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Characterisers of Teaching in a Mathematics Problem Posing Lesson in Preschool Education

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Abstract: Problem posing is rarely seen in Spanish classrooms, least of all in preschool education. The lack of research, coupled with a lack of guidance, has resulted in teachers simply not considering the approach as something that could enrich their work. This study examines a problem-posing lesson given by a preschool teacher to a class of four-year-olds. It focuses in particular on the organisational principles and the factors to be taken into consideration in applying the strategy. Through the application of a bottom-up analysis, seven characterisers are identified, four associated with task design and planning: Characteristics of the resources, type of problem, expression of problem formulation, and type of problem formulation; and three associated with lesson management: Stages in task execution, degree of liberty, and role of the pupils' drawings of the problem.

Keywords: problem posing; preschool education; *bottom-up* approach; mathematics teacher

1. Introduction

The traditional approach to mathematics teaching is for learners to be presented with tasks and problems that have been selected beforehand. Responsibility for providing these usually falls to the teacher, who either devises his or her own problems, or draws on textbooks and other sources for problems that can be used directly or adapted if necessary. The result is that pupils have a passive role to play in the creation of problems. Nevertheless, there is considerable evidence that giving students a more active role in the creation of problems not only develops mathematical thinking, but also enhances creative abilities [1–3]. Indeed, problem posing is a genuinely mathematical activity, facilitating thorough understanding of the various elements involved in the process [4]. Following Stoyanova and Ellerton, we understand problem posing to be "*the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems*" [5] (p. 519).

In addition to their potential for developing creativity and constructing mathematical knowledge, problem posing tasks can provide a valuable means of formative assessment [6], as well as foregrounding a variety of cognitive styles [7]. Silver and Cai identify three criteria for teachers to bear in mind when doing problem posing with their pupils: Fluency, which refers to the number of solvable problems generated which meet the parameters of the problem; originality, which concerns the mathematical creativity demonstrated; and complexity, which considers the sophistication of the mathematical relationships invoked, the degree of difficulty, the cognitive demands, and aspects relating to the complexity of the expression of the problem [8].

In general, there are four fundamental parts to a problem, which need to be coherently articulated by the person formulating it: The context, the mathematical environment, information, and requirement [9]. There are different approaches to setting students to formulate problems, among which two stand out: Basing the problem on one which has already been given or basing it on a situation [2]. Both

approaches involve adding or modifying elements of an established problem so as to create a new problem, which is validated when the other students, with the application of some degree of reflection, are able to find the solution. In the case that concerns us here, in which the students are given the context, three kinds of problem posing tasks are considered [5,10]: Free, in which no restrictions are placed on how to formulate the problem; semi-structured, in which students formulate their own problems based on ones they have been given; and structured, in which the students pose problems by reformulating ones they have already solved, or by making minor variations to the information or requirements given in the problem.

Although it is understood in educational research that problem posing is a useful strategy for promoting children's imagination in preschool education, and encouraging them to develop metaphors to link content areas with their out-of-class experience [11], there is very little research into mathematics education with this age group. One of the few studies available is the research by Pálmer and van Bommel into how three groups of preschool children (27 six-year-olds in total) in the Swedish education system posed new problems similar to one they had recently attempted, which involved calculating the number of cubes required to build a simple three-dimensional structure from a two-dimensional drawing in which some of the blocks were obscured by others, and were hence 'hidden' [12]. Because of the pupils' young age, no assumptions were made about their reading and writing abilities, and the task was planned so as not to require any writing on their part. Nevertheless, most of the pupils managed to pose problems that recognised the mathematical nature of the task, requiring some kind of counting of blocks in a construction. For his part, Lowrie, also working with six-year-olds, describes a framework for analysing the open-ended problems they generate. Although the study recognises that the one-to-one learning environment enjoyed by the researcher might be difficult to replicate, it foregrounds the importance of the teacher's role in in guiding the pupils to produce increasingly sophisticated problems within a short space of time [13].

Young children are especially sensitive to the context in which mathematical notions are presented to them, especially in the earlier stages of primary education compared with the later stages [14]. The importance of context is captured very well in the incident described by Baroody of a young boy who was incapable of carrying out simple operations such as '66 + 4' or '30 − 8' when these were presented as decontextualized additions or subtractions, but who had no problem in providing the correct answers when the same sums were connected to a real-life situation the boy could recognise [15]. The role of context, then, is essential for providing the meaning of the operation in question, as in the example, or more generally, of whatever mathematical notion is being invoked in the activity being carried out [16,17].

Another aspect of preschool education that teachers need to take into account is the children's tacit informal mathematical knowledge [15,18,19]. This is the kind of knowledge they acquire through playing games at home and at school, and is strongly linked practical, concrete experiences. Thus, in the incident described by Baroody above, the young boy's informal knowledge of basic arithmetic is another factor, which, connected to the context of the situation, accounts for his correct answer [15]. One way in which play can be usefully introduced into the classroom is when the teacher is able to make everyday routines an opportunity to develop mathematical skills, such as taking the register, which combines establishing a bijection with enumeration, along with all the subtasks this implicitly involves [20], or arranging the tables and chairs, which requires spatial awareness and the ability to estimate distances. This is one of the aspects of this stage that marks out the particular expertise of infant teachers in comparison with those in primary school [18,19,21].

Given how important it is for pupils (including those in preschool education) to pose mathematical problems, and aware of the limited research in the area, this study focuses on the teacher, and aims to identify characterisers for teaching practices that promote problem posing among their students.

2. Materials and Methods

Given that the aim of this study was to identify and understand the elements that lead to successful implementation of problem posing, the research paradigm it took was interpretative [22], and the methodological approach was that of an instrumental case-study [23], specifically the classroom practices of a preschool teacher. By this means, it was hoped to establish a series of categories for interpreting the rationale of the teacher's everyday practices when planning and carrying out problem posing with her pupils. The teacher, who will be referred to by the pseudonym Rosa, was chosen for her long experience in preschool education, as well as for her willingness and availability to collaborate. In fact, Rosa was a frequent participant in collaborative group projects with researchers and colleagues and was thus very familiar with the dynamics of qualitative research requiring a high degree of verbalisation [24], which brought a welcome degree of fluency to the research process. Rosa's linguistic and cultural background is similar to their students' context, implying that she is familiar with their day to day experiences.

The study was based on a transcribed video recording of a session with a group of 4-year-olds to which Rosa was tutor. Rosa's aim in the session was for the children to learn how to pose problems.

Rosa routinely asks her pupils questions in order to get them to articulate what they are doing while they are carrying out activities, and, as is typical in preschool education, she gets them to represent the outcome of any activity through the medium of a drawing. Because of the size of the classroom in which she works, Rosa usually divides the 24 pupils into three equal groups. For the session in question, she prepared four different scenarios and gave one to each group, keeping the fourth for the group which was first to finish. By this means, she was able to rotate the scenarios so that each group posed a problem in each scenario with very little time lost between scenarios. She also decided that each pupil should work individually within their group in order to ensure that all of them posed a mathematical problem.

In addition to the data supplied by the video-recording and the corresponding transcription, an audio recording was also made, and duly transcribed, of a semi-structured follow-up interview. This allowed the researchers to clarify various questions that could not be answered from viewing the recording, and to complete information deriving from the video-recording. In this respect, the interview was particularly useful for obtaining information relating to the planning stage of the task.

The process of data analysis followed a bottom-up approach [25] using the constant comparison method [26,27], which involves the continuous assignation of data items to conceptual categories created for the purpose. By means of this procedure, the transcriptions were divided into units of meaning, which were then assigned to categories [28]. The categories were under constant revision, bearing in mind the need for them to be disjoint and self-contained, and were given a provisional name pending alterations. The process was considered to have finalised when sufficient theoretical coverage had been achieved [27]. At this point, 7 categories had been established (which we called characterisers), which subsequent analysis suggested could be grouped into two broad organisational groups according to their role in the session: Planning or delivery.

3. Results

3.1. Task Design

In order to achieve her objective of getting the pupils to notice the key features of a problem and then pose similar ones, Rosa considered various aspects of the activity she intended to use in class. In the first instance, the mode she chose for carrying out the activity was that of presenting the pupils with different scenarios featuring items from their everyday experience that they could manipulate, and which would inspire them to pose a problem. The choice of items also took into account the kind of problems that the pupils would be able to formulate from them. The four scenarios she prepared were as follows:

- Dolls and sweets. This scenario featured two dolls and six sweets (Figure 1).
- Transport. This scenario featured a selection of three modes of transport, each a single colour (yellow, green, or red): Car, lorry, and plane. There was no correspondence between colour and type of transport, such that one mode might be represented in two or even three different colours (Figure 2).
- Variegated balls. This scenario featured balls of different sizes, colours, and textures. In this instance, no two balls had more than one characteristic in common (Figure 3).
- Geometric solids. This scenario featured a large range of geometric solids, each in a single bright colour: Prisms, cylinders, cones, and pyramids. The prisms were of two types, those with a rectangular base and those with a square base. As with the types of transport, each solid was available in at least two different colours (Figure 4).

Figure 1. Scenario featuring two dolls and six sweets.

Figure 2. Scenario featuring different kinds of transport.

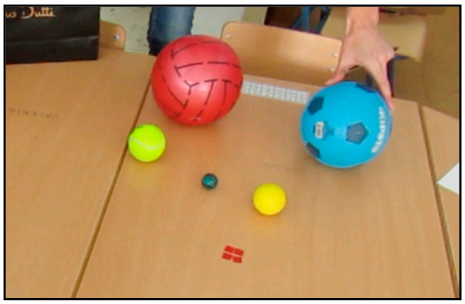

Figure 3. Scenario featuring balls of different textures, sizes, and colours.

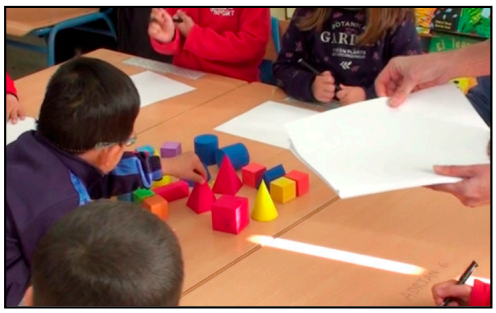

Figure 4. Scenario featuring geometric solids.

As mentioned above, in addition to ensuring that the objects were familiar to the children and small enough for them to grasp and change position by sliding them, flipping them over, or performing some other movement with them, Rosa had considered the kind of problem that each scenario might generate, this being an essential criterion in her choice:

> Interviewer: Did you expect a certain kind of problem in each of the scenarios?
> Rose: Yes, I wanted there to be different kinds of problems.

In the scenario featuring dolls and sweets, Rosa expected the children to pose problems involving arithmetic of some description, such as addition, subtraction or sharing out.

> R: Well, in this one [the scenario featuring the dolls] what I was aiming for was for them to do adding or subtracting . . . or that they would share them out equally or not equally.
> I: Did you expect just sharing?

> R: No, [also] the adding and subtracting, along the lines of "they eat this many", or "one has four and the other has two". I saw this one [scenario] as generating numerical problems.

The other scenarios were chosen to vary the kinds of problems and avoid the repetition of numerical problems. In the case of the scenario featuring types of transport, Rosa was nudging the children to pose classification problems:

> I: In this one (scenario), were you expecting the pupils to pose classification problems and arithmetic problems, addition . . . ?
> R: Classification, either by colour or say, "this one goes in the sky and this one . . . " or say, "this one has got two wheels and this one four". Otherwise they would all have been numerical problems.

Rosa also targeted classification problems in the scenario featuring geometric solids, in terms of their shape and/or colour, or according to the different heights of the shapes. Finally, in the scenario featuring different kinds of ball, she was hoping that, in addition to problems involving sequencing, the pupils would devise problems of classification, not only in terms of the different sizes and colours, but also the different textures of the material from which they were made:

> I: Did you think they could have classified them for other characteristics?
> R: Yes. This one and this one are plastic, the tennis ball is more . . . harder even, while the other is softer . . . I was hoping that they might classify them according to texture, say, or some other characteristic of the balls.

> I: Yes, so that would lead to . . .
> R: Right, that they could do . . . whatever came to mind.

Nevertheless, despite her intentions in this scenario (to foster problem posing related to the geometrical elements she included), in the follow-up interview, she recognised that the likelihood of the pupils devising a problem of classification or sequencing was possibly hindered by there being three variables (colour, size, and texture) on which to focus. By contrast, all the problems posed in this scenario involved numbers, principally counting in order to compare different numbers of balls.

> R: That's what I was aiming for. I thought they'd be able to do it . . . (. . .) Perhaps, had I found some balls that were the same, just different sizes, then putting them in order of size would have a been a viable problem. (. . .) And then perhaps you could say "well, this one has stripes, this one has . . . that way they could do . . . [different problems]".

In this extract, Rosa is of the opinion that had she given her pupils balls that were identical in colour but different in size, they would have been more likely to have come up with problems involving ordering items. However, this would have drained a degree of richness from her intended plan, which, as it was, allowed for a greater variety of problem types, not only ordering problems, but also a range of classification problems involving different criteria for making the sets of balls.

Hence, analysis of the aspects that Rosa considers while planning a problem posing task indicates how, first of all, she designs each scenario to take into account the kind of problem that can be formulated from it.

It can also be noted that it is not by chance that Rosa uses a given scenario. Her decisions with regard to the scenarios are a key element for ensuring that her pupils manage to formulate a problem and identify the elements of which it is composed. This aspect is essential given that the age of the pupils restricts the kind of resources that she can use and requires her to provide all the contextual information pertaining to each scenario.

Also worthy of note with respect to resources is that Rosa used materials that had two specific characteristics. On the one hand, they involved items familiar to the pupils and hence the problems were based on contexts that the pupils understood well, and on the other hand, they could be manipulated by the pupils, who could thus interact with them.

Finally, because of her awareness of the limitations of this age group, which meant that they could not present their problems in written form, Rosa decided to ask the pupils to draw a picture of their problem. She thus planned the activity in such a way that her pupils had to express the problems they posed in a pictorial representation.

In summary, Rosa took various factors into account in the planning phase, directed at helping the pupils to recognise the component elements of a problem, and to pose their own problems, from which it can be observed that four characterisers emerge:

- Type of formulation, in terms of Silver's classification according to which a problem is posed based on a previously given problem, or based on given scenarios, contexts, or experiences [2].
- Characteristics of the materials employed in the scenarios, regarding the familiarity for the pupils and the manipulability.
- The kind of problem in which the material is used in each scenario prompt, with regard to the kind of task required: Ordering, classifying, counting, and so on.
- Characteristics of the pupils with respect to the educational stage, which conditions the mode in which the pupils can formulate their problems, which could be verbal, pictorial, or written.

3.2. Task Management

Once the task for learning how to pose problems has been planned, the implementation of it was of special interest to the study as it provided data on how Rosa managed the task in class. The observation of emergent characteristics in the management of the problem posing activity allowed us to gain awareness of the characterisers that underpinned Rosa's session during the management of the task.

After giving each group the items comprising their scenario, Rosa gave a presentation of the activity for each of them according to the respective scenario. In so doing, she intended that her pupils understood what was involved in each scenario. This presentation consisted of a detailed description of the scenario. For example, in the scenario featuring the dolls and sweets, she said:

> R: Look: one and two children. They're dolls but let's imagine they're real. And here there are one, two, three, four, five, six sweets. We're going to draw this problem. Think what problem we might have here.
> Or in the scenario featuring the balls:
> R: Look, here I've got five balls. Invent a problem.

If any pupils failed to understand the scenario initially, she would ask him or her questions to guide them, such as "Let's see. What are these?" (referring to the different-sized balls) until she had made sure they had understood her.

Once she had presented the activity and described the scenarios to each group, Rosa began to establish conversations with her pupils in order to learn at what point in the formulation they found themselves. In these conversations, Rosa put the emphasis of her questions on understanding which elements of the scenario were going to take the role of elements in the problem (data and context). To do this, she used questions that directed the pupils to identify the number of elements (data) that there were in the scenario, with the intention of establishing a relationship between these elements and a specific context. Hence, she asked questions such as, "How many balls are there?" or "How many [sweets] are there?" with the aim that her pupils should identify the number of items in each scenario and so focus their attention on understanding the items as data in the problem and making connections between them.

After this, Rosa endeavoured to get the pupils to use the data and context she had just identified and to make connections and conditions between them. This moment is related to the time she allows the pupils to think about how to articulate the problem taking into account the elements mentioned above. She guides them with suggestions such as "think about what you want to do, what problem do you want to set?" when she encourages one of her pupils, who had managed to identify the data and context of the problem, to make a problem based on these. In another instance, Rose asks a pupil who had identified the data but had not managed to articulate a problem with them, "What can we do with these sweets?" leading him to establish connections that would enable him to continue the process of problem posing. In the exchange below, a pupil (P1) briefly describes a situation, but without a question. Rosa encourages him to formulate a question for the problem:

> Pupil 1: A man has to collect some money.
> R: And what does he want to do with the money?

After spending some time encouraging the pupils to think about the different elements of the problem, the next stage that Rosa aimed for her pupils to tackle was that of articulating it. This marked the close of the problem formulation stage in which the teacher helps the pupils to go over the problem and check that it contains all the necessary elements to be considered as such. In this way, Rosa makes sure that all the problems posed by her pupils include data, contexts, and questions. Although the pupils managed to articulate a statement concerning the data comprising the problem, many of them failed to actually pose a question about an unknown element, the formulation of the problem thus remaining incomplete. In these cases, Rosa intervened to rectify the situation, following the pattern exemplified in the case of the scenario of the dolls and sweets which lacked a question: "First is 'There are two children and six sweets'. And now the question is 'How many . . . ?'".

In the example below, deriving from the same scenario, another child (P2) takes the problem statement as completed without realising that he has to formulate a question. The episode begins when the pupil, assuming the problem posing to have been completed, mentions that the two children in the scenario are going to eat all the sweets:

R: Who is going to eat them all?
Pupil 2: Him and him.
R: They're going to eat all six sweets between the two of them?
P2: Yes.
R: OK. So what can we ask?

After assuring herself that the pupil distinguishes the data from the context, Rosa interacts with him so as to lead him to pose a question. She thus shows her interest for getting the pupils to formulate problems containing all the required elements, including a question. It was not always the question that was lacking in the incomplete formulations; Rosa also intervened when other elements had been omitted, such as the question statement, which caused her to say, "Come on, let's invent the story [of this one]."

In this kind of intervention, Rosa checks the extent to which both the aim of the problem posing is achieved, and the elements of the problem are identified. It also illustrates another aspect of the organisation of the task—the need to go back over things to check that the formulation contains all the elements of the problem, even when it appears to be well formulated.

The final step that Rosa incorporated into the process of problem posing was that of making sure the problem was a plausible one. In order to do this, once the pupils had formulated their problems, they attempted to solve them to see whether they were coherent in terms of structure, data, conditions, and question. Once again, Rosa's interventions take the form of questions about the scenario on which the problem is based. For example, in the scenario featuring the dolls and sweets, Rosa asks one of the pupils whether the operation employed for solving the problem was sharing them out: "So what happened? Did you share them out?" She also asks about the final outcome of the problem, trying to get the pupils to check whether their problem has a plausible solution, as in the case of the scenario featuring five balls: "And how many were left? If there were five . . . " Another example occurs again in the scenario featuring the dolls and sweets, in which the following exchange takes place between Rosa and a pupil (P3):

R: How many has this one got?
Pupil 3: Three.
R: And how many has this other one got?
P3: Three.
R: Very good, three and three.

To round off the session, once the three groups of children had posed a problem for each of the three scenarios, Rosa led a closure session in which she asked them what they had learned from the activity.

R: What have we learned?
Some Ps: ¡To do problems!
Other Ps: ¡Mathematics!
R: Have you learned to do problems, P1?
P1: (nods) (up and down)
P2: Me, too.
R: What else have we learned?
P3: I know what I've learned, I know what problems are for.
R: You know what problems are for?
P3: The children who know problems show them to others and then the other one knows.
R: Ah, look. ¿Did you hear what he said? Say it again, what are problems for?
P3: So that the children who know—and the children who don't known—show it to the those who don't know so that the ones that didn't know, now they know.
R: Very good.

The pupils who are first to respond to Rosa's question say in unison that what they have learned is to do problems. The verb they use—*do*—should not be confused with other forms of referring to the task, such as *solve*. The lesson focused on problem posing and for the children this problem posing is what they *did*. It would seem, then, that most of the pupils achieved Rosa's aim of learning to pose (do) problems.

At the same time, Rosa takes advantage of P3's contribution to have him explain to his classmates what he had learned from the activity. Although the actual phrasing might appear convoluted, Rosa recognises that his reflection encapsulates a notion that is well worth underlining to the rest of the class. Problem solving, as the pupil points out, is not simply an activity to be learnt, but also a means of showing things to others that he might know which they do not. Essentially, his intervention affirms that problem posing enables pupils to provide their classmates with new knowledge.

Up to this point, we have focused on the four steps Rosa employs to guide her pupils towards posing problems, based on identifying the composite elements:

- Understanding the nature of the objects making up the different scenarios that provide the context to the problem to be formulated.
- Making explicit the problem data and the question to be answered.
- Articulating the problem and all its elements (context, conditions, data, question).
- Checking the plausibility of the problem by attempting to solve it.

In the section below, we focus our attention on a particular aspect of Rosa's task management, which she brings into play in the course of the problem posing activity. This concerns the way she restricted her pupils' responses to the scenarios, not allowing any changes to the parameters, so that their problems were in alignment with those she had originally projected for each scenario.

At various points in the lesson, Rosa became aware of pupils using data or elements that were not strictly part of the scenarios she had set up. If we focus on the elements that Rosa introduces into the scenarios from which the pupils were to formulate their problems, we can see that the pupils could focus their attention on both the nature of the objects (balls, dolls, sweets, vehicles, and geometric bodies), and the data (number of balls, dolls, sweets, and vehicles; characteristics of the balls, vehicles, and geometric bodies, and so on). From here, the pupils could pose their problems using or not the context of the objects and their qualities.

Nevertheless, in this particular activity, Rosa intended the problem to be formulated using only the scenario and the data provided, as long as the result did not deviate too much from the kind of problem envisaged for each scenario. For example, at one point, she noticed that a pupil had drawn more balls on his paper than had originally been specified, and urged him to use the same number as in the scenario, the datum she expected:

R: Why are you drawing so many balls? How many balls are there?

Rosa encouraged her pupil in this way to pose his problem using only the scenario and data provided.

On another occasion, in the scenario featuring the dolls and sweets, Rosa noticed that a pupil had used seven sweets in her formulation rather than six. She approached the girl and asked her the following:

R: This is a sweet, isn't it?

Pupil 4: Yes.

R: And how many sweets were there?

A: Six.

R: Fine, it's OK. If you want you can cross this one out.

Here, Rosa intends to underline the need to use only the information available from the given scenario. The correction comes about as a result of Rosa's original framing of the problem, which was to involve sharing the sweets equally, hence the two dolls and even number of sweets. Rosa feared that a change in the starting data would undermine the formulation of a problem on the basis of sharing things equally [*if that girl was going to share things out it was going to be far easier with six sweets than with seven, as she wouldn't have one left over*], which is why she suggested crossing out the extra sweet, so as to leave open the possibility of posing a problem about sharing.

However, not in all cases does Rosa suggest the pupils reconsider the data they propose that is not explicit in the scenarios. This happens in the scenario featuring the different plastic toys representing different types of transport when a pupil (P5) draws more yellow cars than there actually are in the scenario.

In the drawing illustrated in Figure 5, it can be seen that Rosa has written the colour of each car next to the corresponding picture following the pupil's instructions. In total, there are five cars labelled as yellow in the drawing, despite there only being four yellow vehicles in the set of toys comprising the original scenario, two cars and two aeroplanes. In the follow-up interview, Rosa explained the apparent inconsistency of allowing this pupil to continue formulating his problem after she had previously disallowed the alteration of data:

> R: I guessed they were going to do a classification problem, and hence that boy was doing a problem of classifying according to colours, so it really didn't matter if there were two or three yellow ones. What I clearly saw was that he had intuited was that it was a classification problem.

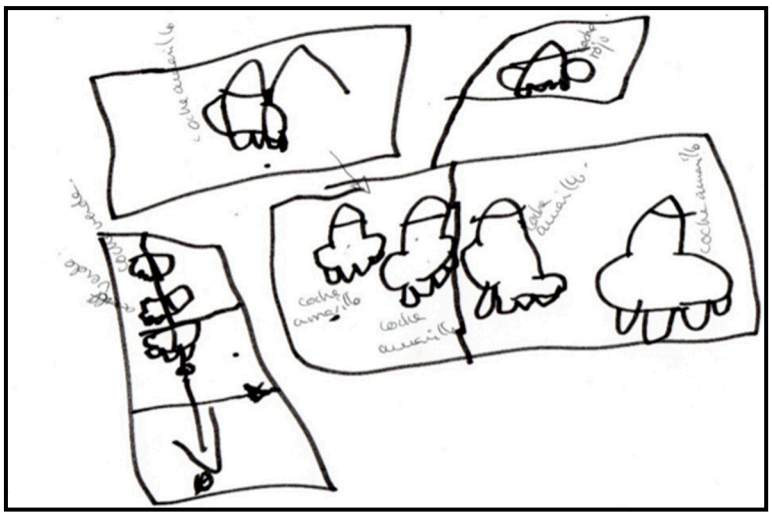

Figure 5. Drawing of different modes of transport (P5).

What distinguishes this response to the situation from her previous ones is Rosa's perception of what might bring about a change in the kind of problem that can be posed. If, in her estimation, an alteration in the original data given in the scenario might place a limitation on the kind of problem that can then be formulated, she redirects the pupil back to the original context in order to retain the potential it was designed to offer.

Rosa's vigilance of the scenarios is not limited to ensuring that the elements involved are properly deployed, she also makes sure that her pupils pose their problems adequately. Being aware that the majority of her pupils have not yet learned to read and write, she gives each child individual attention to ensure that the problem is properly formed, which proves to be especially valuable in this activity. Rosa summarises the procedure thus:

R: The children had to draw a picture of the solution to their problem, and to do so, they had to formulate the problem in their mind and try to capture this in their picture. What is important to note is that their drawings represent the end point.

In order to correctly interpret what is being represented, Rosa needs to ask each child about their drawing to explore the thought processes that lie behind it, and to discover the kind of problem each child was aiming to pose. For example, in the scenario featuring the balls, one pupil (P6) drew the balls in order of size. To make the drawing, the pupil drew circles representing each of the balls, doing his best to draw each subsequent circle smaller than the previous (Figure 6).

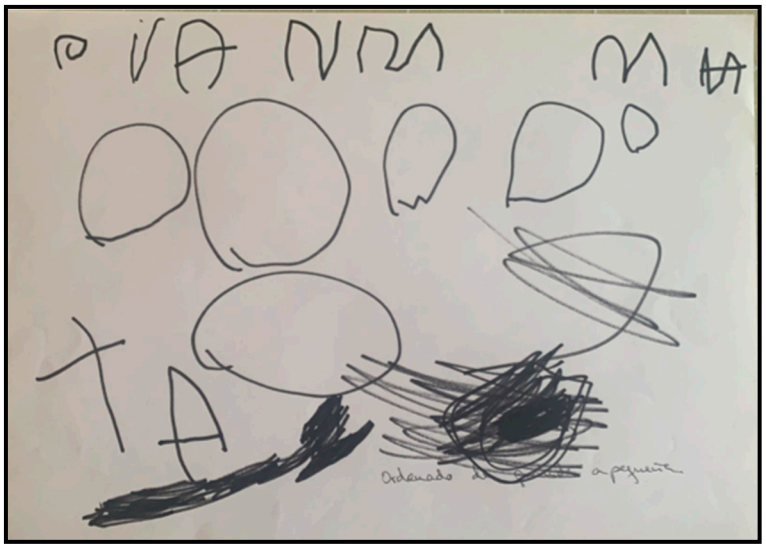

Figure 6. Problem for the scenario featuring various balls: Put them in order from large to small (P6).

However, although the pupil's intention was to draw the balls in order of size, it can be seen above that some of the circles are slightly smaller or slightly larger than he intended, with the result that the picture does not fully represent what he wanted. This was likely due to the fact that the children were still at a stage where their fine motor skills were developing. Realising that this was the case, Rosa questioned him to find out his thinking in drawing the picture that way, and the kind of problem it was intended to represent.

R: Have you put the balls in order? How have you done it?
Pupil 6: Yes.
R: Why?
P6: Because another ball goes here (pointing to a space next to the smallest circle).
R: And you've put them in order, first you've put the big one and at the end you've put the small one?
P6: (nods) (up and down)

In this brief exchange, Rosa gains confirmation of the procedure the pupil has followed to arrive at his representation of the solution to the problem of putting the balls in order that he came up with for this scenario, and is able to dismiss the possibility that the drawing was unrelated to the formulation of the problem. This provides her with implicit information about the problem posed in the pupils' drawing of the solution. At the same time, as well as learning whether he had managed to come up with a problem, Rosa wanted to know the kind of problem the pupil had posed. In this instance, through her questioning, she is able to establish that it is a sequencing problem.

Another example of how Rosa questions her pupils so as to find out information about the problem they have posed and the kind of problem they intended can be seen in the scenario featuring the different modes of transport. In this instance, Rosa questions the pupil (P5) who drew the picture represented in Figure 5.

First of all, she asks the following questions to find out the kind of problem the pupil has posed: "So how have you put them together? All the yellow ones together and all the green ones together?" and also, "What have you done and why?" She followed these questions up with another about the pupil's thought processes when going about his drawing.

It would appear that, at first glance, Rosa took the drawing to represent a classification problem, and so expected the answer to her question to confirm that it consisted of two vehicles of the same colour in each of the squares. However, further inquiry revealed that the pupil had drawn another yellow car separate from the others, and so she asked him to explain why he had done so:

> R: The yellow ones are here, aren't they? Why have you put this one over here? (pointing to another item which the pupil says is a yellow car)
> Pupil 5: Because it has to smash that rock.
> R: What do you want, to break this rock so that this yellow car can be with the other yellow cars?
> P5: Yes.

In sum, the pupil explains that in his problem, there is a yellow car that is separate from the group of yellow cars, and that needs to join them. What initially to Rosa appeared to be a classification problem, turned out, after a little careful probing, to be an addition problem ("There are four yellow cars in one place, and another yellow car in another place. If the yellow car which is by itself joined the others, how many cars would there be in total?"), albeit disguised as a classification problem in its pictorial representation.

Although with a little digging Rosa uncovers an additive structure to the problem, she continues to suspect that the pupil's drawing started life as a picture of a classification problem under the heading "group the cars according to colour". The reason for this, she said, was that the pupil in question was "a fidgety child who lived in his own world." In her opinion "he tried to do a problem of classification by colour," but, being this particular pupil, he used more yellow cars than there were in the scenario, and when he realised this as a result of the teacher's questions, "he rectified his error by saying 'this one is going over there.'"

Returning to the lesson, and continuing with the individualised monitoring of the pupils' progress, Rosa questions them in order to check whether they have achieved the learning objective; that is, whether they have managed to articulate problem statements that include all the required elements (story: Data, conditions, and context; and question). This is exemplified by the problem of one pupil (P7), represented in Figure 7:

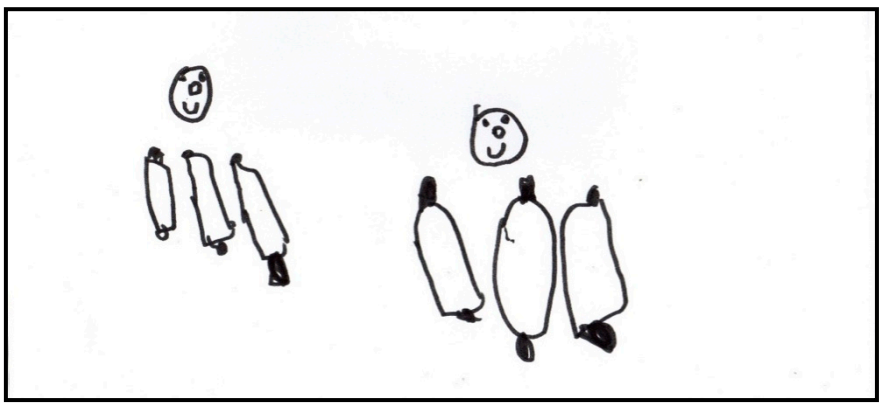

Figure 7. Problem for the scenario featuring the dolls and the sweets.

For her picture of the problem, the pupil drew two faces representing the dolls, and underneath each one drew three sweets. Rosa interpreted this configuration as representing the following problem: *There are two children and six sweets. How many sweets are there to eat* [for each child]? (P7). Although in this instance the formulation of the problem would seem to be amply evident from the drawing of the solution, Rosa wanted to check whether the pupil was able to articulate it for herself and include all the required elements, and so asked her some questions, the resultant exchange of which is reproduced below:

R: So, what happened? You've shared them out, haven't you?
Pupil 7: Yes
R: Come on, what is your story? The story is . . . there are two children, aren't there? What else is there?
P7: Three sweets. Three sweets for one child and three sweets for the other child.
R: But that's the end. The first thing is, there are two children and six sweets. And now we need the question . . . What will that be? How many . . .
P7: . . . sweets are there?
R: ¿Are there to . . . ?
P7: . . . to eat.

This exchange illustrates Rosa's concern to find out whether her pupil has taken into consideration all the elements making up the problem, there being various perceivable elements.

The first element that Rosa sets out to check has been included is that of the story. As the pupil refers to just one item of information (the two children), she asks what else [what other information] there is in the problem, aiming to find out whether the pupil's problem has more information or not. To this, the pupil replied that there were three sweets for one child and three for the other, demonstrating that she had also considered children as part of the information for the problem, although she stated them in the form of the solution. Having obtained a new item of information and being aware that the solution to the problem has also been prematurely provided, Rosa guides the pupil to supply, in addition to the context and data, the formulation of the question. She does this by asking the pupil further questions to redirect her and get her to finally formulate a problem containing all the elements. Further, although the pupil does not explicitly state who will eat the sweets, Rosa takes it as understood, given the pupil's previous comments (*Three sweets. Three sweets for one child and three sweets for the other child*) and her drawing (Figure 7), that when the pupil formulates the question as "how many sweets are there to eat?" she is referring to the number for each child to eat.

Finally, another aspect which Rosa checks is whether the pupils have really understood their own formulations, to which she asks about the results, such as in the exchange below:

Pupil 8: There were five [balls] and the woman wants two and two.
R: And how many left? If there were five . . .
P8: Zero.

In this instance, Rosa realised that the problem the pupil had come up with was not coherent, and so questioned her to find out her thinking behind the formulation. In the above exchange, she asks the pupil to give her the solution to the problem she had posed. From the result, Rosa is able to check whether or not the pupil's problem makes sense. Her doubt about this pupil might derive from the use the pupil makes of the number "four" as "two and two". For this reason, so as to check that the pupil has understood the problem, she asks how many balls are left in the end. However, although the correct answer to the question would be "one" (the result of the operation 5-4), the pupil answers "zero", thus confirming Rosa's doubt and allowing her to redirect the pupil's formulation of the problem.

The description of the activity, together with the actions carried out by the teacher, illustrate different aspects that we can take into consideration and that will allow us to generate the characterisers on which the management of the activity was based.

As we have seen, the pupils' responses to the activity were governed by the way it was presented and how they were allowed to interact with the scenarios. In this respect, it is possible to identify a degree of liberty in terms of the freedom granted by the teacher to the pupils with respect to the data, the context, and the type of problem in each of the starting scenarios. Hence, the problem to be formulated has a greater or lesser degree of similarity with the scenario according to the leeway permitted by Rosa. This leeway varies depending on whether she regards the pupil as being capable of identifying the data and context of their scenario, along with the kind of problem implicit in each situation. Where she considers that the pupil is not capable of doing so, she prevents them from straying from the original configuration.

Analysis of the complete implementation of the problem posing task shows Rosa's lesson to be structured according to different stages, which emerge from observing the way in which she manages the task over the course of different types of interaction with the pupils. We have identified five such stages: *Understanding the situation*, the aim of which is to ensure that the pupils understand the underlying situation of each scenario; *Identifying the elements of the problem*, directed at getting the pupils to identify the information pertaining to each situation; *Designing the problem*, whereby the pupils are given a time limit to find appropriate connections between the information and the context which will enable them to construct a problem; *Articulating the problem*, by which the pupils, once they have constructed their problem, become aware of whether it contains all the required elements; and *Checking the problem*, aimed at solving the problem so as to check its coherence and plausibility. Furthermore, although the ultimate goal of working through each of these stages is the formulation of a problem, the progression does not have to be linear, and pupils can cycle back through previous stages when necessary.

At the same time, with respect to the pupils' use of the scenarios, the teacher guides them towards a strict reproduction of the original parameters described by the scenario. There is, however, a *Degree of liberty* that varies with the teacher's intentions and objectives. In this case, she consistently directed her pupils to use the information derived from the scenario, and only allowed pupils to vary elements when they had no influence over the kind of problem that could be formulated. In this regard, it could be considered a degree of *conditional* liberty.

The final characteriser that emerges from the problem posing task is that of the *Role of the pupils' drawings*. According to the teacher, these drawings illustrate the solution to the problem the pupil is trying to formulate. The teacher's intervention regarding these drawings is crucial, directed on the one hand at finding out the *Type of problem* each pupil has posed for a given scenario, and on the other hand at identifying the *Elements of the problem* that have been taken into account in its formulation. Both characterisers are connected with the teacher's previous intentions. The *Type of problem* (classification, ordering, sharing out, addition and subtraction, and comparison) is linked to the configuration of each scenario, while for its part the *Elements of the problem* (context, data, conditions, question, and result) is linked to the aim of teaching the pupils what a problem consists of.

In summary, the implementation of Rosa's lesson provided the following three emergent characterisers:

- Stages of task development: These are the five stages into which the teacher structures the problem posing activity.
- Degree of liberty: This refers to the leeway offered (or denied) by the teacher for pupils to use information outside that included in the original configuration of the scenarios.

- Role of the drawings: This concerns the importance of the pupils' drawings in determining the kind of problem pupils pose, and the elements of the problem used in its formulation.

4. Discussion

We have approached problem posing from the perspective of the role of the teacher in planning and executing a lesson on how to formulate problems.

From careful observation of the teacher's practice, we have identified a series of characterisers that provide structure to the process of formulating the problem. Some of these patterns are contingent upon the peculiarities of the educational stage in which the children find themselves: The pupils are just beginning to develop important cognitive and fine motor skills, their knowledge is mainly the one they have built through play and in their families, and they have yet to master reading and writing. We have shown the influence of these peculiarities on the management of a problem posing class, and the way the task is carried out, such as problem statements implicitly encoded in a drawing illustrating the solution, and the need for the teacher to constantly probe different elements of the formulation through asking key questions.

In our approach to understanding various elements characterising the teacher's work (encouraging problem posing among her pupils), we have considered two organisers: The planning or design of the problem-posing task, and the classroom execution of this. The process of analysing the planning and carrying out of this lesson, along with a follow-up interview with the teacher, led us to identify four characterisers in the planning phase, and three in the execution phase. In turn, these characterisers, render their own sub-characterisers (Table 1).

It is useful at this point to distinguish the nature or orientation of the characterisers we have identified. Those associated with the task design correspond to what the teacher takes into account with respect to resources (the teaching materials used) [2,5,10], the mathematical content in play (the type of problem), the medium used to communicate the formulation of the problem (the form of expression), and the configuration of the starting conditions on which the problem is to be based (type of problem formulation) [18,19,21]. Conversely, the characterisers associated with the lesson management are associated with the teacher's aim for the pupils to learn to pose problems consistent with the demands of the starting situation (stages in task execution), as well as its limitations (degree of liberty), and the role conferred on the pupils' drawings (role of the pupils' drawings) so as to obtain information about the mathematical content of the problem (problem type, as in task design), and the elements constituting the problem statement (elements of the problem) [9]. With respect to these elements, contemplated by Malaspina [9], it is worth underlining that Rosa places considerable importance on checking the result as, in some instances, it enables her to learn whether the pupil's formulation of the problem is plausible or not. In terms of the mathematical environment, it can be seen that Rosa prepares different scenarios in anticipation of the kind of problem the pupils might devise in each. Consequently, the mathematical environment is associated with the kind of problem the teacher expects her pupils to mathematical environment formulate in each of the scenarios. Likewise, it can be noted that the scenarios are based on semi-structured situations, which present the pupils with a context that they then have to associate with previously acquired mathematical elements in order to pose a problem based on the given situation [5,10].

Table 1. Summary of characterisers.

Organisers	Characterisers	Sub-Characterisers
Task design and planning	Characteristics of the resource	Familiarity Manipulability
	Type of problem amenable to the resource	Classification Ordering Sharing out Comparison Addition and subtraction
	Form of expression of problem formulation	Pictorial Spoken Written
	Type of problem formulation	Structured Semi-structured Free
Lesson management	Stages in task execution	Understanding the situation Identification of the elements of the problem Problem design Articulation of the problem Checking
	Degree of liberty	Strict (information must be exactly as stated in the scenario) Conditional (some information can be changed if appropriate) Free (external information can be used alongside information provided in the scenario)
	Role of the pupils' drawings	Problem type Ordering Classification Comparison Sharing out Addition or subtraction Elements of the problem Data Context Conditions Question Result

5. Conclusions

This study has explored some of the features to be taken into account when planning and delivering problem posing tasks with preschool children. The adaptations that Rosa makes to the problem-posing methodology typically used in later educational stages are closely bound up with the age group she is working with. Among these can be found, for example, the need to use semi-structured situations so that the pupils can begin to formulate their own problems based on a given situation. One remarkable issue is that, even though the main aim of Rosa is to develop the knowledge of the pupils about problems' parts, the students mobilize knowledge about addition and division, which is related to children's potential to face mathematical situations [15]. Another aspect that Rosa takes into consideration in her planning is the involvement of the pupils with the various scenarios they are presented with. This aspect proved to be essential in encouraging the pupils to pose their own problems by providing them with familiar scenarios which, in addition, were highly tangible and could easily be manipulated by them. For her part, throughout the lesson, Rosa restricted the information on which to base the problems to that given in the scenario. Only in very specific cases did she allow certain pupils to modify this information in any way. Her strictness in this regard can again be accounted for by the age and cognitive development of her charges. It cannot be concluded that these considerations

are exclusive to the preschool stage, but it is very likely that at subsequent stages, where the children are more developed, these methodological restrictions can be relaxed. In future developments of this research, it could be interesting to explore how Rosa's noticing and aims have implications in her decisions about the information given to her students.

Unsurprisingly, working on problem posing with preschool children might be considered a priori a tall order, given their cognitive level. Nevertheless, this study demonstrates that not only is it possible, but that different organisational aspects can be taken into account both in the planning and delivery phases of the lesson so as to maximise the pupils' engagement and learning. This study contributes to the work of previous research by providing a perspective that focuses on elements that are not purely mathematical but rather didactic-mathematical. It thus foregrounds the need to broaden the scope of research into problem posing so as to go beyond the predominant focus by which teachers at all educational stages are considered solely as subject specialists for teaching content.

We hold the view that problem posing is a valuable approach at all levels of schooling. It merits not just a place on the curriculum, but specialised training to help teachers incorporate the problem posing dynamic into their lessons. In this regard, the analysis and categorisation of the organisers relating to problem posing tasks could be a useful tool to include in training programmes for preschool education. The characterisers presented in this paper represent an initial cluster of elements on which to focus in the training of prospective preschool teachers, providing them with tools for the planning and management of this kind of lesson, and enabling them to bring cognitively demanding work into their classrooms. This, in turn, would help to lay the foundations in terms of knowledge and creativity for the pupils to face mathematical tasks at future stages. Parallel to relating this research to teacher education, this paper opens a research line into characterising teachers practices on problem posing in preschool, that can be approached, for example, through future elaborations and refinements of the characterisers here developed.

In connection with this study, and a broader-based research project, we are currently studying the kind of knowledge that is mobilised when teachers take a problem-posing approach in their classes. Research into this knowledge (including beliefs and conceptions) is fundamental to understanding the complexity of teaching, and to ensuring that teacher training (initial and in-service) remains relevant and beneficial, not only to the prospective teachers, but also to their future pupils.

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