

On a Class of Orthogonal Polynomials as Corrections in Lienard Differential System: Applications

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Abstract: In this paper we demonstrate some specialized modules for investigating the dynamics of differential models, an integral part of a planned much more general Web-based application for scientific computing. As “corrections” in the Lienard differential system is presented a class of orthogonal polynomials (also known as “shell polynomials”). We will note that some specifics of the amplitudes of these polynomials open up the possibility of modeling signals from the field of antenna-feeder techniques. Algorithms and modules have been consistently used for: automatic generation of a theorem on the number and type of limit cycles (in the light of Melnikov’s considerations); study of the Hamiltonian of the system and “level curves”; for the study of catastrophic surfaces (in the light of Zeeman’s considerations), etc. Similar studies have been carried out for associated polynomials. Numerical examples, illustrating our results using CAS MATHEMATICA are given.

Keywords: Lienard differential system; polynomials $P_n(x)$ as “corrections” in the Lienard system; level curves; emitting chart

MSC: 65L07; 34A34



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1. Introduction

122 years ago, Hilbert [1] proposed 23 mathematical problems, of which the second part of the 16th one is to find the maximal number of limit cycles and their relative locations for polynomial vector fields. To date, several thousand articles and studies have been devoted to this problem. The study of dynamical systems includes bifurcation theory with branch catastrophe theory [2]. See also model in the light of Zeeman’s approach [3]. Arnold [4] discussed the catastrophes of the ADE classification, because of their relation with the Lie groups. For more details of the existing important results on the generalized polynomial Lienard differential systems and the limit-cycle bifurcations of some generalized polynomial Lienard systems, see [5–30] (where the reader can to discover a substantial additional bibliography). Some of our previous research [31–33] on this issue encouraged us to begin developing specialized modules as part of a much more general Web-based application for scientific computing. In Section 2, we demonstrate some algorithms and modules for: automatically generating a theorem for the number and type of limit cycles (in the light of Melnikov’s considerations); study of the Hamiltonian of the system and “level curves”; for the study of catastrophic surfaces (in the light of Zeeman’s considerations); generation and simulation of antenna factor, etc. Concluding remarks are placed in Section 3.

Consider the orthogonal polynomials

$$(n + 1)P_{n+1}(x) = (n + a)xP_n(x) - P_{n-1}(x)$$

$$P_0(x) = 1; P_1(x) = ax.$$

Polynomials are a special class of “shell polynomials” [34]. For more details see [35–37].

In this article we consider the Lienard system [6]

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = Poly(x) + \epsilon F(x)y \end{cases} \tag{1}$$

where $0 \leq \epsilon < 1$; $F(x)$ are specially chosen polynomials, and $Poly(x)$ are the polynomials $P_n(x)$ —see Figure 1. The level curves are studied. Some applications of the polynomials are also given.

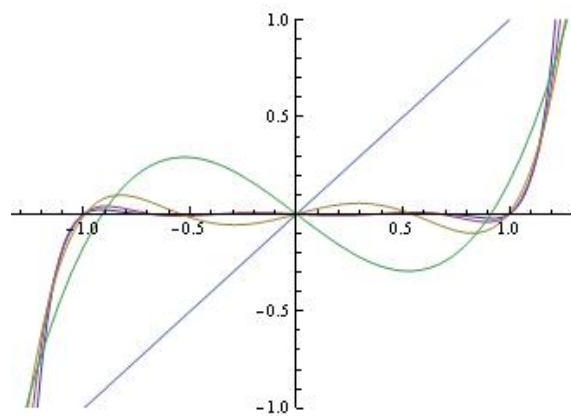


Figure 1. The polynomials $P_n(x)$ for fixed $a = 1$; $n = 1, 3, 5, 7, 9$.

2. Main Results

Let $a = 1$. Then

$$\begin{cases} P_3(x) = x^3 - \frac{5}{6}x \\ P_5(x) = x^5 - \frac{77}{60}x^3 + \frac{7}{24}x \\ P_7(x) = x^7 - \frac{223}{140}x^5 + \frac{59}{90}x^3 - \frac{1}{16}x \\ P_9(x) = x^9 - \frac{4609}{2520}x^7 + \frac{10,219}{10,080}x^5 - \frac{2519}{12,960}3x^3 + \frac{11}{1152}x \\ P_{11}(x) = x^{11} - \frac{55,991}{27,720}x^9 + \frac{22,711}{16,800}x^7 - \frac{67,171}{181,440}x^5 + \frac{2041}{51,840}x^3 - \frac{13}{11,520}x. \end{cases}$$

Let $Poly(x)$ (in (1)) coincides with polynomials $P_n(x)$. Without going into details, we will note some more interesting level curves.

The level curves

The case (1): $n = 9$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} - \frac{1}{10}x^{10} + \frac{4609}{20,160}x^8 - \frac{10,219}{60,480}x^6 + \frac{2519}{51,840}x^4 - \frac{11}{2304}x^2.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted at Figure 2.

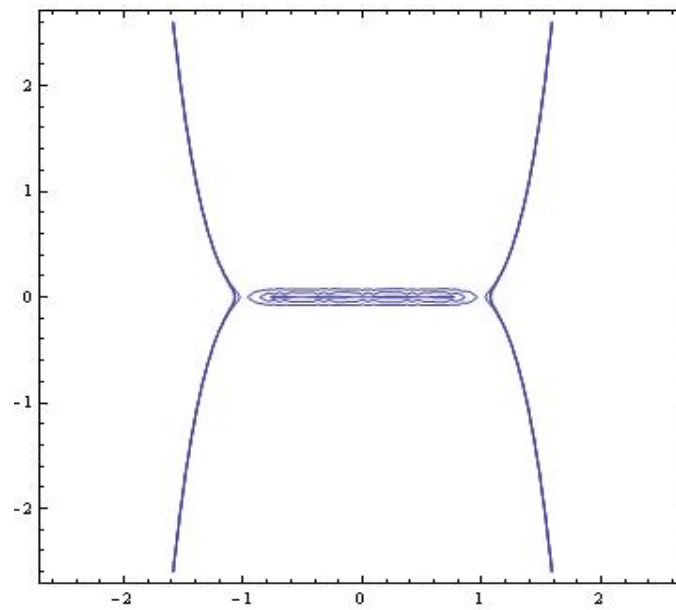


Figure 2. Level curves (the case 1).

The case (2): $n = 7$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} - \frac{1}{8}x^8 + \frac{223}{840}x^6 - \frac{59}{360}x^4 + \frac{1}{32}x^2.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted at Figure 3.

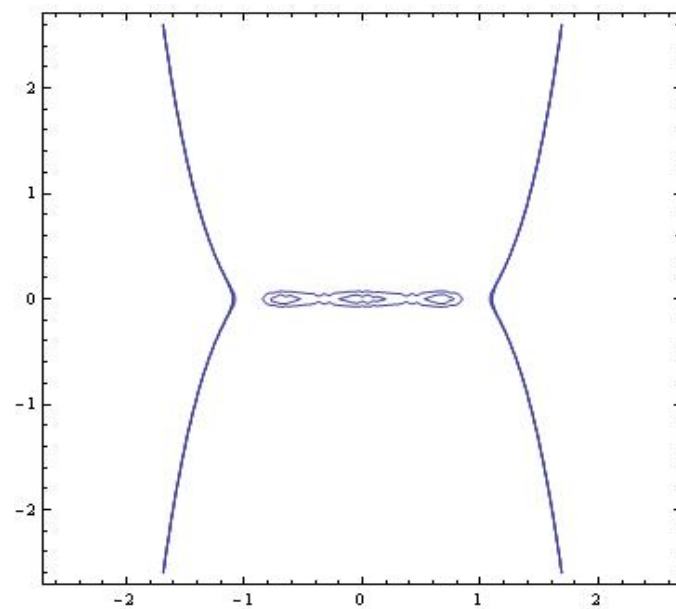


Figure 3. Level curves (the case 2).

The model in the light of Melnikov's considerations

The Melnikov polynomial [5] for the system

$$\begin{cases} \frac{dx}{dt} = y - \epsilon(a_1x + a_2x^2 + \dots + a_{2n+1}x^{2n+1}) \\ \frac{dy}{dt} = -x \end{cases} \quad (2)$$

is defined as

$$M(r^2, n) = \frac{a_1}{2} + \frac{3}{8}a_3r^2 + \dots + \left(\frac{2n + 2}{n + 1} \right) \frac{a_{2n+1}}{2^{2n+2}}r^{2n}. \tag{3}$$

It is known [7,8] that the system for sufficiently small $\epsilon \neq 0$ has at most n limit cycles asymptotic to circles of radii $r_j, j = 1, 2, \dots, n$ as $\epsilon \rightarrow 0$ if and only if the n th degree polynomial $M(r^2, n)$ has n positive roots $r^2 = r_j^2, j = 1, 2, \dots, n$.

The case $n = 7$.

Consider the model

$$\begin{cases} \frac{dx}{dt} = y - \epsilon(x^7 - \frac{223}{140}x^5 + \frac{59}{90}x^3 - \mu x) \\ \frac{dy}{dt} = -x \end{cases} \tag{4}$$

where $\mu > 0, \epsilon > 0$.

The following is valid.

Proposition 1. The Lienard-type system for $n = 7$, and for all sufficiently small $\epsilon \neq 0$.

- (a) for $\mu = \frac{1}{16} = 0.0625$ has three hyperbolic limit cycles 0.444513, 0.727036 and 1.04606.
- (b) for $\mu = 0.0735880318901$ has a simple limit cycle 1.06318 and limit cycle 0.587389 with multiplicity two.

Proof. For the Melnikov polynomial in r^2 (see Figure 4) we have:

$$M(r^2, 3) = -\frac{\mu}{2} + \frac{359}{720}r^2 - \frac{223}{448}r^4 + \frac{35}{128}r^6. \tag{5}$$

Evidently, for example $\mu = 0.0735880318901$ we have a simple limit cycle and cycle with multiplicity two. \square

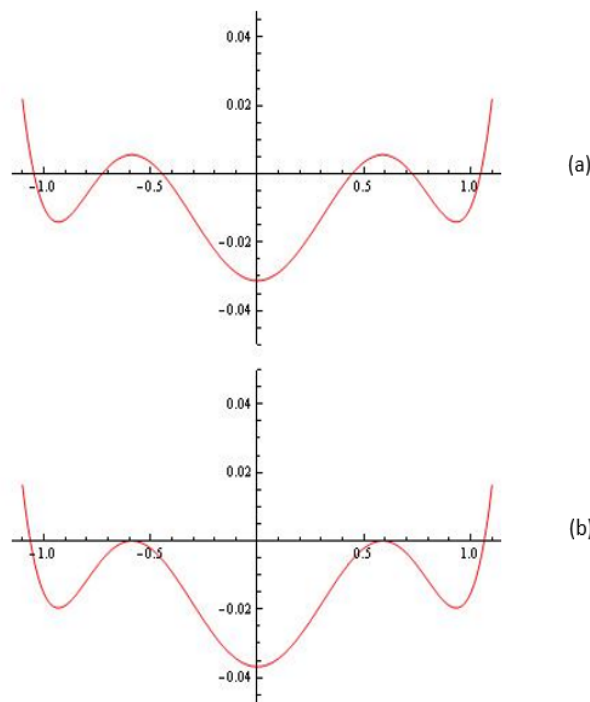


Figure 4. (a) The Melnikov polynomial $M(r^2, 3)$ for $n = 7$ and $\mu = \frac{1}{16} = 0.0625$ (three limit cycles); (b) The Melnikov polynomial $M(r^2, 3)$ for $n = 7$ and $\mu = 0.0735880318901$ (simple limit cycle 1.06318 and limit cycle 0.587389 with multiplicity two).

The case $n = 11$.
 Consider the model

$$\begin{cases} \frac{dx}{dt} = y - \epsilon(x^{11} - \frac{55,991}{27,720}x^9 + \frac{22,711}{16,800}x^7 - \frac{67,171}{181,440}x^5 + \frac{2041}{51,840}x^3 - \mu x) \\ \frac{dy}{dt} = -x \end{cases}$$

where $\mu > 0, \epsilon > 0$.

The following is valid.

Proposition 2. The Lienard-type system for $n = 11$, and for all sufficiently small $\epsilon \neq 0$.

- (a) for $\mu = \frac{13}{11,520} = 0.00112847222221$ has five hyperbolic limit cycles 0.25518, 0.442797, 588203, 0.729379 and 1.03167.
- (b) for $\mu = 0.0012953041893$ has simple limit cycles 0.639205, 0.706901, 1.03221 and limit cycle 0.338942 with multiplicity two.

Proof. For the Melnikov polynomial in r^2 (see Figure 5) we have:

$$M(r^2, 5) = -\frac{\mu}{2} + \frac{6123}{414,720}r^2 - \frac{335,855}{2,903,040}r^4 + \frac{794,885}{2,150,400}r^6 - \frac{3,527,433}{7,096,320}r^8 + \frac{231}{1024}r^{10}.$$

Evidently, for example $\mu = 0.0012953041893$ we have three simple limit cycles and limit cycle with multiplicity two. \square

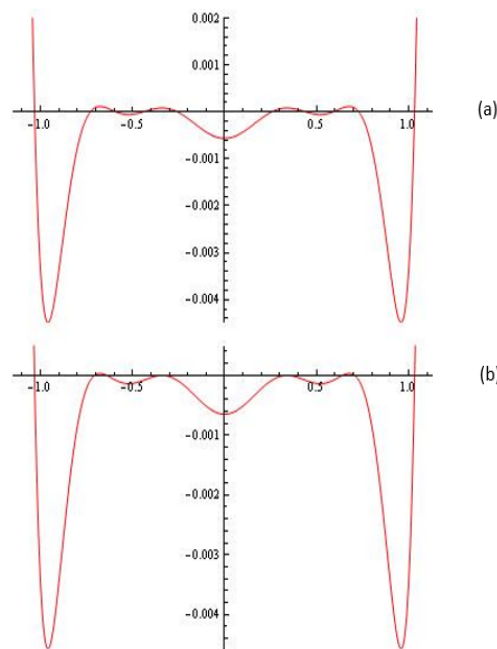


Figure 5. (a) The Melnikov polynomial $M(r^2, 5)$ for $n = 11$ and $\mu = \frac{13}{11,520}$ (five limit cycles); (b) The Melnikov polynomial $M(r^2, 5)$ for $n = 11$ and $\mu = 0.0012953041893$ (simple limit cycles 0.639205, 0.706901, 1.03221 and limit cycle 0.338942 with multiplicity two).

Numerical methods for finding zeros of polynomials can be found in [38–40].

Some simulations

The simulations on the Lienard–type system:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -P_9(x) + \epsilon F(x)y \end{cases} \quad (6)$$

where $F(x) = x - x^3 + x^5 - \frac{1}{7}x^7$

- (1) with $x_0 = 1.1, y_0 = 0.7, b = 1, c = 0.995$;
 (2) with $x_0 = 1.1, y_0 = 0.08, b = 1, c = 0.001$

are depicted on Figures 6 and 7.

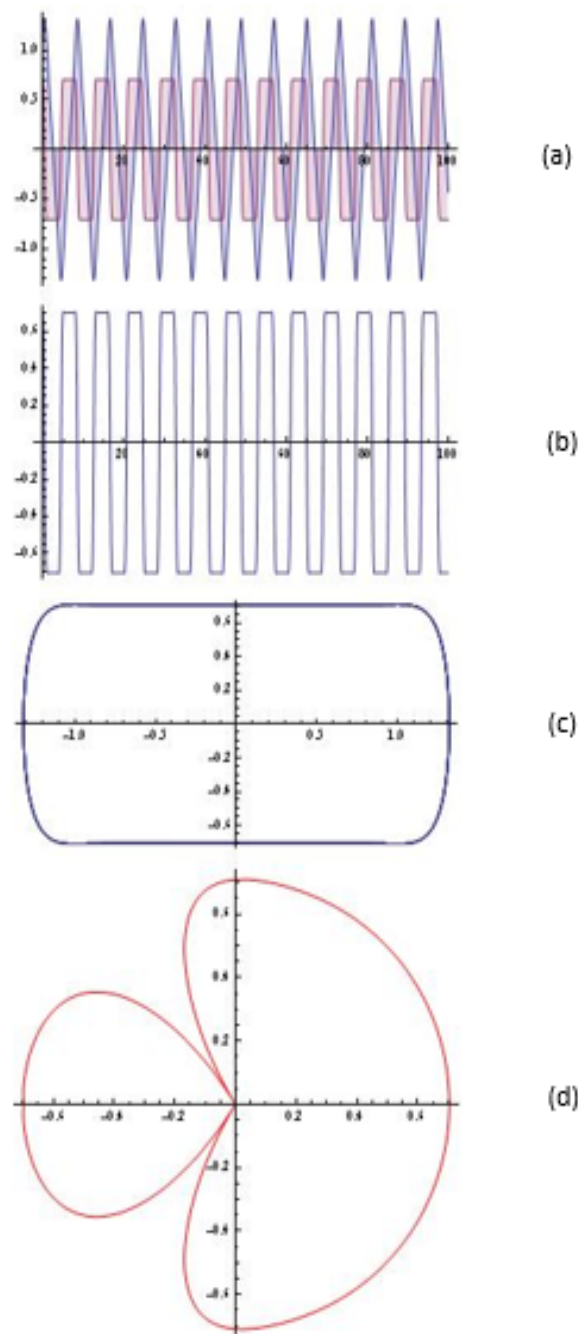


Figure 6. The simulations (system (6)) for $x_0 = 1.1, y_0 = 0.7, b = 1, c = 0.995; \epsilon = 0.0001$: (a) the solution of the system; (b) y -component of the solution; (c) the portrait; (d) emitting chart.

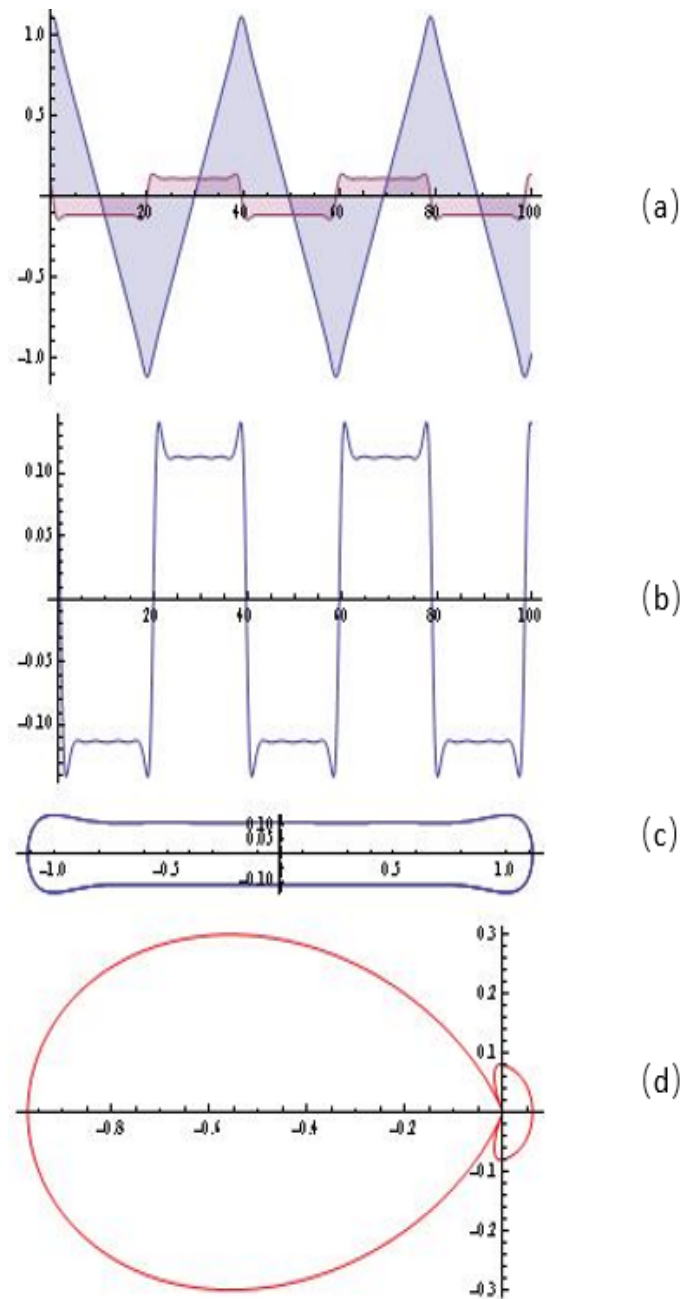


Figure 7. The simulations (system (6)) for $x_0 = 1.1, y_0 = 0.08, b = 1, c = 0.001; \epsilon = 0.0001$: (a) the solution of the system; (b) y -component of the solution; (c) the portrait; (d) emitting chart.

We will note that some specifics of the amplitudes of these polynomials open up the possibility of modeling signals from the field of antenna-feeder technology.

It is easy to take into account that the change of the variable t with $t = b \cos \theta + c$ (θ is the azimuthal angle and c is the phase difference) in the $y(t)$ -component of the solution of the system (6) leads to radiation diagrams [41,42].

Consider the following model in the light of Zeeman’s approach:

$$\begin{cases} \frac{dx}{dt} = c(G(x) - y) \\ \frac{dy}{dt} = \frac{1}{c}x \end{cases} \tag{7}$$

with $c > 0$ and

$$G(x) = x^9 - \frac{4609}{2520}x^7 + \frac{10,219}{10,080}x^5 - \frac{2519}{12,960}3x^3 + qx.$$

The catastrophe surfaces $(x, y, p) = G(x) - y$ ($q = 5, 10, 15$) for the model is depicted on Figure 8.

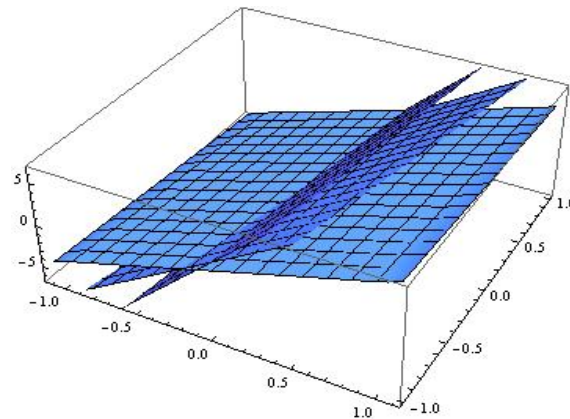


Figure 8. The catastrophe surfaces in the light of Zeeman considerations.

Consider the model (7) with $c > 0$ and

$$G(x) = x^{11} - \frac{55,991}{27,720}x^9 + \frac{22,711}{16,800}x^7 - \frac{67,171}{181,440}x^5 + \frac{2041}{51,840}x^3 - qx.$$

The catastrophe surfaces $(x, y, p) = G(x) - y$ ($q = 1, 2, 3, 4, 5$) for the model is depicted on Figure 9.

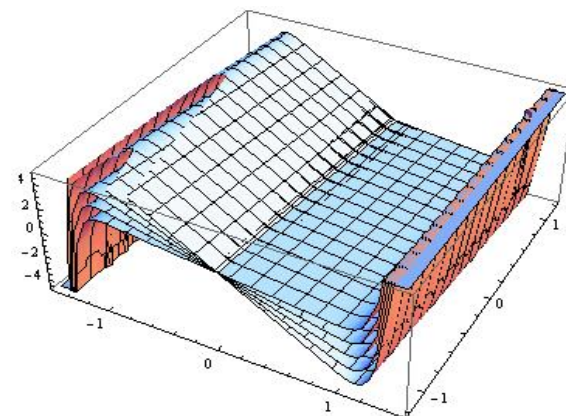


Figure 9. The catastrophe surfaces in the light of Zeeman considerations.

Associated polynomials— $P_n^(x)$.*

We will explicitly note that for modeling specific radiation diagrams, the orthogonal polynomials associated with $P_n(x) - P_n^*(x)$ can also be successfully used, generating from the following recursion [35]:

$$(n + c + 1)P_{n+1}^*(x) = (n + a + c)xP_n^*(x) - P_{n-1}^*(x)$$

$$P_0^*(x) = 1; \quad P_1^*(x) = \frac{a+c}{c+1}x.$$

The polynomials $P_n^*(x)$ for $n = 1, 3, 5, 7, 9$ and fixed $a = 1, c = 3$

$$\begin{cases} P_3^*(x) = x^3 - \frac{11}{30}x \\ P_5^*(x) = x^5 - \frac{533}{840}x^3 + \frac{25}{336}x \\ P_7^*(x) = x^7 - \frac{2131}{2520}x^5 + \frac{7351}{37,800}x^3 - \frac{107}{10,080}x \\ P_9^*(x) = x^9 - \frac{28,271}{27,720}x^7 + \frac{184,463}{554,400}x^5 - \frac{3451}{89,100}3x^3 + \frac{223}{190,080}x \end{cases}$$

are shown in Figure 10.

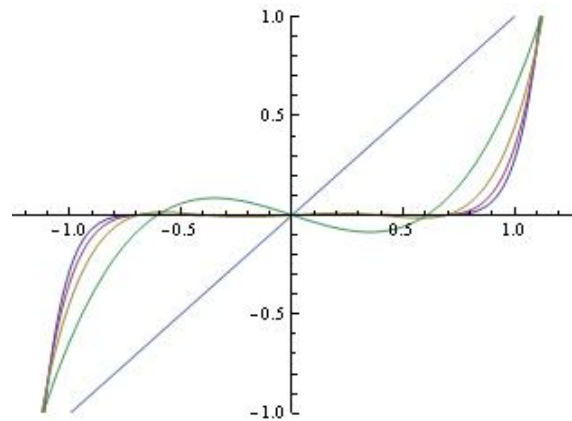


Figure 10. The polynomials $P_n^*(x)$ for fixed $a = 1; c = 3; n = 1, 3, 5, 7, 9$.

Some simulations

The simulations on the Lienard-type system:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -P_5^*(x) + \epsilon F(x)y \end{cases} \tag{8}$$

where

$$F(x) = x - x^3 + x^5 - \frac{1}{7}x^7$$

with $x_0 = 0.8, y_0 = 0.6, b = 0.9, c = 0.7809$ are depicted on Figure 11.

All experiments and algorithms were carried out using our own module in CAS Mathematica.

We define the normalized factor as follows

$$X(\theta) = \frac{|y(b \cos \theta + c)|}{N}$$

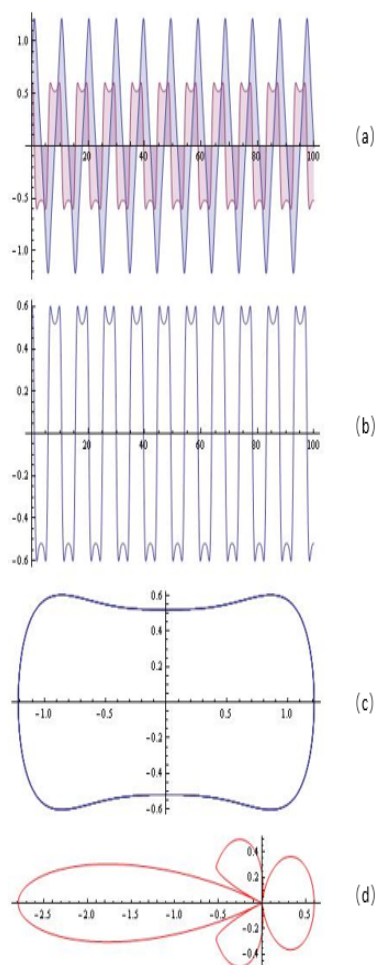


Figure 11. The simulations (system (8)) for $x_0 = 0.8, y_0 = 0.6, b = 0.9, c = 0.7809; \epsilon = 0.0001$: (a) the solution of the system; (b) y -component of the solution; (c) the portrait; (d) emitting chart.

3. Calculator Software Application

We are developing a high-scalable, cloud-based software calculator using serverless architecture [43]. The serverless architecture enables automatic scaling of the system during high load. Furthermore, it can be used to parallelize suitable computations for higher efficiency. Where possible, we employ various optimization techniques for high-performance calculations, including multi-processor and multi-threading calculations, and hardware intrinsics [44–46]. The system is exposing the implemented algorithms using industry-standard application programming interface using HTTP and REST, with data being serialized in JSON and XML formats.

We are developing mobile, native and web-based clients (intellectual property at this stage) to provide an end-user experience for researchers. The following capabilities has been implemented: the user sets: the polynomials $Poly_i(x)$ (see (1)), which can be arbitrary orthogonal polynomials and their associated polynomials (such as associated Hermite polynomials, associated Gegenbauer polynomials, associated Legendre polynomials, associated Lommel polynomials, q -Lommel polynomials associated with the Jackson q -Bessel function, continuous and bivariate q -Hermite polynomials, extended Gegenbauer polynomials and their q -analogues, associated Jacoby polynomials, Chebyshev and Gegenbauer polynomials of higher kind etc.); the function $F(x)$.

The API can be used by reporting and analytics systems like PowerBI and Excel to further investigate the results [47].

The application provides an opportunity for research in two directions—the study of the dynamics of differential systems and the generation of special classes of radiation diagrams.

Some of the algorithms used in this paper

1. In the Melnikov polynomials $M(r^2, n)$, the coefficients are fractional numbers, and with a high degree of the polynomial, the user is faced with solving a “classic problem with imprecise data”. This requires as a first step the use of an algorithm to approximately find the multiplicities of the zeros of the polynomial, then proceed to an algorithm with a user-fixed rate of convergence to determine all the zeros of the polynomial.
2. Specialized algorithm for detailed Hamiltonian study of system (1) and visualization of “level curves” (assuming implementation of software tools in a user-selected computer-algebraic system for scientific calculations).
3. Algorithm for matching the initial approximations when solving the differential system (1), given its interesting specificity and behavior of the solution in confidential time intervals.
4. Algorithm for control and visualization of the “antenna factor” (with a possibly user-set value of the lateral radiation).

4. Concluding Remarks

Some of our previous research on this issue encouraged us to start developing specialized modules, part of a much more general Web-based application for scientific computing. We mention the above algorithms because some of them are hidden from the user. Determining the number and type of cycles, as we have already mentioned, is a complicated task (with inaccurate data—Algorithm 1). Our proposed module automatically generates theorems in light of Melnikov’s considerations (see e.g., Propositions 1 and 2). This is very important for the user to take further steps in the detailed study of the dynamic model (for example, “level curves”—Algorithm 2) with the corrective corrections set by him in the Lienard differential system of the type of arbitrary high-order orthogonal polynomials or their associated or appropriate classes of Morse-type polynomials, etc. We will explicitly note that the user does not have to be a mathematician! What information about it would be, for example, the standard comment from existing computer algebra platforms: “The solution of the differential system lacks consistency under your chosen set of initial approximations”. After all, we have to provide the user with a satisfactory solution to the task (typical example—the hidden Algorithm 3). Another algorithm hidden from the user is checking the conditions in Lienard’s theorem for the existence of a limit cycle at all! Another example of a hidden algorithm is the recurrent generation of the polynomials under study at a user-fixed degree n . Algorithm 4 is extremely complex (and at this stage—insufficiently specified and developed). The use of $y(\theta)$ -the solution component of the corresponding Lienard differential system as an antenna factor is very complicated. As far as the Dolph-Chebyshev technique for synthesis of power pattern end filter prototypes is well known, we note that by analogy we can define hypothetical transmitting functions based on “van Doorn polynomials” (object of consideration in this article). The experts have a word! Of course, research on the “van Doorn array” can be carried out in the light of Soltis considerations [48], but this is the subject of future development and will be omitted here. We fully understand that the construction of such an ambitious Web-based platform for scientific computing can only be realized with the active participation of specialists from various branches of scientific knowledge.

Author Contributions: Conceptualization, V.K.; methodology, N.K.; software, V.K., A.I. and A.R.; validation, N.K., A.R. and A.I.; formal analysis, N.K. and A.I.; investigation, V.K., A.R. and N.K.; resources, A.I., V.K. and N.K.; data curation, N.K. and V.K.; writing—original draft preparation, V.K., N.K. and A.I.; writing—review and editing, A.R.; visualization, V.K.; supervision, N.K.; project administration, N.K.; and funding acquisition, A.R. and A.I. All authors have read and agreed to the published version of the manuscript.

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