft Comput.* **2019**, *23*, 10385–10392. [CrossRef]
14. Arefi, M. Testing statistical hypotheses under fuzzy data and based on a new signed distance. *Iran. J. Fuzzy Syst.* **2018**, *15*, 153–176.
15. Hung, J.-L.; Chen, C.-C.; Lai, C.-M. Possibility Measure of Accepting Statistical Hypothesis. *Mathematics* **2020**, *8*, 551. [CrossRef]
16. Chukhrova, N.; Johannssen, A. Fuzzy hypothesis testing for a population proportion based on set-valued information. *Fuzzy Sets Syst.* **2020**, *387*, 127–157. [CrossRef]
17. Chukhrova, N.; Johannssen, A. Randomized versus non-randomized hypergeometric hypothesis testing with crisp and fuzzy hypotheses. *Stat. Pap.* **2018**, *61*, 1–37. [CrossRef]
18. Filzmoser, P.; Viertl, R. Testing hypotheses with fuzzy data: The fuzzy *p*-value. *Metrika* **2004**, *59*, 21–29. [CrossRef]

19. Parchami, A.; Taheri, S.M.; Mashinchi, M. Fuzzy *p*-value in testing fuzzy hypotheses with crisp data. *Stat Pap.* **2010**, *51*, 209. [CrossRef]
20. Mylonas, N.; Papadopoulos, B. Fuzzy *p*-value of Hypotheses Tests with Crisp Data Using Non-Asymptotic Fuzzy Estimators. *J. Stoch. Anal.* **2021**, *2*, 1. [CrossRef]
21. Couso, I.; Sanchez, L. Mark-recapture techniques in statistical tests for imprecise data. *Internat. J. Approx. Reason.* **2011**, *52*, 240–260. [CrossRef]
22. Grzegorzewski, P.; Hryniewicz, O. Soft methods in hypotheses testing. In *Soft Computing for Risk Evaluation and Management: Applications in Technology, Environment and Finance, in Studies in Fuzziness and Soft Computing*; Ruan, D., Kacprzyk, J., Fedrizzi, M., Eds.; Physica-Verlag: Heidelberg, Germany, 2001; Volume 76, pp. 55–72.
23. Hryniewicz, O. Possibilistic decisions and fuzzy statistical tests. *Fuzzy Sets Syst.* **2006**, *157*, 2665–2673 [CrossRef]
24. Hryniewicz, O. Statistical properties of the fuzzy *p*-value. *Internat. J. Approx. Reason.* **2018**, *93*, 544–560. [CrossRef]
25. Berkachy, R.; Donze, L. Testing hypotheses by fuzzy methods: A comparison with the classical approach. In *Applying Fuzzy Logic for the Digital Economy and Society, in Fuzzy Management Methods*; Meier, A., Portmann, E., Teran, L., Eds.; Springer: Cham, Switzerland, 2019; pp. 1–22.
26. Parchami, A. Fuzzy decision making in testing hypotheses: An introduction to the packages FPV and fuzzy.p.value with practical examples. *Iran. J. Fuzzy Syst.* **2020**, *17*, 67–77.
27. Chukhrova, N.; Johannssen, A. Fuzzy hypothesis testing: Systematic review and bibliography. *Appl. Soft Comput.* **2021**, *106*, 107331. [CrossRef]
28. Buckley, J.J. Fuzzy statistics: Hypotheses testing. *Soft Comput.* **2005**, *9*, 512–518. [CrossRef]
29. Buckley, J.J. *Fuzzy Statistics*; Springer: Berlin/Heidelberg, Germany, 2004.
30. Mylonas, N.; Papadopoulos, B. Fuzzy hypotheses tests for crisp data using non-asymptotic fuzzy estimators and a degree of rejection or acceptance. *Evol. Syst.* **2021**, 1–18. [CrossRef]
31. Sfiris, D.; Papadopoulos, B. Non-asymptotic fuzzy estimators based on confidence intervals. *Inf. Sci.* **2014**, *279*, 446–459. [CrossRef]
32. Dubois, D.; Foulloy, L.; Mauris, G.; Prade, H. Probability-possibility transformations, triangular fuzzy-sets and probabilistic inequalities. *Reliab. Comput.* **2004**, *10*, 273–297.:REOM.0000032115.22510.b5. [CrossRef]
33. Kaya, I.; Kahraman, C. A new perspective on fuzzy process capability indices: Robustness. *Expert Syst. Appl.* **2010**, *37*, 4593–4600. [CrossRef]
34. Kaya, I.; Kahraman, C. Fuzzy Process Capability Analysis and Applications. In *Production Engineering and Management under Fuzziness, Studies in Fuzziness and Soft Computing*; Kahraman, C., Yavuz, M., Eds.; Springer: Berlin/Heidelberg, Germany, 2010; Volume 252, pp. 483–513.
35. Senvar, O.; Tozan, H. Process Capability and Six Sigma Methodology Including Fuzzy and Lean Approaches. In *Products and Services from R and D to Final Solutions*; Fuerstner, I., Ed.; IntechOpen, 2010. Available online: https://www.intechopen.com/books/products-and-services--from-r-d-to-final-solutions/process-capability-and-six-sigma-methodology-including-fuzzy-and-lean-approaches (accessed on 1 January 2020). [CrossRef]
36. Dubois, D.; Prade, H. Ranking of fuzzy numbers in the setting of possibility theory. *Inf. Sci.* **1983**, *30*, 183–224. [CrossRef]
37. Hellenic National Meteorological Service. Available online: http://www.hnms.gr/emy/el/climatology/climatologymonth (accessed on 1 January 2020).