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A Symbolic Method for Solving a Class of Convolution-Type Volterra–Fredholm–Hammerstein Integro-Differential Equations under Nonlocal Boundary Conditions

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Abstract: Integro-differential equations involving Volterra and Fredholm operators (VFIDEs) are used to model many phenomena in science and engineering. Nonlocal boundary conditions are more effective, and in some cases necessary, because they are more accurate measurements of the true state than classical (local) initial and boundary conditions. Closed-form solutions are always desirable, not only because they are more efficient, but also because they can be valuable benchmarks for validating approximate and numerical procedures. This paper presents a direct operator method for solving, in closed form, a class of Volterra–Fredholm–Hammerstein-type integro-differential equations under nonlocal boundary conditions when the inverse operator of the associated Volterra integro-differential operator exists and can be found explicitly. A technique for constructing inverse operators of convolution-type Volterra integro-differential operators (VIDEs) under multipoint and integral conditions is provided. The proposed methods are suitable for integration into any computer algebra system. Several linear and nonlinear examples are solved to demonstrate the effectiveness of the method.

Keywords: integro-differential equations; Volterra; Volterra–Fredholm; nonlinear; nonlocal conditions; boundary value problems; convolution; symbolic computations; exact solution

MSC: 45J05; 47G20; 34B10

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1. Introduction

Volterra–Fredholm integro-differential equations (VFIDEs) are encountered in many different branches of science and engineering [1–4]. As they are extremely difficult and in some cases even impossible to solve in exact closed form, many different techniques have been developed to construct approximate and numerical solutions. Some of these methods are mentioned below.

Two well-known, traditional techniques are the series solution method [1,5] and the variational iteration method [1]. A powerful and widely used method for solving linear and nonlinear integro-differential equations including VFIDEs is the Adomian decomposition method (ADM) and its variant, known as modified ADM [6,7]. Methods employing Taylor polynomial expansions were developed and applied to find the approximate solution of high-order linear VFIDEs under mixed conditions [8] and high-order nonlinear VFIDEs [9,10]. Collocation methods based on Haar wavelets and Legendre wavelets for the numerical solution of nonlinear VFIDEs were presented in [3,11], respectively. An approach using hybrid Legendre polynomials and block–pulse functions was also reported in [12]. The authors of [13] applied the polynomial least squares method to construct approximate analytical solutions for a very general class of nonlinear VFIDEs. A moving least squares (MLS) method was developed in [14]. The Tau method was also used for the numerical solution of the general form of linear VFIDEs [15]. In [16], the reproducing kernel Hilbert space method was employed to numerically solve first-order periodic VFIDEs. A method based on triangular functions and the relative operational matrix to determine the numerical solution of specific nonlinear VFIDEs was proposed in [17]. Fixed-point techniques for the approximate solution of linear and first-order nonlinear VFIDEs were discussed in [18,19], respectively. A quadrature and iterative technique for solving a nonlinear VFIDE with weakly singular kernels was developed in [20]. Numerical techniques for solving fractional VFIDEs were proposed in [21–23] using second Chebyshev wavelets (SCWs), fractional-order Bernoulli functions (FBFs) and Lucas wavelets (LWs), respectively. The numerical solution of multipoint boundary value problems for linear VFIDEs was studied using Legendre polynomials [24,25], and Bernstein polynomials [26,27]. A semi-analytical numerical method for solving multipoint boundary value problems for linear VFIDEs of the neutral type with linear functional arguments was presented in [28]. Finally, for other related types of problems, such as ordinary differential equations under multipoint and integral conditions and nonlinear Volterra equations with loads, see [29,30].

This paper deals with the exact closed-form solution of a class of Volterra–Fredholm integro-differential equations of the Hammerstein type. It is emphasized that due to the complexity involved in solving integro-differential equations (IDEs) and in contrast to numerical methods, exact analytical techniques cannot have a general purpose but are designed and applied to a specific class of IDEs. Methods for solving explicitly different types of Volterra and Fredholm IDEs are discussed in detail in [31]. A method for the exact solution of systems of first-order linear Fredholm IDEs under multipoint and integral conditions was presented in [32]. The exact solution of linear and nonlinear convolution-type Volterra–Fredholm integral equations (VFIEs) was efficiently treated in [33]. In [4], the existence and uniqueness criteria and the closed-form solution of the convolution-type linear VFIDEs of the following form were investigated:

$$\sum_{i=0}^{n} a_i u^{(n-i)}(x) + \sum_{i=0}^{n} \int_0^x k_i(x-t) u^{(n-i)}(t)dt - \sum_{j=1}^{m} \int_0^b \bar{k}_j(x,t) \sum_{i=0}^{n} a_i u^{(n-i)}(t)dt = f(x), \tag{1}$$

subject to the boundary conditions

$$\Phi_i(u) = 0, \quad i = 1,2,\ldots,n, \tag{2}$$

where a_i, $i = 0,1,\ldots,n$ ($a_0 \neq 0$) and $b > 0$ are given real constants; $k_i(x)$, $i = 0,1,\ldots,n$, $\bar{k}_j(x,t)$, $j = 1,2,\ldots,m$, and $f(x)$ are given continuous functions; $u(x)$ is the unknown function assumed n times continuously differentiable; and $u^{(i)}(x) = \frac{d^i u}{dx^i}$ and Φ_i, $i = 1,2,\ldots,n$, are given linear bounded functionals. This technique was used to solve the second-order VFIDEs encountered in the bending analysis of elastic Euler–Bernoulli beams in Eringen's two-phase nonlocal integral model in the case of a distributed transverse load and boundary conditions for a simply supported beam, cantilever beam, clamped pinned beam and clamped beam [4,34].

In this article, we extend the technique to Volterra–Fredholm integro-differential equation in the Hammerstein form

$$\sum_{i=0}^{n} a_i u^{(n-i)}(x) - \sum_{i=0}^{n} \int_a^x k_i(x,t) u^{(n-i)}(t)dt - \sum_{j=1}^{m} \int_a^b q_j(x,t) \varphi_j\Big(t, u(t), u'(t), \ldots, u^{(n)}(t)\Big)dt = f(x), \tag{3}$$

under the general nonlocal boundary conditions

$$\Phi_i(u) - \sum_{j=1}^{l} \nu_{ij}\Theta_j(u) = 0, \quad i = 1,2,\ldots,n, \tag{4}$$

where a_i, $i = 0,1,\ldots,n$, $(a_0 \neq 0)$, a and b are given real constants; $k_i(x,t)$, $i = 0,1,\ldots,n$, $q_j(x,t)$, $\varphi_j(\cdot)$, $j = 1,2,\ldots,m$, and $f(x)$ are given continuous functions; $u(x)$ is the unknown function assumed n times continuously differentiable with $u^{(i)}(x) = \frac{d^i u}{dx^i}$; Φ_i, $i = 1,2,\ldots,n$, are some special, linear bounded functionals; Θ_j, $j = 1,2,\ldots,l$, are given general linear bounded functionals involving values at fixed points and definite integrals of $u^{(i-1)}(x)$, $i = 1,2,\ldots,n$, in the interval $[a,b]$; and ν_{ij}, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,l$, are real constants.

The outline of the paper is as follows: In Section 2, a direct operator method for solving nonlocal boundary value problems for nonlinear VFIDEs is presented. In Section 3, existence and uniqueness criteria and a solution formula for linear VFIDEs are given. Section 4 deals with the inversion of linear Volterra integro-differential operators (VIDEs) of convolution type and the solution of the corresponding VFIDEs. The proposed methods are tested by solving several examples in Section 5. Finally, conclusions and future directions are discussed in Section 6.

2. Formulation and Solution of Nonlinear VFIDEs

First, we formulate the nonlocal boundary problem (3), (4) in a convenient operator form.

Let $X = C[a,b]$, $a,b \in \mathbb{R}$, and let the operators $A, K : X \to X$ defined as

$$Au = \sum_{i=0}^{n} a_i u^{(n-i)}(x), \tag{5}$$

$$Ku = \sum_{i=0}^{n} \int_a^x k_i(x,t) u^{(n-i)}(t)dt, \tag{6}$$

be an nth-order linear differential operator and a linear Volterra-type integral operator, respectively, where $n \in \mathbb{N}$, $a_0 \neq 0$, a_i, $i = 1,2,\ldots,n$, are real constants; $u(x) \in C^n[a,b]$; and kernel functions $k_i(x,t) \in X \times X$, $i = 0,1,\ldots,n$.

Furthermore, let the nonlinear Fredholm.type integral operator $\bar{K} : X \to X$ be defined as

$$\bar{K}u = \sum_{j=1}^{m} \int_a^b q_j(x,t)\varphi_j\Big(t, u(t), u'(t), \ldots, u^{(n)}(t)\Big)dt, \tag{7}$$

where $q_j(x,t)$, $\varphi_j(\cdot)$, $j = 1,2,\ldots,m$, are continuous functions. We assume that kernels $q_j(x,t)$ are degenerate, i.e., without loss of generality of the form

$$q_j(x,t) = g_j(x)h_j(t), \quad j = 1,2,\ldots,m,$$

where $g_j(x)$, $h_j(t) \in X$. In this case, Equation (7) becomes

$$\bar{K}u = \sum_{j=1}^{m} g_j(x) \int_a^b h_j(t)\varphi_j\Big(t, u(t), u'(t), \ldots, u^{(n)}(t)\Big)dt. \tag{8}$$

By defining the row vector of functions

$$g = \begin{pmatrix} g_1 & g_2 & \cdots & g_m \end{pmatrix}, \quad g_j = g_j(x), \quad j = 1,2,\ldots,m,$$

and the column vector of functionals

$$\Psi(u) = \begin{pmatrix} \Psi_1(u) \\ \Psi_2(u) \\ \vdots \\ \Psi_m(u) \end{pmatrix}, \quad \Psi_j(u) = \int_a^b h_j(t)\varphi_j\Big(t, u(t), u'(t), \ldots, u^{(n)}(t)\Big)dt, \tag{9}$$

where Ψ_j, $j = 1, 2, \ldots, m$, are nonlinear functionals defined on X, Equation (8) is written compactly as

$$\bar{K}u = \sum_{j=1}^{m} g_j \Psi_j(u) = g\Psi(u). \tag{10}$$

Let the column vectors of linear bounded functionals

$$\Phi(u) = \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \\ \vdots \\ \Phi_n(u) \end{pmatrix}, \quad \Theta(u) = \begin{pmatrix} \Theta_1(u) \\ \Theta_2(u) \\ \vdots \\ \Theta_l(u) \end{pmatrix},$$

and the $m \times \ell$ constant matrix

$$\mathbf{N} = \begin{pmatrix} \nu_{11} & \nu_{12} & \cdots & \nu_{1l} \\ \nu_{21} & \nu_{22} & \cdots & \nu_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n1} & \nu_{n2} & \cdots & \nu_{nl} \end{pmatrix},$$

where ν_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, l$, are real constants, and write the nonlocal boundary conditions in the matrix form

$$\Phi(u) - \mathbf{N}\Theta(u) = \mathbf{0}, \tag{11}$$

where $\mathbf{0}$ denotes the zero column vector.

By employing (5), (6) and (11) define the linear Volterra-type integro-differential operator $V : X \to X$,

$$\begin{aligned} Vu &= Au - Ku, \\ \mathcal{D}(V) &= \{u : u \in C^n[a,b],\ \Phi(u) - \mathbf{N}\Theta(u) = \mathbf{0}\}, \end{aligned} \tag{12}$$

and then, by means of (10), the nonlinear Volterra–Fredholm-type integro-differential operator $T : X \to X$,

$$Tu = Vu - g\Psi(u), \quad \mathcal{D}(T) = \mathcal{D}(V).$$

Thus, the boundary value problem (3), (4) can be written in the operator form

$$Tu = f, \quad f \in X.$$

We now state the following theorem.

Theorem 1. *If linear operator V is bijective on X and its inverse is denoted by V^{-1}, then the exact solution to the problem $Tu = f$ for all $f \in X$ is given by*

$$u = V^{-1}f + V^{-1}g\mathbf{c}^*, \tag{13}$$

for every vector $\mathbf{c}^* = \Psi(u)$ that is a solution of the nonlinear algebraic (transcendental) system

$$\mathbf{c} - \Psi\Big(V^{-1}f + V^{-1}g\mathbf{c}\Big) = \mathbf{0}. \tag{14}$$

Proof. By multiplying the nonlinear equation

$$Tu = Vu - g\Psi(u) = f,$$

by inverse operator V^{-1}, we obtain

$$u = V^{-1}f + V^{-1}g\Psi(u). \tag{15}$$

By applying vector Ψ on both sides, we have

$$\Psi(u) = \Psi\Big(V^{-1}f + V^{-1}g\Psi(u)\Big).$$

We set $\mathbf{c} = \Psi(u)$ and write

$$\mathbf{c} = \Psi\Big(V^{-1}f + V^{-1}g\mathbf{c}\Big),$$

which is a nonlinear algebraic (transcendental) system with vector $\mathbf{c}$ unknown. Let $\mathbf{c}^*$ be a solution of this system. By substituting $\mathbf{c}^*$ into (15), we obtain the solution in (13). □

3. Solution of Linear VFIDEs

In the case in which $\varphi_j\Big(t, u(t), u'(t), \dots, u^{(n)}(t)\Big)$ is a linear function in $u(t), u'(t), \dots,$ $u^{(n)}(t)$, namely,

$$\varphi_j\Big(t, u(t), u'(t), \dots, u^{(n)}(t)\Big) = \sum_{i=0}^{n} \varphi_{ji}(t) u^{(n-i)}(t), \quad j = 1, 2, \dots, m, \tag{16}$$

where $\varphi_{ji}(t) \in X$, $j = 1, 2, \dots, m$, $i = 1, 2, \dots, n$, Equation (8) is simplified as

$$\bar{K}u = \sum_{j=1}^{m} g_j(x) \int_a^b h_j(t) \sum_{i=0}^{n} \varphi_{ji}(t) u^{(n-i)}(t) dt,$$

and Equation (9) is written as

$$\Psi(u) = \begin{pmatrix} \Psi_1(u) \\ \Psi_2(u) \\ \vdots \\ \Psi_m(u) \end{pmatrix}, \quad \Psi_j(u) = \int_a^b h_j(t) \sum_{i=0}^{n} \varphi_{ji}(t) u^{(n-i)}(t) dt,$$

where now Ψ_j, $j = 1, 2, \dots, m$, are linear bounded functionals defined on $C^n[a, b]$ (see [35] (Theorem 3, p. 480)).

We introduce the $m \times m$ matrix

$$\Psi(g) = \begin{pmatrix} \Psi_1(g_1) & \cdots & \Psi_1(g_m) \\ \vdots & \ddots & \vdots \\ \Psi_m(g_1) & \cdots & \Psi_m(g_m) \end{pmatrix},$$

where element $\Psi_i(g_j)$ is the value of functional Ψ_i on element g_j. It is noted that for a $m \times k$ constant matrix $\mathbf{C}$,

$$\Psi(g\mathbf{C}) = \Psi(g)\mathbf{C}. \tag{17}$$

Define the linear Volterra–Fredholm integro-differential operator $S : X \to X$ as

$$\begin{aligned} Su &= Vu - g\Psi(u), \\ \mathcal{D}(S) &= \mathcal{D}(V) = \{u : u \in C^n[a, b],\ \Phi(u) - \mathbf{N}\Theta(u) = \mathbf{0}\}, \end{aligned}$$

and write the linear boundary value problem (3), (4) and (16) in the symbolic form

$$Su = f, \quad f \in X.$$

We now give the following theorem, where we use the notation I_m to denote the $m \times m$ identity matrix.

Theorem 2. *If linear operator V is bijective on X, then linear operator S is injective if and only if*

$$\det \bar{\mathbf{W}} = \det\left[I_m - \Psi\left(V^{-1}g\right)\right] \neq 0,$$

and the unique solution of the boundary value problem

$$Su = f, \quad \text{for all} \quad f \in X, \tag{18}$$

is given by

$$u = V^{-1}f + V^{-1}g\bar{\mathbf{W}}^{-1}\Psi\left(V^{-1}f\right). \tag{19}$$

Proof. Suppose that $\det \bar{\mathbf{W}} \neq 0$ and $u \in \ker S$. Then, $Su = Vu - g\Psi(u) = 0$; hence,

$$u = V^{-1}g\Psi(u). \tag{20}$$

By applying vector Ψ on both sides and using (17), we have

$$\Psi(u) = \Psi\left(V^{-1}g\Psi(u)\right) = \Psi\left(V^{-1}g\right)\Psi(u),$$

and thus

$$\left[I_m - \Psi\left(V^{-1}g\right)\right]\Psi(u) = \bar{\mathbf{W}}\Psi(u) = \mathbf{0},$$

from where it follows that $\Psi(u) = \mathbf{0}$. Then, from (20), it is implied that $u = 0$, i.e., $\ker S = \{0\}$; therefore, operator S is injective. Conversely, we assume that S is injective, and we will show that $\det \bar{\mathbf{W}} \neq 0$, or equivalently, we suppose that $\det \bar{\mathbf{W}} = 0$, and we will show that S is not injective. Let $\mathbf{c}$ be a nonzero constant vector $\mathbf{c}$ such that $\bar{\mathbf{W}}\mathbf{c} = \mathbf{0}$, and let element $v = V^{-1}g\mathbf{c} \in X$, which is different from zero; otherwise,

$$\bar{\mathbf{W}}\mathbf{c} = \left[I_m - \Psi\left(V^{-1}g\right)\right]\mathbf{c} = \mathbf{c} - \Psi\left(V^{-1}g\right)\mathbf{c} = \mathbf{c} - \Psi\left(V^{-1}g\mathbf{c}\right) = \mathbf{c} - \Psi(v) = \mathbf{c} = \mathbf{0},$$

which is a contradiction. Then,

$$Sv = Vv - g\Psi(v) = g\mathbf{c} - g\Psi(V^{-1}g\mathbf{c}) = g\mathbf{c} - g\Psi(V^{-1}g)\mathbf{c} = g\left[I_m - \Psi\left(V^{-1}g\right)\right]\mathbf{c} = 0,$$

which means that $\ker S \neq \{0\}$; therefore, S is not injective.

To find the unique solution of the boundary value problem (18), we multiply (18) by operator V^{-1} to obtain

$$u = V^{-1}f + V^{-1}g\Psi(u). \tag{21}$$

By applying vector Ψ as above and solving with respect to $\Psi(u)$, we obtain

$$\Psi(u) = \left[I_m - \Psi\left(V^{-1}g\right)\right]^{-1}\Psi\left(V^{-1}f\right) = \bar{\mathbf{W}}^{-1}\Psi\left(V^{-1}f\right). \tag{22}$$

By substituting (22) into (21), we obtain

$$u = V^{-1}f + V^{-1}g\bar{\mathbf{W}}^{-1}\Psi\left(V^{-1}f\right),$$

which is the solution in (19). □

4. Volterra Integro-Differential Operators of Convolution Type

In the previous two sections, we assume that inverse linear operator V^{-1} is known a priori. However, finding the inverse of a Volterra integro-differential operator is not always an easy task.

In this section, we look at the special case where $X = C[0,b]$, the Volterra integral operator in (6) is of convolution type, namely,

$$Ku = \sum_{i=0}^{n} \int_0^x k_i(x-t)u^{(n-i)}(t)dt,$$

and the vector of functionals $\Phi(u)$ in (11) is as follows:

$$\Phi(u) = \text{col}(\Phi_1(u), \Phi_2(u), \ldots, \Phi_n(u)) = \text{col}\Big(u(0), u'(0), \ldots, u^{(n-1)}(0)\Big). \tag{23}$$

In this case, the inverse operator V^{-1} of the linear Volterra integro-differential operator

$$\begin{aligned} Vu &= Au - Ku, \\ \mathcal{D}(V) &= \{u : u \in C^n[0,b],\ \Phi(u) - \mathbf{N}\Theta(u) = \mathbf{0}\} \end{aligned} \tag{24}$$

can be constructed analytically using the Laplace transform for certain classes of functions as it is shown below.

First, we recall the following definitions and properties about the Laplace transform: For a function $f(x) \in C[0,\infty)$ of exponential order, that is, there exist real constants γ and M such that $|f(x)| \leq Me^{\gamma x}$ for all $x \geq 0$, the Laplace transform of f defined by

$$F(s) = \mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)dx,$$

exists for all $s > \gamma$. Laplace transform operator $\mathcal{L}$ is linear and injective. For any two functions $f(x)$ and $g(x)$ of exponential order and any real constants a and b,

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\},$$

and

$$f(x) = \mathcal{L}^{-1}\{F\}.$$

The convolution of f and g is defined as

$$(f * g)(x) = \int_0^x f(x-t)g(t)dt,$$

and holds

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}).$$

For a function $f(x) \in C^n[0,\infty)$ and its derivatives $f^{(i)}(x) \in C[0,\infty)$, $i = 1,2,\ldots,n$, of exponential order, the Laplace transform of the nth derivative $f^n(x)$ exists and is given by

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - \sum_{\ell=1}^{n} s^{n-\ell} f^{(\ell-1)}(0), \tag{25}$$

see, for example, [36] and some new developments in [37].

We prove the following lemma.

Lemma 1. *Let $u^{(i)}(x)$, $k_i(x)$, $i = 0,1,\ldots,n$, and $f(x)$ be continuous functions of exponential order. Then, the equation*

$$Au - Ku = f, \tag{26}$$

is equivalent to

$$u = \hat{p}\Phi(u) + \hat{f}, \tag{27}$$

where $\Phi(u)$ *is the functional defined in* (23)*, and*

$$\begin{aligned} F(s) &= \mathcal{L}\{f\}, \quad K_i(s) = \mathcal{L}\{k_i\}, \; i = 0, 1, \ldots, n, \\ Q(s) &= \frac{1}{\sum_{i=0}^{n}(a_i - K_i(s))s^{n-i}}, \\ P_\ell(s) &= \sum_{i=0}^{n-\ell}(a_i - K_i(s))s^{n-\ell-i}, \quad \ell = 1, 2, \ldots, n, \\ \hat{f} &= \mathcal{L}^{-1}\{Q(s)F(s)\}, \\ \hat{p} &= \begin{pmatrix} \hat{p}_1 & \hat{p}_2 & \cdots & \hat{p}_n \end{pmatrix}, \quad \hat{p}_\ell = \mathcal{L}^{-1}\{Q(s)P_\ell(s)\}, \quad \ell = 1, 2, \ldots, n. \end{aligned} \tag{28}$$

Proof. By applying the Laplace transform operator to (26), we obtain

$$\begin{aligned} \mathcal{L}\{Au - Ku\} &= \mathcal{L}\{Au\} - \mathcal{L}\{Ku\} \\ &= \sum_{i=0}^{n} a_i \mathcal{L}\{u^{(n-i)}\} - \sum_{i=0}^{n} \mathcal{L}\left\{\int_0^x k_i(x-t)u^{(n-i)}(t)dt\right\} \\ &= \sum_{i=0}^{n} a_i \mathcal{L}\{u^{(n-i)}\} - \sum_{i=0}^{n} \mathcal{L}\{k_i\}\mathcal{L}\{u^{(n-i)}\} \\ &= \sum_{i=0}^{n}(a_i - \mathcal{L}\{k_i\})\mathcal{L}\{u^{(n-i)}\} \\ &= \mathcal{L}\{f\}. \end{aligned} \tag{29}$$

From (25), we have

$$\mathcal{L}\{u^{(n-i)}\} = \left(s^{n-i}\mathcal{L}\{u\} - \sum_{\ell=1}^{n-i} s^{n-i-\ell}u^{(\ell-1)}(0)\right),$$

and after substituting into (29), we obtain

$$\begin{aligned} \mathcal{L}\{Au - Ku\} &= \sum_{i=0}^{n}(a_i - \mathcal{L}\{k_i\})\left(s^{n-i}\mathcal{L}\{u\} - \sum_{\ell=1}^{n-i} s^{n-i-\ell}u^{(\ell-1)}(0)\right) \\ &= \sum_{i=0}^{n}(a_i - \mathcal{L}\{k_i\})s^{n-i}\mathcal{L}\{u\} \\ &\quad - \sum_{i=0}^{n}(a_i - \mathcal{L}\{k_i\})\left(\sum_{\ell=1}^{n-i} s^{n-i-\ell}u^{(\ell-1)}(0)\right) \\ &= \left[\sum_{i=0}^{n}(a_i - \mathcal{L}\{k_i\})s^{n-i}\right]\mathcal{L}\{u\} \\ &\quad - \sum_{\ell=1}^{n}\left[\sum_{i=0}^{n-\ell}(a_i - \mathcal{L}\{k_i\})s^{n-\ell-i}\right]u^{(\ell-1)}(0) \\ &= \mathcal{L}\{f\}, \end{aligned}$$

or

$$\left[\sum_{i=0}^{n}(a_i - K_i(s))s^{n-i}\right]U(s) - \sum_{\ell=1}^{n}\left[\sum_{i=0}^{n-\ell}(a_i - K_i(s))s^{n-\ell-i}\right]u^{(\ell-1)}(0) = F(s),$$

where $U(s) = \mathcal{L}\{u\}$ and $F(s) = \mathcal{L}\{f\}$. By solving for U and using relations (28), we obtain

$$U(s) = \sum_{\ell=1}^{n} Q(s)P_\ell(s)u^{(\ell-1)}(0) + Q(s)F(s).$$

By applying then the inverse operator, we obtain

$$\mathcal{L}^{-1}\{U(s)\} = \sum_{\ell=1}^{n} \mathcal{L}^{-1}\{Q(s)P_\ell(s)\}u^{(\ell-1)}(0) + \mathcal{L}^{-1}\{Q(s)F(s)\},$$

and thus

$$u = \sum_{\ell=1}^{n} \hat{p}_l u^{(\ell-1)}(0) + \hat{f} = \hat{p}\Phi(u) + \hat{f},$$

which is Equation (27). □

From Lemma 1, the next two results follow.

Lemma 2. *The linear Volterra integro-differential operator $V_0 : X \to X$ defined as*

$$\begin{aligned} V_0 u &= Au - Ku, \\ \mathcal{D}(V_0) &= \{u : u \in C^n[0,b],\ \Phi(u) = \mathbf{0}\} \end{aligned}$$

is injective, and the unique solution of the initial value problem $V_0 u = f$, for any $f \in X$, is $u = \hat{f}$.

Lemma 3. *The Volterra integro-differential operator $V_1 : X \to X$ defined as*

$$\begin{aligned} V_1 u &= Au - Ku, \\ \mathcal{D}(V_1) &= \{u : u \in C^n[0,b],\ \Phi(u) = \mathbf{c}\}, \end{aligned}$$

where $\mathbf{c}$ is a vector of arbitrary real constants, is injective, and the unique solution of initial value problem $V_1 u = f$, for any $f \in X$, is $u = \hat{p}\mathbf{c} + \hat{f}$.

We now state the following theorem for finding the inverse (V^{-1}) of linear operator V in (24).

Theorem 3. *Let $u^{(i)}(x)$, $k_i(x)$, $i = 0, 1, \ldots, n$, and $f(x)$ be continuous functions of exponential order. Then, the linear Volterra integro-differential operator*

$$\begin{aligned} Vu &= Au - Ku, \\ \mathcal{D}(V) &= \{u : u \in C^n[0,b],\ \Phi(u) - \mathbf{N}\Theta(u) = \mathbf{0}\}, \end{aligned} \tag{30}$$

is injective if

$$\det \hat{\mathbf{W}} = \det[I_n - \mathbf{N}\Theta(\hat{p})] \neq 0, \tag{31}$$

and the unique solution of the nonlocal boundary value problem $Vu = f$, for any $f \in X$, is given by

$$u = \hat{p}\hat{\mathbf{W}}^{-1}\mathbf{N}\Theta(\hat{f}) + \hat{f}. \tag{32}$$

Proof. From Equation (30) and Lemma 1, we have that $Au - Ku = f$ is equivalent to

$$u = \hat{p}\Phi(u) + \hat{f}, \tag{33}$$

where $\Phi(u)$ is to be determined.

By applying vector Θ on (33), we obtain

$$\Theta(u) = \Theta(\hat{p}\Phi(u)) + \Theta(\hat{f}) = \Theta(\hat{p})\Phi(u) + \Theta(\hat{f}).$$

By substituting $\Theta(u)$ into
$$\Phi(u) - \mathbf{N}\Theta(u) = \mathbf{0},$$
after some algebra, we obtain
$$[I_n - \mathbf{N}\Theta(\hat{p})]\Phi(u) = \mathbf{N}\Theta(\hat{f}). \tag{34}$$

If condition (31) holds true, then Equation (34) can be solved uniquely with respect to $\Phi(u)$ to obtain
$$\Phi(u) = [I_n - \mathbf{N}\Theta(\hat{p})]^{-1}\mathbf{N}\Theta(\hat{f}). \tag{35}$$
The substitution of (35) into (33) yields
$$u = \hat{p}[I_n - \mathbf{N}\Theta(\hat{p})]^{-1}\mathbf{N}\Theta(\hat{f}) + \hat{f},$$
which is Equation (32). □

Finally, for completeness, we mention that Theorem 3 can be extended to problems with nonhomogeneous boundary conditions. For this, let operator $\tilde{V} : X \to X$ be defined as
$$\begin{aligned} \tilde{V}u &= Au - Ku, \\ \mathcal{D}(\tilde{V}) &= \{u : u \in C^n[0, b],\ \Phi(u) - \mathbf{N}\Theta(u) = \mathbf{b}\}. \end{aligned} \tag{36}$$
where K is as usual the convolution-type integral operator; the linear bounded functionals $\Phi(u) = \mathrm{col}(u(0), u'(0), \ldots, u^{(n-1)}(0))$; and $\mathbf{b}$ is a column vector of n real constants. It is emphasized that $\tilde{V}$ is not a linear operator since domain $\mathcal{D}(\tilde{V})$ is a nonlinear set. With the same arguments as above, the following theorem can be proved.

Theorem 4. *Under the same conditions as in Theorem 3, the Volterra integro-differential operator $\tilde{V}$ is injective, and the unique solution of the nonlocal boundary value problem $\tilde{V}u = f$, for any $f \in X$, is given by*
$$u = \hat{p}\hat{\mathbf{W}}^{-1}\left(\mathbf{N}\Theta(\hat{f}) + \mathbf{b}\right) + \hat{f}.$$

5. Examples

In this section, we solve four examples representative of the presented theory to demonstrate the ease of implementation and effectiveness of the method. The first example concerns a Volterra integro-differential equation (VIDE) under various types of nonhomogeneous boundary conditions. The remaining examples are one linear and two nonlinear boundary value problems for VFIDEs under nonlocal boundary conditions.

5.1. Example 1

Consider the first-order linear Volterra integro-differential equation
$$u'(x) + a_1 u(x) - \int_0^x e^{-(x-t)}u(t)dt = f(x), \quad x \in [0, 1], \tag{37}$$
where $a_1 \in \mathbb{R}$, $f(x)$ is the input or load function assumed to be a continuous function of exponential order, subject to the following:

(i) Initial condition
$$u(0) = b_1, \tag{38}$$
(ii) Two-point boundary condition
$$u(0) - \nu_{11}u(x_1) = b_1, \tag{39}$$

(iii) Multipoint and integral boundary condition

$$u(0) - \nu_{11}u(x_1) - \nu_{12}u(x_2) - \nu_{13}\int_{\xi_1}^{\xi_2} u(t)dt = b_1, \tag{40}$$

where ν_{1l}, $l = 1,2,3$, and b_1 are real constants, $0 \le x_1 < x_2 \le 1$ and $0 \le \xi_1 < \xi_2 \le 1$.

We take $n = 1$ and

$$\begin{aligned} Au &= u'(x) + a_1u(x), \\ Ku &= \int_0^x k_1(x-t)u(t)dt, \quad k_1(x) = e^{-x}, \\ Vu &= Au - Ku, \quad \mathcal{D}(V) = \Big\{u : u \in C^1[0,1],\ \Phi(u) - \mathbf{N}\Theta(u) = \mathbf{b}\Big\}, \end{aligned}$$

where

(i) $\Phi(u) = \begin{pmatrix} u(0) \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \end{pmatrix}$,

(ii) $\Phi(u) = \begin{pmatrix} u(0) \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} \nu_{11} \end{pmatrix}$, $\Theta(u) = \begin{pmatrix} u(x_1) \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \end{pmatrix}$,

(iii) $\Phi(u) = \begin{pmatrix} u(0) \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} \nu_{11} & \nu_{12} & \nu_{13} \end{pmatrix}$, $\Theta(u) = \begin{pmatrix} u(x_1) \\ u(x_2) \\ \int_{\xi_1}^{\xi_2} u(t)dt \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \end{pmatrix}$.

We note that k_1 is a continuous function of exponential order; therefore, we can apply Theorem 4. Thus, the solution of the initial value problem (37), (38) for $f(x) = 0$ and $b_1 = 1$ turns out to be

$$u(x) = e^{-\frac{a_1+1}{2}x}\left(\cosh(cx) + \frac{1-a_1}{2c}\sinh(cx)\right),$$

where $c = \frac{1}{2}\sqrt{a_1^2 - 2a_1 + 5}$. This problem was studied in [38,39].

The solution of the two-point boundary value problem (37), (39) for $a_1 = 2$, $f(x) = 2e^{-x} + x^2 + 3x - 5$, $\nu_{11} = -1$, $b_1 = 0$, and $x_1 = 1$ is

$$u(x) = x^2 - x - 1.$$

Finally, the solution of the nonlocal boundary value problem (37), (40) for

$$\begin{aligned} a_1 &= -1, \quad f(x) = -(x+2)e^{-x}, \\ \nu_{11} &= -1, \quad \nu_{12} = e^{-1/2}, \quad \nu_{13} = 1, \quad b_1 = 1 + e^{-\frac{1}{4}} - e^{-\frac{3}{4}}, \end{aligned}$$

and the points $x_1 = 0.25$, $x_2 = 0.5$, $\xi_1 = 0.75$ and $\xi_2 = 1$ is

$$u(x) = e^{-x}.$$

5.2. Example 2

Solve the second-order linear VFIDE

$$\begin{aligned} u''(x) + a_1u'(x) + a_2u(x) - \int_0^x \cos(x-t)u'(t)dt - \int_0^x \sin(x-t)u(t)dt \\ - \int_0^1 (x^2 + xt)u(t)dt = f(x), \quad x \in [0,1], \end{aligned} \tag{41}$$

subject to homogeneous boundary conditions

$$u(0) = 0, \quad u(1) = 0. \tag{42}$$

First, we note that $n = 2$, $m = 2$ and $\ell = 2$ and define

$$
\begin{aligned}
Au &= u''(x) + a_1 u'(x) + a_2 u(x), \\
Ku &= \int_0^x k_1(x-t)u'(t)dt + \int_0^x k_2(x-t)u(t)dt, \quad k_1(x) = \cos x, \quad k_2(x) = \sin x, \\
\bar{K}u &= x^2 \int_0^1 u(t)dt + x \int_0^1 tu(t)dt, \\
\Phi(u) &= \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad \Theta(u) = \begin{pmatrix} u'(0) \\ u(1) \end{pmatrix}.
\end{aligned}
$$

Next, we set the vectors

$$
\begin{aligned}
g &= \begin{pmatrix} g_1(x) & g_2(x) \end{pmatrix} = \begin{pmatrix} x^2 & x \end{pmatrix}, \\
\Psi(u) &= \begin{pmatrix} \int_0^1 h_1(t)u(t)dt \\ \int_0^1 h_2(t)u(t)dt \end{pmatrix} = \begin{pmatrix} \int_0^1 u(t)dt \\ \int_0^1 tu(t)dt \end{pmatrix},
\end{aligned}
$$

and define the linear operators

$$
\begin{aligned}
Vu &= Au - Ku, \quad \mathcal{D}(V) = \Big\{ u : u \in C^2[0,1],\ \Phi(u) - \mathbf{N}\Theta(u) = 0 \Big\}, \\
Su &= Vu - g\Psi(u), \quad \mathcal{D}(S) = \mathcal{D}(V).
\end{aligned} \tag{43}
$$

We observe that k_1, k_2, g_1 and g_2 are continuous functions of exponential order. Therefore, we use Theorems 2 and 3 to construct the unique solution of the boundary value problem (41), (42).

Specifically, for $a_1 = -1$, $a_2 = 1$ and $f(x) = -x^2 + x$, we obtain the unique solution

$$
u(x) = \frac{12}{745e - 4611}\Big(238e^x - (25e+5)x^3 - (71e-81)x^2 - (142e-162)x - 238\Big).
$$

For $f(x) = e^x$, we obtain

$$
\begin{aligned}
u(x) = \frac{1}{745e - 4611}\{&[\,(745e - 4611)x + 2111e - 836\,]e^x \\
&+ \Big(-300e^2 + 340e + 1180\Big)x^3 + \Big(-852e^2 + 2406e - 672\Big)x^2 \\
&+ \Big(-1704e^2 + 4812e - 1344\Big)x - 2111e + 836\Big\}.
\end{aligned}
$$

5.3. Example 3

Let the second-order nonlinear VFIDE of Hammerstein type

$$
u''(x) - \int_0^x \cos(x-t)u'(t)dt - 120x\int_0^1 u(t)u'(t)dt = f(x), \quad x \in [0,1], \tag{44}
$$

subject to multipoint boundary conditions

$$
u(0) = 2u'(\frac{1}{2}) - 3u(1), \quad u'(0) = -u'(1). \tag{45}
$$

We start by setting $n = 2$, $m = 1$ and $\ell = 3$, and

$$\begin{aligned}
Au &= u''(x), \\
Ku &= \int_0^x \cos(x-t)u'(t)dt, \\
\bar{K}u &= 120x\int_0^1 u(t)u'(t)dt, \\
\Phi(u) &= \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 2 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Theta(u) = \begin{pmatrix} u'(\frac{1}{2}) \\ u(1) \\ u'(1) \end{pmatrix}.
\end{aligned}$$

Next, we define the vectors

$$g = \begin{pmatrix} g_1(x) \end{pmatrix} = (120x), \quad \Psi(u) = \begin{pmatrix} \int_0^1 u(t)u'(t)dt \end{pmatrix},$$

and the operators

$$\begin{aligned}
Vu &= Au - Ku, \quad \mathcal{D}(V) = \left\{u : u \in C^2[0,1],\ \Phi(u) - \mathbf{N}\Theta(u) = 0\right\}, \\
Tu &= Vu - g\Psi(u), \quad \mathcal{D}(T) = \mathcal{D}(V).
\end{aligned}$$

Theorems 1 and 3 are applicable. Let $f(x) = -\frac{x}{2}$. It follows that (14) is a nonlinear algebraic quadratic equation in c that has the following two real solutions:

$$c_{1,2} = \frac{2351 \pm 24\sqrt{8922}}{149{,}520}.$$

Consequently, we obtain the following two solutions of the nonlinear boundary value problem (44), (45):

$$u_{1,2}(x) = \frac{72 \pm \sqrt{8922}}{598{,}080}\left(96x^5 + 1504x^3 - 2496x + 3\right).$$

5.4. Example 4

As a last example, we consider the first-order nonlinear VFIDE of Hammerstein type

$$u'(x) - \int_0^x \cos(t-x)u(t)dt - 2\int_0^1 (x+t)u^2(t)dt = f(x), \quad x \in [0,1], \tag{46}$$

subject to the boundary condition

$$u(0) = u(1). \tag{47}$$

It is easily seen that $n = 1$, $m = 2$ and $\ell = 1$, and

$$\begin{aligned}
Au &= u'(x), \\
Ku &= \int_0^x \cos(t-x)u(t)dt, \\
\bar{K}u &= 2\int_0^1 (x+t)u^2(t)dt, \\
\Phi(u) &= \begin{pmatrix} u(0) \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 1 \end{pmatrix}, \quad \Theta(u) = \begin{pmatrix} u(1) \end{pmatrix}.
\end{aligned}$$

Define the vectors

$$\begin{aligned}
g &= \begin{pmatrix} g_1(x) & g_2(x) \end{pmatrix} = \begin{pmatrix} 2x & 1 \end{pmatrix}, \\
\Psi(u) &= \begin{pmatrix} \int_0^1 h_1(t)u^2(t)dt \\ \int_0^1 h_2(t)u^2(t)dt \end{pmatrix} = \begin{pmatrix} \int_0^1 u^2(t)dt \\ \int_0^1 2tu^2(t)dt \end{pmatrix},
\end{aligned}$$

and then the operators

$$\begin{aligned} Vu &= Au - Ku, \quad \mathcal{D}(V) = \Big\{u : u \in C^1[0,1],\ \Phi(u) - \mathbf{N}\Theta(u) = 0\Big\}, \\ Tu &= Vu - g\Psi(u), \quad \mathcal{D}(T) = \mathcal{D}(V). \end{aligned}$$

The solution of $Tu = f$ follows using Theorems 1 and 3. Let

$$f(x) = -7\sin x - 4\cos x - \frac{331}{105}x + \frac{125}{28}.$$

It turns out that the nonlinear algebraic system in (14) consists of two quadratic equations with unknown vector $\mathbf{c}$. It possesses two real and two complex roots, of which only the real root

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{331}{210} \\ \frac{43}{28} \end{pmatrix}$$

is retrieved in exact form. In this case, the solution of the nonlinear boundary value problem (46), (47) is

$$u(x) = x^3 - 3x^2 + 2x + 1.$$

We programmed the proposed methods into the free software Maxima computer algebra system (CAS) running on Microsoft Windows. We used the available built-in functions *laplace*, *ilt* and *solve* to compute the Laplace transform, the inverse Laplace transform and the solution of linear and nonlinear systems of algebraic equations, respectively.

6. Discussion and Conclusions

In general, there are not many procedures available in the literature for solving boundary value problems with nonlocal boundary conditions in closed form.

In this paper, a method is presented to solve in closed form a class of linear and nonlinear VFIDEs under homogeneous nonlocal boundary conditions. The method requires the existence of the explicit form of the inverse operator of the associated linear Volterra integro-differential operator. A procedure for constructing this inverse operator in the case of convolution-type VIDEs under homogeneous and nonhomogeneous nonlocal boundary conditions is also given. The methods are implemented in a computer algebra system, and several examples are solved.

The main advantage of the technique is that it is straightforward, does not require approximations or numerical procedures and provides the exact solution in closed form. Moreover, in the nonlinear case, unlike the numerical solutions, it can give all possible solutions that can be calculated analytically. It was proved to be easy to use and very effective. Its main drawback is that its applicability is limited to a class of continuous functions of exponential order for which the direct and inverse Laplace transform exist. It relies on the available facilities for the inverse Laplace transform and explicit integration in the computer algebra software used.

Finally, the solution of VFIDEs subject to nonhomogeneous nonlocal boundary conditions is a more challenging task that needs a different treatment that could be addressed in a separate article in the future.

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