A Sustainable Green Inventory System with Novel Eco-Friendly Demand Incorporating Partial Backlogging under Fuzziness

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Abstract: Environmentally friendly goods are market-oriented goods that create less environmental damage. Their manufacture is related to a product development process designed to consider the environmental consequences that might develop throughout their life cycle. In reality, the global demand for herbal goods is expanding since herbal products are manufactured from plant extracts such as leaves, roots, flowers, and seeds, among others, and cause less environmental destruction. This study introduces a novel, eco-friendly demand determined by the usage of herbal and chemical substances in products. In this context, companies producing these products are encouraged. Firms are interested in producing eco-friendly products while keeping an eye on carbon emissions. This paper presents a sustainable inventory model of non-instantaneous decaying items that follow this eco-friendly demand under partially backlogged shortages. In this study, emission releases due to inventory setup, degradation, and holding were estimated, as were carbon emissions under cap and tax policies. This approach invests in green and preservation technologies to reduce carbon emissions and deterioration. To address the imprecision of the model's cost parameters, we converted them to Pythagorean fuzzy numbers. The optimum profit of the inventory model with carbon emissions is estimated by considering the time that the inventory level takes to reach zero and the replenishment time as decision variables. Numerical examples and a sensitivity analysis of significant parameters have been conducted to examine the effect of variation in the optimal inventory policy.

Keywords: sustainable inventory system; environmental emission reduction; deterioration; green technology; eco-friendly demand

1. Introduction

In recent years, one of the world's biggest concerns became global warming and climate change. As much as 90% of the world's population breathes polluted air [1]. The essential cause behind global warming is the emission of greenhouse gases (GHGs), such as ozone, carbon dioxide, nitrous oxide, methane, etc. Of these GHGs, carbon dioxide is the most widespread and dangerous greenhouse gas (GHG) considering its role in global warming. The primary reasons behind the increase in carbon dioxide in the atmosphere are the oil and gases used in the industrial sector, the extensive burning of coal, and deforestation. We can control carbon emissions using green technology. Green technology is an eco-friendly concept through which we can make our environment safer and pollution-free by conserving natural resources. Nowadays, companies focus on green technology in order to reduce costs and increase brand value since customers are interested in buying eco-friendly products. Researchers have been implementing this model in a sustainability environment in recent years. Recently, research has focused on sustainability and reducing carbon emissions. Many researchers (Singh and Mishra [2], Hasan et al. [3]) implemented their models in a sustainable environment. We develop this sustainable model by considering carbon emissions under a carbon cap and tax policy with green technology investment.

By making a few changes, we can make the environment eco-friendlier without much difficulty. More straightforward ways to make the environment greener include reducing waste, recycling waste, renewing energy, and refusing plastic, as shown in Figure 1.

Figure 1. Green environment perspective of proposed sustainable inventory system.

This article is structured as follows: Section 2 offers a review of the literature that highlights the study's aim and the research gap. In Section 3, we provide both inventory and green parameter notations and assumptions. Following that, we provide a schematic diagram and model formulation of green environments. In Section 4, we provide a new ranking method and a de-fuzzified value for the triangular Pythagorean fuzzy cost parameters. In Section 5, we present a numerical example to understand the model and examine the sensitivity of the demand parameters and other inventory and green parameters with managerial implications. Finally, in the conclusion section, we discuss the outcome and benefits of the proposed sustainable inventory model.

2. Literature Review

Nowadays, people are conscious about their health, so they have started avoiding products containing chemical ingredients. At the same time, they are showing interest in buying products containing herbal ingredients. The demand that depends on eco-friendly ingredients of substances is known as eco-friendly demand. This paper was developed by considering a novel eco-friendly demand. Many people are interested in buying herbal products, mainly regarding skincare, body care, and hair care. Those types of products' demand depend on the usage percentage of herbal and chemical ingredients in the product. Suppose the usage percentage of chemical ingredients in the product is higher; in that case, some people are not interested in buying the product. I the same way, if the usage percentage of herbal ingredients in the product is high, people will show interest in buying those products. In the present scenario, most of the product sales depend upon advertisement. Advertisement plays a crucial role in marketing products. The product can reach the customer through advertisement. The demand rate of the product also depends upon the number of advertisements. The selling price and stock also play a key role in product sales and demand. Das et al. [4] developed an inventory model by considering price-dependent demand under partial backlogging. Pervin et al. [5] developed an inventory model for considering price- and stock-dependent demand. Pal et al. [6] proposed a production quantity model of ramp-type demand. Cardenas et al. [7] proposed an inventory model by examining non-linear stock-dependent demand. Mishra et al. [8] presented an inventory

model considering hybrid stock- and price-dependent demand. Chen et al. [9] presented an inventory model by examining time-varying price-dependent demand. Modak and Peter [10] presented a supply chain inventory model by assuming price- and delivery-time-dependent stochastic demand. Manna et al. [11] developed a production inventory model by exploring advertisement-related demand. Rad et al. [12] developed an inventory model by considering price- and advertisement-sensitive demand. Ranveer et al. [13] proposed a two-warehouse inventory model by considering how the demand rate of the items increases the selling price. We have examined how the demand rate of the product of this model is sensitive to selling price, and also how it depends on the usage percentage of herbal and chemical ingredients in the product, stock, and advertisement.

In previous research, many researchers considered the deterioration rate under preservation (Hsieh and Dye [14], Yang et al. [15], Yang et al. [16]). Chen and Dye [17] presented an inventory model by taking preservation investment as a decision variable. We have assessed deterioration of the items under preservation.

In a real market, many of the parameters are imprecise in nature. Interval-valued and fuzzy numbers are the mathematical tools that handle uncertainties. Many inventory and supply chain systems were developed under different uncertain environments Sharma et al. [18] and Mallick et al. [19] proposed an inventory model by considering the fuzzy lead time. Yadav et al. [20] proposed a supply chain by considering cost parameters as fuzzy numbers. Khatua et al. [21] developed a production control model under a fuzzy environment. Govindan et al. [22] developed a green supply chain management problem under an intuitionistic fuzzy environment. Banu and Mondal [23] implemented an inventory model by considering fuzzy demand. Moghdani et al. [24] developed a multi-item inventory model under a fuzzy environment. Arockia et al. [25] implemented an inventory model by considering ordering and import costs as fuzzy numbers. Adak and Mahapatra [26] developed an inventory model under a fuzzy environment. Shaikh et al. proposed [27] an inventory model under an interval-valued environment. Sahoo et al. [28] developed an inventory model by taking various fuzzy numbers. Govindan et al. [29] developed a closed-loop supply chain under a fuzzy uncertain environment. Zulqarnain et al. [30] proposed an application for sustainable supply chain management under the Pythagorean fuzzy environment. We have taken the cost parameters of the inventory model as triangular Pythagorean fuzzy numbers.

3. Model Formulation

This section presents a green inventory system with novel eco-friendly demand incorporating partial backlogging. Later, this proposed system will be discussed in a fuzzy environment.

Notations:

Inventory parameters:

p—Selling price of unit item.

B_0—Ordering cost for each replenishment.

C_{h_1}—Holding costs of unit item per unit time.

C_p, C_s—Purchasing cost and Shortage cost of unit item, respectively.

C_l—Opportunity cost due to lack of inventory.

θ_o—Deterioration rate without preservation.

C_{d_1}—Cost of each deteriorated item per unit time.

ξ—Preservation investment cost per unit time.

Environment parameters and indices:

Y—Carbon emissions cap (kg/year).

k—Deteriorated inventory weight in the warehouse (kg/unit).

d—Space required for each unit of product (meter/unit).

B_1—Carbon emission unit associated with ordering cost (kg/year).

C_{h_2}—Carbon emission for holding inventory (kg/year).

C_{d_2}—Carbon emission for inventory obsolescence (kg/year).

ϕ—Carbon tax ($/kg).

m—Efficiency of greener technology in reducing emission.
G—Green investment cost per unit time.
OCE—Total emission associated with ordering cost (kg/order/year).
HCE—Total emission for holding inventory (kg/year).
DCE—Total emission of deterioration for inventory (kg/year).

Decision variables:

T—Cycle time in each replenishment.
t_1—The time at which the inventory level reaches zero.

This paper aims to introduce realistic features such as deterioration, partial backlogging, impreciseness, eco-friendly demand, etc., based on real market conditions. At present, many people are aware of buying a product containing chemical ingredients. At the same time, they show interest in buying products containing herbal ingredients because herbal products are produced from plant extracts. We can make herbal products without harming the environment by consuming natural raw materials from plants. We have assumed the demand rate depends on the usage percentage of chemical and herbal ingredients in the product. Preservation controls the deterioration, reducing the product's spoilage and making the inventory model more eco-friendly, so we have assumed deterioration of the items under preservation. We consider this study of inventory management in a sustainable environment. Generally, green investment controls carbon emissions. We have taken the cost parameters as triangular Pythagorean fuzzy numbers to make the inventory model more realistic. Pythagorean fuzzy sets are the extension of fuzzy and intuitionistic fuzzy sets and deal with a huge number of uncertainties compared to fuzzy and intuitionistic fuzzy sets. Table 1 indicates the innovation of the proposed inventory system compared to previous literature ("NC" stands for "Not Considered").

Table 1. Contribution of this article compared to previous studies.

Articles	Demand	Preservation Technology	Carbon Emission	Environment
Pal et al. [31] (2015)	Ramp type	NC	NC	Fuzzy
Mishra [32] (2016)	Constant	Considered	NC	Crisp
Mahapatra et al. [33] (2019)	Price, advertisement & stock	NC	NC	Fuzzy
Li et al. [34] (2019)	Price-dependent	Considered	NC	Crisp
MdMashud et al. [35] (2020)	Price & advertisement	Considered	NC	Crisp
Pervin et al. [36] (2020)	Price & stock	Considered	NC	Crisp
De et al. [37] (2020)	Stock dependent	Considered	NC	Fuzzy
Mashud et al. [38] (2020)	Price-sensitive	Considered	Considered	Crisp
Das et al. [39] (2021)	Selling price & stock level	Considered	NC	Crisp
Mohammad et al. [40] (2021)	Price & Stock	NC	NC	Crisp
Mishra [41] (2021)	Trade credit & price sensitive	Considered	Considered	Crisp
Rahman et al. [42] (2021)	Price & Stock	Considered	NC	Crisp
Xu et al. [43] (2020)	Time-varying	NC	Considered	Crisp
Ruidas et al. [44] (2021)	Price-sensitive	Considered	Considered	Interval-valued
Present research work	Price, advertisement, stock & eco-friendly	Considered	Considered	Pythagorean Fuzzy

In the context of a triangular Pythagorean fuzzy cost pattern with green technology investment, the contribution of this work is determining the optimal cycle time in order to increase the overall profit of the inventory management system. To the best of our knowledge, this study explores all the concepts listed, which have never been studied together in earlier research literature.

Assumptions:

1. This model introduces a novel eco-friendly demand. The demand rate of the model increases with stock, the number of advertisements (*A*), and the usage percentage of herbal

ingredients in the product (α), and the demand rate decreases with price (p) and the usage percentage of chemical ingredients in the product (β).

2. The demand rate of this model depends upon stock, price, the number of advertisements, and the usage percentage of herbal and chemical ingredients in the product, that is, $D(q(t), A, p, \alpha, \beta) = (a + bq(t))A^{\nu}\gamma e^{(\alpha - \beta)\rho}p^{-\delta}$, where a, b, ρ, $\gamma > 0$, $A > 1$, and $0 \leq \alpha, \beta, \nu, \delta \leq 1$. Here, A is the number of advertisements in the cycle and ν is the impact of the advertisement. α, β is the usage percentage of herbal and chemical ingredients in the product, respectively.

3. The deteriorating items in the inventory model are constant and controllable under preservation technology, i.e., $\theta = \theta_o e^{-a_1\xi}$, where $0 < \theta_0 < 1$ and $a_1 > 0$.

4. Cost parameters of this model are taken as triangular Pythagorean fuzzy numbers.

5. Shortages are allowed, and the partially backlogged and backlogging rate (f) is constant.

6. A single item is considered in each replenishment.

7. Carbon emission occurs due to the setup, deterioration, and inventory holding.

8. In this model, tax is imposed to deal with carbon emissions. The tax amount is to be imposed under the carbon cap and tax policy. This is an alternate strategy in which a tax is imposed on a specific quantity of emissions instead of the whole amount of emissions. This occurs when the regulatory authority imposes a carbon emission cap on companies and levies a tax solely on the quantity of emissions that exceed the cap. The situation might alternatively be seen as the company purchasing carbon emission permits from other companies when their quantity exceeds the quota.

9. The inventory system plans to move towards a greener inventory system by investing in advanced technology, energy-efficient machinery, non-conventional energy, and so on. The fraction of reduction of average emissions $E = \eta\left(\frac{mG}{1+mG}\right)$ because η is the amount of carbon emission when green technology is invested in, and m alters the ability of green technology to decrease emissions. The fraction of reduction E = 0, when G = 0, and tends to η when $G \rightarrow \infty$. The parameter m reflects the efficiency of greener technology in reducing emissions. With an investment G, we can reduce the per unit emission by investing G. The investment cost function, E(G), is continuously differentiable with $E'(G) > 0$, $E''(G) < 0$.

By considering the above assumption, we formulated an inventory model. The starting stock level of the inventory system is Q_1. The stock level of the inventory system decreases due to the demand and decay of goods. At the time t_1, the stock level of the inventory system reaches zero. At the time interval (t_1, T), shortages occur, and unsatisfied demand partially backlogs with the backlogging rate f. At shortages, the inventory level is Q_2. The total reorder quantity of the inventory system is $Q = Q_1 + Q_2$ as shown in Figure 2.
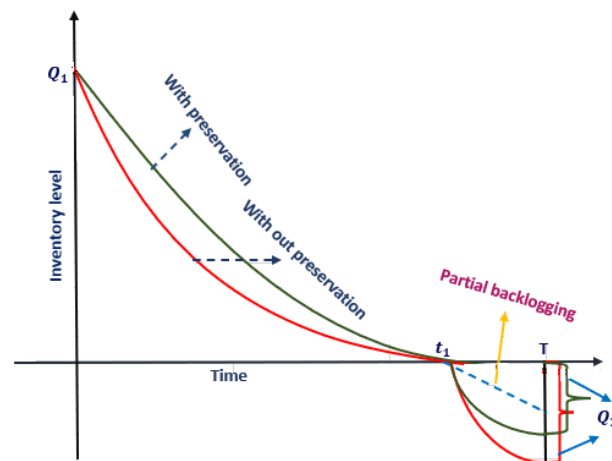

Figure 2. Graphical representation of inventory model.

Let *q(t)* be the inventory level at time t, and the following differential equations describe the decrease in stock levels of the inventory systems due to demand and deterioration at the time interval $(0, t_1)$.

$$\frac{dq(t)}{dt} + \theta(\theta_o, \xi)q(t) = -(a + bq(t))A^v \gamma e^{(\alpha - \beta)\rho} p^{-\delta}, 0 \leq t \leq t_1. \tag{1}$$

$$\frac{dq(t)}{dt} = -fap^{-\delta} A^v \gamma e^{(\alpha - \beta)\rho} p^{-\delta}, t_1 \leq t \leq T. \tag{2}$$

with boundary conditions $q(0) = Q_1, q(t_1) = 0$ and $q(T) = -Q_2$.

The solution of Equations (1) and (2) using boundary conditions is given by

$$q(t) = \frac{aF(e^{(\theta + bF)(t_1 - t)} - 1)}{\theta + bF}, 0 \leq t \leq t_1. \tag{3}$$

$$q(t) = afF(t_1 - t), t_1 \leq t \leq T. \tag{4}$$

where $F = A^v \gamma e^{(\alpha - \beta)\rho} p^{-\delta}$.

Inserting $t = 0$ in Equation (3), we obtain

$$Q_1 = \frac{aF(e^{(\theta + bF)t_1} - 1)}{\theta + bF}. \tag{5}$$

$$Q_2 = afF(T - t_1). \tag{6}$$

Since the ordering quantity is $Q = Q_1 + Q_2$,

$$Q = \frac{aF(e^{(\theta + bF)t_1} - 1 + (\theta + bF)f(T - t_1))}{\theta + bF} \tag{7}$$

3.1. Mathematical Analysis of the Proposed Inventory System

The ordering cost of the discussed inventory system, including transportation cost, is given by

$$OC = B_0. \tag{8}$$

Holding costs are the added costs for holding and maintaining inventory. The inventory system is under stock in the time interval $(0, t_1)$ so, the holding cost of the proposed inventory system in the interval $(0, t_1)$ is

$$HC = C_{h_1} \int_0^{t_1} I(t)dt = \frac{aFC_{h_1}}{(\theta + bF)^2} [e^{(\theta + bF)t_1} - (\theta + bF)t_1 - 1] \tag{9}$$

Deterioration costs occur due to decaying or deteriorating stock. At the time interval $(0, t_1)$, the total stock of the inventory system is Q_1. The stock Q_1 decreases due to demand, and the remaining quantity decreases due to deterioration in the interval $(0, t_1)$, so the deterioration cost of the items in the inventory system is given by

$$DC = C_{d_1} \left[Q_1 - \int_0^{t_1} D(q(t), A, p, \alpha, \beta)dt \right] = \frac{aFCd_1\theta}{(\theta + bF)^2} [e^{(\theta + bF)t_1} - (\theta + bF)t_1 - 1] \tag{10}$$

The cost of purchasing items in the inventory system is given as

$$PC = C_p Q = \frac{C_1 aF}{\theta + bF} [(e^{(\theta + bF)t_1} - 1 + (\theta + bF)f(T - t_1))]. \tag{11}$$

At the time interval (t_1, T), the inventory was overstocked, so shortages occurred due to the non-availability of the stock in the time period (t_1, T). The cost that is added due to the lack of stock is given by

$$SC = -C_s \int_{t1}^{T} q(t)dt = \frac{CsfaF}{2}(T + t_1)^2 \tag{12}$$

At the time interval (t_1, T) shortages occur due to that retailer's lost sales. The cost that occurs due to loss of sales is termed the opportunity cost and is given as follows

$$OP = C_l \int_{t_1}^{T} (1 - f)aF dt = C_l aF(1 - f)(T - t_1) \tag{13}$$

Sales revenue for selling items in the time interval $(0, T)$ is

$$\begin{aligned} \text{SR} &= p \int_0^{t_1} D(q(t), A, p, g(\alpha, \beta))dt + pQ_2 \\ &= \frac{paf}{(\theta+bF)^2}[(\theta + bF)^2(T - t_1)f + \theta^2 t_1 + bF(e^{(\theta+bF)t_1} + \theta t_1 - 1)] \end{aligned} \tag{14}$$

Advertisement cost (*AC)* of the inventory system is the product of the number of advertisements to the cost of each advertisement (g)

$$AC = gA. \tag{15}$$

Preservation technology investment (*PTI*) per cycle is

$$PTI = \xi T. \tag{16}$$

3.2. Analysis of the Inventory System under the Green Environment

Emissions are produced in the inventory system due to holding the commodities, deteriorating items, and setting up and transporting inventory. Using green technology, we can control the emissions in the inventory system. For example, we can reduce the deterioration rate by recycling items to reduce emissions from the deteriorating products. By using solar power and upgrading high-efficiency lighting, we can reduce high holding costs of the inventory system, and by using electric vehicles, we can reduce transportation costs. Investing money to develop green technology is known as a green technology investment.

The green technology investment per cycle is given by

$$GTI = GT \tag{17}$$

The total carbon allowance cost is calculated by the carbon tax ϕ with the total carbon emission cap Y, so the total carbon allowance cost per cycle is

$$CA = \phi YT. \tag{18}$$

The emissions impact inventory ordering costs due to the transportation of goods and set up of the instruments. The total emission corresponding ordering cost per cycle is equal to

$$OCE = B_1. \tag{19}$$

Emissions are produced due to holding the inventory, and the total emissions produced by holding the items in the inventory per cycle is equal to

$$HCE = \frac{C_{h_2}daF}{(\theta + bF)^2}[e^{(\theta+bF)t_1} - (\theta + bF)t_1 - 1]. \tag{20}$$

In the inventory system, the emissions are produced due to the disposal of deteriorating items, so the total emissions produced due to deteriorating items per cycle is equal to

$$DCE = \frac{aFC_{d_2}\,k\theta}{(\theta + bF)^2}[e^{(\theta + bF)t_1} - (\theta + bF)t_1 - 1].$$

(21)

The total carbon emission cost per cycle is equal to

$$TEC = \phi(CE) = \phi\left(B_1 + \frac{aF\theta}{(\theta + bF)^2}\left(C_{d_2}\theta k + C_{h_2}d\right)\left(e^{(\theta + bF)t_1} - (\theta + bF)t_1 - 1\right)\right).$$

(22)

where *CE* is the total emission produced per cycle, equal to

$$CE = B_1 + \frac{aF\theta}{(\theta + bF)^2}\left(C_{d_2}\theta k + C_{h_2}d\right)\left(e^{(\theta + bF)t_1} - (\theta + bF)t_1 - 1\right).$$

(23)

4. Proposed Inventory Model under Pythagorean Fuzziness

The Pythagorean fuzzy set is used to demonstrate the fuzziness of the suggested inventory system. We have implemented the inventory system in a green environment with ambiguous cost parameters assuming a Triangular Pythagorean Fuzzy Number (TPFN), i.e., $C_{h1} = \{(h_1, h_2, h_3), (h'_1, h'_2, h'_3)\}, C_{d1} = \{(d_1, d_2, d_3), (d'_1, d'_2, d'_3)\}, C_l = \{(l_1, l_2, l_3), (l'_1, l'_2, l'_3)\}, C_s = \{(s_1, s_2, s_3), (s'_1, s'_2, s'_3)\}, C_p = \{(p_1, p_2, p_3), (p'_1, p'_2, p'_3)\}$ and $B_0 = \{(b_1, b_2, b_3), (b'_1, b'_2, b'_3)\}$. The proposed novel de-fuzzification approach for the Pythagorean fuzzy number (PFN) for the aforementioned fuzzy cost parameters assesses the de-fuzzified values of the cost parameters, which are indicated by $\widetilde{C}_{h_1}, \widetilde{C}_{d_1}, \widetilde{C}_l, \widetilde{C}_s, \widetilde{C}_p$ and $\widetilde{B}_0$.

In Appendix B, we have presented the Pythagorean fuzzy sets and their operations required for further calculations. We have provided unique de-fuzzification procedures for Pythagorean fuzzy sets based on the notion of Pythagorean fuzzy sets described in Appendix B.

4.1. Defuzzification of Pythagorean Fuzzy Number

The Mellin transform $M_X(t)$ of a probability density function $f(x)$, where x is positive, is defined as

$$M_X(t) = \int_0^\infty x^{t-1} f(x) dx.$$

(24)

We know that if the non-membership value decreases, the acceptance of the PFN increases. As a consequence, if the value of the complement of the non-membership function increases, the acceptance possibility of the PFN increases. Considering those concepts, the membership function and the non-membership are taken into account to rank the PFNs.

Now the complement of the non-membership function of a normal generalized TPFN A_{PF} is given by

$$\left[1 - \nu_{A_{PF}}(x)\right] = \begin{cases} 1 - \sqrt{\frac{a_2 - x}{a_2 - a'_1}} & \text{when } a'_1 \leq x < a_2 \\ 1 - \sqrt{\frac{x - a_2}{a'_3 - a_2}} & \text{when } a_2 \leq x \leq a'_3 \\ 0, & \text{otherwise} \end{cases}$$

(25)

4.2. Proportional Probability Distribution

We consider a probability density function $f(x) = c\mu_{A_{PF}}(x)$ associated with $A_{PF}(x)$, where c is a constant, obtained by using the property of probability density function, that is, $\int_\infty^{-\infty} f(x)dx = 1$ implies that $\int_\infty^{-\infty} c\mu_{A_{PF}}(x)dx = 1$.

Here, we define two probability density functions, $g_1(x)$ and $g_2(x)$ corresponding to the membership function $\mu_{A_{PF}}(x)$ and the complement of the non-membership function $\left[1 - \nu_{A_{PF}}(x)\right]$, respectively. Let $g_1(x) = c_1\mu_{A_{PF}}(x)$. Then, the constant c_1 in $g_1(x)$ is given

as $c_1 = \frac{3}{2(a_3 - a_1)}$ and $g_1(x) = \begin{cases} \frac{3}{2(a_3 - a_1)} \sqrt{\frac{x - a_1}{a_2 - a_1}} & for\ a_1\ \leq\ x < a_2 \\ \frac{3}{2(a_3 - a_1)} \sqrt{\frac{a_3 - x}{a_3 - a_2}} & for\ a_2\ \leq\ x\ \leq a_3 \\ 0,\ otherwise \end{cases}$ Similarly, considering

$g_2(x) = c_2 [1 - \nu_{A_{PF}}(x)]$, the constant c_2 in the function $g_2(x)$ is obtained as $c_2 = \frac{3}{(a'_3 - a'_1)}$

and $g_2(x) = \begin{cases} \frac{3}{(a'_3 - a'_1)} \left(1 - \sqrt{\frac{a_2 - x}{a_2 - a'_1}} \right) & when\ a'_1\ \leq\ x < a_2 \\ \frac{3}{(a'_3 - a'_1)} \left(1 - \sqrt{\frac{x - a_2}{a'_3 - a_2}} \right) & when\ a_2\ \leq\ x\ \leq a_3 \\ 0,\ otherwise \end{cases}$.

With the help of a convex combination of two functions $g_1(x)$ and $g_2(x)$, we define $h(x)$ as the probability density function corresponding to the normal GTPFN A_{PF}, which is defined as $h(x) = \lambda g_1(x) + (1 - \lambda) g_2(x)$, where $0 \leq \lambda \leq 1$.

As we know, any probability density function with finite support is associated with an expected value. The Mellin transform is used to find its expected value of the random variable *X* with probability density function *h(x)* given by $E(X) = \int_{\infty}^{-\infty} x f(x) dx$.

Using definition (24), the expected value of *X* is given by

$$E(X) = M_X(2) = \frac{1}{10} \left[3(a'_3 + a'_1) + 4a_2 + \lambda \left(4(a_1 + a_3) - 2a_2 - 3(a'_3 + a'_1) \right) \right]. \qquad (26)$$

The total profit of the inventory system per unit time with carbon emissions under the Pythagorean fuzzy environment is defined as follows:

$$TAP(t_1, T) = \frac{1}{T} [SR - OC - HC - DC - PC - SC - OP - PTI - AC - GTI + \phi (YT - CE\{1 - \eta (mG/(1 + mG))\})] \qquad (27)$$

The mathematical expression for the total profit incorporating all the derived computation is given by:

$$\begin{aligned} TAP(t_1, T) = \frac{1}{T} \big[& \frac{paF}{(\theta + bF)^2} t[(\theta + bF)^2 (T - t_1)f + \theta^2 t_1 + bF(e^{(\theta + bF)t_1} + \theta t_1 - 1] - \widetilde{B}_0 \\ & - \widetilde{C}_{h_1} \frac{aF}{(\theta + bF)^2} [e^{(\theta + bF)t_1} - (\theta + bF)t_1 - 1] - \frac{aF\widetilde{C}_{d_1}\theta}{(\theta + bF)^2} [e^{(\theta + bF)t_1} - (\theta + bF)t_1 - 1] \\ & - \widetilde{C}_1 aF(1 - f)(T - T_1) - \frac{\widetilde{C}_p aF}{\theta + bF} [(e^{(\theta + bF)t_1} - 1 + (\theta + bF)f(T - t_1))] + \phi (YT \\ & - (B_1 + \frac{aF}{(\theta + bF)^2} (C_{d_2}\theta k + C_{h_2}d)(e^{(\theta + bF)t_1} - (\theta + bF)t_1 - 1)) \{1 - \eta (\frac{mG}{1 + mG})\}) \\ & - \frac{\widetilde{C}_s faF}{2} (T + t_1)^2 - \xi T - gA - GT \big] \end{aligned} \qquad (28)$$

To find the optimum total profit of the proposed inventory system, we need to optimize the above Expression (28) to the two vital decision parameters. The first one is the time at an inventory level of zero, and the second is the cycle length of replenishment. Based on Theorem A1 stated in Appendix A, the optimum solution of the proposed model in a Pythagorean and green environment is determined.

5. Results of an Empirical Study

5.1. Results and Discussion

In a company's newly launched eco-friendly product, the demand rate of the product depends on the number of advertisements, price, and stock. The demand rate of the product also depends on the usage percentage of herbal and chemical ingredients in the product. The selling price per unit item is $p = \$55$ per unit item, the price-sensitive parameter is $\delta = 0.6$, the usage percentage of herbal ingredients in the product is $\alpha = 0.8$, the usage percentage of chemical ingredients in the product is $\beta = 0.2$, the number of advertisements per cycle is $A = 50$ and each advertisement cost is $g = \$10$, the impact of the number of advertisements is $\nu = 0.4$, the holding cost per unit item per unit time is $\widetilde{C}_{h_1} = \$14.28$, the ordering cost per order is $\widetilde{B}_0 = \$200$, the purchasing cost per unit item is $\widetilde{C}_p = \$36.4$, the shortage cost per unit item per unit time is $\widetilde{C}_s = \$30$, the loss of opportunity per unit item is $\widetilde{C}_l = \$25$, the initial deterioration rate of the item is $\theta_o = 0.1$, the deterioration cost per unit time is

$\widetilde{C}_{d_1}$ = \$23.6, the preservation cost per unit time is ξ = \$5 and the controlling preservation parameter is a_1 = 0.5, the backlogging rate is f = 0.9, the green investment amount per unit time is G = \$6, the carbon cap is given by Y = 800 (kg/year), the tax per kg carbon emission is φ = \$0.43, the amount of emission associated with setup, holding, and ordering costs are, respectively, $B_1 = 150\left(\frac{kg}{setup}\right)$, $C_{h_2} = 12\left(\frac{\frac{kg}{unit}}{year}\right)$ $C_{d_2} = 14\left(\frac{\frac{kg}{unit}}{year}\right)$, and the other parameters are $a = 150$, $b = 0.1$ $k = 0.2\left(\frac{kg}{unit}\right)$, $d = 0.4\left(\frac{meter}{unit}\right)$, $m = 0.8$, $\eta = 0.2$, $\gamma = 2$, $\rho = 1.2$ and $\lambda = 0.6$, where $C_{h_1} = \{(10, 15, 18), (8, 15, 20)\}$, $C_{d_1} = \{(12, 20, 38), (10, 20, 40)\}$, $C_l = \{(15, 25, 35), (13, 25, 37), C_s = \{(20, 30, 40), (18, 30, 42)\}$, $C_p = \{(22, 40, 48), (20, 40, 50)\}$ and $B_0 = \{(100, 200, 300), (100, 200, 300)\}$. The parameter input values are partially taken from previous literature [45].

In Table 2, we can observe that the total profit is high in a sustainable environment. The benefit of developing models in a sustainable environment is that we can control items' waste and emissions. This inventory system allows a carbon emission cap to release a limited amount of emission. The concave representation of the total profit of the inventory system is shown in Figure 3.

Table 2. Total profit of inventory system under various environments.

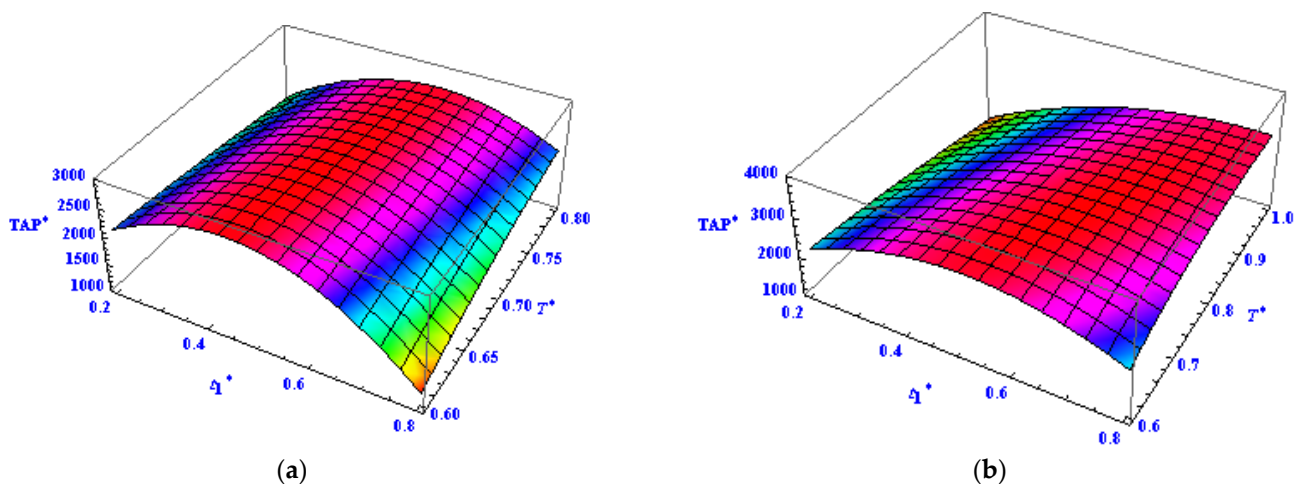| Environment	t_1^* **(Year)**	T^* **(Year)**	$TAP^*\left(t_1^*, T^*\right)$	Q_1^*	Q_2^*	Q^*
Green and preservation	0.597481	0.744975	3063.43	168.197	35.3348	203.582
Without green and preservation environment	0.473401	0.670989	2714.41	134.663	47.3356	181.998

Figure 3. Graphical depiction of concavity of optimal total profit: (**a**) Without preservation; (**b**) under preservation.

Table 3 shows the characteristics of the green investment parameter G. Suppose we increase the green investment amount of the inventory system; in that case, the total profit of the system also increases up to the maximum investment of G. As seen in Figure 4, the overall profit declines after the maximum investment of G. We can also observe that for this given numerical example, the maximum amount of G we can invest is \$6 per unit of time.

In Table 4, we can observe changes in the total profit for various values of λ, where λ is the parameter of the de-fuzzified function and the value of λ ∈ [0, 1]. If we increase the value of λ, the total profit of the system also increases. Based on the actual market conditions, the decision maker can choose the value of λ.

Table 3. Total profit of inventory system under various values of G.

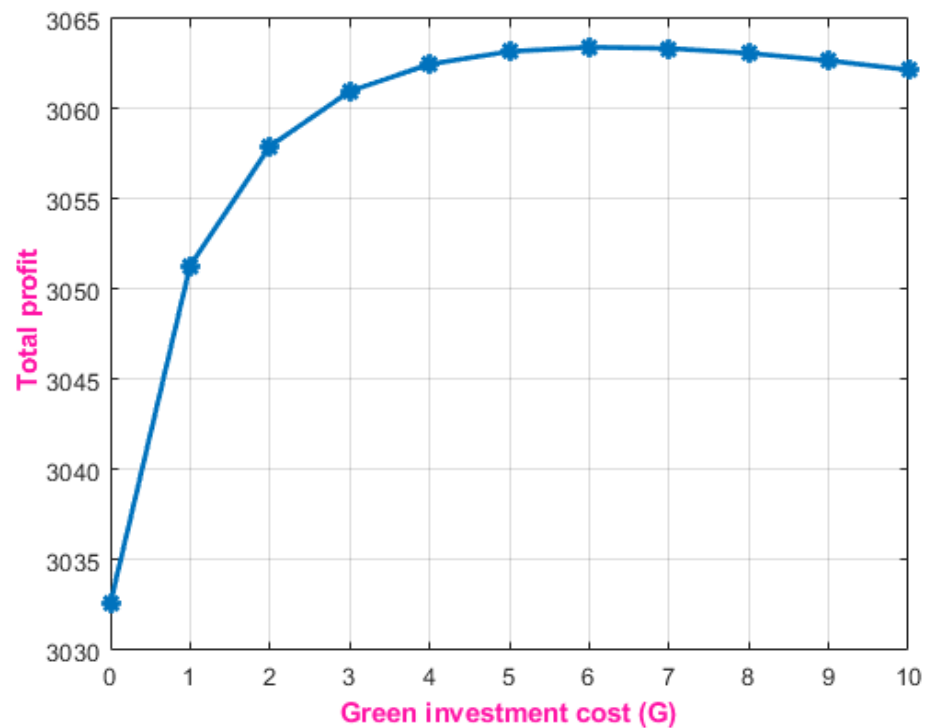| G	t_1^* (Year)	T^* (Year)	$TAP^*\left(t_1^*,T^*\right)$
0	0.592295	0.744918	3032.58
1	0.595060	0.744936	3051.32
2	0.596135	0.744950	3057.94
3	0.596708	0.744960	3060.99
4	0.597063	0.744967	3062.49
5	0.597305	0.744972	3063.20
6	0.597481	0.744975	3063.43
7	0.597614	0.744978	3063.37
8	0.597718	0.744980	3063.10

Figure 4. Total profit of the inventory system under green investment parameter G.

Table 4. Total profit of inventory system under various values of λ.

Parameter	t_1^* (year)	T^* (year)	$TAP^*\left(t_1^*,T^*\right)$	Q_1^*	Q_2^*	Q^*
$\lambda = 0$	0.590854	0.742191	2892.08	166.227	36.2553	202.483
$\lambda = 0.2$	0.593049	0.743111	2949.15	166.880	32.9498	202.830
$\lambda = 0.4$	0.595258	0.744039	3006.27	167.536	35.6430	203.179
$\lambda = 0.6$	0.597481	0.744975	3063.43	168.197	35.3348	203.582
$\lambda = 0.8$	0.599717	0.745919	3120.63	168.862	35.0252	203.888
$\lambda = 1$	0.601967	0.746871	3177.88	169.532	34.7142	204.246

By employing eco-friendly demand, three cases will appear. The first case is $\omega < 1$; in this case, the usage percentage of herbal ingredients in the product is less than the usage percentage of chemical ingredients. Suppose the usage percentage of herbal ingredients in

the product is equal to the percentage of chemical ingredients, or some customers do not consider the ingredients in the product. In that case, these two scenarios come under case $\omega = 1$. The third case is $\omega > 1$ case. In this case, the percentage of herbal ingredients in the product is more than the chemical ingredients. Table 5 presents the total profit changes in all these cases.

Table 5. Total profit of inventory system under various values of α and β.

$\omega\left(\frac{\alpha}{\beta}\right)$	t_1^* (year)	T^* (year)	$TAP^*\left(t_1^*, T^*\right)$	Q_1^*	Q_2^*	Q^*
$\omega = 0.1$	0.929017	1.31569	652.618	80.5255	29.2729	109.798
$\omega = 0.5$	0.857248	1.19576	858.821	90.2861	31.0508	121.337
$\omega = 1$	0.777642	1.06138	1185.44	104.554	33.0861	137.640
$\omega = 2$	0.708321	0.94251	1637.00	121.683	72.2574	193.941
$\omega = 4$	0.677252	0.88837	1918.93	131.567	35.2865	166.854

Observations: In the $\omega > 1$ case, the demand rate of the products will increase due to higher herbal ingredients in the product, so sales of the product increase. Due to rising sales, the total profit of the inventory system also increases.

In the $\omega = 1$ case, the products' demand rate neither increases nor decreases due to the herbal and chemical ingredients. The total profit will also not be affected by herbal and chemical ingredients. In the $\omega < 1$ case, the product's demand rate will decrease due to the higher chemical ingredients, so sales of the product will decrease. Due to the decrease in sales, the total profit of the inventory system also drops.

5.2. Sensitivity Analysis

Observations: From Table 6, we can notice the following things:

1. The total inventory profit is highly effective in the selling price p. If the item's price increases, then the total profit, t_1 and T, also increases, as shown in Figure 5.

2. If the purchasing cost $(\widetilde{C}_p)$ of the inventory system increases, then the total profit of the inventory system decreases. It is natural that when the purchasing cost is higher, the cycle length and total profit decrease.

3. The total inventory profit is moderately sensitive to the parameter $\widetilde{C}_{h_1}$. It shows that if the holding cost of the product increases, the total profit of the system decreases but is not high, and only shows a medium impact. The cycle length T and t_1 decrease if we increase the holding cost.

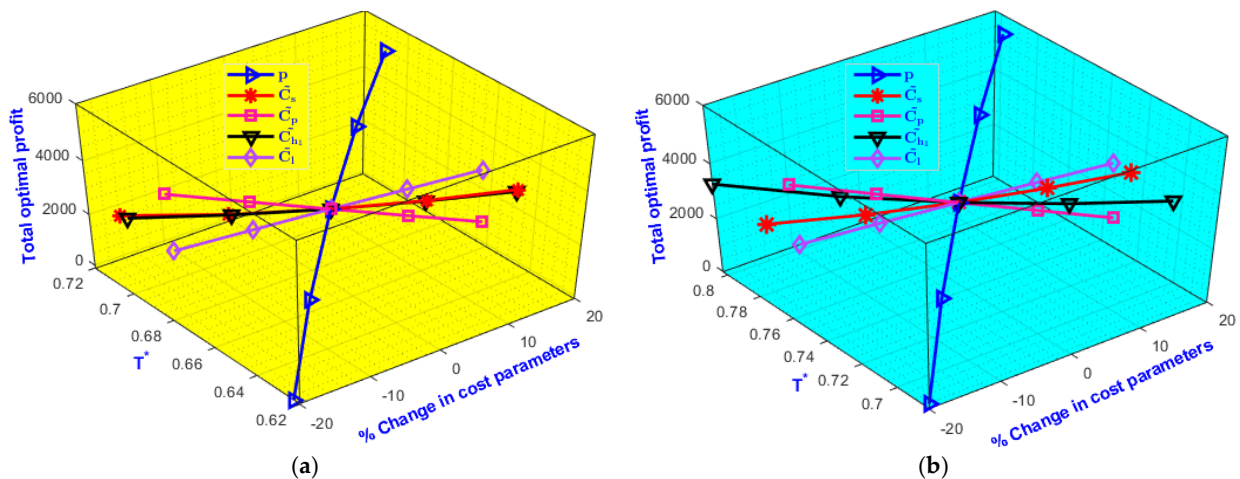4. The inventory system shows less impact on the other inventory cost parameters.

Figure 5. Graphical representation of optimal profit under various inventory cost parameters: (**a**) Without preservation; (**b**) under preservation.

Table 6. Sensitivity analysis for inventory-related cost parameters.

Parameter	% Change	t_1^* (Year)	% Change in t_1^*	T^* (Year)	% Change in T^*	$TAP^*\left(t_1^*,T^*\right)$	% Change in Optimal Profit
	−20	0.506489	−15.23	0.681956	−8.46	211.998	−93.08
p	−10	0.552553	−7.52	0.714726	−4.06	1725.64	−43.67
	10	0.641394	7.35	0.772970	3.76	4264.10	39.19
	20	0.684374	14.54	0.798899	7.24	5355.06	74.81
	−20	0.590467	−1.17	0.752651	1.03	3090.95	0.90
$\widetilde{C}_l$	−10	0.594049	−0.57	0.748929	0.53	3076.90	0.44
	10	0.600759	0.55	0.740787	−0.56	3050.55	−0.42
	20	0.603883	1.07	0.736359	−1.16	3038.28	−0.82
	−20	0.684918	14.63	0.803316	7.83	3272.55	6.73
$\widetilde{C}_{h_1}$	−10	0.637576	6.71	0.771261	3.53	3162.68	3.24
	10	0.562924	−5.78	0.722983	−2.95	2973.13	−2.95
	20	0.532723	−10.84	0.704282	−5.46	2890.49	−5.65
	−20	0.591066	−1.07	0.771057	3.50	3088.60	0.82
$\widetilde{C}_s$	−10	0.594565	−0.49	0.756679	1.57	3074.87	0.37
	10	0.599948	0.41	0.735260	−1.30	3053.74	−0.32
	20	0.602064	0.77	0.727063	−2.40	3045.43	−0.59
	−20	0.598489	0.17	0.745626	0.09	3066.00	0.08
$\widetilde{C}_{d_1}$	−10	0.597984	0.08	0.745300	0.04	3064.71	0.04
	10	0.596978	−0.08	0.744651	−0.04	3062.15	−0.04
	20	0.596476	−0.17	0.744327	−0.09	3060.87	−0.08
	−20	0.644074	7.80	0.755922	1.47	5063.67	65.29
$\widetilde{C}_p$	−10	0.620338	3.83	0.750409	0.73	4060.68	32.55
	10	0.575438	−3.69	0.739597	−0.72	2071.64	−32.38
	20	0.554156	−7.25	0.734258	−1.44	1085.04	−64.58

Observations: From Table 7, we can notice the following things:

1. The total inventory profit is highly effective in the price-sensitive parameter δ. If the price sensitivity of the item is high, then the total profit of the inventory system reduces. The reason behind this is if the price sensitivity of the system is high, then most customers are not interested in buying, and the demand rate of the product decreases due to the cycle length T and the time t_1 of the inventory system increases, as shown in Figure 6.

2. The total inventory profit is moderately sensitive to the parameters ν and a. It shows that if the impact of the advertisement increases due to the product sales increasing, the demand rate and profit of the system increase but not to a large extent; it only shows a medium impact. If we increase parameter a, the demand rate of the product moderately increases due to the profit of the inventory system increasing moderately. The cycle length T and t_1 decreases moderately due to the increase in parameters. The total inventory profit is moderately sensitive to the parameters ν and a. It shows that if the impact of the advertisement increases because the product sales increase, the demand rate and profit of the system increase, but it only shows a medium impact. If we increase parameter a, the product's demand rate moderately increases because the profit of the inventory system increases moderately. The cycle length T and t_1 decrease moderately due to parameters ν and a. The remaining parameters show less of an impact on the total profit.

3. Suppose the carbon tax ϕ of the inventory system increases under green technology investment. In that case, the total profit of the inventory system also increases. It concludes that high carbon tax is always beneficial for the environment. If the amount of emission when green technology is invested (η) increases, then the total profit also increases, but not by much, which shows the realistic nature of the presented model.

Table 7. Sensitivity analysis for inventory parameters.

| Parameter	% Change	t_1^* (Year)	% Change in t_1^*	T^* (Year)	% Change in T^*	$TAP^*\left(t_1^*,T^*\right)$	% Change in Optimal Profit
	−20	0.665700	11.42	0.868054	16.52	2041.38	−33.36
ν	−10	0.629738	5.40	0.803960	7.92	2503.65	−18.27
	10	0.568860	−4.79	0.690794	−7.27	3740.76	22.11
	20	0.543883	−8.97	0.641169	−13.93	4560.23	48.86
	−20	0.657762	10.09	0.838376	12.54	2326.92	−24.04
a	−10	0.625239	4.65	0.787939	5.77	2692.04	−12.12
	10	0.573400	−4.03	0.707768	−5.00	3440.25	12.30
	20	0.552230	−7.57	0.675106	−9.38	3821.84	24.76
	−20	0.602878	0.90	0.745145	0.02	3032.20	−1.02
ϕ	−10	0.600157	0.45	0.745046	0.01	3047.76	−0.51
	10	0.594846	−0.44	0.744933	−0.01	3079.20	0.51
	20	0.592253	−0.88	0.744918	−0.02	3095.07	1.03
	−20	0.596431	−0.18	0.744955	−0.003	3056.03	−0.24
η	−10	0.596955	−0.09	0.744965	−0.001	3059.73	−0.12
	10	0.598008	0.09	0.744987	0.002	3067.14	0.12
	20	0.598537	0.18	0.745000	0.003	3070.85	0.24
	−20	0.521267	−12.76	0.592795	−20.43	5632.57	83.86
δ	−10	0.555001	−7.11	0.663614	−10.92	4160.98	35.83
	10	0.648565	8.55	0.837693	12.45	2244.62	−26.73
	20	0.708557	18.59	0.942920	26.57	1635.08	−46.63
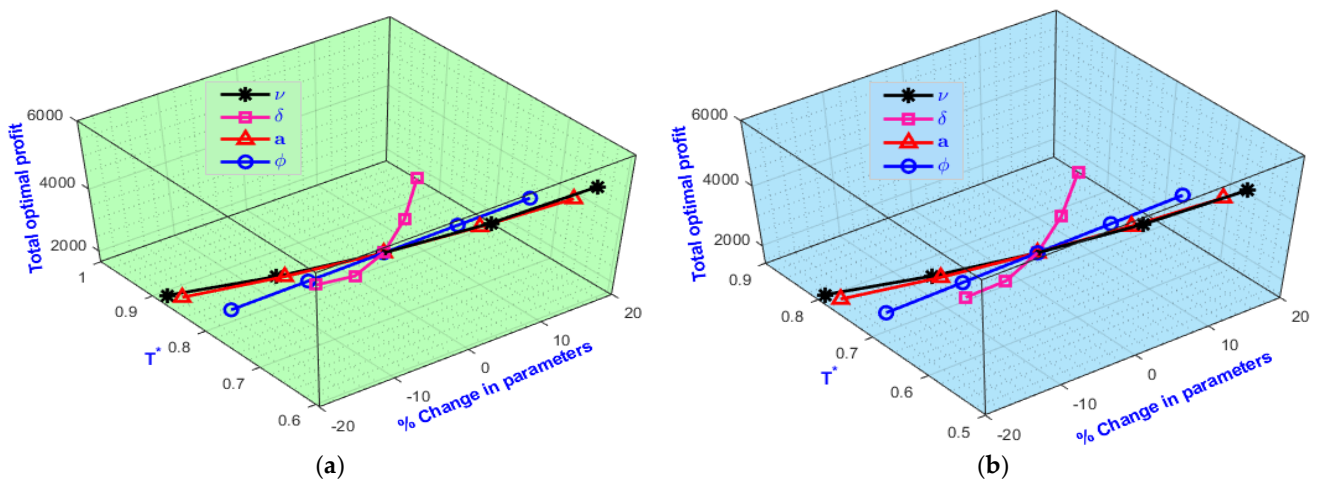
Figure 6. Graphical representation of optimal profit under various parameters: (**a**) Without preservation; (**b**) under preservation.

From Figure 7, we can observe the following changes:

1. Suppose the carbon emission associated with the setup process (B_1) increases; the optimal cycle length increases, and optimal profit decreases. The setup process is less sensitive to the total optimal solution.

2. Suppose the carbon emission amount associated with items $\left(C_{d_2}\right)$ decaying increases. In that case, the optimal cycle length and profit decrease. The parameter $\left(C_{d_2}\right)$ does not show much effectiveness in the total optimal solution.

3. Suppose the carbon emission amount associated with the inventory process $\left(C_{h_2}\right)$ increases, then the optimal cycle length and profit decrease. It is quite natural that when the inventory holding cost is higher, the company tries to finish the on-hand inventory as early as possible, so the cycle length decreases. Here, the parameter $\left(C_{h_2}\right)$ is less effective in the total optimal solution.
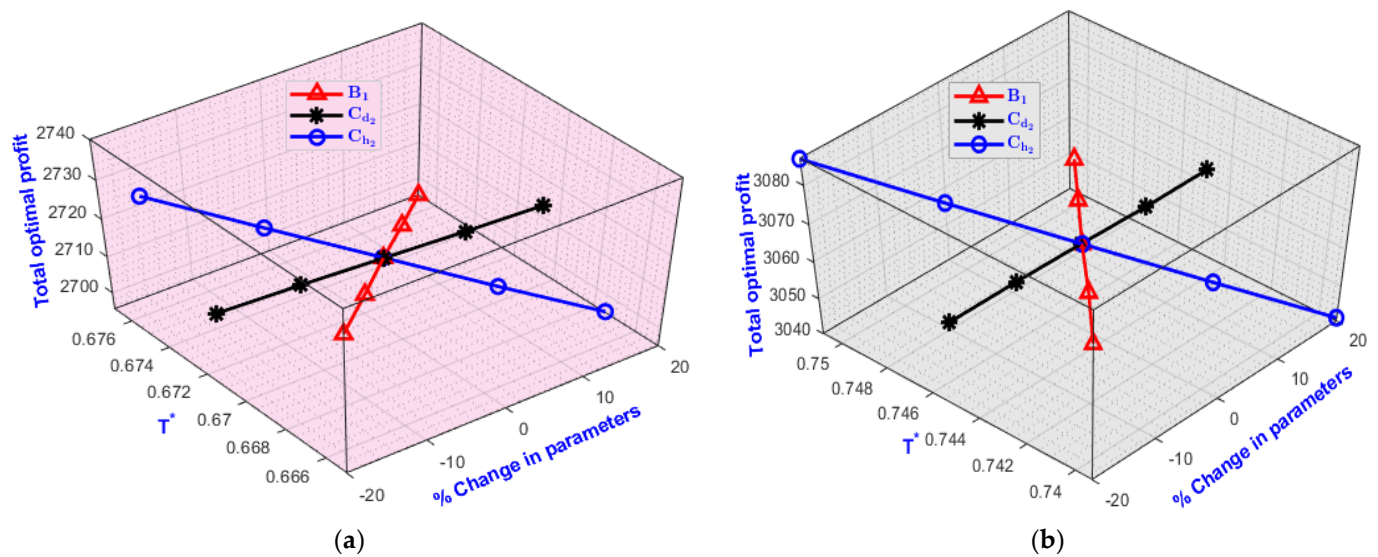

Figure 7. Graphical representation of optimal profit under various emission parameters: (**a**) Without preservation; (**b**) under preservation.

5.3. Managerial Implication

Environmentally conscious inventory models are more beneficial than ordinary inventory models. A carbon tax is levied on the carbon emissions required to produce commodities and services. Under a cap-and-trade system, the supply of GHG allowances is limited by the mandated cap. This study provides the strategy and critical implication for managing a sustainable inventory system with partial backlogging shortages when incorporating carbon emissions.

This model is highly sensitive to price-sensitive parameters and the selling price. So, the manager/decision-maker has to concentrate on those parameters. Pythagorean fuzzy cost parameters help decision-makers estimate the costs and obtain high profits. In this model, deterioration costs show less of an impact on the total profit under preservation. Preservation technology reduces deterioration and helps decision-makers reduce waste and make a profit. The advantage of using this model is that it optimizes both preservation and green investment costs, which allows decision-makers to make the maximum profit. This model was developed in a sustainable environment, which helps managers/decision-makers to make a profit because, in recent days, most customers are interested in buying eco-friendly products and also encourage products made by green environmental firms.

6. Conclusions

In recent years, people have been showing more interest in purchasing products containing herbal components. Similarly, the demand for a product drops as the percentage of chemical substances used increases. This model was created by considering new eco-

friendly demand, which is determined by the ratio of herbal and chemical substances in the product. Furthermore, because the demand rate is affected by advertising, stock, and pricing, we may use this model to introduce new and environmentally friendly items to the market. Table 6 demonstrates that the overall profit of the inventory system is extremely sensitive to the price-sensitive parameter, implying that the suggested model helps regulate the item's selling price. The advantages of the proposed model are that it was created in a sustainable and uncertain environment. Section 5 concludes from Table 2 that the total inventory profit is high under preservation and a green environment because preservation controls the degradation of items and the green environment lowered emissions and emission costs in the proposed model. Reducing waste and emissions helps to keep the environment green and pollution-free. Therefore, implementing this model has the advantages of making our environment emission-free and creating a high inventory profit. Emissions are released in the retailer's warehouse during the holding, ordering, and degradation of commodities. This model is subject to a carbon tax and cap policy. If we raise the carbon tax under a green environment, the total profit will rise as well. This indicates that a high carbon price benefits both the environment and the economy.

In the study of inventory management, the imprecise cost parameters were represented as Pythagorean fuzzy numbers. Pythagorean fuzzy numbers are extensions of fuzzy and intuitionistic fuzzy numbers. Consequently, the decision-maker may choose appropriate values for cost parameters to estimate profit while dealing with unpredictable market conditions. The proposed model is developed by considering the fixed backlog and deterioration rates. In real market situations, the backlog and deterioration rates are not constant for all items. These are the main limitations of the model. So, we can further develop this model by considering variable backlog and deterioration rates. As a further development of this study, one can implement the above concepts for production inventory models and supply chain inventory models in future research.

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Appendix A

Any function of the form, $\Gamma(t) = \frac{I(t)}{J(t)}$, $t \in R^n$ is strictly pseudo-concave if I(t) is not only differentiable but also strictly concave; however, J(t) is infinite and strict as well. Using the result of $\Gamma(t)$, we can show that the objective function is a pseudo-concave function

Theorem A1. *If* $\left[\left(\widetilde{C}_{h_1} + \widetilde{C}_p(\theta + bF) + \widetilde{C}_{d_1}\theta + \phi\left(C_{h_2}d + C_{d_2}k\theta\right) \right) \right] > pbF + \left(\frac{mG}{1+mG} \right)\eta\phi$ $\left(C_{h_2}d + C_{d_2}k\theta\right)$ *the Hessian matrix for* $TAP(t_1, T)$ *is always negative definite, and hence* $TAP(t_1, T)$ *attains the global maximum at the point* $\left(t_1^*, T^*\right)$ *and the point* $\left(t_1^*, T^*\right)$ *is unique.*

Proof. Let $I(t_1, T) = TAP1(t_1, T)$ and $J(t_1, T) = T$

$$TAP(t_1, T) = [\frac{paF}{(\theta+bF)^2}[(\theta+bF)^2(T-t_1)f + \theta^2 t_1 + bF(e^{(\theta+bF)t_1} + \theta t_1 - 1] - \widetilde{B}_0$$
$$-\widetilde{C}_{h_1}\frac{aF}{(\theta+bF)^2}[e^{(\theta+bF)t_1} - (\theta+bF)t_1 - 1] - \frac{aF\widetilde{C}_{d_1}\theta}{(\theta+bF)^2}[e^{(\theta+bF)t_1} - (\theta+bF)t_1 - 1]$$
$$-\widetilde{C}_1 aF(1-f)(T-T_1) - \frac{\widetilde{C}_p aF}{\theta+bF}[(e^{(\theta+bF)t_1} - 1 + (\theta+bF)f(T-t_1))] + \phi(YT$$
$$-(B_1 + \frac{aF}{(\theta+bF)^2}(C_{d_2}\theta k + C_{h_2}d)(e^{(\theta+bF)t_1} - (\theta+bF)t_1 - 1))\{1 - \eta(\frac{mG}{1+mG})\})$$
$$-\frac{\widetilde{C}_s faF}{2}(T+t_1)^2 - \xi T - gA - GT] \tag{A1}$$

$$\frac{\partial^2 TAP1(t_1,T)}{\partial t_1^2} = aF[-\widetilde{C}_s f - e^{(\theta+bF)t_1}(\widetilde{C}_{h_1} + \widetilde{C}_p(\theta+bF) - pbF + \widetilde{C}_{d_1}\theta + (1 - (\frac{mG}{1+mG})\eta)\phi(C_{h_2}d + C_{d_2}k\theta))]. \tag{A2}$$

$$\frac{\partial^2 TAP1(t_1,T)}{\partial T \partial t_1} = a\widetilde{C}_s fF. \tag{A3}$$

$$\frac{\partial^2 TAP1(t_1,T)}{\partial t_1 \partial T} = a\widetilde{C}_s fF. \tag{A4}$$

The second principal minor is >0 only if $\left[\left(\widetilde{C}_{h_1} + \widetilde{C}_p(\theta+bF) + \widetilde{C}_{d_1}\theta + \phi(C_{h_2}d + C_{d_2}k\theta)\right)\right]$ $> pbF + \left(\frac{mG}{1+mG}\right)\eta\phi\left(C_{h_2}d + C_{d_2}k\theta\right)$. Since $\frac{\partial^2 TAP1(t_1,T)}{\partial T^2} < 0$ and $H_{2,2} > 0$, the function $I(t_1,T)$ is strictly concave and differentiable. Additionally, $J(t_1, T) = T$ strictly positive and an affine function. Now on the basis of Theorem 3.2.5 in Cambini and Martein [46], $TAP(t_1,T)$ archives the global maximum value at the unique point, which is obtained from the necessary conditions, i.e., (t_1^*, T^*). This completes the proof. $\square$

Appendix B

Pythagorean fuzzy set and its operation:

Pythagorean fuzzy set: Let X be a universal set a Pythagorean fuzzy set P in X defined as follows: $P = \{(x, \mu_P(x), \nu_P(x)/x \in X\}$, where $\mu_P(x), \nu_P(x)$ are mappings from X to [0, 1]. For all $x \in X$, $\mu_P(x)$, is the membership function of $x \in P$ and $\nu_P(x)$ is th non-membership function of $x \in P$ with the condition $0 \leq \mu_P^2(x) + \nu_P^2(x) \leq 1$ and $\pi_P(x) = \sqrt{1 - \mu_P^2(x) - \nu_P^2(x)}$ is called the degree of indeterminacy of $x \in P$.

Triangular Pythagorean fuzzy Number (TPFN): A TPFN (A_{PF}) is defined as $A_{PF} = \{(a_1, a_2, a_3; a_1', a_2, a_3')\}$ with the following membership function and non-membership function:

$$\mu_{A_{PF}}(x) = \begin{cases} \sqrt{\frac{x-a_1}{a_2-a_1}}, & when \ a_1 \leq x < a_2 \\ \sqrt{\frac{a_3-x}{a_3-a_2}}, & when \ a_2 \leq x \leq a_3 \\ 0, & otherwise \end{cases}$$

$$\nu_{A_{PF}}(x) = \begin{cases} \sqrt{\frac{a_2-x}{a_2-a_1'}}, & when \ a_1' \leq x < a_2 \\ \sqrt{\frac{x-a_2}{a_3'-a_2}}, & when \ a_2 \leq x \leq a_3' \\ 1, & otherwise \end{cases}$$

where $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_3'$, $0 \leq \mu_{A_{PF}}(x)^2 + \nu_{A_{PF}}(x)^2 \leq 1$.

Example: Let $A_{PF} = \{20, 50, 90; 10, 50, 100)\}$ be a TPFN with the membership function and graphical representation of membership function we can seen in Figure A1.

$$\mu_{A_{PF}}(x) = \begin{cases} \sqrt{\frac{x-20}{30}}, & when \ 20 \leq x \leq 50 \\ \sqrt{\frac{90-x}{40}}, & when \ 50 \leq x \leq 90 \\ 0, & otherwise \end{cases}$$

$$\nu_{A_{PF}}(x) = \begin{cases} \sqrt{\frac{50-x}{40}}, & when\ 10\ \leq\ x\ \leq 50 \\ \sqrt{\frac{x-50}{50}}, & when\ 50\ \leq\ x\ \leq 100 \\ 1, & otherwise \end{cases}$$

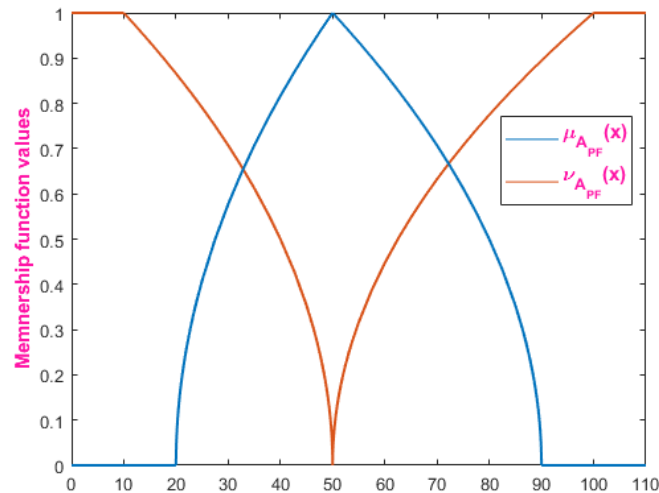where $0 \leq \mu_{A_{PF}}(x)^2 + \nu_{A_{PF}}(x)^2 \leq 1$

Figure A1. Graphical representation of membership functions of Triangular Pythagorean fuzzy number.

Some arithmetic operations on TPFNs:

Addition: Let $A_{PF} = \left\{ \left(a_1, a_2, a_3; a_1', a_2, a_3' \right) \right\}$ and $B_{PF} = \left\{ \left(b_1, b_2, b_3; b_1', b_2, b_3' \right) \right\}$ be two TPFNs, then $\widetilde{C}_{PF} = A_{PF} \oplus B_{PF}$ is also a TPFN: $\widetilde{C}_{PF} = \{ (a_1 + b_1, a_2 + b_2, a_3 + b_3; a_1' + b_1', a_2 + b_2, a_3' + b_3') \}$.

Multiplication: Let $A_{PF} = \left\{ \left(a_1, a_2, a_3; a_1', a_2, a_3' \right) \right\}$ and $B_{PF} = \left\{ \left(b_1, b_2, b_3; b_1', b_2, b_3' \right) \right\}$ be two TPFNs, then $\widetilde{C} = A_{PF} \otimes B_{PF}$ can approximately be defined as follows: $\widetilde{C} \cong \left\{ \left(a_1 b_1, a_2 b_2, a_3 b_3; a_1' b_1', a_2 b_2, a_3' b_3' \right) \right\}$.

Let $A_{PF} = \{20, 50, 90; 10, 50, 100)\}$, $B_{PF} = \{10, 50, 80; 10, 50, 90)\}$ TPFNs be defined as follows $A_{PF} \oplus B_{PF} = \{30, 100, 170; 20, 100, 190)\}$ and $A_{PF} \otimes B_{PF} = \{200, 2500, 7200; 100, 2500, 9000)\}$.

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