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Examples of Problem-Solving Strategies in Mathematics Education Supporting the Sustainability of 21st-Century Skills

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Abstract: The overall aim of education is to train proactive, motivated, and independent citizens to face and overcome continuous challenges. Critical thinking—finding solutions to problems—is of primary importance in the 21st century to handle challenging situations and deal with obstacles in careers. A critical literature review approach was used to assess, critique, synthesizes, and expand the theoretical foundation of the topic. Teaching mathematical problem-solving is an efficient way to develop 21st-century skills and to give cross-curricular experiences with real-world meaning to learners. Concrete examples were presented to prove that Pólya's heuristic could be used in a broader context to help learners acquire the modern skills needed to succeed in their careers. By including in the learning process and practicing specific methods for solving mathematical problems, students could learn a way of thinking to approach and solve problems successfully in a broader context in life. The paper's outcome provides teachers and educators with methods, learning models, and strategies for developing 21st-century skills in students at all levels during classroom activities.

Keywords: problem-solving strategy; critical thinking; 21st-century skills; heuristic; sustainability; mathematical education; cross-curricular integration; transversal skills

MSC: 97C70; 97D50

1. Introduction

In the 1980s and 1990s, many research papers predicting that future growth will depend on new competencies, abilities, and skills at the workplace were published. The statement made by Davis in his research paper from 1988 came true and portrays today's society [1] (p. 139): "As compared to the medieval world or the world of antiquity, today's world is characterized as being scientific, technological, rational, and mathematized." This "mathematized world" in which we live today is dynamic and rapidly changing. The expansion of the web in the last two decades from the web of things to the web of thought presents a real provocation across economies and demands a qualified workforce. Hence, different organizations, educators, businesses, governments, researchers, and professionals worldwide have started to identify the skills needed and to analyze when and how they can be taught to provide appropriate pedagogical methods. Scientific expertise grows at warp speed, which leads to questioning of what skills are essential for students in order to face work challenges and to become successful citizens.

Sustainable development needs innovation to drive competitiveness, and it demands technological infrastructure to create an environment where the latest technology and knowledge can blend to trigger new approaches and solutions. It is widely acknowledged that 21st-century knowledge is an enabler of growth, that performance is redefined, that economic growth, sustainable economic development, social cohesion, and equality of opportunity depend on workforce skills, and that abilities need to meet the information-age requirements. Therefore, sustainable economic growth requires new qualities and different features, and solicits educational systems to equip learners and students with skills and competencies that help them manage change and generate and execute ideas through flexibility and initiative. These qualities are judged to be vital for navigating the complexities and uncertainties of the modern world; thus, they are labelled as 21st-century skills.

The present paper aims to demonstrate through concrete examples how problem solving based on mathematical content can foster the development of 21st-century skills, and it proves that teaching problem-solving strategies through mathematics can develop complex problem-solving skills that strengthen critical thinking. The goal of the paper is not to analyze the challenges of mathematical problem solving in-depth, but to prove the relevance of the Pólya method in the context of 21st-century problem solving, to provide ways to develop concepts and strategies, and, thus, to develop competences through mathematics education at universities. Its outcome provides teachers and educators with methods, learning models, and strategies for developing 21st-century skills in students during classroom activities.

The research methodology adopted is a secondary qualitative methodology, led by the following questions: What are 21st-century skills? How are they defined? How could mathematics education help in developing 21st-century problem-solving skills? What methodology and methods could be introduced in teaching and learning mathematics to enhance the cross-curricular integration of problem solving?

An integrative literature review was conducted to evaluate the theory, provide an overview of the knowledge base, and examine the validity and accuracy of certain mathematical theories in order to answer the research questions. To make the greatest contribution to the paper, we used the following criteria: reliable sources of content (international institutions and scientific literature databases), relevance to the topic, and novelty and innovation in the field. The primary keywords used during the research were 21st-century skills, problem solving, 21st-century skill learning strategies, 21st-century skill learning models, and teacher readiness to implement 21st-century skills.

The article is organized as follows: Section 2 defines what 21st-century skills are and why are they are important; Section 3 presents a short overview of mathematical problem solving and reveals its connection to 21st-century skills. Section 4 contains some relevant methodological examples, learning strategies, methods, and models of the applications of problem-solving strategies in mathematics and shows how the Pólya heuristic principles work in the examples presented, proving their relevance and that it could become a tool to develop 21st-century skills. Lastly, Section 5 presents the discussions, conclusions, and further research possibilities.

2. Scope of 21st-Century Skills

Technological development is broadly recognized as a driver of change, and it is assumed in [2] that "information technologies are affecting jobs, skills, wages, and the economy." In this sense, technology requires the elevation of the cognitive abilities of the workforce. Furthermore, globalization demands international competitiveness and is deemed as another change driver. The modern workplace needs new skills to an even greater extent, namely the ability to solve non-routine problems, complex communication competencies, and verbal and quantitative literacy [3].

At the "Meeting of Organization for Economic Cooperation and Development (OECD) Education Ministers" in 2001, topics on "investing in competencies for all" were debated, and it was highlighted that the knowledge economy is rapidly changing the demand for skills and competencies. As background material for this meeting, a special edition of the Education Policy Analysis (EPA) [4] was published, which explores the competencies required in the knowledge economy. Chapter 4 lists the "workplace competencies" needed in a knowledge economy: "communication skills, problem-solving skills, the ability to work in teams, and information and communication technology (ICT) skills", with the mention that these "are becoming important and complementary to the basic core or foundation skills." Thus, the new transversal skills are communication, problem solving, teamwork, and decision-making [4]. Additionally, the working paper published by the OECD presents the issues related to the teaching and assessment of the 21st-century skills and competencies, proposing a new framework for thinking about these in terms of three dimensions: information, communication, and ethics and social impact. Concerning the information dimension, the following critical skills are identified [5] (p. 9): "research and problem-solving as they both involve at some point defining, searching for, evaluating, selecting, organizing, analyzing, and interpreting information."

The New Skills Agenda for Europe [6] emphasizes the strategic importance of skills "for sustaining jobs, growth, and competitiveness" and indicates that "formal education and training should equip everyone with a broad range of skills ... these include transversal skills such as ... critical thinking, problem-solving." The European Skills Agenda for sustainable competitiveness, social fairness, and resilience published in 2020 is built on the Recovery Plan for Europe [6] (p. 4–5), and highlights that [7] (p. 13): "Beyond technical skills, the labor market needs transversal skills like working together, critical thinking, and creative problem solving."

The European Skills, Competences, and Occupations (ESCO) outlines eight broad categories, each comprising a number of specific skills sets. In the first place is the category consisting of "communication, collaboration, and creativity" to which belong 16 skills, including "solving problems", defined as [8]: "developing and implementing solutions to practical, operational, or conceptual problems which arise in the execution of work in a wide range of contexts." The subset of skills that belong to it are solving problems, developing solutions, and implementing new procedures or processes [8].

There are various definitions, debates, initiatives, teaching methods, and assessments presented and examined by different interested groups (educational researchers, policymakers, teachers, employers), such as the P21 Partnership for 21st-century learning [9], National Research Council (NRC) [10], Assessment and Teaching of 21st-century skills (ATC21) [11], and Applied Educational System (AES) [12]. ATC21 [11] defined ten 21st-century skills and grouped them in four categories under the name KSAVE (knowledge, skills, attitudes, values, and ethics). The first category includes ways of thinking (creativity and innovation, critical thinking, problem solving, decision-making, learning to learn, metacognition), the second one includes ways of working, the third one includes tools for doing work and, lastly, ways of living in the world. The ACT21 white papers were published by Springer Science and Business Media, and the volume was edited in 2012 [13]. Since 1994, the "Generation Ready" organization [14] has helped educators to train students to be ready for life's challenges. Moreover, to assist students and learners in staying competitive in the changing world and labor market, the AES identified 12 essential skills that must be acquired during education. It is claimed that these "21st-century" skills are needed to prevail and succeed in careers through the information age. The considered skills were grouped into three categories, namely: learning skills (critical thinking—finding solutions to problems, creativity, collaboration, communication), literacy skills (information literacy, media literacy, technology literacy), and life skills (flexibility, leadership, initiative, productivity, social skills). These skills are universal requirements for the 21st century; however, their importance varies from case to case, depending on each individual's career aspirations.

The National Science Teacher Association (NSTA) reinforces the definition of the 21st-century skills provided by P21 and NRC, which incorporates core subject knowledge, learning and innovation skills, information, media, and technology skills, life and career skills, adaptability, complex communication/social skills, non-routine problem solving, self-management/self-development, and system thinking. According to Ouedraogo et al. [15], problem-solving skills have an impact on innovativeness: “Creative problem-solving affects innovation outcomes and non-financial performance when innovative culture is high.”

The NSTA observed a strong correlation between 21st-century skills and science education, thus recommending them to be included and taught in core disciplines. The NSTA Board of Directors adopted the Declaration for Quality Science Education in June 2011 [16].

Furthermore, digital technologies are rated as one of the most powerful driving forces in the modern economy, making the workplace a mathematized environment. Davis argued that, in everyday life, citizens need to understand the ways mathematics influence and shape their life. “Computerization represents the effective means for the realization of current mathematizations as well as an independent driving force toward the installation of an increasing number of mathematizations” [1] (p. 139). In this context, it can be understood why core subjects listed in the P21 framework include mathematics and why it is considered essential for all students in the 21st century. The P21 [9] was founded in 2001 in Washington, D.C., and the organization, in collaboration with businesses, educators, governments, and parents, aims to ensure that all students are qualified for career readiness and global citizenship.

There is no doubt that 21st-century skills can be acquired and taught through mathematics, since teachers can actively engage students and learners in complex mathematical tasks and develop students’ strategic thinking, which impacts both personal and professional life circumstances.

Since cross-curricular thinking is required at modern workplaces in the information age to encourage collaboration in different fields, it is recommended to promote and enhance the cross-curricular integration of mathematics.

Nevertheless, math teachers have to use innovative, sustainable teaching strategies, and in this respect, they need to define problems and assignments differently.

The P21 Math Map is a fundamental toolkit that helps students recognize the utility of mathematics and better understand the world. The 21st-century skills map was released in collaboration with the nation’s math educators and includes examples of how, through mathematics teaching in K–12 education, 21st-century skills can be developed [17].

Jang investigated and studied the STEM (Science, Technology, Engineering, and Mathematics) education and how it covers the 21st-century skills required in performing work-related tasks [18]. He identified relevant 21st-century STEM competencies using data from the workplace, which he consequently classified. By being aware of the required skills, STEM educators can develop reformed curricula that connect education and work, thus helping them train better-suited graduates for their future careers. “First, STEM workers appear to be required to have higher-order thinking skills, such as critical thinking, complex problem solving, and judgment and decision making. Specifically, they are required to solve problems using skills of mathematics and science. STEM workers need to use logic and reasoning to identify the strengths and weaknesses of alternative solutions” [18]. Therefore, STEM graduate experts recognize cognitive skills (non-routine problem solving, system thinking, and critical thinking) as increasingly important in the labor market. Moreover, according to Koenig, cognitive skills require reasoning skills [19].

The critical review of the literature presented above reveals that problem solving is one of the most valuable skills progressively expected from job seekers in the modern workplace.

3. A Short Overview of Mathematical Problem Solving

According to Pehkonen [20], the primary role of teaching mathematics in many countries across all age groups is to foster the students’ understanding of mathematical structures and to develop mathematical thinking [21]. Moreover, he claims that teaching has to support the learners’ mathematical thinking and to provide a basic understanding of mathematical concepts and constructs that will give them a foundation to deal with information and solve problems. Thus, it means that it is not enough to practice mechanical skills in mathematics, but to practice mathematical thinking and to solve real-life challenges are equally critical. In many countries, this is the primary purpose of school teaching, and one great example is the United States [22]. Hence, the goal of teaching should be to train active independent citizens who are proactive, motivated, and able to think critically to help them overcome obstacles that they will encounter later in their life.

According to Rosli et al. [23], the shift in learning theory from behaviorism to constructivism has had a significant impact on the teaching and learning of mathematics. Many mathematics educators developed numerous constructivist theories to gain knowledge in the subject, such as Hejny [24].

If students have the possibility to develop their mathematical knowledge, and they do not only see quick mathematical solutions, then their know-how can be expanded and can become sustainable. It is vital, according to Mason [25], to cultivate and sustain mathematical thinking in students; thus, the tools used in this regard can make them sensitive to their learning experiences in mathematics education.

The first results on this subject were published with the title "How to solve it" by Pólya. In his most cited book [26] published in 1945, the Pólya heuristic method developed for mathematical problem solving had a significant impact on mathematical thinking and methodology development. Pólya's strategy can be considered an accumulation point of mathematical thinking and had a powerful influence on the research done by his followers who implemented his approach in their work.

In 1980, Schoenfeld introduced a framework [27] for analyzing how and why some people are successful in problem solving and others not. A review of the evolution of problem solving and recent trends is presented in the papers by Voskoglou [28,29]. According to Kalmykova [30] (p. 49), Bogolyubov considers "both the analytic and synthetic method negatively if operating in isolation, contrasting with the case if the processes of analysis and synthesis interact." In contrast, Pólya's strategy is an attempt to combine the two approaches. The authors consider that the first step, "understanding the problem", is, in fact, the primary analytic breakdown of the problem, while the steps "make a plan" and "execute the plan" are related to the synthetic breakdown. The final stage, "feedback", is rooted analytically, so the analytic–synthetic method is closer to Pólya's strategy [26].

Mathematics is, in fact, a complex problem-solving activity, not just linear thinking. Walle stated that, "Teaching mathematics through problem-solving generally means that children solve problems to learn new mathematics, not just to apply mathematics after it has been learned. This helps children develop relational understanding" [31] (p.12). Moreover, "The importance of problem solving in learning mathematics comes from the belief that mathematics is primarily about reasoning, not memorization. Hence, problem solving allows students to develop the understanding and to explain the processes used to arrive at solutions, rather than remembering and applying a set of procedures" [32].

In 2016, a literature review of the critical debates on mathematical problem solving was published [2]. According to English and Gainsburg [2], the practice of problem solving develops unique abilities in learners.

The authors examine ways how it can be taught to make students successful problem solvers in the 21st-century and how mathematical content can foster general skills. Some remarks on the principles presented in their paper [2] are:

- Since additional mathematics courses are not an optimal solution, it is more useful to implement the teaching of problem-solving strategies in the existing mathematics education.
- Specific noncognitive and general skills (that are typically underpromoted in education) are highly expected in work. Many of them belong to a high cognitive level, so it is worth paying attention to their improvement during mathematics lessons.
- It is essential to study different facets of some mathematical notions, since they help to understand and design conceptual models that represent the basis of processes and systems in IT fields.
- Mathematics should not be an isolated discipline. Students have to be able to connect knowledge from different mathematical disciplines; for example, to connect algebra and geometry to other knowledge.
- Employers highly favor the ability to apply one's knowledge and experience to novel, unfamiliar situations. This is presumed to be most effectively fostered when learning occurs in work-based contexts or when replicated in schools. International studies such as the TIMSS (Trends in

International Mathematics and Science Study) and PISA (PISA is the OECD's Program for International Student Assessment) emphasize this dimension in mathematics education using nonroutine tasks from real life.

Rahman [33] describes the concept of problem solving and a theoretical framework of problem-solving skills with a literature review in his paper from 2019.

Evans and Swan [34] present the value of critiquing alternative problem-solving strategies, advising a change in the "triple X" teaching—exposition, examples, exercises—with learning mathematics by understanding, critiquing, comparing, and discussing multiple approaches to a problem.

Lester and Cai [35] provided an analysis of 30 years of research on teaching problem solving, and highlighted that problem-solving skills cannot be taught separately; they must be incorporated in mathematics training. Moreover, teaching problem solving through mathematics involves giving math depth and illustrating concepts and problem-solving techniques [35] (p. 120): "Developing students' ability to solve problems is not an isolated instructional act or a topic that is covered separately from the rest of the math curriculum...Learning of substantive mathematical content and developing problem-solving skills cannot be separated from one another; problem-solving should be infused into all aspects of mathematics learning."

Schoenfeld [36] stated that problem solving is a goal-oriented "acting in the moment" activity like "teaching, cooking, and brain surgery." Furthermore, Hung and Seokhee [37] indicate that students and learners can grow and nurture their mathematical thinking by themselves by expanding the complexity of their mathematical skills through problem choices that match their interests. Any idea or solution gained by students through independent internalization will remain in the long term as an active part of their knowledge and mathematical arguing. The nonroutine problem-solving skills and strategies acquired through a sustainable mathematical education can be used to solve real-life problems at the workplace.

By analyzing the literature on 21st-century skills and conceptual frameworks, the main finding was that problem-solving skills will be in demand and highly sought after in the future. Thus, this paper concentrates on finding ways to develop them through mathematical problem solving to bridge the gap between solving math problems and real-life problems.

Moreover, the success of the development of 21st-century skills through education also depends on teachers' readiness [38]. Research papers in the field of mathematics education identify a gap between the effectiveness of developing 21st-century skills through learning and the teachers' preparedness. Therefore, educators and teachers need various learning models, methods, learning strategies, and formative assessments to be able to enhance the quality 21st-century skills in teaching [38,39]. Susilo et al. [38] affirm that, "Teachers need training on models and learning methods that can develop 21st-century skills."

References [17] and [40] contain materials on teachers' preparation, classroom instruction, and assessments. However, Bai and Song [41] argue in their work that, generally, books and publications fail to cover these areas.

The present paper's purpose is to address this failure and provide a tool for teachers that helps them develop strategies, models, and methods to infuse 21st-century skills in their teaching in the classroom.

4. Problem-Solving Activities and Strategies to Develop 21st-Century Skills

Problem-solving is the core component of mathematics education. Thus, in the practice of teaching mathematics, students are helped to solve problems; problem solving represents a powerful approach to expanding mathematical concepts and skills.

The authors agree with Novoselov, who states, "There is no general method for solving different problems. Problems vary in their conditions and difficulty, and demand to use different methods" [30]. Through selected examples, this paper intends to emphasize the applicability of Pólya's strategy for solving different problems. The chosen examples serve as a model for university-level in-service and future teachers on how to create their own models in the teaching process. Although the examples solve word problems and elementary arithmetic, the strategy behind them and exemplified in the paper could be adapted for various subjects and levels in mathematical problem solving and beyond, thus serving the cross-curricular aspect of 21st-century skills.

In this section, examples of problem-solving activities are presented from a methodical point of view, which shows how Pólya's heuristic and structure facilitate learners' acquisition of 21st-century skills. It is essential to improve these skills in order to master the phases of problem solving, to learn how complex choices can be made, to formulate and share relevant arguments, and to judge others' arguments and opinions. Moreover, they help learners to build up a diagram, a scheme, or a map of the problem, which sets the ground for dynamic and collaborative discussions.

Pólya's heuristic and structure not only give a cross-curricular experience and real-world meaning for students and learners, but they help teachers to facilitate fruitful discussions to develop problem-solving skills. The examples presented provide broader cultural and historical points of view, thus serving a more in-depth understanding. Dishon and Gilead [42] indicate that, "The cultivation of 21st-century skills should be embedded within local histories rather than generalized models."

4.1. Examples and Discussions. How Pólya's Heuristic Principles Work in the Case of a Special Problem

The problem, known as "Finding One Coin of 12 in Three Steps", and some different versions can be found in many places (see, e.g., the Math Forum [43]). The formulation of the problem on the cited Math Forum page is: "There is a pile of twelve coins, all of equal size. Eleven are of equal weight. One is of a different weight. In three weighings find the unequal coin and determine if it is heavier or lighter."

The twelve coins problem is a well-known problem, but somehow, it has a strange, long-lasting effect. The twelve coins problem was presented as a puzzle in the New York Times [44]: "My Dad proposed the coin puzzle when I was, like, 10 years old. After working on it for weeks, I gave up and asked him for the answer. 'Sorry,' he said, 'I just know the problem, not the solution.'" The problem appears on different social media pages, which proves its relevance; for instance: GeekForGeeks [45] deals with the generalization and theoretical background of the problem, offering ideas for the application of trees for those who wish to explore the type of problem. Nigel Coldwell's page [46] reflects how the different solutions are discussed by interested people, including several theoretical approaches as well.

The choice of the twelve coins problem is partially due to the high interest expressed by the students of a recent instructional event during a CEEPUS (Central European Exchange Program for University Studies) Network CIII-HU-0028 Summer University activity. As a practical application, more than 30 participants, most of them students training to become teachers, worked out the solution of the twelve coins problem together, which was unknown to them, except for two of the lecturers, who led the discussions. Moreover, inspired by Ágnes Tuska's [47] lecture about "George Pólya's influence on mathematics competitions in the USA", a group of students from Partium University [48] decided to prepare a presentation reflecting on how they would implement the steps of Pólya's model. Furthermore, during the event, a preliminary investigation was conducted with 34 teachers to assess whether they knew about 21st-century skills, what they are, whether mathematics has a role in developing them, and if there were methods or strategies known to them that they could use in this respect. The primary outcomes were that half of them had heard about 21st-century skills, but only a third could define them correctly, giving examples of what skills are included. Moreover, only a third of the asked teachers knew what methods and strategies could be used in mathematics to develop these 21st-century skills. In panel discussions on how 21st-century skills could be integrated into mathematics, the main conclusion was that open-ended, real-life, interdisciplinary problems, problems that connect mathematics and other natural sciences (STEM), word problems, and visualization of solutions could be possible approaches, since they encourage creativity and have a long-lasting effect.

The findings from the literature review, the outcomes from the preliminary investigation, and the authors' extensive didactical experience led the authors to choose the problems that appear in this paper. Although some of the problems are known in the context of math education, the presented solutions are unique and belong to the authors. They developed them based on comprehensive teaching experience and feedback from students from high school to university, and they were created specifically to develop 21st-century skills.

The detailed description is aimed at offering a model, a tool, which can be adapted by teachers to any level of education above the first elementary classes to develop 21st-century skills. When planning a teaching session, the teacher should consider the age and mathematical background of the learners. For simplicity, the problem will be mentioned as the 12 coins problem throughout the paper.

Step 1. Understanding the Problem

Let us cite Pólya [26] (p. 23): "First of all, the verbal statement of the problem must be understood. The teacher can check this up to a certain extent; he/she has to ask the student to repeat the statement, and the students should be able to state the problem fluently. The student must be able to point out the principal parts of the problem, the unknown, the data, the condition. Hence, the teacher can seldom afford to miss the questions: What is the unknown? What are the data? What is the condition?"

In similar problems of weighing, one can understand a measurement using an "old-fashioned balance", sometimes called a mathematical balance, and by weigh, one can compare the weights of two sets of coins by putting equal numbers of coins on both sides of the scale. The result of a measurement is that they are either balanced or not.

"Where should I start? Start from the statement of the problem", as Pólya formulated in [26] (p. 38).

The student has to understand what a measurement means and to think about what conclusions can be formulated using the result—the outcome of the weighing. The best way is to ask the students questions in order to guide them to formulate all possible information to be obtained after a measurement. For this reason, it is good to bear in mind Pólya's hint to read the statement of the problem over and over again to completely understand the question [26] (p. 38): "Where should I start? Start again from the statement of the problem. Start when this statement is so clear for you and so well impressed on your mind that you may lose sight of it for a while without fear of losing it altogether."

During this step, the students need to get acquainted with the details of the problem and work together with the teacher for better understanding [26] (p. 38):

"What can I do? Isolate the principal parts of your problem. The hypothesis and the conclusion are the principal parts of the problem. Go through the principal parts of your problem, consider them one by one, consider them in turn, consider them in various combinations, relating each detail to other details, and each to the whole of the problem."

In our case, the teacher must achieve (eventually by auxiliary questions) the formulation of statements by the students themselves, such as:

- If the weighing result is that the scales do balance, it means that all coins on the scales are good, and the other coins not used for the given weighing remain suspect.
- If the scales do not balance, it means that all other coins that were not weighted are good, while those weighted are suspect.
- In this latter case, the students need to formulate an additional idea and name this key idea.
- If the scales do not balance, it is needed to distinguish the suspect coins on the two scales and separate them as suspected of being heavier and suspected of being lighter.

At this stage, teachers can develop the learning skills (communication and collaboration) and social skills (networking with others for mutual benefit) of learners.

Step 2. Making a Plan

To get the maximum possible information, teachers should ask students to analyze and formulate all statements related to the possible outcome of a given weighing. An auxiliary but important element is to introduce notations reflecting this information. Let us denote in the following G_i as the good coins and S_i as the suspect ones, while H_i or L_i are those suspected of being heavier or, respectively; the index *i* should vary from 1 to 12.

Planning the First Weighing

When starting to think about the problem, for planning the first weighing, one can easily conclude that starting by measuring one coin against one, two coins against two, or five coins against is not the right step, while measuring six against six is useless, as one does not get enough information, except separating them as suspected of being heavier (six coins) or of being lighter (the other six coins).

As only two cases remain—to start with weighing three against three or four against four coins—it is good to start by analyzing the case of planning the first weighing by comparing three coins with another three coins of the 12. The tables below contain a special notation to distinguish the many possible cases; a triplet of numbers of the form N-N-N will be used, where the first N means the number of the weighing (1 to 3), the second will be the number of coins put on each of the two scales (theoretically 1 to 6, but in reality, this is 1 to 4), while the last number indicates the possible outcomes (2 or 3 depending on the need to distinguish the cases when they do not balance).

Possible outcomes of weighing three coins against three and the information received from it are listed below (see Table 1).

Table 1. Possible outcomes of weighing three coins against three and the information received from it.

Case	Outcome of the Weighing	Information	How Can One Use It?
1-3-1	They balance	The six coins on the scales of the balance are all good, and the other six are suspect.	One can separate them, and denote them if needed by G_1, G_2, . . . , G_6, and S_1, S_2, . . . , S_6.
1-3-2	They do not balance	The six coins on the balance will be suspect—three of them suspected of being heavier and three of them suspected of being lighter—and the other six are good.	One will introduce the notations: H_1, H_2, and H_3 for those suspected of being heavier, or L_1, L_2, and L_3 for those suspected of being lighter, and G_1, G_2, . . . , G_6 for the good ones.

The second case is to analyze the possible outcomes of a first weighing starting with comparing four coins against four, and the information we get from this weighing is summarized below (see Table 2).

Table 2. The possible outcomes of weighing four coins against four as a first step, and the information received from them.

Case	Outcome of the Weighing	Information	How Can One Use It?
1-4-1	They balance	The eight coins on the scales of the balance are all good, and the other four are suspect.	One can separate them and denote them if needed by G_1, G_2, . . . , G_8, and S_1, S_2, . . . , S_4.
1-4-2	They do not balance	The eight coins on the balance will be suspect—four of them suspected of being heavier, and four of them suspected of being lighter—the other four that were not measured are good.	One will use the notations: H_1, H_2, H_3, and H_4 for those suspected of being heavier, or L_1, L_2, L_3, and L_4 for those suspected of being lighter, while G_1, G_2, . . . , G_4 will denote the good ones.

Planning the Second Weighing

As seen in the previous two tables (Tables 1 and 2), the two logical possibilities are those when starting weighing coins either three by three or four by four.

It is easy to deduce that out of the first cases, denoted as 1-3-1 and 1-3-2, only one, the case 1-3-2, can be finished in the remaining two weighings. In the case of 1-3-1, one will get stuck; thus, this first case, measuring coins three by three first, must be given up. Reasoning could include the analysis of possible measurements; there are six suspect and six good coins, but only two possible weighings remain.

- There is no use in measuring suspect coins three by three; they will surely not balance, and the last weighing has to be used for three coins suspected of being lighter and three coins suspected of being heavier.
- There is no use in measuring suspect coins two by two; e.g., in case they balance, one remains with two suspect coins, but one does not know if they are heavier or lighter.
- There is also no use in measuring suspect coins one by one.
- If one tries to analyze other cases, they will get stuck as well. Possible cases are: three suspect against three good coins, or four suspect against four good coins.

This part of making up the plan offers the teacher the most possibility to ask questions to control the way of understanding the problem and to work out the right steps for the planning of the solution.

Remember [26] (p. 38): "What can I gain by doing so? You should prepare and clarify details which are likely to play a role afterwards."

The right conclusion is that, for the first weighing, only the case of measuring coins four by four remains.

The Second Weighing in Case 1-4-1.

In this case, one will remain with four suspect coins. The best idea is to measure three of the suspect ones, say S_1, S_2, and S_3, against three good ones, say G_1, G_2, and G_3. The possible outcomes are shown in Table 3.

Table 3. The possible outcomes of the second weighing in case 1-4-1.

Case	Outcome of the Weighing	Information	How Can One Use It?
2-3-1	They balance	The six coins on the scales of the balance are all good, the last suspect, and S_4 remains suspect.	One can separate S_4.
2-3-2	They do not balance	The three suspect coins on the balance will remain suspect, but extra information is obtained if they are suspected of being heavier or lighter.	One will change the notations, according to the result of the weighing, to H_1, H_2, and H_3 or L_1, L_2, and L_3.

The Second Weighing in Case 1-4-2

In this case, one has eight suspect coins, but something more is known: As the result of the previous weighing, one can distinguish four as being suspected of being heavier and the other four as suspected of being lighter; this fact is recorded using the notation H_1, H_2, H_3, and H_4 and L_1, L_2, L_3, and L_4, while for the good ones, G_1, G_2, ... , G_4 are used. One now has to measure four by four again, using the following distribution: on one of the scales, H_1, H_2, H_3, and L_1 will be put, while H_4, G_1, G_2, and G_3 will be on the other scale. The reason for organizing the second weighing in this way is to obtain maximum information. If they balance, one has to continue with L_2, L_3, and L_4; if they do not balance and the first scale goes down, one has to continue with H_1, H_2, and H_3 if it goes up with L_1 and H_4. The possible outcomes are presented in Table 4.

Table 4. The possible outcomes of the second weighing in case 1-4-2.

Case	Outcome of the Weighing	Information	How Can One Use It?
2-4-1	They balance	The five suspect coins on the scales of the balance are all good, and the last suspect ones are L_2, L_3, and L_4	One can separate L_2, L_3, and L_4
2-4-2	They do not balance, and the scale with H_1, H_2, H_3, and L_1 goes down	The three suspect coins are H_1, H_2, and H_3	One can separate H_1, H_2, and H_3
2-4-3	They do not balance, and the scale with H_1, H_2, H_3, and L_1 goes up	The two suspect coins are L_1 and H_4	One can separate L_1 and H_4

Planning the Third Weighing

As has been shown, there are four different possible outcome cases resulting from the second weighing:

A. One can separate S_4
B. One can separate L_2, L_3, and L_4
C. One can separate H_1, H_2, and H_3
D. One can separate L_1 and H_4

Two of them, cases 2 and 3, are symmetrical, and one can use them in a similar way.

Case A. S_4 has been separated, and it will be compared with one of the good coins. They cannot balance, as the 11 other coins are good and are not suspect anymore. The steps are planned according to Table 5 as usual.

Table 5. The possible outcomes of the last weighing in case A.

Case	Outcome of the Weighing	Information, Result
A3-1-1	They do not balance, and the scale with S_4 moves up	The last suspect coin S_4 on the scale is lighter
A3-1-2	They do not balance, and the scale with S_4 moves down	The last suspect coin S_4 on the scale is heavier

Case B. L_2, L_3, and L_4 have been separated and L_2 and L will be compared. The possible outcomes are presented in Table 6.

Table 6. The possible outcomes of the last weighing in case B.

Case	Outcome of the Weighing	Information, Result
B3-1-1	They do balance	The suspect coin L_4 is lighter
B3-1-2	They do not balance; the scale with L_2 moves up, and L_3 moves down	The suspect coin L_2 is lighter
B3-1-3	They do not balance; the scale with L_2 moves down, and L_3 moves up	The suspect coin L_3 is lighter

Case C. H_1, H_2, and H_3 have been separated, and H_1 and H_2 will be compared. The possible outcomes are in Table 7.

Table 7. The possible outcomes of the last weighing in case C.

Case	Outcome of the Weighing	Information, Result
C3-1-1	They do balance	The suspect coin H_3 is heavier
C3-1-2	They do not balance; the scale with H_1 moves down, and H_2 moves up	The suspect coin H_1 is heavier
C3-1-3	They do not balance; the scale with H_1 moves up, and H_2 moves down	The suspect coin H_2 is heavier

Case D. L_1 and H_4 have been separated, and L_1 will be compared with one of the good coins. The possible outcomes are presented in Table 8.

Table 8. The possible outcomes of the last weighing in case D.

Case	Outcome of the Weighing	Information, Result
D3-1-1	They do balance	The suspect coin H_4 is heavier
D3-1-2	They do not balance, and the scale with L_1 moves up	The suspect coin L_1 is lighter

This step offers the possibility for teachers to promote life skills, such as analyzing the different options when looking at solutions. Moreover, it develops critical thinking and creativity, and encourages initiative and the ability to create and deliver a plan.

Step 3. Executing the Plan

An example is given in Appendix A on how the flowchart of the complete solution can be made.

At this step, teachers help learners acquire the 4C skills (critical thinking, creativity, communication, and collaboration) through networking and design thinking. At the same time, this encourages learners to achieve goals collaboratively.

The usage of schema whenever possible is underlined by Lein et al. [49], who proved that "intervention effects for schema-based transfer instruction were larger than those for schema-based instruction", illustrating that mathematical word-problem-solving interventions are suitable for students with learning disabilities and/or mathematics difficulties as well.

Step 4. Feedback

Once the students have understood and could solve the problem, it is good to conclude by verbalizing all key steps of the reasoning in order to sum up the methods used to solve the problem, the strategy worked out to tackle the problem, and the way all possible cases were taken into account. By doing so, the teacher contributes to the formation of the skills necessary for the generalization of the method. The teachers are given a tool to develop the strategic thinking of learners, to benefit from working together to gain a better understanding of the problem, and to debate and assess solutions to find the best one.

The experience of problem-solving activities mentioned above shows how Pólya's heuristic method supports learners in acquiring critical-thinking and problem-solving skills, in learning how they can make complex choices, in constructing realizable/applicable arguments to sustain their choice, in sharing their own arguments, in criticizing other arguments, in building up a diagram or a scheme, in creating a possibility for dynamic, collaborative discussions, and in developing students'/learners' life skills, such as flexibility, leadership, initiative, and social skills. In addition, it shows how teachers can facilitate a fruitful discussion to develop the necessary skills of their students.

4.2. Examples and Discussions. Problems for Training Pre-Service and In-Service Teachers

There are many elementary arithmetic courses for future primary school teachers. Many researchers in mathematics education have developed problem-solving strategies. Kalmykova [30] presents a practical application of neuropsychological studies, which leads to parallel processing of analytic and synthetic activity in the problem solving of learners. Classroom activities in the field of problem solving are approached by Kalmykova [30] using five "auxiliary" methods: (1) concretization, (2) abstraction, (3) modification, (4) graphical analysis, and (5) analogy.

This approach is similar to the stages formulated by Pólya [26]—understanding the problem, devising a plan, carrying out the plan, and looking back. If learners understand the problem, then they need concrete conditions and important input information. In the stage of devising a plan, the mathematical abstractions and graphical analyses of schemas lead to developing a plan for the solution. Modification and analogy are useful for the last two stages—carrying out the plan and looking back. Word tasks for weak and good students presented by studies from Kalmykova [30] develop algorithmic thinking via active schemas using operations with natural numbers, such as multiplication and addition. Algorithmic thinking, according to Dagiene et al. [50], is thinking in terms of sequences and rules—executing an algorithm and creating an algorithm.

Next, two examples are given as models for primary teachers that not only develop 21st-century skills to solve problems encountered in life, but also, their ethnomathematical and cross-curricular aspects incorporate mathematics history in mathematics education. The first problem enables students to learn about math in various cultures, thus developing their social and cross-cultural skills as well.

There is a curricular reform in many countries, and it is expected that mathematics will be positioned as a part of society and culture. The case of Slovakian curricula is presented in Appendix C. The chosen examples enable divergent thinking and support the possibility that one problem can have more correct solutions. The use of models (paper cards or computer animation) supports design thinking.

Implementing Pólya's problem-solving structure enables the rethinking of the art of the teaching process by learners in the above-mentioned topic.

4.2.1. Multiplication of Two Natural Numbers

The first example is devoted to the multiplication of two natural numbers as an operation, which has its own properties. This example is chosen based on the argumentation of Menchinskaya and Moro [51]: "The formation of the concept of numbers and of the arithmetical operations is a difficult and crucial task of arithmetic instruction." Pólya's structure serves here as a tool for developing mathematical inquiry in learners. In the description of the steps, Pólya's questions will be cited.

Step 1. Understanding the Problem

Pólya said [26] (p. 23): "The student must be able to point out the principal parts of the problem, the unknown, the data, the condition. Hence, the teacher can seldom afford to miss the questions: What is the unknown? What are the data? What is the condition?"

The teacher discusses the inputs and outputs in the algorithm of multiplication with students through simple examples. Inputs are the factors—in our case, two natural numbers. The output is the product—another natural number. It is possible to start with the idea of multiplication as repeated addition. According to Menchinskaya and Moro [51], it is essential to observe that "not every multiplication can be replaced by multiplication, but only addition of equal addends", and to recommend comparing the examples $3 + 3 + 3 + 3 + 3 =$ and $3 + 3 + 3 + 3 + 3 + 5 =$.

This activity, according to P21 [17], develops critical thinking, since individuals "analyze how parts of a whole interact with each other in mathematical systems."

This step is an opportunity to discuss the operation of multiplication of natural numbers and to check if the students understand its properties, such as associativity, commutativity, distributivity, and so on.

Step 2. Devising the Plan

The teachers' question can be: "Did you use the whole condition?" The multiplication of natural numbers and powers of ten is easy to understand: 2 ⊕ 10 = 20, 56 ⊕ 100 = 5600. The use of the distributive rule helps understanding as well: 56 ⊕ 25 = 56 ⊕ (20+5) = 1120 + 280 = 1400. These rules appear in the "classical" algorithm of multiplication, which can be taught in different ways.

```
  56          214
 ·25        ·2326
 280         1284
1120         428
1400        642
           428
           497764
```

For simplicity, it is common to omit the zeros at the end in the partial sums starting from the second row and to use a technique to leave the place for them out (for example, in the case of 4280 notation, the number 428_ will be used; similarly, for 64,200, it appears as 642_ _, and 428,000 will be 428_ _ _).

Remark: It can be shown that the order of writing the partial sums can differ.

```
56 · 25        56 · 25
  280         112
112_           280
1400            1400
```

It is important to develop the divergent thinking of the students so that they can see the same method by working differently and writing differently, thus developing their creativity in this way.

In Step 2, it is important to analyze other algorithms of multiplication as an answer to the question: "Do you know a related problem?" [26]. Thus, Muhammad ibn Musa al-Khwarizmi's algorithm of multiplication is presented as an alternative method.

Muhammad ibn Musa al-Khwarizmi (780–850) wrote *Arithmetical tractate;* Cajori [52] and Benediktova [53], who were devoted to natural numbers and operations, republished his work as a critical edition [54], which contains the following algorithm of multiplication for the example of 214 · 2326.

In the first step, the numbers will be written in the way seen below; the bold digits must be in the same column.

  2326
21**4**

The second step of the algorithm is the multiplication of the number 21**4** with the first bold digit 2, representing the thousands of the number **2**326. The product, 42**8,** is copied over the numbers used in the first step; the bold digits are again in the same column, as shown below:

42**8**
  2326
21**4**

The third step contains a special joining and a shift. The digit **2** used for the previous multiplication will be replaced by **0,** and the first two numbers, 42**8** and **0**326, will be "added", keeping the column alignment as before, getting 42**8,**326. The number 214 will be shifted one digit to the right.

428**3**26
214

The fourth step: The number 21**4** is multiplicated now with the next unused digit, 3, representing the hundreds of the number 2326, getting 64**2** to be copied over the numbers in step 3; the bold digits must be in the same column, as shown below.

64**2**
428**3**26
214

The fifth step is again a joining and a shift, as in step three. The digit **3** used for the previous multiplication will be replaced by **0,** and the first two numbers, 64**2** and 428,**0**26, will be "added", keeping the column alignment as before, getting 492,**2**26. The number 214 will be shifted one digit to the right again.

492**2**26
214

The sixth step is to multiply the number 21**4** with the next unused digit, 2, representing the tens of the number 2326, and getting 42**8** to be copied over the numbers in step five; the bold digits must be in the same column, as shown below.

42**8**
4922**2**6
214

The seventh step is again a joining and a shift, as in step three and five. The digit **2** used for the previous multiplication will be replaced by **0,** and the first two numbers, 42**8** and 492,2**0**6, will be "added", keeping the column alignment as before, getting 496,4**8**6. The number 21**4** will be shifted one digit to the right again.

4964**8**6
214

The eighth step is to multiply the number 21**4** with the next unused digit, 6, representing the ones of the number 2326, getting 128**4** to be copied over the numbers in step seven; the bold digits must be in the same column, as shown below.

128**4**
49648**6**
214

The final step is shown by the fact that the number 214 does not need to be shifted to the right because the ones are already in the same column.

To finish the algorithm, the first two numbers are joined. The digit **6** is replaced by **0,** and the numbers 128**4** and 496,48**0** are "added", keeping the column alignment, and getting 497,764.

Thus, the result of the multiplication of 214 by 2326 is 497,764.

By using examples from the history of mathematics, media literacy, one of the 21st-century skills [17], can be developed in students as well. Menchinskaya and Moro [51] argue that "it is instructive to analyze how, in the history of the development of human culture", the multiplication of natural numbers has developed, and "how it was continually enriched with new content, and what significance this concept has at the present level of development of mathematics." However, "the modern child should ... imagine the situations and the practical problems for which people first needed numbers" and work with them.

The multiplication of natural numbers appears in a "university diploma work" from 1753 written by Vajkovics [55] (p. 18) (see Appendix B).

Step 3. Carrying Out the Plan

The understanding of the previous algorithm reveals the question: Is this algorithm of multiplication correct like the well-known one? For this stage, helpful questions for students are: "Can you see clearly that the steps in the presented algorithm are correct?" [26] For better understanding, it is recommended to repeat the Al-Khwarizmi algorithm of multiplication with other numbers, checking the result with the classical method.

Step 4. Looking Back

It is important to use the right interpretation of these algorithms and analyze why they work. Questions in this stage are: "Can you check the result of multiplication? Can you see how the algorithm of multiplications works at a glance?" [26]. The previous steps of the Al-Khwarizmi algorithm serve for a better understanding of such rules as associativity, commutativity, and distributivity of addition and multiplication, as well as to discuss the distributive rule for multiplication in the example of 214 · 2326, which can be written in the following way, the bold digits helping the understanding:

$$214 \cdot 2326 = 214 \cdot (\mathbf{2}000+\mathbf{3}00+\mathbf{2}0+\mathbf{6}) = \mathbf{428}{,}000 + \mathbf{64}{,}200 + \mathbf{428}0 + \mathbf{1284} =$$
$$= \mathbf{492{,}2}00 + \mathbf{428}0 + \mathbf{1284} = \mathbf{496{,}48}0 + \mathbf{1284} = \mathbf{497{,}764}$$

The bold letters show that in the al-Khwarizmi algorithm, partial sums were used, which are visible in the above calculating details of the distributivity for the multiplication of 214 by 2326; the latter one is written as sums of the thousands, hundreds, tens, and ones.

The learners will be able to compare by identifying the details of numbers written in bold characters in the Al-Khwarizmi algorithm and the above algebraic formalization.

Using the question "Can you derive the result differently?" Pólya [26], students' reasoning and critical thinking can be developed by showing them that different solutions and algorithms can be identified for the same problem.

While it is interesting to compare more algorithms for multiplication, they also strengthen the understanding. In this regard, different historical alternatives for multiplication can be explored, such as examples from India (Figure 1) and Japan (Figure 2), reconstructed below by Cajori [52].

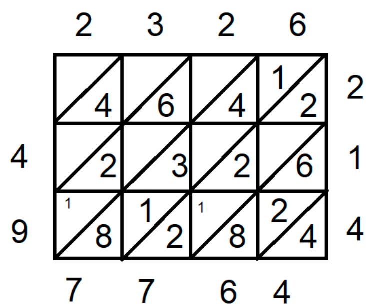

Figure 1. Historical algorithm for multiplication in India (214 · 2326 = 497,764). (Authors' own construction; source: [52].).

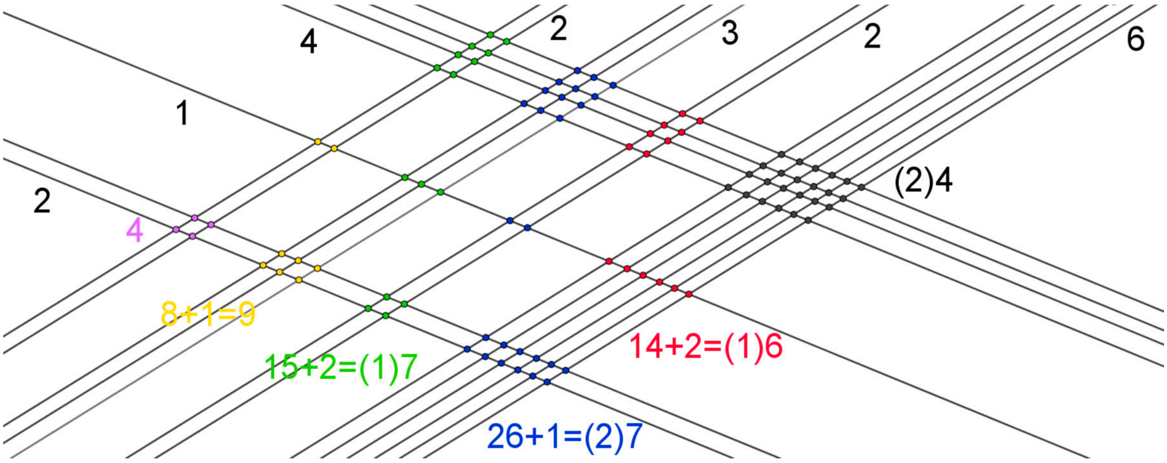

Figure 2. Historical algorithm for multiplication in Japan (214 · 2326 = 497,764). (Authors' own construction in GeoGebra; source: [52]).

This multiple-sourcing approach shows that the algorithm of multiplication is not only a mechanical activity, but it comes from properties of mathematical notions, which can be visualized as well, and can develop learners' information literacy skills.

4.2.2. Introduction of the Binary Number System

The next problem is the authors' own unique problem, which was used in their didactical careers in teaching mathematics. Diverse real-life situations demand the handling of operations in different number systems in mathematics education. Transversal skills and 21st-century skills are developed via applications "in a wide variety of situations and work settings" [56]. For example, time measurement is an example of a combined numbers system (1 h is 60 min, 1 min is 60 s, 1 day is 24 h, 1 year is 365 or 366 days). Another example is the temperature measured in Fahrenheit or Celsius, where the transformation can be described with the relation (0 °C = 32 °F, 100 °C = 212 °F). Another group of examples are currency transformations (1 EUR = 1.0819 CHF (CHF—Swiss Franc); 1 EUR = 358.16 HUF (HUF—Hungarian Forint [57]). The problem of "currency exchange" serves as an introduction to the binary number system using storytelling for children. The suggested teaching method follows Pólya's structure.

Step 1. Understanding of the Problem

This level of teaching requires the problem to be introduced using the following story:

Gulliver is coming to the Binary Island. There is an interesting currency here; the "exchange rates" for 1 EUR is 1 A-gulden, 2 A-guldens are 1 B-gulden, 2 B-guldens are 1 C-gulden, and so on for D- and upper guldens. In the shops, it is possible to use a maximum of one piece of each type of gulden.

The first question is: How can one pay 10 EUR in a Binary Island shop?

The questions reflecting Pólya's [26] ideas ("What is the unknown? What are the data? What is the condition?") are processed via discussions with the students about properties of the special currency on Binary Island. By this means, learning skills (communication and collaboration) and social skills (networking with others for mutual benefit) can be developed.

Step 2. Devising the Plan

In this stage, it is important to compare the difference between the representation of numbers in the decimal number system and the currency system on Binary Island.

Useful questions are: "Here is a problem related to yours and solved before. Could you use it?" [26]. Representation of the number 1234 written in the decimal system is possible via cards:

1000	100	100	10	10	10	1	1	1	1

It is possible to use up to nine pieces of cards with values of 1, 10, 100, 1000, . . . , while on Binary Island, only up to one piece of each kind of gulden can be used. The representation can also be made with cards.

F	E	D	C	B	A

Step 3. Carrying Out the Plan

It is possible to show the solution with the aforementioned cards. "Draw a figure!" [26]. This will be started with A cards, and the first step is to show 10 A-gulden, but more than 1 A-gulden is used.

A	A	A	A	A	A	A	A	A	A

Now, it is possible to replace 2 A-guldens with 1 B-gulden; the result is visualized, but 5 B-guldens are used.

B	B	B	B	B

Again, 2 B-guldens are replaced with 1 C-gulden. Still, two pieces of C-gulden are needed.

C	C	B

The final step is to replace 2 C-gulden with 1 D-gulden, and thus, only one gulden of types D and B is used:

D	B

The following question helps the understanding in this stage: "Can you see clearly that the steps in the presented algorithm are correct?"

Step 4. Looking Back

At this moment, it is helpful to repeat and to explain the algorithm of changing the cards used to the students. This practice helps the acquisition of the "acting in the moment" skills, answering the question: "Can you see it at a glance?" The interpretation of the obtained result belongs to the stage following the question: "Can you use the result, or the method, for some other problem?" [26]. The following table shows the final result in a different, more mathematical way, helping to develop the goal-oriented learning behavior of the students:

D	C	B	A	Binary Island gulden representation
1	0	1	0	Binary number system representation

The order of the digits 0 and 1 in the table represents what is usually written in mathematics: $10_{10} = 1010_2$, meaning that 10 in the decimal number system is written as 1010 in the binary number system.

Further, more complex applications: Use of similar cards is helpful in designing other student activities to follow up the application of the above model; e.g., for a higher-complexity problem, the addition of two numbers in the binary system as an application of the question: "Can you use the result, or the method for some other problem?" [26]. Take the following example:

$$1101_2 + 1011_2 = ?$$

The first step is to exchange the numbers into the cards. The table of the card model is better for visibility:

D	C		A
D		B	A

The algorithm is based on changing of cards from right to left: 2 A-guldens are 1 B-gulden.

D	C	B	
D		B	

It holds in a similar way that 2 B-guldens are 1 C-gulden.

D	C		
D	C		

Now, it is possible to change 2 C-guldens into 1 D-gulden.

D			
D			
D			

The last step is to change 2 D-guldens into 1 E-gulden and to show the final result with the numbers 0 and 1, as a representation in the binary number system.

E	D			
1	1	0	0	0

Thus, the result can be written as: $1101_2 + 1011_2 = 11{,}000_2$.

Step 4 and Pólya's questions give learning strategies for teachers to develop learners' strategic thinking through working together by using a learning game with cards to gain a better understanding of the problem and to find the solution.

5. Discussion and Conclusions

The role of problem solving in learning mathematics is widely known; many studies and research papers have been published on this topic. Nevertheless, in 1945, Pólya was the first to give a clear description of a method for teaching problem solving in mathematics.

The present paper focuses on the future perspectives of problem solving motivated by the following trends:

- The rapidly changing environment due to technological disruption, globalization, and climate change, which requires the development of new transversal skills through education.
- Cross-curricular integration helps students and learners to remember math knowledge better and to link it to real-life situations, allowing them to connect different subject areas.

In the upcoming years, these tendencies will reshape, step by step, the mathematical education at all levels.

Furthermore, it is challenging for teachers to make math engaging, to prove its utility in real life, and to find ways to develop problem-solving skills that equip graduates with the skills needed to thrive in a modern workplace. In this sense, they have to figure out which mathematics contents are sustainable.

The present paper intends to make a case for rethinking how learners acquire knowledge and what future needs they have to face in the rapidly changing world, and, thus, to update the curriculum in this respect. This theoretical research's main finding is that the Pólya method, which is well known and used in mathematical problem solving, could be used to develop 21st-century skills. The examples and strategies presented demonstrate that by being including in the learning process and practicing certain methods of solving mathematical problems, students could learn a way of thinking to approach and solve problems successfully in a broader context later in life. The examples presented prove that Pólya's heuristic could be used in a broader context to help learners acquire the modern skills needed to succeed in their careers. The uniqueness of the problem is the challenge of finding solutions that remain in memory for a long time and can be retrieved for future use. In addition to this challenge, if the solution method is also anchored, then this method can be utilized to solve other problems encountered in life. Thus, the example is a reminder of the method. Regarding the 21st-century skills, the goal is to understand, practice, and adapt the method to everyday life situations. This must be mastered in the context of mathematics education.

English and Sriraman [58] argue that there is scarce information in the literature on how learners solve mathematics problems outside the classroom. Nevertheless, when examining the way people solve non-mathematical problems faced in life, it can be observed that mathematical problem solving has similarities to general problem solving. One approach to authentic problems that appeared in the literature and gained a high degree of interest is episodic future thinking [59]. Maciejewski [60] states that, "Planning in a mathematical situation, therefore, may be analogous to planning in a general, non-mathematical situation, and may involve episodic future thinking."

The critical review of the literature in Section 2 reveals that problem solving is one of the most valuable skills in modern work. In this respect, this paper presents teaching models and strategies that prove that teaching problem-solving thinking in mathematics is an efficient way to develop transversal and cross-curricular skills, such as critical thinking, creativity, collaboration, communication, information, media and technology literacy, and life skills, in order to develop learners' abilities to face non-mathematical challenges.

Section 3 presents a short overview of mathematical problem solving from Pólya and of current problem-solving processes, frameworks, and new tendencies. The role of math teachers is to transfer knowledge to students, to provide problem-solving strategies, and, based on them, to explain and help them build up their own approaches. In this way, learners succeed in acquiring transversal skills that they can put into practice to solve real-life problems at the workplace. The developed mathematical problem-solving strategies can be applied to a variety of situations: They can be used to diagnose an illness, in experimental physics, to solve non-routine problems at the workplace, and in unforeseen emergency cases, power supply shortcuts, crisis management, and others. Moreover, science-based decisions also need problem-solving skills.

Those who have good problem-solving skills can adapt quickly to every situation. They are goal-oriented, find solutions quickly, and make decisions quickly. This ability needs to be continuously developed, and it should start in elementary school. Education should provide the necessary mathematical knowledge; however, besides the usual tasks, assignments that boost problem-solving skills should be added as well.

Section 4 reveals concrete examples and teaching methods to show how problem solving could be used to develop 21st-century skills, and how the cross-curricular integration of mathematics could be achieved.

Depending on the students' engagement and the teachers' readiness, almost everybody can become a good learner of mathematics. By asking appropriate questions, teachers can help students to build up their own goal-oriented strategies for problem solving. The Pólya principle is a crucial element in this education process. Problem solving is not a linear process; learners must acquire abilities to think forward and backward, between and across, to simplify and generalize. A good problem solver has a goal-oriented behavior and can make decisions in a rapidly changing environment.

The presented concrete examples prove that Pólya's heuristic can be used successfully nowadays as well. It can help learners to improve their communication skills by constructing viable arguments and judging others' reasonings in order to develop their creativity, out-of-box thinking, collaboration, flexibility, productivity, and leadership.

The paper's findings, based on the Pólya's heuristic steps [26] (pp. 23–27), including the skills they can develop, are:

Step 1: "Understanding the problem": The first step helps learners to develop their information literacy—to understand the facts, data, and media literacy—to understand methods and be able to distinguish between the ones that are credible and the ones that are not. By formulating suitable questions, teachers can develop learners' social skills.

Steps 2: "Devising a plan": This step promotes life skills, such as flexibility to accept that they always have a lot to learn, to accept other opinions, to understand how to react in different situations, and to analyze the different possibilities when looking at solutions. It also develops creativity and critical thinking, and guides them to have initiative and to be able to create and deliver a plan.

Step 3: "Carrying out the plan": The third step helps learners to acquire learning skills through networking. It boosts critical thinking, creativity, communication, and collaboration. Moreover, it fosters leadership skills and achieving goals collaboratively, and it promotes productivity, an ability to complete the work in an appropriate amount of time. It teaches how learners can become efficient by using prior experiences.

Step 4: "Looking back and further development of the topic": The last step reveals to learners the benefits of working together to gain a better understanding of the problem and to debate and assess solutions to find the best one. Thus, it develops strategic thinking.

The presented examples enhance the technological literacy of learners as well, and the use of the GeoGebra software in the classroom activities strengthens students' understanding. The modeling used in the examples presented reveals the positive effect on the development of problem-solving activities and fosters not only learning, but literacy and life skills as well. Through modeling, the teacher can realize the cross-curricular integration of mathematics.

To enhance critical thinking, it is essential to teach learners to consider and analyze various possible solutions to the problem, to choose one, and to define a set of coherent actions to deliver results. In this respect, in Section 3, different multiplication algorithms are presented.

To teach students to complete work in an appropriate amount of time and to increase their efficiency/productivity, the teacher needs to provide relevant assignments. Learning is possible only through practice and experience; thus, to improve problem-solving skills, lots of problems need to be solved. Teachers have to create problems with different difficulty levels and give them to students gradually until their problem-solving skills are sharpened. Pólya's stages deliver a concrete way to develop algorithmic thinking in learners, which is essential in informatics and programming. It is possible to use the given examples not only during classical teaching, but also in different mathematics circles and when preparing seminars for different kinds of competitions (for instance, Mathematical Olympiad, Kangaroo [61], Bebras [62], Mathematical Duels [63]). These facts show possible developments and applications for non-formal education.

The paper targets current and future teachers and presents them with a method to rethink their teaching method for equipping students with 21st-century skills. Furthermore, it addresses the scientific community to convey the relevance and importance of heuristics in developing 21st-century skills and to stress the need for future research in this context. The number of heuristics conferences worldwide demonstrates that heuristics have been gaining a lot of attention from different disciplines in the last years.

In order to train and strengthen 21st-century skills in learners, when it comes to problem-solving skills, the gap must be recognized between what the teachers learned and what is expected of them to teach. Teacher readiness is a sensitive topic for many countries and needs in-depth analysis. The integration of problem-solving skills into mathematics is currently a challenging theme; how and when it has to be taught, what the curriculum has to contain, and which pedagogical methods have to be applied are important aspects of this matter. According to Hill et al. [64], math anxiety is a negative emotional response of learners to their current or prospective situation involving mathematics (see also [65]). This happens in an educational process when mathematics teachers introduce new topics with insufficient use of suitable models and a low stage of visualization, and when they present new mathematical notions, such as "ready mathematics", without steps that show how the new notions were created. The given examples are suitable for one of the possible constructivist ways to present mathematics concepts via some problem-solving strategy that supports inquiry by learners. In this way, learners have many possibilities to build up their knowledge individually and step-by-step. Future research plans include qualitative and quantitative research in these subjects with international students from CEEPUS partner universities. Future research directions may include the development of the correlation of problem-solving strategies and collaborative and technology-enhanced learning as well.

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Appendix A

The flowchart of the weighings:

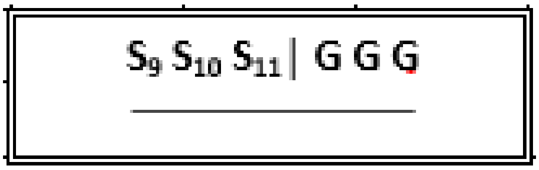
Example for a possible flowchart of the strategy of the solution to the 12 coins problem:

For a better understanding of the flowchart, the above-mentioned notations (G, S, L, and H) will be used, the weighings themselves will be marked in a box, and the simple arrow ↓ will indicate which way to continue the given reasoning.

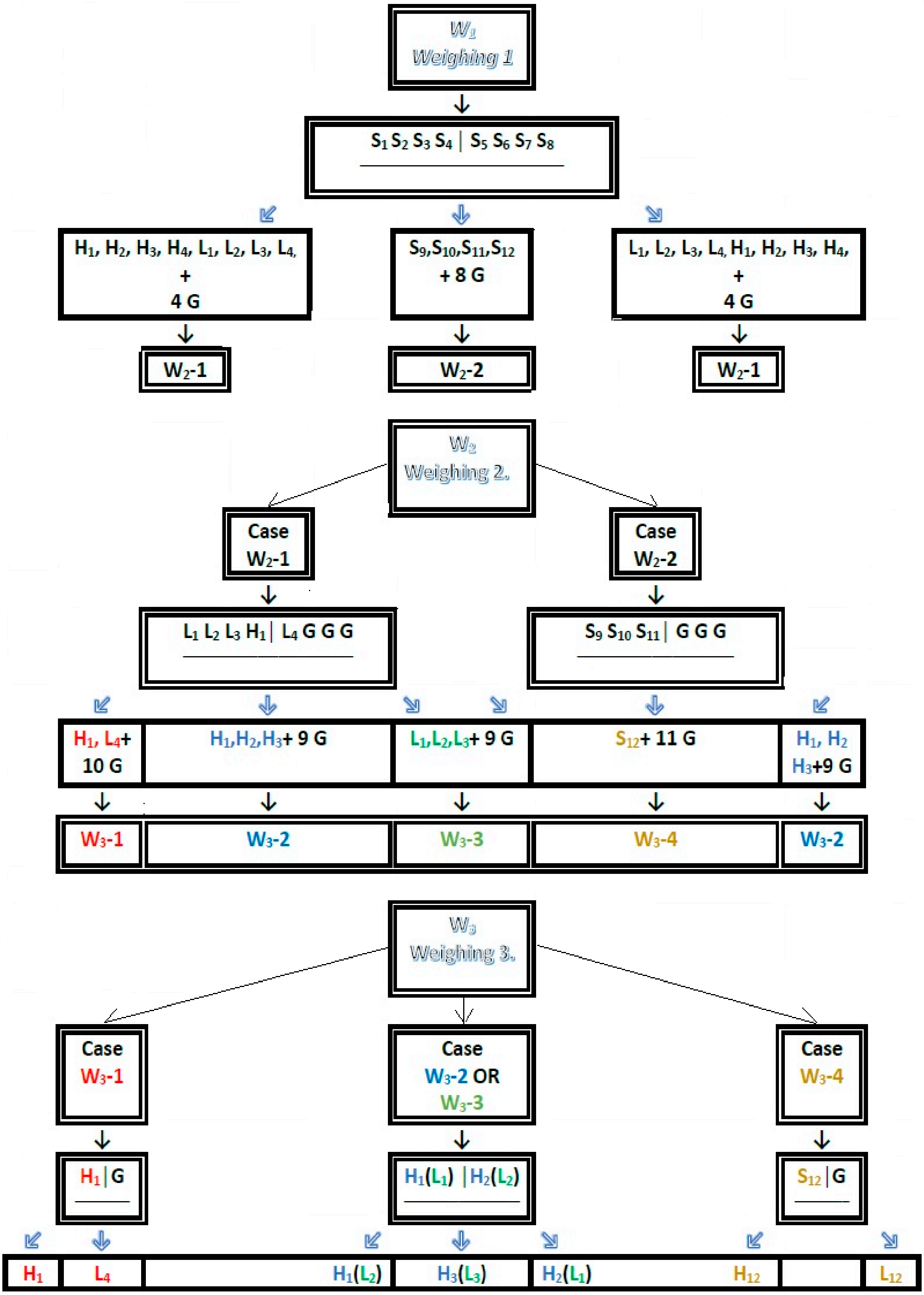
The double arrows ↙, ⇓, ↘ will indicate the direction, as well as the position of the scales and the result of weighing.

The arrow ↙ indicates that the left scale moves down and the arrow ↘ indicates that the right scale moves down, while the arrow ⇓ means that they balance.

The simple rectangular boxes contain the information resulting from a weighing, while double-border boxes contain either the number or case of weighing, or sketch the details of weighing; e.g., the box below indicates a weighing of three suspect coins, numbered S_9, S_{10}, and S_{11}, against three good ones, denoted with G.



For a better understanding of the flowchart and to better follow up the four cases, colors will be used to distinguish the outcomes of cases **W_3-1**, **W_3-2**, W_3-3, and**W_3-4**, as well as the final outcomes and the possible results of Weighing 3.


W1
Weighing 1
S1 S2 S3 S4 | S5 S6 S7 S8
H1, H2, H3, H4, L1, L2, L3, L4, + 4 G
S9,S10,S11,S12 + 8 G
L1, L2, L3, L4, H1, H2, H3, H4, + 4 G
W2-1
W2-2
W2-1
W2
Weighing 2.
Case W2-1
Case W2-2
L1 L2 L3 H1 | L4 G G G
S9 S10 S11 | G G G
H1, L4+ 10 G
H1,H2,H3+ 9 G
L1,L2,L3+ 9 G
S12+ 11 G
H1, H2 H3+9 G
W3-1
W3-2
W3-3
W3-4
W3-2
W3
Weighing 3.
Case W3-1
Case W3-2 OR W3-3
Case W3-4
H1 | G
H1(L1) | H2(L2)
S12 | G
H1
L4
H1(L2)
H3(L3)
H2(L1)
H12
L12

Appendix B

Historical remarks on the multiplication algorithm by Vajkovics (1753).

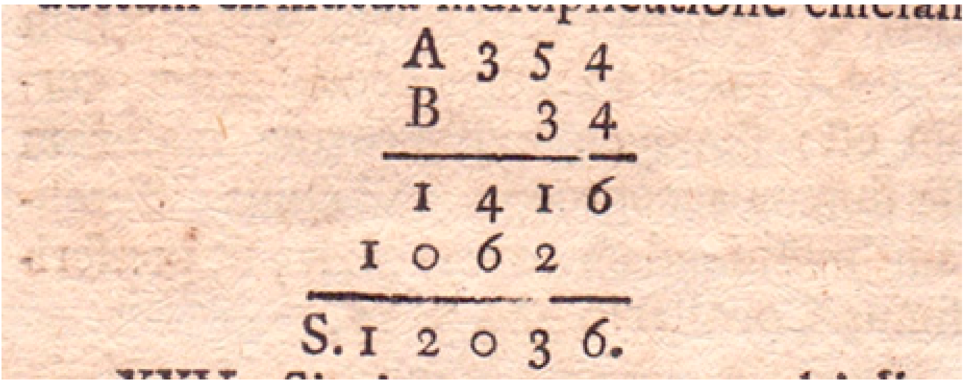
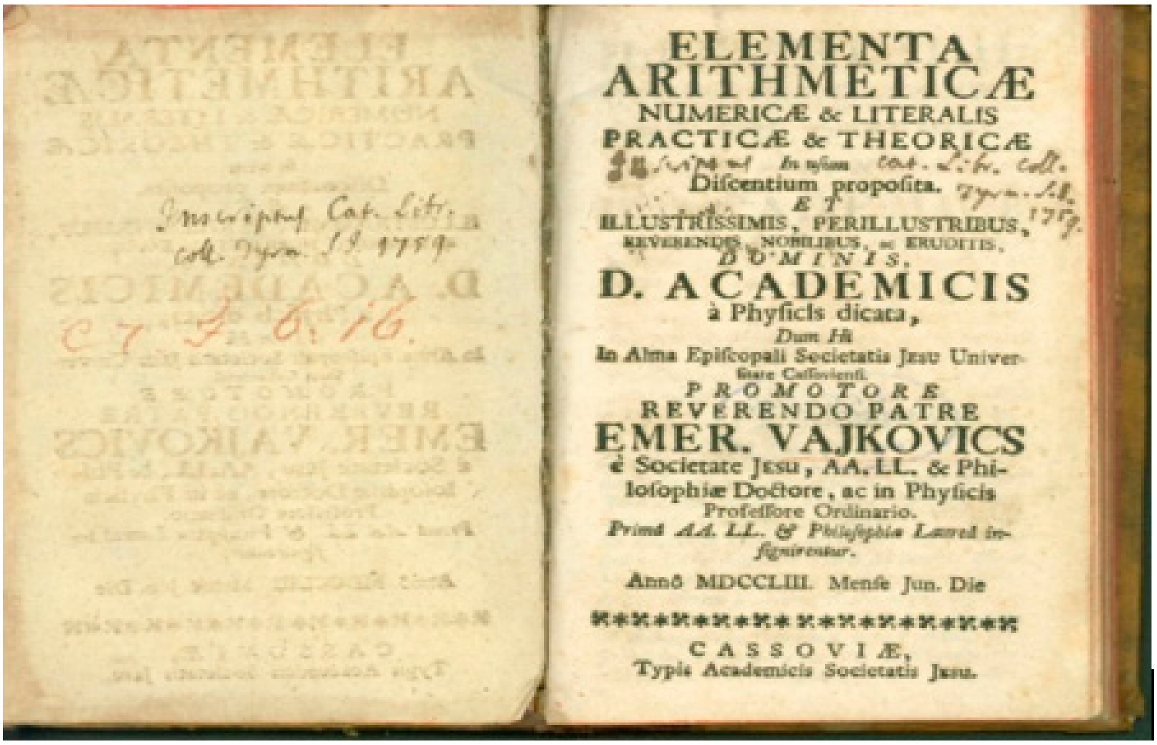


Figure A1. The multiplication algorithm published in [55] (p. 18).



ELEMENTA
ARITHMETICÆ
NUMERICÆ & LITERALIS
PRACTICÆ & THEORICÆ
In usum
Discentium proposita.
ET
ILLUSTRISSIMIS, PERILLUSTRIBUS,
REVERENDIS, NOBILIBUS, ac ERUDITIS,
DOMINIS,
D. ACADEMICIS
à Physicis dicata,
In Alma Episcopali Societatis Jesu Univer-
PROMOTORE
REVERENDO PATRE
EMER. VAJKOVICS
è Societate Jesu, AA. LL. & Phi-
losophiæ Doctore, ac in Physicis
Professore Ordinario.
Primâ AA. LL. & Philosophiæ Laureâ in-
signirentur.
Annô MDCCLIII. Mense Jun. Die
CASSOVIÆ,
Typis Academicis Societatis Jesu.

Figure A2. Title page of *Elementa Arithmeticae* [55].

Appendix C

Part of the Slovakian curriculum for the upper secondary level (translation to English of page 8 of the document made by the authors; see Štátny vzdelávací program Matematika (Vzdelávacia oblasť: Matematika a práca s informáciami, PRÍLOHA ISCED 3A, available on the webpage https://www.statpedu.sk/files/articles/dokumenty/statny-vzdelavaci-program/matematika_isced3a.pdf).

Over course of studies at the upper secondary level, it is necessary to include:

- problem tasks;
- historical notes;
- various small projects supporting interdisciplinary relationships, e.g., mathematics and art, Euclid, mathematics and space, mathematics and center of gravity, mathematics and the game billiards;
- information documenting the current and historical use of mathematics.

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