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Article

Selection of AI Architecture for Autonomous Vehicles Using Complex Intuitionistic Fuzzy Rough Decision Making

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Abstract: The advancement of artificial intelligence (AI) has become a crucial element in autonomous cars. A well-designed AI architecture will be necessary to attain the full potential of autonomous vehicles and will significantly accelerate the development and deployment of autonomous cars in the transportation sector. Promising autonomous cars for innovating modern transportation systems are anticipated to address many long-standing transporting challenges related to congestion, safety, parking, and energy conservation. Choosing the optimal AI architecture for autonomous vehicles is a multi-attribute decision-making (MADM) dilemma, as it requires making a complicated decision while considering a number of attributes, and these attributes can have two-dimensional uncertainty as well as indiscernibility. Thus, in this framework, we developed a novel mathematical framework "complex intuitionistic fuzzy rough set" for tackling both two-dimensional uncertainties and indiscernibility. We also developed the elementary operations of the deduced complex intuitionistic fuzzy rough set. Moreover, we developed complex intuitionistic fuzzy rough (weighted averaging, ordered weighted averaging, weighted geometric, and ordered weighted geometric) aggregation operators. Afterward, we developed a method of MADM by employing the devised operators and investigated the case study "Selection of optimal AI architecture for autonomous vehicles" to reveal the practicability of the devised method of MADM. Finally, to reveal the dominance and supremacy of our proposed work, a benchmark dilemma was used for comparison with various prevailing techniques.

Keywords: artificial intelligence (AI) architecture; autonomous vehicles; complex intuitionistic fuzzy rough set; MADM method

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1. Introduction

For a multitude of reasons, the design of artificial intelligence (AI) systems for autonomous vehicles is critical. With the potential to transform transportation and make it safer, more effective, and environmentally friendly, autonomous vehicles represent a revolutionary technical development. These vehicles' ability to sense their surroundings, make judgments, and maneuver through the intricate and dynamic real-world environment is largely dependent on the design of the AI system. To reduce the chance of accidents, the AI architecture in autonomous vehicles has to be built with redundancy and fail-safes. An autonomous vehicle's AI system needs to be reliable and capable of handling unexpected situations because a mistake might have serious repercussions. To guarantee the safety of both passengers and other road users, safety-critical capabilities such as object identification, collision avoidance, and emergency braking must be incorporated into the AI architecture. Scalability and flexibility are also crucial factors to take into account. To enable the ongoing development of the AI system as it learns from fresh data and experiences, the architecture should provide over-the-air upgrades. Since autonomous technology is anticipated to be incorporated into a wide range of vehicles, including personal automobiles, commercial trucks, and even drones, this flexibility also extends to diverse vehicle platforms. AI may be deployed more easily in a variety of applications with the help of a flexible architecture.

The multi-attribute decision-making (MADM) technique is one of the most valuable and dominant techniques for evaluating awkward and unreliable information in genuine life dilemmas. MADM plays an important role in real-life decision-making (DM) situations; it gives the best options and results for different alternatives under certain attributes that need to be evaluated when making a decision. When evaluating alternatives, sometimes it can be challenging to express the evaluation value of an attribute as a real number. To address this difficulty, Zadeh [1] initiated the mathematical framework known as fuzzy set (FS) theory as an extension of crisp set theory. Pawlak [2] developed the mathematical framework known as the rough set (RS) theory as a means of addressing ambiguity and uncertainty in knowledge representation and data processing. It offers a method for dealing with uncertain or insufficient data without depending on conventional probability theory or fuzzy logic. RS structures consist of upper and lower approximations. It separates a universe of items into sets with varying degrees of indiscernibility, where indiscernibility is the inability to tell one object from another based on the available information. This theory was first put out in 1982. Later, as a fuzzy expansion of RSs, Dubois and Prade [3] suggested the idea of fuzzy RSs (FRSs) by substituting fuzzy relations for binary relations. Using the idea of truth grade from fuzzy logic in the RS framework, an FRS offers a more potent framework. Later on, Cornelis et al. [4] introduced the notion of intuitionistic FRSs (IFRSs) by combining the concepts of intuitionistic fuzzy sets (IFSs) and RSs.

1.1. Motivation and Contribution

An autonomous car relies heavily on AI architecture to perceive the environment, make decisions, and control the car. Selecting an AI structure for autonomous vehicles is a complex system that involves integrating various techniques, algorithms, and systems. Over the years, we have seen companies debating over which AI architecture should be used. There are a lot of high-level overviews of the key components in an autonomous vehicle AI architecture, such as types of sensors, perception and object detection, localization and high-definition mapping, DM and motion-planning frameworks, control systems and machine learning approaches, safety and ethical considerations, scalability, and flexibility. A detailed discussion of these key components is provided as follows:

- Data fusion, perception, and object detection: These refer to combining the data from multiple sensors (vision system, lidar system (light detection and ranging), radar system (detection of objects and their speed, even in poor visibility), and ultrasonic sensor for short-range object detection) to produce a more accurate and reliable representation of the vehicle's surrounding. The AI architecture should be able to use these sensors' data efficiently and develop a fusion algorithm to improve perception and DM. By combining the information from these multiple sensors, autonomous vehicles can create an exceptionally flexible and redundant cognitive framework, thereby reducing the potential risk associated with wrong decisions in critical DM scenarios. In real-time, the AI structures need to be able to detect and accurately identify different objects, such as pedestrians, road hurdles, speed breakers, other cars, lane markings and traffic signs, and other static elements in the environment. For this, sophisticated computer vision and deep learning algorithms are needed.
- Localization and mapping: The localization system in the vehicles uses a combination of a global positioning system (GPS) and inertial measurement units (IMUs) for precise positioning. Autonomous vehicles often use high-definition (HD) maps that provide detailed information about the road network, pedestrians, road hurdles, speed breakers, lane markings, and traffic signs in the environment. These maps are combined with real-time sensor data to build a comprehensive understanding of the vehicle's surroundings. The AI architecture should be able to accurately localize the car within its environment and maintain a detailed map of the environment, including static and dynamic elements, to support efficient navigation and DM.

The below figure, Figure 1, represents the data fusion algorithm of autonomous vehicles.

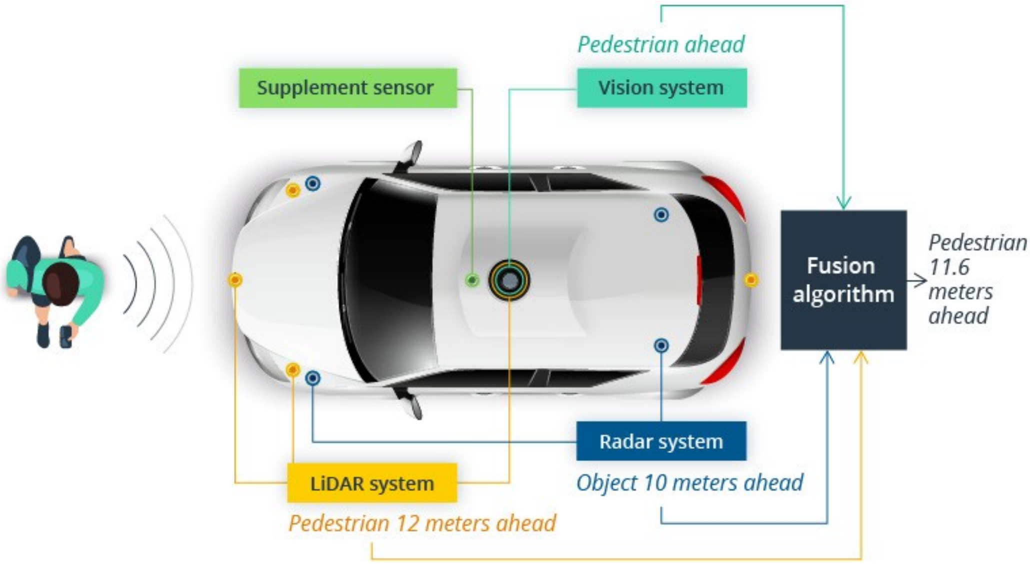

Figure 1. Data fusion algorithm.

- Decision-making and motion-planning framework: The DM system is the brain of autonomous vehicles. It uses the information from the perception, localization, and mapping systems to plan the vehicle's trajectory and make decisions about when to accelerate, brake, or change lanes. This system relies on advanced algorithms to navigate the vehicle safely and efficiently.
- Control system and machine learning approach: The AI DM system interconnects with the vehicle control system, which translates the high-level decisions into precise commands that control the vehicle's steering, acceleration, and brakes. Machine learning is the backbone of autonomous vehicles; the perception and DM modules, including deep neural networks, rely on advanced machine learning techniques to improve their performance and adapt to new situations.

The below figure, Figure 2, represents the AI architecture of an autonomous vehicle.

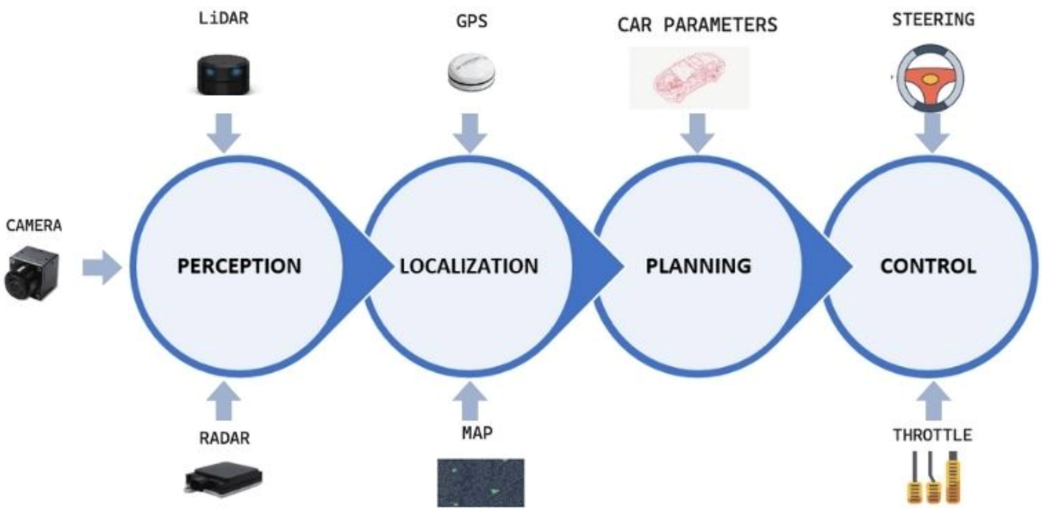

Figure 2. Basic 4 pillars of AI architecture for autonomous vehicle.

- Safety, reliability, and scalability: The AI architecture must be designed with safety and ethical principles in mind, ensuring that autonomous vehicle actions and decisions prioritize the safety of pedestrians and other road users. AI systems develop rigorous testing and validation processes, including simulation and real-world trials, and they ensure that robust fail-safe mechanisms are in place to handle unexpected situations.

The selection of the best AI architecture for autonomous vehicles involves a holistic approach that prioritizes safety, performance, and adaptability. By focusing on these key components, developers can create more dynamic and efficient autonomous systems. Despite a considerable amount of research efforts, there are still challenging issues in AI structures for autonomous car perception, object detection, localization and high-definition mapping, DM, and motion-planning frameworks. Many AI methods have been applied to improve the AI structures of autonomous cars and help them make human-like decisions in different situations, such as following other cars, steering, and path planning. However, such applications are inherently limited by the data's availability, quality, complexity, and uncertainty. If the data collected from different sensors of autonomous vehicles contain extra fuzzy information along with roughness, then there is no tool that can handle that type of information. Then, the information collected from these sensors can be noisy, incomplete, and affected by different environmental factors. Moreover, from the literature review, we observed that the FS, FRS, and IFRS theories are restricted by their limitations and conditions. For decision makers, an IFRS is the best tool for handling truth-grade (TG) and false-grade (FG) information in the form of lower and upper approximations, but IFRSs cannot handle the additional fuzzy information (second dimension) in the form of lower and upper approximations. Similarly, other existing theories cannot model the information that contains additional fuzzy information. Therefore, this era requires the development of a new mathematical framework that can handle second-dimension (additional fuzzy information or complex fuzzy information) TG and FG information in the form of lower and upper information. Motivated by this research gap, in this article, we developed the concept of complex IFRSs (CIFRSs), and then, based on this newly defined relation, we devised a new novel mathematical framework called a CIFRS that can not only handle the TG and FG in its structure, but can also tackle two-dimensional uncertainties and indiscernibility. Moreover, we developed elementary operations such as the complement, union, intersection, and algebraic operations (sum, product) of the deduced CIFRSs. For comparing the two complex intuitionistic fuzzy rough numbers (CIFRNs), we developed the accuracy and score function. As aggregation is a fundamental mathematical tool to convert the overall information into a single value, based on this observation, we developed complex intuitionistic fuzzy rough (CIFR) weighted averaging (CIFRWA), CIFR ordered weighted averaging (CIFROWA), CIFR weighted geometric (CIFRWG), and CIFR ordered weighted geometric aggregation operators (AOs), and investigated their properties in detail. According to the application point of view, we used the MADM technique based on CIFR information for the selection of optimal AI architecture for autonomous vehicles.

1.2. Aims and Objectives

This study aimed to develop a new framework, "CIFRSs", to overcome the existing problems in the theory of IFRS for the selection of the finest AI architecture for autonomous cars by using CIFR AOs to effectively combine and evaluate various performance metrics and criteria. The specific objectives were as follows:

- To define and develop the theory of CIFRSs.
- To define and develop different operators, such as the average AOs and geometric AOs, in the framework of CIFRSs.
- To apply a newly defined framework to assess multiple AI architectures based on a wide range of performance indicators, and utilize the AOs to combine diverse evaluation criteria into a single, meaningful value.
- To select a suitable MADM method based on the problem characteristics and data availability.
- To develop an MADM algorithm for aggregation and evaluation to handle large datasets and complex AI architectures.

- To examine the comparative study of the proposed methods with the present notions to show the supremacy and efficiency of the established work.

By achieving these objectives, this research will contribute to the development of a systematic and reliable approach for the selection of optimal AI architectures, leading to the improved performance and safety of autonomous vehicles.

1.3. Study Framework

The remaining article is designed as follows. In Section 2, we review the background study and notion of IFSs, complex FSs (CFSs), FRSs, and IFRSs and discuss their related operations. In Section 3, we demonstrate the definition of a complex intuitionistic fuzzy relation, which will help us further demonstrate the definition of a CIFRS. Section 3 is focused on the basic operating rules for CIFRNs. In Section 4, we develop several new AOs based on CIFRSs and discuss their properties using basic operations. Section 5 is based on an MADM algorithm and an application for delivered work to show the reliability and functioning of the developed notions. Section 6 is about the comparative study, and some conclusions for further research are given in Section 7.

2. Background of the Study

Security and data privacy are two very important issues. Large volumes of data are gathered by autonomous vehicles concerning their environment, occupants, and internal processes. Strong security features are required in the architecture to guard against cyber-attacks and unwanted access to these data. To increase public confidence in autonomous vehicles, privacy protection for its users is crucial. Another important component is inter-operability. Autonomous vehicles need to connect with infrastructure, other vehicles, and traffic control systems to function. By providing standardized protocols and interfaces for communication, the AI architecture should be created to support a more organized and effective transportation environment. Ma et al. [5] devised various applications of AI in the creation of autonomous vehicles. For autonomous vehicles, Khayyam et al. [6] used AI and the IoT. Pereira et al. [7] originated architecture for autonomous vehicles. Kurzidem et al. [8] discussed a methodology to examine architectures in autonomous vehicles. The challenges, opportunities, and applications of intelligent automation and autonomous vehicles were examined by Bathla et al. [9]. Zong et al. [10] devised an architecture design for an autonomous vehicle. By employing blockchain and AI, Bendiab et al. [11] discussed the challenges and solutions to the security of autonomous vehicles.

In crisp set theory, there are only two possibilities in the form of yes or no, or (0) or (1), for each element from the universe of discourse, but FSs include the TG and have more possibilities from a unit closed interval. With time, many researchers have given their ideas in different fields of life using the FS theory framework. Esogbue et al. [12] demonstrated the application of the FS theory to the optimal flood control problem arising in water resource systems. Guiffrida and Nagi [13] developed FS theory applications in production management research. Driankoy and Saffiotti [14] created various techniques based on fuzzy logic for autonomous vehicles. Wang et al. [15] investigated the lateral control of autonomous vehicles by employing fuzzy logic. Awad et al. [16] developed a model by employing fuzzy logic for the path tracking of autonomous vehicles. FSs ignore FG characteristics and instead address uncertainty through TGs. This restriction falls short of capturing all the uncertainty that exists in real-world situations. To overcome this limitation, an IFS was developed by Atanassov [17] that includes FGs in addition to TGs, and their sum should be in the unit closed interval. This allows for a more thorough depiction of uncertainty in DM and the modeling of uncertain information. Later on, different valuable applications in IFSs have been explored by researchers in different fields. Dengfeng et al. [18] proposed new similarity measures of IFSs and their application to pattern recognition. De et al. [19] developed an application of IFSs in medical diagnoses. Garg and Rani [20] introduced novel distance measures for IFSs based on various triangle centers of isosceles triangular fuzzy numbers and their applications. Xu [21] introduced IF

AOs and Xu and Yager [22] proposed some geometric AOs based on IFSs. Jia and Wang [23] introduced Choquet integral-based IF arithmetic AOs in multi-criteria DM. An extended MAIRCA method for coronavirus vaccine selection in the age of COVID-19 based on IFSs was proposed by Ecer [24]. Even though FSs and IFSs are very useful and dominant, they do have some limitations because they do not cover the two-dimensional information in a single set. Later on, Ramot et al. [25] derived the major idea of CFSs, which contain the TGs in the form of complex numbers and cover the two-dimensional information in a single set. The complex fuzzy (CF) function contains two main terms, the phase term and the amplitude term, whose range is in the complex plane's unit circle. Tamir et al. [26] also invented the notion of CFSs in the cartesian structure, where TGs have real and unreal parts that are placed in the complex plane's unit square. A few operations for CFSs were developed by Zhang et al. [27]. Hu et al. [28] devised distances of CFSs and the continuity of their operations. Rehman [29] investigated probability AOs under the setting of CFSs. Cornelis et al. [4] introduced the notion of intuitionistic FRSs (IFRSs) by combining the concepts of IFSs and RSs. To examine intuitionistic fuzzy rough (IFR) approximation operators, Zhou and Wu [30] constructed a generic framework that used both constructive and axiomatic techniques. Also, Zhou and Wu [31] developed the idea of rough IFSs and IFRSs, and demonstrated their logical study in detail. The idea of an IF relation was defined by Bustince and Burillo [32]. By using the IF relation and the idea of two universes, Zhang et al. [33] examined the general framework of IFRSs. By using topology, Yun and Lee [34] defined and examined some properties of the IFR approximation operator and IF relations. Zhang [35] proposed the generalized IFR approximation operators based on IF coverings by combining the theories of RSs and IFSs. Yahya et al. [36] developed a novel approach to the IFR frank AO-based and evolution-based distance from the average solution (EDAS) method for MADM. Chinram et al. [37] examined the EDAS method for MADM based on IFR AOs. Ahmmad et al. [38] utilized the IFR Aczel–Alsina average AOs and their applications in medical diagnoses. Mahmood et al. [39] examined confidence level AOs based on IFRSs with an application in medical diagnoses. Mahmood et al. [40] derived the analysis and prioritization of the factors of the robotic industry with the assistance of the EDAS technique based on IFR Yager AOs.

Preliminaries

In this sequel, we recalled some basic notions linked to IFSs, CFSs, FRSs, and IFRSs. Also, their related operations and properties are discussed.

Definition 1 [17]**.** *Let U be a universal set; IFS € on set U is given as*

$$€ = \left\{ \left(\check{s}^*, \mathrm{M}_€\left(\check{s}^*\right), \mathrm{N}_€\left(\check{s}^*\right) \right) \middle| \check{s}^* \in U \right\}, \quad (1)$$

in which $\mathrm{M}_€ : U \to [0,1]$ represents the TG and $\mathrm{N}_€ : U \to [0,1]$ represents the FG of $\check{s}^* \in U$ to the set € such that $0 \leq M_€(\check{s}^*) + N_€(\check{s}^*) \leq 1$. For easiness, the IF number (IFN) is symbolized by $€ = (\mathrm{M}_€, \mathrm{N}_€)$.

Definition 2 [17]**.** *For two IFNs, $€_1 = \left(M_{€_1}, N_{€_1}\right)$ and $€_2 = \left(M_{€_2}, N_{€_2}\right)$, the following are true:*

1. $€_1 \cup €_2 = \left(\max\left(\mathrm{M}_{€_1}\left(\check{s}^*\right), \mathrm{M}_{€_2}\left(\check{s}^*\right)\right), \min\left(\mathrm{N}_{€_1}\left(\check{s}^*\right), \mathrm{N}_{€_2}\left(\check{s}^*\right)\right)\right)$;
2. $€_1 \cap €_2 = \left(\min\left(\mathrm{M}_{€_1}\left(\check{s}^*\right), \mathrm{M}_{€_2}\left(\check{s}^*\right)\right), \max\left(\mathrm{N}_{€_1}\left(\check{s}^*\right), \mathrm{N}_{€_2}\left(\check{s}^*\right)\right)\right)$;
3. $€_1^c = \left(\mathrm{N}_{€_1}, \mathrm{M}_{€_1}\right)$, where $€_1^c$ represents the complement of $€_1$;
4. $€_1 \oplus €_2 = \left(\mathrm{M}_{€_1} + \mathrm{N}_{€_2} - \mathrm{M}_{€_1}\mathrm{N}_{€_2}, \mathrm{M}_{€_1}\mathrm{N}_{€_2}\right)$;
5. $€_1 \otimes €_2 = \left(\mathrm{M}_{€_1}\mathrm{N}_{€_2}, \mathrm{M}_{€_1} + \mathrm{N}_{€_2} - \mathrm{M}_{€_1}\mathrm{N}_{€_2}\right)$;
6. $\lambda €_1 = \left(1 - \left(1 - \mathrm{M}_{€_1}\right)^\lambda, \mathrm{N}_{€_1}^\lambda\right)$;
7. $€_1^\lambda = \left(\mathrm{N}_{€_1}^\lambda, 1 - \left(1 - \mathrm{M}_{€_1}\right)^\lambda\right)$.

Definition 3 [26]. *Let U be a universal set; CFS € on set U is given as*

$$€ = \left\{ \left(š^*, M_€\left(š^*\right) \right) \middle| š^* \in U \right\} = \left\{ \left(š^*, \varphi_€\left(š^*\right) + \iota ƀ_€\left(š^*\right) \right) \middle| š^* \in U \right\} \tag{2}$$

in which $M_€(š^*)$ *represents the complex TG and* $\varphi_€, ƀ_€ \in [0,1]$, $\iota = \sqrt{-1}$. *For easiness, the CFN is symbolized by* $€ = (\varphi_€ + \iota ƀ_€)$.

Definition 4 [26,29]. *For two CFNs,* $€_1 = (\varphi_{€_1} + \iota ƀ_{€_1})$ *and* $€_2 = (\varphi_{€_2} + \iota ƀ_{€_2})$, *the following are true:*

1. $€_1 \cup €_2 = \left(\max\left(\varphi_{€_1}, \varphi_{€_2}\right) + \iota\max\left(ƀ_{€_1}, ƀ_{€_2}\right)\right)$;
2. $€_1 \cap €_2 = \left(\min\left(\varphi_{€_2}, \varphi_{€_2}\right) + \iota\min\left(ƀ_{€_1}, ƀ_{€_2}\right)\right)$;
3. $€_1^c = \left(\left(1 - \varphi_{€_1}\right) + \iota\left(1 - ƀ_{€_1}\right)\right)$, where $€_1^c$ represents the complement of $€_1$;
4. $€_1 \oplus €_2 = \left(\left(\varphi_{€_1} + \varphi_{€_2} - \varphi_{€_1}\varphi_{€_2}\right) + \iota\left(ƀ_{€_1} + ƀ_{€_2} - ƀ_{€_1}ƀ_{€_2}\right)\right)$;
5. $€_1 \otimes €_2 = \left(\left(\varphi_{€_1}\varphi_{€_2}\right) + \iota\left(ƀ_{€_1}ƀ_{€_2}\right)\right)$;
6. $\lambda €_1 = \left(\left(1 - \left(1 - \varphi_{€_1}\right)^\lambda\right) + \iota\left(1 - \left(1 - ƀ_{€_1}\right)^\lambda\right)\right)$;
7. $€_1^\lambda = \left(\left(\varphi_{€_1}\right)^\lambda + \iota\left(ƀ_{€_1}\right)^\lambda\right)$.

Definition 5 [3]. *Let* $(U,\ R_e^*)$ *be a fuzzy approximation space and let set A be the FS in U. Then, the upper and lower approximation of A w.r.t.* $(U,\ R_e^*)$ *is denoted and defined by*

$$\overline{R_e^*}(A) = \left\{ \left(š^*, M_{\overline{R_e^*}}\left(š^*\right) \right) \middle| š^* \in U \right\}$$
$$\underline{R_e^*}(A) = \left\{ \left(š^*, M_{\underline{R_e^*}}\left(š^*\right) \right) \middle| š^* \in U \right\}$$

where

$$M_{\overline{R_e^*}}\left(š^*\right) = \bigvee\nolimits_{ɨ^* \in U}\left[e\left(š^*, ɨ^*\right) \wedge \varphi_A\left(ɨ^*\right)\right]$$
$$M_{\underline{R_e^*}}\left(š^*\right) = \bigwedge\nolimits_{ɨ^* \in U}\left[\left(1 - e\left(š^*, ɨ^*\right)\right) \vee \varphi_A\left(ɨ^*\right)\right]$$

Then, the pair $R_e^*(A) = \left(\overline{R_e^*}(A),\ \underline{R_e^*}(A)\right)$ *is called the FRS.*

Definition 6 [37]. *Let U be a nonempty and finite universe of discourse and* R_e^* *be an IF relation on U; then, we can say that* $(U,\ R_e^*)$ *defines the IF approximation space. Then, for a set* $A = \{(ɨ^*,\ \varphi_A(ɨ^*), \mathcal{L}_A(ɨ^*)) | ɨ^* \in U\} \in IFS(U)$, *we can denote and define the upper and lower approximation of A w.r.t.* $(U,\ R_e^*)$ *as*

$$\overline{R_e^*}(A) = \left\{ \left(š^*, M_{\overline{R_e^*}}\left(š^*\right), N_{\overline{R_e^*}}\left(š^*\right) \right) \middle| š^* \in U \right\}$$
$$\underline{R_e^*}(A) = \left\{ \left(š^*, M_{\underline{R_e^*}}\left(š^*\right), N_{\underline{R_e^*}}\left(š^*\right) \right) \middle| š^* \in U \right\}$$

where

$$M_{\overline{R_e^*}}\left(š^*\right) = \bigvee\nolimits_{ɨ^* \in U}\left[a\left(š^*, ɨ^*\right) \vee \varphi_A\left(ɨ^*\right)\right]$$
$$M_{\underline{R_e^*}}\left(š^*\right) = \bigwedge\nolimits_{ɨ^* \in U}\left[\left(a\left(š^*, ɨ^*\right)\right) \wedge \varphi_A\left(ɨ^*\right)\right]$$
$$N_{\overline{R_e^*}}\left(š^*\right) = \bigwedge\nolimits_{ɨ^* \in U}\left[\left(b\left(š^*, ɨ^*\right)\right) \wedge \mathcal{L}_A\left(ɨ^*\right)\right]$$
$$N_{\underline{R_e^*}}\left(š^*\right) = \bigvee\nolimits_{ɨ^* \in U}\left[b\left(š^*, ɨ^*\right) \vee \mathcal{L}_A\left(ɨ^*\right)\right]$$

where $0 \le M_{\overline{R_e^*}} + N_{\overline{R_e^*}} \le 1$, *and* $0 \le M_{\underline{R_e^*}} + N_{\underline{R_e^*}} \le 1$. $\overline{R_e^*}(A)$ *and* $\underline{R_e^*}(A)$ *are IFSs. Then, the pair* $R_e^*(A) = \left(\overline{R_e^*}(A), \underline{R_e^*}(A)\right) = \left\{ \left(š^*, < M_{\overline{R_e^*}}\left(š^*\right), N_{\overline{R_e^*}}\left(š^*\right) >, < M_{\underline{R_e^*}}\left(š^*\right), N_{\underline{R_e^*}}\left(š^*\right) > \right) \middle| š^* \in U \right\}$ *is called an IFRS with respect to* (U, R_e^*). *For simplicity, we will say that* $R_e^*(A) = \left(\overline{R_e^*}(A), \underline{R_e^*}(A)\right) = \left(< M_{\overline{R_e^*}}\left(š^*\right), N_{\overline{R_e^*}}\left(š^*\right) >, < M_{\underline{R_e^*}}\left(š^*\right), N_{\underline{R_e^*}}\left(š^*\right) > \right)$ *represents the IFRN.*

Definition 7 [37]**.** *For two IFRNs,* $R_e^*(A) = \left(\overline{R_e^*}(A),\ \underline{R_e^*}(A)\right)$ *and* $R_e^*(B) = \left(\overline{R_e^*}(B),\ \underline{R_e^*}(B)\right)$*, the following are true:*

1. $R_e^*(A) \cup R_e^*(B) = \left(\overline{R_e^*}(A) \cup \overline{R_e^*}(B), \underline{R_e^*}(A) \cup \underline{R_e^*}(B)\right)$;
2. $R_e^*(A) \cap R_e^*(B) = \left(\overline{R_e^*}(A) \cap \overline{R_e^*}(B), \underline{R_e^*}(A) \cap \underline{R_e^*}(B)\right)$;
3. $R_e^*(A)^c = \left(\overline{R_e^*}(A)^c, \underline{R_e^*}(A)^c\right)$, where $\overline{R_e^*}(A)^c$ and $\underline{R_e^*}(A)^c$ represent the complement of the IF rough approximation operators $\overline{R_e^*}(A)$ and $\underline{R_e^*}(A)$;
4. $R_e^*(A) \oplus R_e^*(B) = \left(\overline{R_e^*}(A) \oplus \overline{R_e^*}(B), \underline{R_e^*}(A) \oplus \underline{R_e^*}(B)\right)$;
5. $R_e^*(A) \otimes R_e^*(B) = \left(\overline{R_e^*}(A) \otimes \overline{R_e^*}(B), \underline{R_e^*}(A) \otimes \underline{R_e^*}(B)\right)$;
6. $\lambda R_e^*(A) = \left(\lambda \overline{R_e^*}(A), \lambda \underline{R_e^*}(A)\right)$;
7. $(R_e^*(A))^\lambda = \left(\left(\overline{R_e^*}(A)\right)^\lambda, \left(\underline{R_e^*}(A)\right)^\lambda\right)$.

3. Complex Intuitionistic Fuzzy Rough Set

In this section, we demonstrate the definition of complex intuitionistic fuzzy (CIF) relations that will help us further demonstrate the definition of a CIFRS. In the overall discussion throughout the article, U represents the universal set and R_e^* represents the CIF relation.

Definition 8. *Assume that* U *is the universal set; then, any CIF subset* R_e^* *of* $U \times U$ *is called a CIF relation and is given by* $R_e^* = \{(\check{s},\ ł),\ (\mathfrak{W}(\check{s},\ ł), \text{N}(\check{s},\ ł)) | \mathfrak{W}(\check{s},\ ł) = e(\check{s},\ ł) + \iota f(\check{s},\ ł),\ \text{N}(\check{s},\ ł) = g(\check{s},\ ł) + \iota h(\check{s},\ ł)\ \}$*. Here,* $\mathfrak{W}(\check{s},\ ł)$ *and* $\text{N}(\check{s},\ ł)$ *are respectively called the TG and FG, and* $\mathfrak{W}(\check{s},\ ł) : U \times U \to [0,\ 1] + \iota[0,\ 1]$ *and* $\text{N}(\check{s},\ ł) : U \times U \to [0,\ 1] + \iota[0,\ 1]$ *satisfy* $0 \le e(\check{s},\ ł) + g(\check{s},\ ł) \le 1$ *and* $0 \le f(\check{s},\ ł) + h(\check{s},\ ł) \le 1$.

Definition 9. *Let* U *be a nonempty and finite universe of discourse and* R_e^* *be a CIF relation on* U*; then, we can say that* $(U,\ R_e^*)$ *defines the CIF approximation space. Then, for a set* $A = \left\{\left(ł^*,\ \varphi_A(ł^*) + \iota \mathfrak{b}(ł^*), \mathcal{L}_A(ł^*) + \iota \wp_A(ł^*)\right) \middle| ł^* \in U\right\} \in CIFS(U)$*, we can define the upper and lower approximation of* A *w.r.t.* $(U,\ R_e^*)$*, denoted by*

$$\begin{array}{l}\overline{R_e^*}(A) = \left\{\left(\check{s}^*, \text{M}_{\overline{R_e^*}}\left(\check{s}^*\right), \text{N}_{\overline{R_e^*}}\left(\check{s}^*\right)\right) \middle| \check{s}^* \in U\right\} \\ \underline{R_e^*}(A) = \left\{\left(\check{s}^*, \text{M}_{\underline{R_e^*}}\left(\check{s}^*\right), \text{N}_{\underline{R_e^*}}\left(\check{s}^*\right)\right) \middle| \check{s}^* \in U\right\}\end{array}$$

where

$$\begin{array}{l}\text{M}_{\overline{R_e^*}}\left(\check{s}^*\right) = \bigvee_{ł^* \in U}\left[e\left(\check{s}^*, ł^*\right) \vee \varphi_A\left(ł^*\right)\right] + \iota \bigvee_{ł^* \in U}\left[f\left(\check{s}^*, ł^*\right) \vee \mathfrak{b}_A\left(ł^*\right)\right] = \varphi_{\overline{R_e^*}} + \iota \mathfrak{b}_{\overline{R_e^*}} \\ \text{M}_{\underline{R_e^*}}\left(\check{s}^*\right) = \bigwedge_{ł^* \in U}\left[e\left(\check{s}^*, ł^*\right) \wedge \varphi_A\left(ł^*\right)\right] + \iota \bigwedge_{ł^* \in U}\left[f\left(\check{s}^*, ł^*\right) \wedge \mathfrak{b}_A\left(ł^*\right)\right] = \varphi_{\underline{R_e^*}} + \iota \mathfrak{b}_{\underline{R_e^*}} \\ \text{N}_{\overline{R_e^*}}\left(\check{s}^*\right) = \bigwedge_{ł^* \in U}\left[g\left(\check{s}^*, ł^*\right) \wedge \mathcal{L}_A\left(ł^*\right)\right] + \iota \bigwedge_{ł^* \in U}\left[h\left(\check{s}^*, ł^*\right) \wedge \wp_A\left(ł^*\right)\right] = \mathcal{L}_{\overline{R_e^*}} + \iota \wp_{\overline{R_e^*}} \\ \text{N}_{\underline{R_e^*}}\left(\check{s}^*\right) = \bigvee_{ł^* \in U}\left[g\left(\check{s}^*, ł^*\right) \vee \mathcal{L}_A\left(ł^*\right)\right] + \iota \bigvee_{ł^* \in U}\left[h\left(\check{s}^*, ł^*\right) \vee \wp_A\left(ł^*\right)\right] = \mathcal{L}_{\underline{R_e^*}} + \iota \wp_{\underline{R_e^*}}\end{array}$$

where $\le \varphi_{\overline{R_e^*}} + \mathcal{L}_{\overline{R_e^*}} \le 1, 0 \le \varphi_{\underline{R_e^*}} + \mathcal{L}_{\underline{R_e^*}} \le 1, 0 \le \mathfrak{b}_{\overline{R_e^*}} + \wp_{\overline{R_e^*}} \le 1$ *and* $0 \le \mathfrak{b}_{\underline{R_e^*}} + \wp_{\underline{R_e^*}} \le 1$. *As* $\overline{R_e^*}(A)$ *and* $\underline{R_e^*}(A)$ *are CIFRSs, then the pair*

$$R_e^*(A) = \left(\overline{R_e^*}(A), \underline{R_e^*}(A)\right) = \left\{\left(\check{s}^*, < \text{M}_{\overline{R_e^*}}\left(\check{s}^*\right), \text{N}_{\overline{R_e^*}}\left(\check{s}^*\right) >, < \text{M}_{\underline{R_e^*}}\left(\check{s}^*\right), \text{N}_{\underline{R_e^*}}\left(\check{s}^*\right) >\right) \middle| \check{s}^* \in U\right\} \quad (3)$$

is called the CIFRS with respect to $(U,\ R_e^*)$ *.For simplicity, we will say that* $R_e^*(A) = (\overline{R_e^*}(A),\ \underline{R_e^*}(A)) = (< \text{M}_{\overline{R_e^*}}(\check{s}^*), \text{N}_{\overline{R_e^*}}(\check{s}^*) >, < \text{M}_{\underline{R_e^*}}(\check{s}^*), \text{N}_{\underline{R_e^*}}(\check{s}^*) >) = ((\varphi_{\overline{R_e^*}} + \iota \mathfrak{b}_{\overline{R_e^*}}, \mathcal{L}_{\overline{R_e^*}} + \iota \wp_{\overline{R_e^*}}), (\varphi_{\underline{R_e^*}} + \iota \mathfrak{b}_{\underline{R_e^*}}, \mathcal{L}_{\underline{R_e^*}} + \iota \wp_{\underline{R_e^*}}))$ *represents the CIFRN.*

Remark 1. *A CIFRS is signified by a TG* $M_{R_e^*} = \varphi_A(ɟ^*) + \iota ƀ(ɟ^*)$ *and FG* $N_{R_e^*} = \mathcal{L}_A(ɟ^*) + \iota\wp_A(ɟ^*)$. *If we remove the imaginary parts from both the TG and FG, then the CIFRS will convert into an IFRS. Also, if we remove the FG, then the IFRS will convert into an FRS. This shows that the CIFRS is a modification of the FRS.*

Example 1. *Let* $U = \{\check{s}_1^*, \check{s}_2^*, \check{s}_3^*, \check{s}_4^*\}$ *be a universal set; CIFR is defined in Table 1 as follows:*

Table 1. CIFR.

R_e^*	$\check{s}_1^*$	$\check{s}_2^*$	$\check{s}_3^*$	$\check{s}_4^*$
$\check{s}_1^*$	$\begin{pmatrix}(0.3+\iota 0.4),\\(0.5+\iota 0.2)\end{pmatrix}$	$\begin{pmatrix}(0.6+\iota 0.4),\\(0.3+\iota 0.5)\end{pmatrix}$	$\begin{pmatrix}(0.3+\iota 0.4),\\(0.2+\iota 0.3)\end{pmatrix}$	$\begin{pmatrix}(0.2+\iota 0.6),\\(0.5+\iota 0.3)\end{pmatrix}$
$\check{s}_2^*$	$\begin{pmatrix}(0.4+\iota 0.2),\\(0.5+\iota 0.6)\end{pmatrix}$	$\begin{pmatrix}(0.2+\iota 0.3),\\(0.7+\iota 0.2)\end{pmatrix}$	$\begin{pmatrix}(0.5+\iota 0.6),\\(0.5+\iota 0.4)\end{pmatrix}$	$\begin{pmatrix}(0.7+\iota 0.8),\\(0.2+\iota 0.1)\end{pmatrix}$
$\check{s}_3^*$	$\begin{pmatrix}(0.6+\iota 0.3),\\(0.3+\iota 0.7)\end{pmatrix}$	$\begin{pmatrix}(0.3+\iota 0.4),\\(0.5+\iota 0.4)\end{pmatrix}$	$\begin{pmatrix}(0.4+\iota 0.3),\\(0.3+\iota 0.6)\end{pmatrix}$	$\begin{pmatrix}(0.3+\iota 0.4),\\(0.6+\iota 0.3)\end{pmatrix}$
$\check{s}_4^*$	$\begin{pmatrix}(0.7+\iota 0.4),\\(0.2+\iota 0.6)\end{pmatrix}$	$\begin{pmatrix}(0.5+\iota 0.7),\\(0.4+\iota 0.2)\end{pmatrix}$	$\begin{pmatrix}(0.5+\iota 0.6),\\(0.4+\iota 0.3)\end{pmatrix}$	$\begin{pmatrix}(0.5+\iota 0.6),\\(0.5+\iota 0.3)\end{pmatrix}$

Now, we assume that $A = \{(\check{s}_1^*, 0.2+\iota 0.5, 0.4+\iota 0.3), (\check{s}_2^*, 0.3+\iota 0.4, 0.4+\iota 0.3), (\check{s}_3^*, 0.4+\iota 0.7, 0.3+\iota 0.2), (\check{s}_4^*, 0.5+\iota 0.6, 0.3+\iota 0.2)\}$ is a CIFS over U.

$$\begin{aligned}\overline{R_e^*}(A) &= \left\{\left(\check{s}^*, \mathrm{M}_{\overline{R_e^*}}\left(\check{s}^*\right), \mathrm{N}_{\overline{R_e^*}}\left(\check{s}^*\right)\right)\middle|\check{s}^* \in U\right\}\\ \underline{R_e^*}(A) &= \left\{\left(\check{s}^*, \mathrm{M}_{\underline{R_e^*}}\left(\check{s}^*\right), \mathrm{N}_{\underline{R_e^*}}\left(\check{s}^*\right)\right)\middle|\check{s}^* \in U\right\}\end{aligned}$$

Now, to find $\overline{R_e^*}(A)$ and $\underline{R_e^*}(A)$, we have

$$\begin{aligned}\mathrm{M}_{\overline{R_e^*}}\left(\check{s}_1^*\right) = \bigvee_{ɟ^* \in U} \quad & \left[e\left(\check{s}^*, ɟ^*\right) \vee \varphi_A\left(ɟ^*\right)\right] + \iota \bigvee_{ɟ^* \in U}\left[f\left(\check{s}^*, ɟ^*\right) \vee ƀ_A\left(ɟ^*\right)\right]\\ & = (0.3 \vee 0.2) \vee (0.6 \vee 0.3) \vee (0.3 \vee 0.4) \vee (0.2 \vee 0.5)\\ & + \iota(0.4 \vee 0.5) \vee (0.4 \vee 0.4) \vee (0.4 \vee 0.7) \vee (0.6 \vee 0.6)\\ & = 0.6 + \iota 0.7\end{aligned}$$

$$\begin{aligned}\mathrm{N}_{\overline{R_e^*}}\left(\check{s}_1^*\right) = \bigwedge_{ɟ^* \in U} \quad & \left[\left(g\left(\check{s}^*, ɟ^*\right)\right) \wedge \mathcal{L}_A\left(ɟ^*\right)\right] + \iota \bigwedge_{ɟ^* \in U}\left[\left(h\left(\check{s}^*, ɟ^*\right)\right) \wedge \wp_A\left(ɟ^*\right)\right]\\ & = ((0.5) \wedge 0.4) \wedge ((0.3) \wedge 0.4) \wedge ((0.2) \wedge 0.3) \wedge ((0.5) \wedge 0.3)\\ & + \iota((0.2) \wedge 0.3) \wedge ((0.5) \wedge 0.3) \wedge ((0.3) \wedge 0.2) \wedge ((0.3) \wedge 0.2)\\ & = 0.2 + \iota 0.2\end{aligned}$$

In the same way, we can obtain the other values:

$$\begin{aligned}\mathrm{M}_{\overline{R_e^*}}\left(\check{s}_2^*\right) = 0.7 + \iota 0.8, \mathrm{M}_{\overline{R_e^*}}\left(\check{s}_3^*\right) = 0.6 + \iota 0.7, \mathrm{M}_{\overline{R_e^*}}\left(\check{s}_4^*\right) = 0.7 + \iota 0.7,\\ \mathrm{N}_{\overline{R_e^*}}\left(\check{s}_2^*\right) = 0.2 + \iota 0.1, \mathrm{N}_{\overline{R_e^*}}\left(\check{s}_3^*\right) = 0.3 + \iota 0.2, \mathrm{N}_{\overline{R_e^*}}\left(\check{s}_4^*\right) = 0.2 + \iota 0.2\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{M}_{\underline{R_e^*}}\left(\check{s}_1^*\right) = 0.2 + \iota 0.4, \mathrm{M}_{\underline{R_e^*}}\left(\check{s}_2^*\right) = 0.2 + \iota 0.2, \mathrm{M}_{\underline{R_e^*}}\left(\check{s}_3^*\right) = 0.2 + \iota 0.3, \mathrm{M}_{\underline{R_e^*}}\left(\check{s}_4^*\right) = 0.2 + \iota 0.4,\\ \mathrm{N}_{\underline{R_e^*}}\left(\check{s}_1^*\right) = 0.5 + \iota 0.5, \mathrm{N}_{\underline{R_e^*}}\left(\check{s}_2^*\right) = 0.7 + \iota 0.6, \mathrm{N}_{\underline{R_e^*}}\left(\check{s}_3^*\right) = 0.6 + \iota 0.7, \mathrm{N}_{\underline{R_e^*}}\left(\check{s}_4^*\right) = 0.5 + \iota 0.6\end{aligned}$$

Then, the upper complex intuitionistic fuzzy approximation is

$$\overline{R_e^*}(A) = \left\{\begin{array}{c}\left(\check{s}^*{}_1, 0.6 + \iota 0.7, 0.2 + \iota 0.2\right), \left(\check{s}^*{}_2, 0.7 + \iota 0.8, 0.2 + \iota 0.1\right), \left(\check{s}^*{}_3, 0.6 + \iota 0.7, 0.3 + \iota 0.2\right),\\ \left(\check{s}^*{}_4, 0.7 + \iota 0.7, 0.3 + \iota 0.2\right)\end{array}\right\}$$

And the lower complex intuitionistic fuzzy approximation is

$$\underline{R_e^*}(A) = \left\{ \begin{array}{c} \left(\check{s}^*{}_1, 0.2+\iota 0.4, 0.5+\iota 0.5\right), \left(\check{s}^*{}_2, 0.2+\iota 0.2, 0.7+\iota 0.6\right), \left(\check{s}^*{}_3, 0.2+\iota 0.3, 0.6+\iota 0.7\right), \\ \left(\check{s}^*{}_4, 0.2+\iota 0.4, 0.5+\iota 0.6\right) \end{array} \right\}$$

Hence, it is seen that $\left(\overline{R_e^*}(A), \underline{R_e^*}(A)\right)$ is a CIFRS.

Basic Operations and Properties of Complex Intuitionistic Fuzzy Rough Set

Definition 10. *Let* $R_e^*(A) = \left(\overline{R_e^*}(A), \underline{R_e^*}(A)\right)$ *and* $R_e^*(B) = \left(\overline{R_e^*}(B), \underline{R_e^*}(B)\right)$ *be two CIFRNs; then,*

1. Complement

$$\begin{array}{c} R_e^*(A)^c = \left(\overline{R_e^*}(A)^c, \underline{R_e^*}(A)^c\right) \\ \overline{R_e^*}(A)^c = \left\{ \left(\check{s}^*, \mathrm{N}_{\overline{R_e^*}}\left(\check{s}^*\right), \mathrm{M}_{\overline{R_e^*}}\left(\check{s}^*\right)\right) \middle| \check{s}^* \in U \right\} \\ \underline{R_e^*}(A)^c = \left\{ \left(\check{s}^*, \mathrm{N}_{\underline{R_e^*}}\left(\check{s}^*\right), \mathrm{M}_{\underline{R_e^*}}\left(\check{s}^*\right)\right) \middle| \check{s}^* \in U \right\} \end{array} \tag{4}$$

2. Union

$$\begin{array}{c} R_e^*(A) \cup R_e^*(B) = (\overline{R_e^*}(A) \cup \overline{R_e^*}(B), \underline{R_e^*}(A) \cup \underline{R_e^*}(B)) \\ \overline{R_e^*}(A) \cup \overline{R_e^*}(B) = \{\check{s}^*, (\max[\varphi_{\overline{R_e^*}}(A), \varphi_{\overline{R_e^*}}(B)] + \iota\max[\mathfrak{b}_{\overline{R_e^*}}(A), \mathfrak{b}_{\overline{R_e^*}}(B)]\ ), (\min[\mathcal{L}_{\overline{R_e^*}}(A), \mathcal{L}_{\overline{R_e^*}}(B)] \\ +\iota\min[\wp_{\overline{R_e^*}}(A) + \wp_{\overline{R_e^*}}(B)]\ )\} \\ \underline{R_e^*}(A) \cup \underline{R_e^*}(B) = \{\check{s}^*, (\max[\varphi_{\underline{R_e^*}}(A), \varphi_{\underline{R_e^*}}(B)] + \iota\max[\mathfrak{b}_{\underline{R_e^*}}(A), \mathfrak{b}_{\underline{R_e^*}}(B)]\ ), (\min[\mathcal{L}_{\underline{R_e^*}}(A), \mathcal{L}_{\underline{R_e^*}}(B)] \\ +\iota\min[\wp_{\underline{R_e^*}}(A) + \wp_{\underline{R_e^*}}(B)]\ )\} \end{array} \tag{5}$$

3. Intersection:

$$\begin{array}{c} R_e^*(A) \cap R_e^*(B) = (\overline{R_e^*}(A) \cap \overline{R_e^*}(B),\ \underline{R_e^*}(A) \cap\ \underline{R_e^*}(B)) \\ \overline{R_e^*}(A) \cap \overline{R_e^*}(B) = \{\check{s}^*, (\min[\varphi_{\overline{R_e^*}}(A), \varphi_{\overline{R_e^*}}(B)] + \iota\min[\mathfrak{b}_{\overline{R_e^*}}(A), \mathfrak{b}_{\overline{R_e^*}}(B)]\ ), (\max[\mathcal{L}_{\overline{R_e^*}}(A), \mathcal{L}_{\overline{R_e^*}}(B)] \\ +\iota\max[\wp_{\overline{R_e^*}}(A) + \wp_{\overline{R_e^*}}(B)]\ )\} \\ \underline{R_e^*}(A) \cap \underline{R_e^*}(B) = \{\check{s}^*, (\min[\varphi_{\underline{R_e^*}}(A), \varphi_{\underline{R_e^*}}(B)] + \iota\min[\mathfrak{b}_{\underline{R_e^*}}(A), \mathfrak{b}_{\underline{R_e^*}}(B)]\ ), (\max[\mathcal{L}_{\underline{R_e^*}}(A), \mathcal{L}_{\underline{R_e^*}}(B)] \\ +\iota\max[\wp_{\underline{R_e^*}}(A), \wp_{\underline{R_e^*}}(B)]\ )\} \end{array} \tag{6}$$

Definition 11. *Let* $R_e^*(A) = \left(\overline{R_e^*}(A), \underline{R_e^*}(A)\right)$ *and* $R_e^*(B) = \left(\overline{R_e^*}(B), \underline{R_e^*}(B)\right)$ *be two CIFRNs; then,*

1. Algebraic sum:

$$\begin{array}{c} R_e^*(A) \oplus R_e^*(B) = (\overline{R_e^*}(A) \oplus \overline{R_e^*}(B), \underline{R_e^*}(A) \oplus \underline{R_e^*}(B)) \\ \overline{R_e^*}(A) \oplus \overline{R_e^*}(B) = \{(x, (\varphi_{\overline{R_e^*}}(A) + \varphi_{\overline{R_e^*}}(B) - \varphi_{\overline{R_e^*}}(A).\varphi_{\overline{R_e^*}}(B),\ \mathfrak{b}_{\overline{R_e^*}}(A) + \mathfrak{b}_{\overline{R_e^*}}(B) - \\ \mathfrak{b}_{\overline{R_e^*}}(A).\mathfrak{b}_{\overline{R_e^*}}(B)), (\mathcal{L}_{\overline{R_e^*}}(A).\mathcal{L}_{\overline{R_e^*}}(B), \wp_{\overline{R_e^*}}(A).\wp_{\overline{R_e^*}}(B)))\} \\ \underline{R_e^*}(A) \oplus\ \underline{R_e^*}(B) = \{(x, (\varphi_{\underline{R_e^*}}(A) + \varphi_{\underline{R_e^*}}(B) - \varphi_{\underline{R_e^*}}(A).\varphi_{\underline{R_e^*}}(B),\ \mathfrak{b}_{\underline{R_e^*}}(A) + \mathfrak{b}_{\underline{R_e^*}}(B) \\ -\ \mathfrak{b}_{\underline{R_e^*}}(A).\mathfrak{b}_{\underline{R_e^*}}(B)), (\mathcal{L}_{\underline{R_e^*}}(A).\mathcal{L}_{\underline{R_e^*}}(B), \wp_{\underline{R_e^*}}(A).\wp_{\underline{R_e^*}}(B)))\} \end{array} \tag{7}$$

2. Algebraic product:

$$\begin{array}{c} R_e^*(A) \otimes R_e^*(B) = (\overline{R_e^*}(A) \otimes \overline{R_e^*}(B),\ \underline{R_e^*}(A) \otimes\ \underline{R_e^*}(B)) \\ \overline{R_e^*}(A) \otimes \overline{R_e^*}(B) = \{(x, (\varphi_{\overline{R_e^*}}(A).\varphi_{\overline{R_e^*}}(B),\ \mathfrak{b}_{\overline{R_e^*}}(A).\mathfrak{b}_{\overline{R_e^*}}(B)), (\mathcal{L}_{\overline{R_e^*}}(A) + \mathcal{L}_{\overline{R_e^*}}(B) - \mathcal{L}_{\overline{R_e^*}}(A).\mathcal{L}_{\overline{R_e^*}}(B), \wp_{\overline{R_e^*}}(A) \\ +\wp_{\overline{R_e^*}}(B) - \wp_{\overline{R_e^*}}(A).\wp_{\overline{R_e^*}}(B)))\} \\ \underline{R_e^*}(A) \otimes\ \underline{R_e^*}(B) = \{(x, (\varphi_{\underline{R_e^*}}(A).\varphi_{\underline{R_e^*}}(B),\ \mathfrak{b}_{\underline{R_e^*}}(A).\mathfrak{b}_{\underline{R_e^*}}(B)), (\mathcal{L}_{\underline{R_e^*}}(A) + \mathcal{L}_{\underline{R_e^*}}(B) - \mathcal{L}_{\underline{R_e^*}}(A).\mathcal{L}_{\underline{R_e^*}}(B), \wp_{\underline{R_e^*}}(A) + \\ \wp_{\underline{R_e^*}}(B) - \wp_{\underline{R_e^*}}(A).\wp_{\underline{R_e^*}}(B)))\} \end{array} \tag{8}$$

Next, we defined some other operations on CIFRNs.

Definition 12. *Let $R_e^*(A) = \left(\overline{R_e^*}(A),\ \underline{R_e^*}(A)\right)$ for a CIFRS and $\lambda > 0$; then,*

1.

$$\lambda R_e^*(A) = \left(\lambda \overline{R_e^*}(A), \lambda \underline{R_e^*}(A)\right)$$

$$\begin{aligned}\lambda\overline{R_e^*}(A) &= \left\{\left(x, \left(1-\left(1-\varphi_{\overline{R_e^*}}(A)\right)^{\lambda}, 1-\left(1-\mathfrak{b}_{\overline{R_e^*}}(A)\right)^{\lambda}\right), \left(\left(\mathcal{L}_{\overline{R_e^*}}(A)\right)^{\lambda}, \left(\wp_{\overline{R_e^*}}(A)\right)^{\lambda}\right)\right)\right\} \\ \lambda\underline{R_e^*}(A) &= \left\{\left(x, \left(1-\left(1-\varphi_{\underline{R_e^*}}(A)\right)^{\lambda}, 1-\left(1-\mathfrak{b}_{\underline{R_e^*}}(A)\right)^{\lambda}\right), \left(\left(\mathcal{L}_{\underline{R_e^*}}(A)\right)^{\lambda}, \left(\wp_{\underline{R_e^*}}(A)\right)^{\lambda}\right)\right)\right\}\end{aligned} \tag{9}$$

2.

$$\left(R_e^*(A)\right)^{\lambda} = \left(\left(\overline{R_e^*}(A)\right)^{\lambda}, \left(\underline{R_e^*}(A)\right)^{\lambda}\right)$$

$$\begin{aligned}\left(\overline{R_e^*}(A)\right)^{\lambda} &= \left\{\left(x, \left(\left(\varphi_{\overline{R_e^*}}(A)\right)^{\lambda}, \left(\mathfrak{b}_{\overline{R_e^*}}(A)\right)^{\lambda}\right), \left(1-\left(1-\mathcal{L}_{\overline{R_e^*}}(A)\right)^{\lambda}, 1-\left(1-\wp_{\overline{R_e^*}}(A)\right)^{\lambda}\right)\right)\right\} \\ \left(\underline{R_e^*}(A)\right)^{\lambda} &= \left\{\left(x, \left(\left(\varphi_{\underline{R_e^*}}(A)\right)^{\lambda}, \left(\mathfrak{b}_{\underline{R_e^*}}(A)\right)^{\lambda}\right), \left(1-\left(1-\mathcal{L}_{\underline{R_e^*}}(A)\right)^{\lambda}, 1-\left(1-\wp_{\underline{R_e^*}}(A)\right)^{\lambda}\right)\right)\right\}\end{aligned} \tag{10}$$

Definition 13. *The score function for the CIFRN $R_e^*(A) = \left(\overline{R_e^*}(A),\ \underline{R_e^*}(A)\right) = \left(\left(\varphi_{\overline{R_e^*}} + \iota\mathfrak{b}_{\overline{R_e^*}}, \mathcal{L}_{\overline{R_e^*}} + \iota\wp_{\overline{R_e^*}}\right), \left(\varphi_{\underline{R_e^*}} + \iota\mathfrak{b}_{\underline{R_e^*}}, \mathcal{L}_{\underline{R_e^*}} + \iota\wp_{\underline{R_e^*}}\right)\right)$ is given as*

$$S_F(R_e^*(A)) = \frac{1}{8}\left(4 + \varphi_{\overline{R_e^*}} + \mathfrak{b}_{\overline{R_e^*}} + \varphi_{\underline{R_e^*}} + \mathfrak{b}_{\underline{R_e^*}} - \mathcal{L}_{\overline{R_e^*}} - \wp_{\overline{R_e^*}} - \mathcal{L}_{\underline{R_e^*}} - \wp_{\underline{R_e^*}}\right),\ S_F(R_e^*(A)) \in [0, 1]. \tag{11}$$

The accuracy function for the CIFRN is given as

$$A_F(R_e^*(A)) = \frac{1}{8}\left(\varphi_{\overline{R_e^*}} + \mathfrak{b}_{\overline{R_e^*}} + \varphi_{\underline{R_e^*}} + \mathfrak{b}_{\underline{R_e^*}} + \mathcal{L}_{\overline{R_e^*}} + \wp_{\overline{R_e^*}} + \mathcal{L}_{\underline{R_e^*}} + \wp_{\underline{R_e^*}}\right),\ A_F(R_e^*(A)) \in [0, 1]. \tag{12}$$

4. Aggregation Operators Based on CIFRSs

In this section, we develop several new AOs based on CIFRSs and discuss their properties using basic operations.

Definition 14. *Let $R_e^*(A_t) = \left(\overline{R_e^*}(A_t),\ \underline{R_e^*}(A_t)\right)(t = 1, 2, 3 \ldots, \mathfrak{s})$ be a collection of CIFRNs and $\omega = \left(\omega_1,\ \omega_2, \omega_3 \ldots, \omega_{\mathfrak{s}}\right)^T$ be the weight vector (WV) with $\omega_{\mathfrak{t}} \in [0,\ 1]$ such that $\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} = 1$; then, a CIFRWA operator is defined as*

$$\begin{aligned}&\text{CIFRWA}(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})) = \left(\oplus_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \overline{R_e^*}(A_t), \oplus_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}}\ \underline{R_e^*}(A_t)\right) \\ &= \left(\left(\omega_1 \overline{R_e^*}(A_1) \oplus \omega_2 \overline{R_e^*}(A_2) \oplus \ldots . \oplus \omega_{\mathfrak{s}} \overline{R_e^*}(A_{\mathfrak{s}})\right), \left(\omega_1 \underline{R_e^*}(A_1) \oplus \omega_2 \underline{R_e^*}(A_2) \oplus \ldots . \oplus \omega_{\mathfrak{s}} \underline{R_e^*}(A_{\mathfrak{s}})\right)\right)\end{aligned} \tag{13}$$

Based on the above definition, the results for the CIFRWA operator are as follows.

Theorem 1. *By employing the above equation, we obtain the CIFRNs and*

$$\begin{aligned}&CIFRWA(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})) = \left(\oplus_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \overline{R_e^*}(A_t), \oplus_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}}\ \underline{R_e^*}(A_t)\right) \\ &= \left(\begin{array}{c}\left(\begin{array}{c}\left(1-\prod_{\mathfrak{t}=1}^{\mathfrak{s}}\left(1-\varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(1-\prod_{\mathfrak{t}=1}^{\mathfrak{s}}\left(1-\mathfrak{b}_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}}\left(\mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}}\left(\wp_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right)\end{array}\right), \\ \left(\begin{array}{c}\left(1-\prod_{\mathfrak{t}=1}^{\mathfrak{s}}\left(1-\varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(1-\prod_{\mathfrak{t}=1}^{\mathfrak{s}}\left(1-\mathfrak{b}_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}}\left(\mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}}\left(\wp_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right)\end{array}\right)\end{array}\right)\end{aligned} \tag{14}$$

Proof. We proved the above equation by using a well-known method of mathematical induction (MI), assuming that, for $\mathfrak{s} = 2$, we have

$$R_e^*(A_1) = \left(\omega_1 \overline{R_e^*}(A_1), \omega_1 \underline{R_e^*}(A_1)\right) = \left(\begin{array}{c} \left(\begin{array}{c} \left(1-\left(1-\varphi_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\right) + \mathrm{i}\left(1-\left(1-\text{ƀ}_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\right), \\ \left(\left(\mathcal{L}_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\right) + \mathrm{i}\left(\left(\wp_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\right) \end{array}\right), \\ \left(\begin{array}{c} \left(1-\left(1-\varphi_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\right) + \mathrm{i}\left(1-\left(1-\text{ƀ}_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\right), \\ \left(\left(\mathcal{L}_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\right) + \mathrm{i}\left(\left(\wp_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\right) \end{array}\right) \end{array}\right)$$

$$R_e^*(A_2) = \left(\omega_2 \overline{R_e^*}(A_2), \omega_2 \underline{R_e^*}(A_2)\right) = \left(\begin{array}{c} \left(\begin{array}{c} \left(1-\left(1-\varphi_{\overline{R_e^*}}(A_2)\right)^{\omega_2}\right) + \mathrm{i}\left(1-\left(1-\text{ƀ}_{\overline{R_e^*}}(A_2)\right)^{\omega_2}\right), \\ \left(\left(\mathcal{L}_{\overline{R_e^*}}(A_2)\right)^{\omega_2}\right) + \mathrm{i}\left(\left(\wp_{\overline{R_e^*}}(A_2)\right)^{\omega_2}\right) \end{array}\right), \\ \left(\begin{array}{c} \left(1-\left(1-\varphi_{\underline{R_e^*}}(A_2)\right)^{\omega_2}\right) + \mathrm{i}\left(1-\left(1-\text{ƀ}_{\underline{R_e^*}}(A_2)\right)^{\omega_2}\right), \\ \left(\left(\mathcal{L}_{\underline{R_e^*}}(A_2)\right)^{\omega_2}\right) + \mathrm{i}\left(\left(\wp_{\underline{R_e^*}}(A_2)\right)^{\omega_2}\right) \end{array}\right) \end{array}\right)$$

$$\begin{aligned} & CIFRWA(R_e^*(A_1), R_e^*(A_2)) \\ & = \left(\omega_1 \overline{R_e^*}(A_1) \oplus \omega_2 \overline{R_e^*}(A_2), \omega_1 \underline{R_e^*}(A_1) \oplus \omega_2 \underline{R_e^*}(A_2)\right) \\ & = \left(\begin{array}{c} \left(\begin{array}{c} \left(1-\left(1-\varphi_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\right) + \mathrm{i}\left(1-\left(1-\text{ƀ}_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\right), \\ \left(\left(\mathcal{L}_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\right) + \mathrm{i}\left(\left(\wp_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\right) \end{array}\right), \\ \left(\begin{array}{c} \left(1-\left(1-\varphi_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\right) + \mathrm{i}\left(1-\left(1-\text{ƀ}_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\right), \\ \left(\left(\mathcal{L}_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\right) + \mathrm{i}\left(\left(\wp_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\right) \end{array}\right) \end{array}\right) \\ & \oplus \left(\begin{array}{c} \left(\begin{array}{c} \left(1-\left(1-\varphi_{\overline{R_e^*}}(A_2)\right)^{\omega_2}\right) + \mathrm{i}\left(1-\left(1-\text{ƀ}_{\overline{R_e^*}}(A_2)\right)^{\omega_2}\right), \\ \left(\left(\mathcal{L}_{\overline{R_e^*}}(A_2)\right)^{\omega_2}\right) + \mathrm{i}\left(\left(\wp_{\overline{R_e^*}}(A_2)\right)^{\omega_2}\right) \end{array}\right), \\ \left(\begin{array}{c} \left(1-\left(1-\varphi_{\underline{R_e^*}}(A_2)\right)^{\omega_2}\right) + \mathrm{i}\left(1-\left(1-\text{ƀ}_{\underline{R_e^*}}(A_2)\right)^{\omega_2}\right), \\ \left(\left(\mathcal{L}_{\underline{R_e^*}}(A_2)\right)^{\omega_2}\right) + \mathrm{i}\left(\left(\wp_{\underline{R_e^*}}(A_2)\right)^{\omega_2}\right) \end{array}\right) \end{array}\right) \end{aligned}$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} 1-\left(1-\varphi_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\left(1-\varphi_{\overline{R_e^*}}(A_2)\right)^{\omega_2} + \mathrm{i}\left(1-\left(1-\text{ƀ}_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\left(1-\text{ƀ}_{\overline{R_e^*}}(A_2)\right)^{\omega_2}\right), \\ \left(\left(\mathcal{L}_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\left(\mathcal{L}_{\overline{R_e^*}}(A_2)\right)^{\omega_2}\right) + \mathrm{i}\left(\left(\wp_{\overline{R_e^*}}(A_1)\right)^{\omega_1}\left(\wp_{\overline{R_e^*}}(A_2)\right)^{\omega_2}\right) \end{array} \right), \\ \left(\begin{array}{c} 1-\left(1-\varphi_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\left(1-\varphi_{\underline{R_e^*}}(A_2)\right)^{\omega_2} + \mathrm{i}\left(1-\left(1-\text{ƀ}_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\left(1-\text{ƀ}_{\underline{R_e^*}}(A_2)\right)^{\omega_2}\right), \\ \left(\left(\mathcal{L}_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\left(\mathcal{L}_{\underline{R_e^*}}(A_2)\right)^{\omega_2}\right) + \mathrm{i}\left(\left(\wp_{\underline{R_e^*}}(A_1)\right)^{\omega_1}\left(\wp_{\underline{R_e^*}}(A_2)\right)^{\omega_2}\right) \end{array} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(1-\prod_{t=1}^{2}\left(1-\varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(1-\prod_{t=1}^{2}\left(1-\text{ƀ}_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right), \\ \left(\prod_{t=1}^{2}\left(\mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(\prod_{t=1}^{2}\left(\wp_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right) \end{array} \right), \\ \left(\begin{array}{c} \left(1-\prod_{t=1}^{2}\left(1-\varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(1-\prod_{t=1}^{2}\left(1-\text{ƀ}_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right), \\ \left(\prod_{t=1}^{2}\left(\mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(\prod_{t=1}^{2}\left(\wp_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right) \end{array} \right) \end{array} \right)$$

Next, suppose that it is true for $\mathfrak{s} = K$.

$$\text{CIFRWA}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*\left(A_K\right)\right) = \left(\oplus_{t=1}^{K}\omega_t\overline{R_e^*}(A_t), \oplus_{t=1}^{K}\omega_t\underline{R_e^*}(A_t)\right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(1-\prod_{t=1}^{K}\left(1-\varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(1-\prod_{t=1}^{K}\left(1-\text{ƀ}_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right), \\ \left(\prod_{t=1}^{K}\left(\mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(\prod_{t=1}^{K}\left(\wp_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right) \end{array} \right), \\ \left(\begin{array}{c} \left(1-\prod_{t=1}^{K}\left(1-\varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(1-\prod_{t=1}^{K}\left(1-\text{ƀ}_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right), \\ \left(\prod_{t=1}^{K}\left(\mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(\prod_{t=1}^{K}\left(\wp_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right) \end{array} \right) \end{array} \right)$$

Now, we have to show that it is true for $\mathfrak{s} = K + 1$; we have

$$\text{CIFRWA}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*\left(A_K\right), R_e^*\left(A_{K+1}\right)\right) = \left(\oplus_{t=1}^{K+1}\omega_t\overline{R_e^*}(A_t), \oplus_{t=1}^{K+1}\omega_t\underline{R_e^*}(A_t)\right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(1-\prod_{t=1}^{K}\left(1-\varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(1-\prod_{t=1}^{K}\left(1-\text{ƀ}_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right), \\ \left(\prod_{t=1}^{K}\left(\mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(\prod_{t=1}^{K}\left(\wp_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right) \end{array} \right), \\ \left(\begin{array}{c} \left(1-\prod_{t=1}^{K}\left(1-\varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(1-\prod_{t=1}^{K}\left(1-\text{ƀ}_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right), \\ \left(\prod_{t=1}^{K}\left(\mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right) + \mathrm{i}\left(\prod_{t=1}^{K}\left(\wp_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right) \end{array} \right) \end{array} \right)$$

$$
\oplus\begin{pmatrix}
\begin{pmatrix}
\left(1-\left(1-\varphi_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)+\mathrm{i}\left(1-\left(1-\text{ƀ}_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right), \\
\left(\left(\mathcal{L}_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)+\mathrm{i}\left(\left(\wp_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)
\end{pmatrix}, \\
\begin{pmatrix}
\left(1-\left(1-\varphi_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)+\mathrm{i}\left(1-\left(1-\text{ƀ}_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right), \\
\left(\left(\mathcal{L}_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)+\mathrm{i}\left(\left(\wp_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)
\end{pmatrix}
\end{pmatrix}
$$

$$
=\begin{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\left(1-\prod_{t=1}^{K}\left(1-\varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right)+\left(1-\left(1-\varphi_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right) \\
-\left(1-\prod_{t=1}^{K}\left(1-\varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right)\left(1-\left(1-\varphi_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)
\end{pmatrix} \\
+\iota\begin{pmatrix}
\left(1-\prod_{t=1}^{K}\left(1-\text{ƀ}_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right)+\left(1-\left(1-\text{ƀ}_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right) \\
-\left(1-\prod_{t=1}^{K}\left(1-\text{ƀ}_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right)\left(1-\left(1-\text{ƀ}_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)
\end{pmatrix}
\end{pmatrix}, \\
\left(\left(\prod_{t=1}^{K}\left(\mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\left(\mathcal{L}_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)+\mathrm{i}\left(\prod_{t=1}^{K}\left(\wp_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\left(\wp_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)\right)
\end{pmatrix}, \\
\begin{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\left(1-\prod_{t=1}^{K}\left(1-\varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right)+\left(1-\left(1-\varphi_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right) \\
-\left(1-\prod_{t=1}^{K}\left(1-\varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right)\left(1-\left(1-\varphi_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)
\end{pmatrix} \\
+\mathrm{i}\begin{pmatrix}
\left(1-\prod_{t=1}^{K}\left(1-\text{ƀ}_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right)+\left(1-\left(1-\text{ƀ}_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right) \\
-\left(1-\prod_{t=1}^{K}\left(1-\text{ƀ}_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right)\left(1-\left(1-\text{ƀ}_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)
\end{pmatrix}
\end{pmatrix}, \\
\left(\left(\prod_{t=1}^{K}\left(\mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\left(\mathcal{L}_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)+\mathrm{i}\left(\prod_{t=1}^{K}\left(\wp_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\left(\wp_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)\right)
\end{pmatrix}
\end{pmatrix}
$$

$$
=\begin{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\left(1-\prod_{t=1}^{K}\left(1-\varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\left(1-\varphi_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right) \\
+\mathrm{i}\left(1-\prod_{t=1}^{K}\left(1-\text{ƀ}_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\left(1-\text{ƀ}_{\overline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)
\end{pmatrix}, \\
\left(\left(\prod_{t=1}^{K+1}\left(\mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right)+\mathrm{i}\left(\prod_{t=1}^{K+1}\left(\wp_{\overline{R_e^*}}(A_t)\right)^{\omega_t}\right)\right)
\end{pmatrix}, \\
\begin{pmatrix}
\begin{pmatrix}
\left(1-\prod_{t=1}^{K}\left(1-\varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\left(1-\varphi_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right) \\
+\mathrm{i}\left(1-\prod_{t=1}^{K}\left(1-\text{ƀ}_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\left(1-\text{ƀ}_{\underline{R_e^*}}\left(A_{K+1}\right)\right)^{\omega_{K+1}}\right)
\end{pmatrix}, \\
\left(\left(\prod_{t=1}^{K+1}\left(\mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right)+\mathrm{i}\left(\prod_{t=1}^{K+1}\left(\wp_{\underline{R_e^*}}(A_t)\right)^{\omega_t}\right)\right)
\end{pmatrix}
\end{pmatrix}
$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{K}+1} \left(1 - \varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{K}+1} \left(1 - \mathfrak{b}_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(\prod_{\mathfrak{t}=1}^{\mathfrak{K}+1} \left(\mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(\prod_{\mathfrak{t}=1}^{\mathfrak{K}+1} \left(\wp_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) \end{array} \right), \\ \left(\begin{array}{c} \left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{K}+1} \left(1 - \varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{K}+1} \left(1 - \mathfrak{b}_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(\prod_{\mathfrak{t}=1}^{\mathfrak{K}+1} \left(\mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(\prod_{\mathfrak{t}=1}^{\mathfrak{K}+1} \left(\wp_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) \end{array} \right) \end{array} \right)$$

This shows that it holds true for $\mathfrak{s} = \mathfrak{K} + 1$. Hence, it will hold true for all $\mathfrak{s} \geq 0$. □

From the above theorem, $\overline{R_e^*}(A_t)$ and $\underline{R_e^*}(A_t)$ are CIFRNs. So, by Definitions 4 and 5, $\oplus_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \overline{R_e^*}(A_t)$ and $\oplus_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \underline{R_e^*}(A_t)$ are also CIFRNs. Therefore, the CIFRWA is also a CIFRN.

Theorem 2 *(idempotency property). Let* $R_e^*(A_t) = \left(\overline{R_e^*}(A_t), \underline{R_e^*}(A_t)\right)(t = 1, 2, 3 \ldots, \mathfrak{s})$ *be a collection of CIFRNs with WV* $\omega = \left(\omega_1, \omega_2, \ldots, \omega_{\mathfrak{s}}\right)^T$, $\omega_{\mathfrak{t}} \in [0, 1]$, *and* $\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} = 1$. *If* $R_e^*(A_t) = R_e^*(A)\ \forall\ (t = 1, 2, 3 \ldots, \mathfrak{s})$, *where* $R_e^*(A) = \left(\overline{R_e^*}(A), \underline{R_e^*}(A)\right)$, *then*

$$\mathrm{CIFRWA}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) = R_e^*(A) \tag{15}$$

Proof. Let us assume that $R_e^*(A_t) = R_e^*(A) \forall (t = 1, 2, 3 \ldots, \mathfrak{s})$; then,

$$\mathrm{CIFRWA}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) = \left(\oplus_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \overline{R_e^*}(A_t), \oplus_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \underline{R_e^*}(A_t)\right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \mathfrak{b}_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\wp_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) \end{array} \right), \\ \left(\begin{array}{c} \left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \mathfrak{b}_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\wp_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) \end{array} \right) \end{array} \right)$$

For all t, $R_e^*(A_t) = R_e^*(A)$. Therefore,

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(1 - \left(1 - \varphi_{\overline{R_e^*}}(A)\right)^{\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(1 - \left(1 - \text{ƀ}_{\overline{R_e^*}}(A)\right)^{\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}}}\right), \\ \left(\left(\mathcal{L}_{\overline{R_e^*}}(A)\right)^{\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(\left(\wp_{\overline{R_e^*}}(A)\right)^{\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}}}\right) \end{array} \right), \\ \left(\begin{array}{c} \left(1 - \left(1 - \varphi_{\underline{R_e^*}}(A)\right)^{\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(1 - \left(1 - \text{ƀ}_{\underline{R_e^*}}(A)\right)^{\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}}}\right), \\ \left(\left(\mathcal{L}_{\underline{R_e^*}}(A)\right)^{\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(\left(\wp_{\underline{R_e^*}}(A)\right)^{\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}}}\right) \end{array} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(1 - \left(1 - \varphi_{\overline{R_e^*}}(A)\right)\right) + \mathrm{i}\left(1 - \left(1 - \text{ƀ}_{\overline{R_e^*}}(A)\right)\right), \\ \left(\mathcal{L}_{\overline{R_e^*}}(A)\right) + \mathrm{i}\left(\wp_{\overline{R_e^*}}(A)\right) \end{array} \right), \\ \left(\begin{array}{c} \left(1 - \left(1 - \varphi_{\underline{R_e^*}}(A)\right)\right) + \mathrm{i}\left(1 - \left(1 - \text{ƀ}_{\underline{R_e^*}}(A)\right)\right), \\ \left(\mathcal{L}_{\underline{R_e^*}}(A)\right) + \mathrm{i}\left(\wp_{\underline{R_e^*}}(A)\right) \end{array} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\varphi_{\overline{R_e^*}}(A) + \mathrm{i}\text{ƀ}_{\overline{R_e^*}}(A), \mathcal{L}_{\overline{R_e^*}}(A) + \mathrm{i}\wp_{\overline{R_e^*}}(A)\right), \\ \left(\varphi_{\underline{R_e^*}}(A) + \mathrm{i}\text{ƀ}_{\underline{R_e^*}}(A), \mathcal{L}_{\underline{R_e^*}}(A) + \mathrm{i}\wp_{\underline{R_e^*}}(A)\right) \end{array} \right) = R_e^*(A)$$

□

Theorem 3 *(monotonicity property)***.** *Let*

$$R_e^*(A_t) = \left(\overline{R_e^*}(A_t), \underline{R_e^*}(A_t)\right) = \left(\begin{array}{c} \left(\varphi_{\overline{R_e^*}}(A_t) + \mathrm{i}\text{ƀ}_{\overline{R_e^*}}(A_t), \mathcal{L}_{\overline{R_e^*}}(A_t) + \mathrm{i}\wp_{\overline{R_e^*}}(A_t)\right), \\ \left(\varphi_{\underline{R_e^*}}(A_t) + \mathrm{i}\text{ƀ}_{\underline{R_e^*}}(A_t), \mathcal{L}_{\underline{R_e^*}}(A_t) + \mathrm{i}\wp_{\underline{R_e^*}}(A_t)\right) \end{array} \right) \text{ and}$$

$$R_e^*(B_t) = \left(\overline{R_e^*}(B_t), \underline{R_e^*}(B_t)\right) = \left(\begin{array}{c} \left(\varphi_{\overline{R_e^*}}(B_t) + \mathrm{i}\text{ƀ}_{\overline{R_e^*}}(B_t), \mathcal{L}_{\overline{R_e^*}}(B_t) + \mathrm{i}\wp_{\overline{R_e^*}}(B_t)\right), \\ \left(\varphi_{\underline{R_e^*}}(B_t) + \mathrm{i}\text{ƀ}_{\underline{R_e^*}}(B_t), \mathcal{L}_{\underline{R_e^*}}(B_t) + \mathrm{i}\wp_{\underline{R_e^*}}(B_t)\right) \end{array} \right) (\mathfrak{t} = 1, 2, \ldots, \mathfrak{s})$$

be a collection of two CIFRSs, and $\omega = \left(\omega_1, \omega_2, \ldots, \omega_{\mathfrak{s}}\right)^T$ *be the WV with* $\omega_{\mathfrak{t}} \in [0,1]$ *and* $\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} = 1$*. If* $\overline{R_e^*}(A_t) \leq \overline{R_e^*}(B_t)$, $\underline{R_e^*}(A_t) \leq \underline{R_e^*}(B_t)$*, then*

$$\text{CIFRWA}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) \leq \text{CIFRWA}\left(R_e^*(B_1), R_e^*(B_2), \ldots, R_e^*(B_{\mathfrak{s}})\right) \quad (16)$$

Proof. Assume $\varphi_{\overline{R_e^*}}(A_t) \leq \varphi_{\overline{R_e^*}}(B_t)$, $\text{ƀ}_{\overline{R_e^*}}(A_t) \leq \text{ƀ}_{\overline{R_e^*}}(B_t)$, $\mathcal{L}_{\overline{R_e^*}}(A_t) \geq \mathcal{L}_{\overline{R_e^*}}(B_t)$, $\wp_{\overline{R_e^*}}(A_t) \geq \wp_{\overline{R_e^*}}(A_t)$ and $\varphi_{\underline{R_e^*}}(A_t) \leq \varphi_{\underline{R_e^*}}(B_t)$, $\text{ƀ}_{\underline{R_e^*}}(A_t) \leq \text{ƀ}_{\underline{R_e^*}}(B_t)$, $\mathcal{L}_{\underline{R_e^*}}(A_t) \geq \mathcal{L}_{\underline{R_e^*}}(B_t)$, $\wp_{\underline{R_e^*}}(A_t) \geq \wp_{\underline{R_e^*}}(A_t)$ for all $(\mathfrak{t} = 1, 2, \ldots, \mathfrak{s})$; then,

$$1 - \text{ƀ}_{\overline{R_e^*}}(A_t) \geq 1 - \text{ƀ}_{\overline{R_e^*}}(B_t)$$

Then,

$$\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}} \geq \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \varphi_{\overline{R_e^*}}(B_t)\right)^{\omega_{\mathfrak{t}}}$$

$$1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}} \leq 1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \varphi_{\overline{R_e^*}}(B_t)\right)^{\omega_{\mathfrak{t}}}$$

Similarly,

$$1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \text{ƀ}_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}} \leq 1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \text{ƀ}_{\overline{R_e^*}}(B_t)\right)^{\omega_{\mathfrak{t}}}$$

Next,

$$\mathcal{L}_{\overline{R_e^*}}(A_t) \geq \mathcal{L}_{\overline{R_e^*}}(B_t)$$
$$\prod_{t=1}^{\mathfrak{s}}\left(\mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{\omega_t} \geq \prod_{t=1}^{\mathfrak{s}}\left(\mathcal{L}_{\overline{R_e^*}}(B_t)\right)^{\omega_t} \text{ and}$$
$$\prod_{t=1}^{\mathfrak{s}}\left(\wp_{\overline{R_e^*}}(A_t)\right)^{\omega_t} \geq \prod_{t=1}^{\mathfrak{s}}\left(\wp_{\overline{R_e^*}}(B_t)\right)^{\omega_t}$$

Similarly,

$$1-\prod_{t=1}^{\mathfrak{s}}\left(1-\varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_t} \leq 1-\prod_{t=1}^{\mathfrak{s}}\left(1-\varphi_{\underline{R_e^*}}(B_t)\right)^{\omega_t},$$
$$1-\prod_{t=1}^{\mathfrak{s}}\left(1-\mathfrak{b}_{\underline{R_e^*}}(A_t)\right)^{\omega_t} \leq 1-\prod_{t=1}^{\mathfrak{s}}\left(1-\mathfrak{b}_{\underline{R_e^*}}(B_t)\right)^{\omega_t},$$
$$\prod_{t=1}^{\mathfrak{s}}\left(\mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{\omega_t} \geq \prod_{t=1}^{\mathfrak{s}}\left(\mathcal{L}_{\underline{R_e^*}}(B_t)\right)^{\omega_t} \text{and}$$
$$\prod_{t=1}^{\mathfrak{s}}\left(\wp_{\underline{R_e^*}}(A_t)\right)^{\omega_t} \geq \prod_{t=1}^{\mathfrak{s}}\left(\wp_{\underline{R_e^*}}(B_t)\right)^{\omega_t}$$

Hence,

$$\text{CIFRWA}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) \leq \text{CIFRWA}\left(R_e^*(B_1), R_e^*(B_2), \ldots, R_e^*(B_{\mathfrak{s}})\right).$$

□

Theorem 4 *(boundedness property). Let* $R_e^*(A_t) = \left((R_e^*(A_t))^+, (R_e^*(A_t))^-\right)(t = 1, 2, 3 \ldots, \mathfrak{s})$ *be a collection of CIFRNs where* $(R_e^*(A_t))^+ = \left(\min_t \overline{R_e^*}(A_t), \max_t \underline{R_e^*}(A_t)\right)$ *and* $(R_e^*(A_t))^- = \left(\max_t \overline{R_e^*}(A_t), \min_t \underline{R_e^*}(A_t)\right)$*; then,*

$$\left(R_e^*(A_t)\right)^- \leq \text{CIFRWA}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) \leq \left(R_e^*(A_t)\right)^+ \tag{17}$$

Proof. As $\left(\overline{R_e^*}(A_t)\right)^+ = \begin{bmatrix} \left(\left(\max_t \varphi_{\overline{R_e^*}}(A_t) + \imath \max_t \mathfrak{b}_{\overline{R_e^*}}(A_t)\right), \left(\min_t \mathcal{L}_{\overline{R_e^*}}(A_t) + \imath \min_t \wp_{\overline{R_e^*}}(A_t)\right)\right), \\ \left(\left(\max_t \varphi_{\underline{R_e^*}}(A_t) + \imath \max_t \mathfrak{b}_{\underline{R_e^*}}(A_t)\right), \left(\min_t \mathcal{L}_{\underline{R_e^*}}(A_t) + \imath \min_t \wp_{\underline{R_e^*}}(A_t)\right)\right) \end{bmatrix}$ and $\left(\overline{R_e^*}(A_t)\right)^- = \begin{bmatrix} \left(\left(\min_t \varphi_{\overline{R_e^*}}(A_t) + \imath \min_t \mathfrak{b}_{\overline{R_e^*}}(A_t)\right), \left(\max_t \mathcal{L}_{\overline{R_e^*}}(A_t) + \imath \max_t \wp_{\overline{R_e^*}}(A_t)\right)\right), \\ \left(\left(\min_t \varphi_{\underline{R_e^*}}(A_t) + \imath \min_t \mathfrak{b}_{\underline{R_e^*}}(A_t)\right), \left(\max_t \mathcal{L}_{\underline{R_e^*}}(A_t) + \imath \max_t \wp_{\underline{R_e^*}}(A_t)\right)\right) \end{bmatrix}$

Since, for all $t = 1, 2, 3 \ldots \mathfrak{s}$, we have

$$\min_t\{\varphi_{\underline{R_e^*}}(A_t)\} \leq \varphi_{\underline{R_e^*}}(A_t) \leq \max_t\{\varphi_{\underline{R_e^*}}(A_t)\}$$
$$(1-\min_t\{\varphi_{\underline{R_e^*}}(A_t)\}) \geq 1-\varphi_{\underline{R_e^*}}(A_t) \geq (1-\max_t\{\varphi_{\underline{R_e^*}}(A_t)\})$$
$$\prod_{t=1}^{\mathfrak{s}}(1-\min_t\{\varphi_{\underline{R_e^*}}(A_t)\})^{\omega_t} \geq \prod_{t=1}^{\mathfrak{s}}(1-\varphi_{\underline{R_e^*}}(A_t))^{\omega_t} \geq \prod_{t=1}^{\mathfrak{s}}(1-\max_t\{\varphi_{\underline{R_e^*}}(A_t)\})^{\omega_t}$$
$$(1-\min_t\{\varphi_{\underline{R_e^*}}(A_t)\}) \geq (1-\{\varphi_{\underline{R_e^*}}(A_t)\})^{\omega_t} \geq (1-\max_t\{\varphi_{\underline{R_e^*}}(A_t)\})$$
$$1-(1-\min_t\{\varphi_{\underline{R_e^*}}(A_t)\}) \leq 1-\prod_{t=1}^{\mathfrak{s}}(1-\varphi_{\underline{R_e^*}}(A_t))^{\omega_t} \leq 1-(1-\max_t\{\varphi_{\underline{R_e^*}}(A_t)\})$$

Hence,

$$\min_t\left\{\varphi_{\underline{R_e^*}}(A_t)\right\} \leq 1-\prod_{t=1}^{\mathfrak{s}}\left(1-\varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_t} \leq \max_t\left\{\varphi_{\underline{R_e^*}}(A_t)\right\} \tag{18}$$

Similarly,

$$\min_{\mathfrak{t}}\left\{ \mathfrak{b}_{\underline{R_e^*}}(A_t) \right\} \leq 1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \mathfrak{b}_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}} \leq \max_{\mathfrak{t}}\left\{ \mathfrak{b}_{\underline{R_e^*}}(A_t) \right\} \tag{19}$$

Next, for every $t = 1, 2, 3 \ldots \mathfrak{s}$, we have

$$\min_{\mathfrak{t}}\left\{ \mathcal{L}_{\underline{R_e^*}}(A_t) \right\} \leq \mathcal{L}_{\underline{R_e^*}}(A_t) \leq \max_{\mathfrak{t}}\left\{ \mathcal{L}_{\underline{R_e^*}}(A_t) \right\}$$

$$\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\min_{\mathfrak{t}}\left\{ \mathcal{L}_{\underline{R_e^*}}(A_t) \right\} \right)^{\omega_{\mathfrak{t}}} \leq \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\mathcal{L}_{\underline{R_e^*}}(A_t) \right)^{\omega_{\mathfrak{t}}} \leq \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\max_{\mathfrak{t}}\left\{ \mathcal{L}_{\underline{R_e^*}}(A_t) \right\} \right)^{\omega_{\mathfrak{t}}}$$

This implies that

$$\min_{\mathfrak{t}}\left\{ \mathcal{L}_{\underline{R_e^*}}(A_t) \right\} \leq \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\mathcal{L}_{\underline{R_e^*}}(A_t) \right)^{\omega_{\mathfrak{t}}} \leq \max_{\mathfrak{t}}\left\{ \mathcal{L}_{\underline{R_e^*}}(A_t) \right\} \tag{20}$$

Similarly,

$$\min_{\mathfrak{t}}\left\{ \wp_{\underline{R_e^*}}(A_t) \right\} \leq \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\wp_{\underline{R_e^*}}(A_t) \right)^{\omega_{\mathfrak{t}}} \leq \max_{\mathfrak{t}}\left\{ \wp_{\underline{R_e^*}}(A_t) \right\} \tag{21}$$

Similarly, we can show that

$$\min_{\mathfrak{t}}\left\{ \varphi_{\overline{R_e^*}}(A_t) \right\} \leq 1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}} \leq \max_{\mathfrak{t}}\left\{ \varphi_{\overline{R_e^*}}(A_t) \right\} \tag{22}$$

$$\min_{\mathfrak{t}}\left\{ \mathfrak{b}_{\overline{R_e^*}}(A_t) \right\} \leq 1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \mathfrak{b}_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}} \leq \max_{\mathfrak{t}}\left\{ \mathfrak{b}_{\overline{R_e^*}}(A_t) \right\} \tag{23}$$

$$\min_{\mathfrak{t}}\left\{ \mathcal{L}_{\overline{R_e^*}}(A_t) \right\} \leq \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\mathcal{L}_{\overline{R_e^*}}(A_t) \right)^{\omega_{\mathfrak{t}}} \leq \max_{\mathfrak{t}}\left\{ \mathcal{L}_{\overline{R_e^*}}(A_t) \right\} \tag{24}$$

$$\min_{\mathfrak{t}}\left\{ \wp_{\overline{R_e^*}}(A_t) \right\} \leq \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\wp_{\overline{R_e^*}}(A_t) \right)^{\omega_{\mathfrak{t}}} \leq \max_{\mathfrak{t}}\left\{ \wp_{\overline{R_e^*}}(A_t) \right\} \tag{25}$$

From Equations (18)–(25), we have

$$\left(R_e^*(A_t)\right)^{-} \leq \text{CIFRWA}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) \leq \left(R_e^*(A_t)\right)^{+}$$

□

Definition 15. *Let* $R_e^*(A_t) = \left(\overline{R_e^*}(A_t),\ \underline{R_e^*}(A_t)\right)(t = 1, 2, 3 \ldots, \mathfrak{s})$ *be a collection of CIFRNs, and* $\omega = \left(\omega_1,\ \omega_2, \ldots, \omega_{\mathfrak{s}}\right)^T$ *be the WV with* $\omega_{\mathfrak{t}} \in [0,\ 1]$ *and* $\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} = 1$. *Then, a CIFROWA operator is determined as*

$$\mathrm{CIFROWA}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) = \left(\oplus_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \overline{R_e^*}\left(A_{\mathfrak{o}(\mathfrak{t})}\right), \oplus_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \underline{R_e^*}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \varphi_{\overline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \mathfrak{b}_{\overline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\mathcal{L}_{\overline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\wp_{\overline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right) \end{array} \right) \\ , \left(\begin{array}{c} \left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \varphi_{\underline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \mathfrak{b}_{\underline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\mathcal{L}_{\underline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\wp_{\underline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right) \end{array} \right) \end{array} \right) \tag{26}$$

where $(\mathfrak{o}(1), \mathfrak{o}(2), \mathfrak{o}(3), \ldots, \mathfrak{o}(\mathfrak{s}))$ *is a permutation of* $(1, 2, \ldots, \mathfrak{s})$ *such that* $R_e^*\left(A_{\mathfrak{o}(\mathfrak{t}-1)}\right) \geq R_e^*\left(A_{\mathfrak{o}(\mathfrak{t})}\right) \forall \mathfrak{t}$.

Similar to Theorems 1–4, CIFROWA also satisfies the properties of idempotency, monotonicity, and boundedness.

Definition 16. *Let* $R_e^*(A_t) = \left(\overline{R_e^*}(A_t),\ \underline{R_e^*}(A_t)\right)(t = 1, 2, 3 \ldots, \mathfrak{s})$ *be a collection of CIFRNs and* $\omega = \left(\omega_1,\ \omega_2, \omega_3 \ldots, \omega_{\mathfrak{s}}\right)^T$ *be the weight vector (WV) with* $\omega_{\mathfrak{t}} \in [0,\ 1]$ *such that* $\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} = 1$*; then, a CIFRWG operator is defined as*

$$\mathrm{CIFRWG}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) = \left(\otimes_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \left(\overline{R_e^*}(A_t)\right)^{\omega_{\mathfrak{t}}}, \otimes_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \left(\underline{R_e^*}(A_t)\right)^{\omega_{\mathfrak{t}}}\right)$$

$$= \left(\left(\omega_1 \overline{R_e^*}(A_1) \otimes \omega_2 \overline{R_e^*}(A_2) \otimes \ldots.. \otimes \omega_{\mathfrak{s}} \overline{R_e^*}(A_{\mathfrak{s}})\right), \left(\omega_1 \underline{R_e^*}(A_1) \otimes \omega_2 \underline{R_e^*}(A_2) \otimes \ldots.. \otimes \omega_{\mathfrak{s}} \underline{R_e^*}(A_{\mathfrak{s}})\right)\right)$$

Based on the above definition, the results for the CIFRWG operator are as follows:

Theorem 5. *By employing the above equation, we obtain the CIFRNs and*

$$\mathrm{CIFRWG}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) = \left(\otimes_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \left(\overline{R_e^*}(A_t)\right)^{\omega_{\mathfrak{t}}}, \otimes_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \left(\underline{R_e^*}(A_t)\right)^{\omega_{\mathfrak{t}}}\right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\varphi_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\mathfrak{b}_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \wp_{\overline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) \end{array} \right), \\ \left(\begin{array}{c} \left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\varphi_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\mathfrak{b}_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) + i\left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \wp_{\underline{R_e^*}}(A_t)\right)^{\omega_{\mathfrak{t}}}\right) \end{array} \right) \end{array} \right) \tag{27}$$

Proof. Similar to Theorem 1. □

Theorem 6 *(idempotency property)***.** *Let* $R_e^*(A_t) = \left(\overline{R_e^*}(A_t),\ \underline{R_e^*}(A_t)\right)(t = 1, 2, 3 \ldots, \mathfrak{s})$ *be a collection of CIFRNs with WV* $\omega = \left(\omega_1,\ \omega_2, \ldots, \omega_{\mathfrak{s}}\right)^T$, $\omega_{\mathfrak{t}} \in [0,\ 1]$, *and* $\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} = 1$. *If* $R_e^*(A_t) = R_e^*(A)\ \forall\ (t = 1, 2, 3 \ldots, \mathfrak{s})$, *where* $R_e^*(A) = \left(\overline{R_e^*}(A),\ \underline{R_e^*}(A)\right)$, *then*

$$\mathrm{CIFRWG}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) = R_e^*(A) \tag{28}$$

Theorem 7 *(monotonicity property)***.** *Let*

$$R_e^*(A_t) = \left(\overline{R_e^*}(A_t), \underline{R_e^*}(A_t)\right) = \left(\begin{array}{c}\left(\varphi_{\overline{R_e^*}}(A_t) + \mathrm{i}\mathfrak{b}_{\overline{R_e^*}}(A_t), \mathcal{L}_{\overline{R_e^*}}(A_t) + \mathrm{i}\wp_{\overline{R_e^*}}(A_t)\right), \\ \left(\varphi_{\underline{R_e^*}}(A_t) + \mathrm{i}\mathfrak{b}_{\underline{R_e^*}}(A_t), \mathcal{L}_{\underline{R_e^*}}(A_t) + \mathrm{i}\wp_{\underline{R_e^*}}(A_t)\right)\end{array}\right) \text{ and}$$

$$R_e^*(B_t) = \left(\overline{R_e^*}(B_t), \underline{R_e^*}(B_t)\right) = \left(\begin{array}{c}\left(\varphi_{\overline{R_e^*}}(B_t) + \mathrm{i}\mathfrak{b}_{\overline{R_e^*}}(B_t), \mathcal{L}_{\overline{R_e^*}}(B_t) + \mathrm{i}\wp_{\overline{R_e^*}}(B_t)\right) \\ \left(\varphi_{\underline{R_e^*}}(B_t) + \mathrm{i}\mathfrak{b}_{\underline{R_e^*}}(B_t), \mathcal{L}_{\underline{R_e^*}}(B_t) + \mathrm{i}\wp_{\underline{R_e^*}}(B_t)\right)\end{array}\right) (\mathfrak{t} = 1, 2, \ldots, \mathfrak{s})$$

be a collection of two CIFRSs, and $\omega = \left(\omega_1, \omega_2, \ldots, \omega_{\mathfrak{s}}\right)^T$ *be the WV with* $\omega_{\mathfrak{t}} \in [0,1]$ *and* $\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} = 1$. *If* $\overline{R_e^*}(A_t) \leq \overline{R_e^*}(B_t)$, $\underline{R_e^*}(A_t) \leq \underline{R_e^*}(B_t)$, *then*

$$\mathrm{CIFRWG}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) \leq \mathrm{CIFRWG}\left(R_e^*(B_1), R_e^*(B_2), \ldots, R_e^*(B_{\mathfrak{s}})\right) \quad (29)$$

Theorem 8 *(boundedness property)***.** *Let* $R_e^*(A_t) = \left((R_e^*(A_t))^+, (R_e^*(A_t))^-\right)(t = 1, 2, 3 \ldots, \mathfrak{s})$ *be a collection of CIFRNs where* $(R_e^*(A_t))^+ = \left(\min_{\mathfrak{t}} \overline{R_e^*}(A_t), \max_{\mathfrak{t}} \underline{R_e^*}(A_t)\right)$ *and* $(R_e^*(A_t))^- = \left(\max_{\mathfrak{t}} \overline{R_e^*}(A_t), \min_{\mathfrak{t}} \underline{R_e^*}(A_t)\right)$*; then,*

$$\left(R_e^*(A_t)\right)^- \leq \mathrm{CIFRWG}\left(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})\right) \leq \left(R_e^*(A_t)\right)^+ \quad (30)$$

Definition 17. *Let* $R_e^*(A_t) = \left(\overline{R_e^*}(A_t),\ \underline{R_e^*}(A_t)\right)(t = 1, 2, 3 \ldots, \mathfrak{s})$ *be a collection of CIFRNs, and* $\omega = \left(\omega_1,\ \omega_2, \ldots, \omega_{\mathfrak{s}}\right)^T$ *be the WV with* $\omega_{\mathfrak{t}} \in [0,\ 1]$ *and* $\sum_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} = 1$. *Then, a complex intuitionistic fuzzy rough ordered weighted geometric (CIFROWG) operator is determined as*

$$\mathrm{CIFROWG}(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})) = \left(\otimes_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \overline{R_e^*}\left(A_{\mathfrak{o}(\mathfrak{t})}\right), \otimes_{\mathfrak{t}=1}^{\mathfrak{s}} \omega_{\mathfrak{t}} \underline{R_e^*}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)$$
$$= \left(\begin{array}{c}\left(\begin{array}{c}\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\varphi_{\overline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\mathfrak{b}_{\overline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \mathcal{L}_{\overline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \wp_{\overline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right)\end{array}\right), \\ \left(\begin{array}{c}\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\varphi_{\underline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(\prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(\mathfrak{b}_{\underline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right), \\ \left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \mathcal{L}_{\underline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right) + \mathrm{i}\left(1 - \prod_{\mathfrak{t}=1}^{\mathfrak{s}} \left(1 - \wp_{\underline{R_e^*}}\left(A_{\mathfrak{o}(\mathfrak{t})}\right)\right)^{\omega_{\mathfrak{t}}}\right)\end{array}\right)\end{array}\right) \quad (31)$$

where $(\mathfrak{o}(1),\ \mathfrak{o}(2),\ \mathfrak{o}(3), \ldots, \mathfrak{o}(\mathfrak{s}))$ *is a permutation of* $(1,\ 2, \ldots,\ \mathfrak{s})$ *such that* $R_e^*\left(A_{\mathfrak{o}(\mathfrak{t}-1)}\right) \geq R_e^*\left(A_{\mathfrak{o}(\mathfrak{t})}\right) \forall\ \mathfrak{t}$.

Similar to Theorems 1–4, CIFROWA also satisfies the properties of idempotency, monotonicity, and boundedness.

5. A Method of MADM in the Setting of CIFR Information

In the following section, we demonstrate an MADM technique using the AOs in the structure of CIFRNs.

Assume that there are $\bar{\Upsilon}$ alternatives and þ attributes. Let $\breve{A}_{\mathfrak{a}}\left(\mathfrak{a} = 1, 2, \ldots, \bar{\Upsilon}\right)$, $\breve{B}_{\mathfrak{m}}\left(\mathfrak{m} = 1, 2, \ldots, \text{þ}\right)$, and $\omega = \left(\omega_1,\ \omega_2, \ldots, \omega_{\text{þ}}\right)^T$ be the WV of attributes with $\omega_{\text{þ}} \in [0,\ 1]\ \forall\ \text{þ}$ and $\sum_{\mathfrak{m}=1}^{\text{þ}} \omega_{\mathfrak{m}} = 1$. The decision maker or expert will identify the evaluated values of the considered $\bar{\Upsilon}$ alternatives based on the interpreted attributes. These values will be in the structure of CIFRNs, i.e., $\mathcal{M} = \left(\Omega_{ⱺɱ}\right)_{\bar{Ƴ}\times þ} = \left(\left(\overline{\mathrm{M}}_{ⱺɱ}, \overline{\mathrm{N}}_{ⱺɱ}\right), \left(\underline{\mathrm{M}}_{ⱺɱ}, \underline{\mathrm{N}}_{ⱺɱ}\right)\right)_{\bar{Ƴ}\times þ} = \left(\left(\overline{\varphi}_{ⱺɱ} + \mathrm{i}\overline{\mathcal{L}}_{ⱺɱ}, \overline{ƀ}_{ⱺɱ} + \mathrm{i}\overline{\wp}_{ⱺɱ}\right), \left(\underline{\varphi}_{ⱺɱ} + \mathrm{i}\underline{\mathcal{L}}_{ⱺɱ}, \underline{ƀ}_{ⱺɱ} + \mathrm{i}\underline{\wp}_{ⱺɱ}\right)\right)_{\bar{Ƴ}\times þ}$, which will be used to construct a CIFR decision matrix. To tackle this MADM dilemma, we interpreted the underlying algorithm.

5.1. Algorithm

We demonstrated the algorithm to solve MADM problems in the structure of CIFRSs by employing CIFRWA and CIFRWG operators.

Step 1: The attributes may come in two types in each MADM process, a benefit type and a cost type. Thus, there is a requirement for normalization, and for that, the formula below is given.

$$\Omega_{ⱺɱ} = \begin{cases} \left(\left(\overline{\varphi}_{ⱺɱ} + \mathrm{i}\overline{\mathcal{L}}_{ⱺɱ}, \overline{ƀ}_{ⱺɱ} + \mathrm{i}\overline{\wp}_{ⱺɱ}\right), \left(\underline{\varphi}_{ⱺɱ} + \mathrm{i}\underline{\mathcal{L}}_{ⱺɱ}, \underline{ƀ}_{ⱺɱ} + \mathrm{i}\underline{\wp}_{ⱺɱ}\right)\right) \text{for benefit type of attribute} \\ \begin{pmatrix} \left(1 - \overline{\varphi}_{ⱺɱ} + \mathrm{i}\left(1 - \overline{\mathcal{L}}_{ⱺɱ}\right), 1 - \overline{ƀ}_{ⱺɱ} + \mathrm{i}\left(1 - \overline{\wp}_{ⱺɱ}\right)\right), \\ \left(1 - \underline{\varphi}_{ⱺɱ} + \mathrm{i}\left(1 - \underline{\mathcal{L}}_{ⱺɱ}\right), 1 - \underline{ƀ}_{ⱺɱ} + \mathrm{i}\left(1 - \underline{\wp}_{ⱺɱ}\right)\right) \end{pmatrix} \text{for cost type of attribute} \end{cases}$$

Step 2: By employing the

$$\begin{aligned} &\mathrm{CIFRWA}(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})) = \left(\oplus_{t=1}^{\mathfrak{s}} Ꙍ_t \overline{R_e^*}(A_t), \oplus_{t=1}^{\mathfrak{s}} Ꙍ_t \underline{R_e^*}(A_t)\right) \\ &= \begin{pmatrix} \begin{pmatrix} \left(1 - \prod_{t=1}^{\mathfrak{s}}\left(1 - \varphi_{\overline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) + \mathrm{i}\left(1 - \prod_{t=1}^{\mathfrak{s}}\left(1 - ƀ_{\overline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right), \\ \left(\prod_{t=1}^{\mathfrak{s}}\left(\mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) + \mathrm{i}\left(\prod_{t=1}^{\mathfrak{s}}\left(\wp_{\overline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) \end{pmatrix}, \\ \begin{pmatrix} \left(1 - \prod_{t=1}^{\mathfrak{s}}\left(1 - \varphi_{\underline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) + \mathrm{i}\left(1 - \prod_{t=1}^{\mathfrak{s}}\left(1 - ƀ_{\underline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right), \\ \left(\prod_{t=1}^{\mathfrak{s}}\left(\mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) + \mathrm{i}\left(\prod_{t=1}^{\mathfrak{s}}\left(\wp_{\underline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) \end{pmatrix} \end{pmatrix} \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\mathrm{CIFRWG}(R_e^*(A_1), R_e^*(A_2), \ldots, R_e^*(A_{\mathfrak{s}})) \\ &= \left(\otimes_{t=1}^{\mathfrak{s}} Ꙍ_t \left(\overline{R_e^*}(A_t)\right)^{Ꙍ_t}, \otimes_{t=1}^{\mathfrak{s}} Ꙍ_t \left(\underline{R_e^*}(A_t)\right)^{Ꙍ_t}\right) \\ &= \begin{pmatrix} \begin{pmatrix} \left(\prod_{t=1}^{\mathfrak{s}}\left(\varphi_{\overline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) + \mathrm{i}\left(\prod_{t=1}^{\mathfrak{s}}\left(ƀ_{\overline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right), \\ \left(1 - \prod_{t=1}^{\mathfrak{s}}\left(1 - \mathcal{L}_{\overline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) + \mathrm{i}\left(1 - \prod_{t=1}^{\mathfrak{s}}\left(1 - \wp_{\overline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) \end{pmatrix}, \\ \begin{pmatrix} \left(\prod_{t=1}^{\mathfrak{s}}\left(\varphi_{\underline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) + \mathrm{i}\left(\prod_{t=1}^{\mathfrak{s}}\left(ƀ_{\underline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right), \\ \left(1 - \prod_{t=1}^{\mathfrak{s}}\left(1 - \mathcal{L}_{\underline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) + \mathrm{i}\left(1 - \prod_{t=1}^{\mathfrak{s}}\left(1 - \wp_{\underline{R_e^*}}(A_t)\right)^{Ꙍ_t}\right) \end{pmatrix} \end{pmatrix} \end{aligned} \tag{33}$$

operators to the supposed decision information provided in matrix $\mathcal{M}$, all the aggregated values of alternatives $\breve{A}_{ⱺ}$, $\left(ⱺ = 1, 2 \ldots, \bar{Ƴ}\right)$ can be derived.

Step 3: The score values of the aggregated outcomes were investigated by

$$\begin{aligned} S_F(R_e^*(A)) = \tfrac{1}{8}\left(4 + \varphi_{\overline{R_e^*}} + ƀ_{\overline{R_e^*}} + \varphi_{\underline{R_e^*}} + ƀ_{\underline{R_e^*}} - \mathcal{L}_{\overline{R_e^*}} - \wp_{\overline{R_e^*}} - \mathcal{L}_{\underline{R_e^*}} - \wp_{\underline{R_e^*}}\right), \\ S_F(R_e^*(A)) \in [0, 1] \end{aligned} \tag{34}$$

Step 4: In this step, the alternatives were ranked by employing the score values and achieving the optimal alternative.

Step 5: End.

5.2. Case Study

Choosing the optimal AI architecture for autonomous vehicles is essential for a vehicle manufacturing company that specializes in these vehicles. The company's main goal is to develop cutting-edge autonomous vehicles that put superior performance, safety, and efficiency first. Four potential AI architectures and four crucial attributes were determined to help achieve these goals and direct the DM process.

Alternatives (AI architectures)

$\breve{A}_1$: Recurrent neural networks (RNNs): The potential for processing sequential data is provided by RNNs, which is necessary for making accurate trajectory predictions and motion planning in autonomous vehicles.

$\breve{A}_2$: Convolutional neural networks (CNNs): CNNs are well known for their ability to analyze images and identify objects, which is in line with the need for the visual input from cameras and other sensors in autonomous vehicles to be accurately interpreted.

$\breve{A}_3$: Graph neural networks (GNNs): In order to simulate complicated road networks and comprehend the spatial interactions between items on the road—a crucial aspect of autonomous systems—GNNs are made to operate with graph-structured data.

$\breve{A}_4$: Hybrid neural networks (HNNs): By combining many neural network designs, HNNs might possibly offer a comprehensive solution for a range of autonomous driving issues. However, the implementation's complexity needs to be weighed against the goals of the business.

Attributes

฿_1: Accuracy: This is a critical component. Accurate and dependable decision-making abilities of the chosen AI architecture are necessary to guarantee the enhanced performance and safety of autonomous vehicles in a variety of driving conditions.

฿_2: Computational efficiency: The demands of production revolve around computational efficiency. In order to facilitate prompt answers on the road and improve the overall efficiency of autonomous car manufacturing, the AI architecture must handle data effectively.

฿_3: Robustness: This is a crucial need. To ensure the safety of passengers and other road users, autonomous vehicles must exhibit a dependable performance in inclement weather, limited visibility, and the presence of unforeseen road obstructions.

฿_4: Scalability: Being scalable is essential to production. To ensure the scalability of autonomous vehicle manufacturing, the AI architecture must be able to adapt to and handle increasing complexity, facilitate future upgrades, and accommodate a variety of vehicle kinds and configurations.

Further, for each attribute, the considered weights were $(0.3,\ 0.1,\ 0.2,\ 0.4)$. Based on these attributes, the company assessed each AI architecture and gave each alternative an assessment score in the framework of a CIFRS to create a CIFR decision matrix, which is revealed in Table 2.

Table 2. Complex intuitionistic fuzzy rough numbers.

	$₿_1$	$₿_2$	$₿_3$	$₿_4$
$\breve{A}_1$	$((0.3+\imath 0.7),(0.5+\imath 0.2)),((0.4+\imath 0.3),(0.4+\imath 0.2))$	$((0.4+\imath 0.5),(0.3+\imath 0.4)),((0.7+\imath 0.3),(0.1+\imath 0.5))$	$((0.7+\imath 0.2),(0.1+\imath 0.6)),((0.3+\imath 0.3),(0.5+\imath 0.6))$	$((0.7+\imath 0.3),(0.2+\imath 0.7)),((0.8+\imath 0.2),(0.1+\imath 0.8))$
$\breve{A}_2$	$((0.4+\imath 0.1),(0.5+\imath 0.3)),((0.8+\imath 0.4),(0.1+\imath 0.3))$	$((0.2+\imath 0.3),(0.3+\imath 0.4)),((0.4+\imath 0.3),(0.5+\imath 0.4))$	$((0.6+\imath 0.4),(0.3+\imath 0.4)),((0.2+\imath 0.8),(0.7+\imath 0.2))$	$((0.7+\imath 0.6),(0.2+\imath 0.2)),((0.9+\imath 0.5),(0.1+\imath 0.2))$
$\breve{A}_3$	$((0.6+\imath 0.3),(0.2+\imath 0.7)),((0.3+\imath 0.5),(0.4+\imath 0.3))$	$((0.2+\imath 0.4),(0.3+\imath 0.5)),((0.4+\imath 0.4),(0.5+\imath 0.6))$	$((0.6+\imath 0.4),(0.4+\imath 0.2)),((0.1+\imath 0.7),(0.3+\imath 0.2))$	$((0.8+\imath 0.7),(0.2+\imath 0.1)),((0.3+\imath 0.8),(0.6+\imath 0.2))$
$\breve{A}_4$	$((0.8+\imath 0.3),(0.1+\imath 0.6)),((0.4+\imath 0.7),(0.6+\imath 0.3))$	$((0.2+\imath 0.5),(0.6+\imath 0.3)),((0.2+\imath 0.6),(0.5+\imath 0.3))$	$((0.3+\imath 0.1),(0.5+\imath 0.4)),((0.6+\imath 0.2),(0.4+\imath 0.5))$	$((0.7+\imath 0.1),(0.2+\imath 0.1)),((0.8+\imath 0.2),(0.1+\imath 0.6))$

Through the steps below, this MADM was tackled.

Step 1: The data given in Table 2 are of the benefit type, so there was no need to normalize it.

Step 2: Using the CIFRWA operators, the determined aggregated outcome of each alternative was

$$\begin{aligned}
\breve{A}_1 &= (((0.585+i0.460),(0.238+i0.441)),((0.628+i0.262),(0.209+i0.475)))\\
\breve{A}_2 &= (((0.568+i0.415),(0.297+i0.278)),((0.777+i0.545),(0.173+i0.242)))\\
\breve{A}_3 &= (((0.675+i0.524),(0.239+i0.242)),((0.275+i0.681),(0.454+i0.252)))\\
\breve{A}_4 &= (((0.653+i0.213),(0.218+i0.252)),((0.633+i0.444),(0.265+i0.438)))
\end{aligned} \tag{35}$$

Step 3: The obtained score values of $S_F(\breve{A}_\omega)(\omega = 1, 2, 3, 4)$ of the CIFRNs $(\breve{A}_\omega)(\omega = 1, 2, 3, 4)$ were

$S_F(\breve{A}_1) = 0.571$, $S_F(\breve{A}_2) = 0.664$, $S_F(\breve{A}_3) = 0.621$, $S_F(\breve{A}_4) = 0.596$

Step 4: The values $\breve{A}_\omega(\omega = 1, 2, 3, 4)$ were ranked with the following score values $S_F(\breve{A}_\omega)(\omega = 1, 2, 3, 4)$ of the overall CIFRNs:

$$\breve{A}_2 > \breve{A}_3 > \breve{A}_4 > \breve{A}_1.$$

From the ranking, it was observed that $\breve{A}_2$, or the "convolutional neural network", was the optimal AI architecture for autonomous vehicles.

Step 5: End.

If we used the CIFRWG operator, then the results of the above problem were as follows:

Step 1: The data given in Table 2 are of the benefit type, so there was no need to normalize it.

Step 2: Using the CIFRWA operators, the determined aggregated outcome of each alternative was

$$\begin{aligned}
\breve{A}_1 &= (((0.513+i0.375),(0.298+i0.543)),((0.527+i0.255),(0.291+i0.618)))\\
\breve{A}_2 &= (((0.506+i0.302),(0.332+i0.295)),((0.593+i0.488),(0.319+i0.253)))\\
\breve{A}_3 &= (((0.603+i0.459),(0.255+i0.404)),((0.247+i0.631),(0.483+i0.283)))\\
\breve{A}_4 &= (((0.543+i0.163),(0.296+i0.365)),((0.534+i0.325),(0.386+i0.477)))
\end{aligned} \tag{36}$$

Step 3: The obtained score values of $S_F\left(\breve{A}_{\mathbb{Q}}\right)\left(\mathbb{Q}=1,\ 2,\ 3,\ 4\right)$ of the CIFRNs $\left(\breve{A}_{\mathbb{Q}}\right)\left(\mathbb{Q}=1,\ 2,\ 3,\ 4\right)$ were

$S_F\left(\breve{A}_1\right)=0.490$, $S_F\left(\breve{A}_2\right)=0.586$, $S_F\left(\breve{A}_3\right)=0.565$, $S_F\left(\breve{A}_4\right)=0.505$

Step 4: The values $\breve{A}_{\mathbb{Q}}\left(\mathbb{Q}=1,\ 2,\ 3,\ 4\right)$ were ranked with the following score values $S_F\left(\breve{A}_{\mathbb{Q}}\right)\left(\mathbb{Q}=1,\ 2,\ 3,\ 4\right)$ of the overall CIFRNs:

$$\breve{A}_2 > \breve{A}_3 > \breve{A}_4 > \breve{A}_1.$$

From the ranking, it was observed that $\breve{A}_2$, or the "convolutional neural network", was the optimal AI architecture for autonomous vehicles.

Step 5: End.

6. Comparison Analysis

This part presents a comparative study to show the validity, superiority, and effectiveness of our suggested methodologies and proposed work. This is because making comparisons is essential for comprehending the importance and effectiveness of any newly created work. We were unable to differentiate between the good and the terrible without a comparison. As a result, the goal of this study was to compare and investigate the decision-making mechanisms of existing models with our proposed work. In this approach, we used some previously published ideas from the theories of FRSs, IFSs, and IFRSs. Following a discussion of a few established theories, we compared them to our new proposed work.

- ❖ IF AOs by Xu. [21] and geometric AOs based on IFSs by Xu and Yager [22].
- ❖ IFR AOs for MCDM by Chinram et al. [37].
- ❖ IFR Frank AOs for MCDM by Yahya et al. [36].
- ❖ Aczel–Alsina average AOs based on IFRSs by Ahmmad et al. [38].
- ❖ Confidence level AOs based on IFRSs by Mahmood et al. [39].
- ❖ Yager AOs based on IFRSs by Mahmood et al. [40].

Next, we employed the already published work that was considered and the interpreted theory to aggregate and solve the information revealed in Table 2, which is in the framework of the CIFRS. The score values and the ranking order after tackling that information are provided in Table 3.

Table 3. Comparative study between proposed and existing work.

Methods	**Score Values of Alternatives** $\breve{A}_1$ $\breve{A}_2$ $\breve{A}_3$ $\breve{A}_4$	**Rankings**
Xu [18]	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$
Xu and Yager [19]	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$
Chinram et al. [37]	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$
Yahya et al. [36]	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$
Ahmmad et al. [38]	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$
Mahmood et al. [39]	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$
Mahmood et al. [40]	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$	$\dot{\times} \rightleftarrows \dot{\times} \rightleftarrows \dot{\times}$
Interpreted theory	0.571, 0.664, 0.621, 0.596	$\breve{A}_2 > \breve{A}_3 > \breve{A}_4 > \breve{A}_1$
Interpreted theory	0.491, 0.586, 0.565, 0.505	$\breve{A}_2 > \breve{A}_3 > \breve{A}_4 > \breve{A}_1$

Table 3 reveals that only our anticipated theory solved the information presented in Table 2 (i.e., CIFR information), and none of the current work was able to aggregate and cope with this information because of the certain limitations that every existing theory contains. For instance, Xu [21] and Xu and Yager [22] devised AOs within IFSs, which merely aggregated the information containing TGs and FGs, but were not able to aggregate information containing roughness or additional fuzzy information. Chinram et al. [37], Yahya et al. [36], Ahmmad et al. [38], Mahmood et al. [39], and Mahmood et al. [40]

developed various AOs in the framework of IFRSs that could aggregate information containing the roughness and TGs and FGs, and could also aggregate information in the structure of FSs, IFSs, and RSs, but could not aggregate information that contains additional fuzzy information (second dimension). Further, there is no such AO or MADM technique that can aggregate and solve information that contains additional fuzzy information in TGs and FGs along with roughness. This implies that our work is more dominant and valuable than the existing theories. Furthermore, the anticipated theory can reduce the structure of FSs, RSs, IFSs, and IFRSs. In Figure 3, we present a graphical representation of the data given in Table 3, which contained different alternatives in the shape of distinct colors, showing the ranking results.

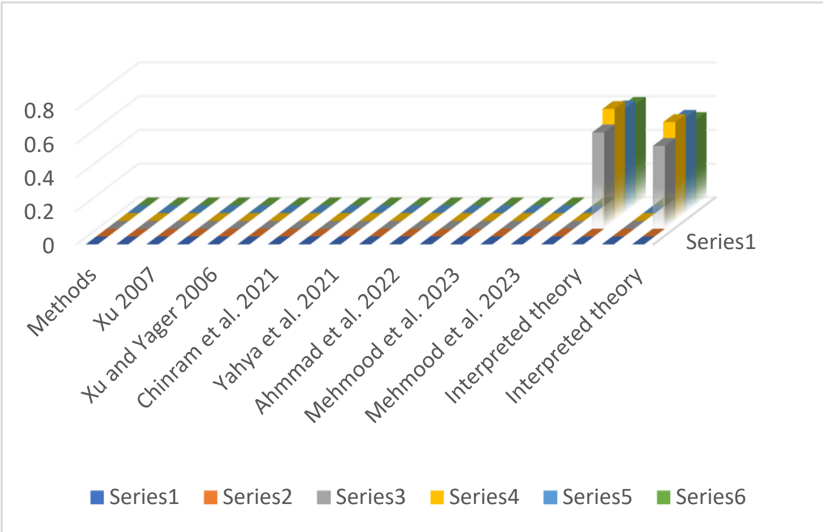

Figure 3. Comparison between proposed work and work established by Xu [21], Xu and Yager [22], Chinram et al. [37], Yahya et al. [36], Ahmmad et al. [38] and Mehmood et al. [39,40].

Theoretical and Practical Implications

Selecting the finest AI architecture for autonomous cars is necessary for a vehicle manufacturing company that specializes in these vehicles. Autonomous cars operate in highly dynamic and uncertain environments. The data collected by sensors are often noisy, incomplete, and subject to varying environmental conditions. CIFRSs provide a valuable framework for addressing these challenges. CIFRSs can be a valuable tool in this context for several reasons. They can help to integrate the data from multiple sensors, with each sensor having its own uncertainties, into a unified representation. This can enhance the car's perception of its surroundings. CIFRSs can improve the accuracy of object classification, whether they are cars, obstacles, or pedestrians, by considering the inherent ambiguity in sensor data. CIFRSs can support DM modules in autonomous cars, such as sensor data fusion, object classification, path planning, fault detection and diagnoses, obstacles, avoidance, risk assessments, and emergency braking. The application of CIFRSs in selecting the finest architecture for autonomous vehicles has significant practical implications. CIFRSs use both complex TGs and FGs in the form of LA and UA and enhances vehicle safety by improving the accuracy of perception and DM. In case of accidents, a strong understanding of uncertainty through CIFRSs can help to reduce or mitigate liability. Therefore, CIFRSs are a valuable tool for researchers to use in the development of new AI technologies and algorithms for the finest AI architecture.

7. Conclusions

Selecting the best AI architecture for autonomous cars is an MADM problem, as it involves making a complex choice while considering several factors, some of which may have two-dimensional ambiguity and/or indiscernibility. CIFRSs have the ability to consider both complex TGs and FGs in the form of LA and UA. Therefore, in this article, we first provided a unique mathematical framework for dealing with two-dimensional uncertainties and indiscernibility: the "CIFRS". The elementary operations, such as a union, intersection, complement addition, multiplication, etc., of the anticipated CIFRS are also provided. After that, we developed certain AOs in the environment of CIFRSs, such as CIFRWA, CIFROWA, CIFRWG, and CIFOWG operators. Then, using the developed operators, we anticipated a technique of MADM in the context of CIFRSs and examined the case study "Selection of optimal AI architecture for autonomous vehicles" to determine whether the developed method of MADM was workable. We juxtaposed our original hypothesis with a few contemporary hypotheses in order to illustrate the superiority and domination.

7.1. Limitations

We noticed that the theory of CIFR information is very dominant and valuable. CIFRSs combine the concepts of CIFSs and RSs to handle the uncertainty, vagueness, and indiscernibility in data. Despite their utility, there are several limitations associated with CIFRSs. This is because, in various situations, they do not work effectively, for example, if a person gives information in the form of complex Pythagorean fuzzy sets, complex picture fuzzy sets, or bipolar complex fuzzy sets.

7.2. Advantages

The proposed AI architecture for autonomous vehicles, using an MADM technique with CIFR aggregation operators, offers numerous advantages over the existing architectures, including an enhanced decision-making exactness, optimal architecture selection, flexibility, the enhanced handling of complex decision variables, robustness in real-life situations, and the complete integration of multiple attributes. These benefits make the anticipated architecture better suited to the vibrant and uncertain atmospheres in which autonomous vehicles operate, potentially leading to much safer and efficient autonomous driving systems across the world.

7.3. Future Work

Our objective for the future is to expand this theory into the framework of bipolar complex fuzzy sets [41–43], graph theory [44–46], and picture fuzzy sets [47,48].

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References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
2. Pawlak, Z. Rough sets. *Int. J. Comput. Inf. Sci.* **1982**, *11*, 341–356. [CrossRef]
3. Dubois, D.; Prade, H. Rough fuzzy sets and fuzzy rough sets. *Int. J. Gen. Syst.* **1990**, *17*, 191–209. [CrossRef]
4. Cornelis, C.; De Cock, M.; Kerre, E.E. Intuitionistic fuzzy rough sets: At the crossroads of imperfect knowledge. *Expert Syst.* **2003**, *20*, 260–270. [CrossRef]
5. Ma, Y.; Wang, Z.; Yang, H.; Yang, L. Artificial intelligence applications in the development of autonomous vehicles: A survey. *IEEE/CAA J. Autom. Sin.* **2020**, *7*, 315–329. [CrossRef]

6. Khayyam, H.; Javadi, B.; Jalili, M.; Jazar, R.N. Artificial intelligence and internet of things for autonomous vehicles. In *Nonlinear Approaches in Engineering Applications: Automotive Applications of Engineering Problems*; Springer: Berlin/Heidelberg, Germany, 2020; pp. 39–68.
7. Pereira, J.L.; Rossetti, R.J. An integrated architecture for autonomous vehicles simulation. In Proceedings of the 27th Annual ACM Symposium on Applied Computing, Trento, Italy, 26–30 March 2012; pp. 286–292.
8. Kurzidem, I.; Saad, A.; Schleiss, P. A systematic approach to analyzing perception architectures in autonomous vehicles. In Proceedings of the Model-Based Safety and Assessment: 7th International Symposium 2020, IMBSA 2020, Lisbon, Portugal, 14–16 September 2020; Proceedings 7; pp. 149–162.
9. Bathla, G.; Bhadane, K.; Singh, R.K.; Kumar, R.; Aluvalu, R.; Krishnamurthi, R.; Kumar, A.; Thakur, R.N.; Basheer, S. Autonomous vehicles and intelligent automation: Applications, challenges, and opportunities. *Mob. Inf. Syst.* **2022**, *2022*, 7632892. [CrossRef]
10. Zong, W.; Zhang, C.; Wang, Z.; Zhu, J.; Chen, Q. Architecture design and implementation of an autonomous vehicle. *IEEE Access* **2018**, *6*, 21956–21970. [CrossRef]
11. Bendiab, G.; Hameurlaine, A.; Germanos, G.; Kolokotronis, N.; Shiaeles, S. Autonomous vehicles security: Challenges and solutions using blockchain and artificial intelligence. *IEEE Trans. Intell. Transp. Syst.* **2023**, *24*, 3614–3617. [CrossRef]
12. Esogbue, A.O.; Theologidu, M.; Guo, K. On the application of fuzzy sets theory to the optimal flood control problem arising in water resources systems. *Fuzzy Sets Syst.* **1992**, *48*, 155–172. [CrossRef]
13. Guiffrida, A.L.; Nagi, R. Fuzzy set theory applications in production management research: A literature survey. *J. Intell. Manuf.* **1998**, *9*, 39–56. [CrossRef]
14. Driankov, D.; Saffiotti, A. (Eds.) *Fuzzy Logic Techniques for Autonomous Vehicle Navigation*; Physica: Berlin/Heidelberg, Germany, 2013; Volume 61.
15. Wang, X.; Fu, M.; Ma, H.; Yang, Y. Lateral control of autonomous vehicles based on fuzzy logic. *Control Eng. Pract.* **2015**, *34*, 1–17. [CrossRef]
16. Awad, N.; Lasheen, A.; Elnaggar, M.; Kamel, A. Model predictive control with fuzzy logic switching for path tracking of autonomous vehicles. *ISA Trans.* **2022**, *129*, 193–205. [CrossRef]
17. Atanassov, K.T.; Stoeva, S. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
18. Dengfeng, L.; Chuntian, C. New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. *Pattern Recognit. Lett.* **2002**, *23*, 221–225. [CrossRef]
19. De, S.K.; Biswas, R.; Roy, A.R. An application of intuitionistic fuzzy sets in medical diagnosis. *Fuzzy Sets Syst.* **2001**, *117*, 209–213. [CrossRef]
20. Garg, H.; Rani, D. Novel distance measures for intuitionistic fuzzy sets based on various triangle centers of isosceles triangular fuzzy numbers and their applications. *Expert Syst. Appl.* **2022**, *191*, 116228. [CrossRef]
21. Xu, Z. Intuitionistic fuzzy aggregation operators. *IEEE Trans. Fuzzy Syst.* **2007**, *15*, 1179–1187.
22. Xu, Z.; Yager, R.R. Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int. J. Gen. Syst.* **2006**, *35*, 417–433. [CrossRef]
23. Jia, X.; Wang, Y. Choquet integral-based intuitionistic fuzzy arithmetic aggregation operators in multi-criteria decision-making. *Expert Syst. Appl.* **2022**, *191*, 116242. [CrossRef]
24. Ecer, F. An extended MAIRCA method using intuitionistic fuzzy sets for coronavirus vaccine selection in the age of COVID-19. *Neural Comput. Appl.* **2022**, *34*, 5603–5623. [CrossRef]
25. Ramot, D.; Milo, R.; Friedman, M.; Kandel, A. Complex fuzzy sets. *IEEE Trans. Fuzzy Syst.* **2002**, *10*, 171–186. [CrossRef]
26. Tamir, D.E.; Jin, L.; Kandel, A. A new interpretation of complex membership grade. *Int. J. Intell. Syst.* **2011**, *26*, 285–312. [CrossRef]
27. Zhang, G.; Dillon, T.S.; Cai, K.Y.; Ma, J.; Lu, J. Operation properties and δ-equalities of complex fuzzy sets. *Int. J. Approx. Reason.* **2009**, *50*, 1227–1249. [CrossRef]
28. Hu, B.; Bi, L.; Dai, S.; Li, S. Distances of complex fuzzy sets and continuity of complex fuzzy operations. *J. Intell. Fuzzy Syst.* **2018**, *35*, 2247–2255. [CrossRef]
29. ur Rehman, U. Selection of Database Management System by Using Multi-Attribute Decision-Making Approach Based on Probability Complex Fuzzy Aggregation Operators. *J. Innov. Res. Math. Comput. Sci.* **2023**, *2*, 1–16.
30. Zhou, L.; Wu, W.Z. On generalized intuitionistic fuzzy rough approximation operators. *Inf. Sci.* **2008**, *178*, 2448–2465. [CrossRef]
31. Zhou, L.; Wu, W.Z. Characterization of rough set approximations in Atanassov intuitionistic fuzzy set theory. *Comput. Math. Appl.* **2011**, *62*, 282–296. [CrossRef]
32. Bustince, H.; Burillo, P. Structures on intuitionistic fuzzy relations. *Fuzzy Sets Syst.* **1996**, *78*, 293–303. [CrossRef]
33. Zhang, X.; Zhou, B.; Li, P. A general frame for intuitionistic fuzzy rough sets. *Inf. Sci.* **2012**, *216*, 34–49. [CrossRef]
34. Yun, S.M.; Lee, S.J. Intuitionistic fuzzy rough approximation operators. *Int. J. Fuzzy Log. Intell. Syst.* **2015**, *15*, 208–215. [CrossRef]
35. Zhang, Z. Generalized intuitionistic fuzzy rough sets based on intuitionistic fuzzy coverings. *Inf. Sci.* **2012**, *198*, 186–206. [CrossRef]
36. Yahya, M.; Naeem, M.; Abdullah, S.; Qiyas, M.; Aamir, M. A novel approach on the intuitionistic fuzzy rough frank aggregation operator-based EDAS method for multicriteria group decision-making. *Complexity* **2021**, *2021*, 5534381. [CrossRef]
37. Chinram, R.; Hussain, A.; Mahmood, T.; Ali, M.I. EDAS method for multi-criteria group decision making based on intuitionistic fuzzy rough aggregation operators. *IEEE Access* **2021**, *9*, 10199–10216. [CrossRef]

38. Ahmmad, J.; Mahmood, T.; Mehmood, N.; Urawong, K.; Chinram, R. Intuitionistic Fuzzy Rough Aczel-Alsina Average Aggregation Operators and Their Applications in Medical Diagnoses. *Symmetry* **2022**, *14*, 2537. [CrossRef]
39. Mahmood, T.; Ahmmad, J.; Ali, Z.; Yang, M.S. Confidence Level Aggregation Operators Based on Intuitionistic Fuzzy Rough Sets With Application in Medical Diagnosis. *IEEE Access* **2023**, *11*, 8674–8688. [CrossRef]
40. Mahmood, T.; Ahmmad, J.; ur Rehman, U.; Khan, M.B. Analysis and Prioritization of the Factors of the Robotic Industry with the Assistance of EDAS Technique Based on Intuitionistic Fuzzy Rough Yager Aggregation Operators. *IEEE Access* **2023**, *11*, 50462–50479. [CrossRef]
41. Mahmood, T.; Ur Rehman, U. A novel approach towards bipolar complex fuzzy sets and their applications in generalized similarity measures. *Int. J. Intell. Syst.* **2022**, *37*, 535–567. [CrossRef]
42. Gwak, J.; Garg, H.; Jan, N. Hybrid integrated decision-making algorithm for clustering analysis based on a bipolar complex fuzzy and soft sets. *Alex. Eng. J.* **2023**, *67*, 473–487. [CrossRef]
43. Gwak, J.; Garg, H.; Jan, N.; Akram, B. A new approach to investigate the effects of artificial neural networks based on bipolar complex spherical fuzzy information. *Complex Intell. Syst.* **2023**, *9*, 4591–4614. [CrossRef]
44. Akram, M.; Sarwar, M.; Dudek, W.A. *Graphs for the Analysis of Bipolar Fuzzy Information*; Springer: Berlin, Germany, 2021; Volume 401, p. 452.
45. Akram, M.; Akmal, R. Application of bipolar fuzzy sets in graph structures. *Appl. Comput. Intell. Soft Comput.* **2016**, *2016*, 5859080. [CrossRef]
46. Akram, M. Bipolar fuzzy graphs. *Inf. Sci.* **2011**, *24*, 5548–5564. [CrossRef]
47. Ozer, O. Hamacher Prioritized Aggregation Operators Based on Complex Picture Fuzzy Sets and Their Applications in Decision-Making Problems. *J. Innov. Res. Math. Comput. Sci.* **2022**, *1*, 33–54.
48. Khan, Q.; Jabeen, K. Schweizer-Sklar Aggregation Operators with Unknown Weight for Picture Fuzzy Information. *J. Innov. Res. Math. Comput. Sci.* **2022**, *1*, 83–106.

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