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Technology of Input–Output Analysis with CES Production: Application for Studying the Kazakhstan Supply Chain during the COVID-19 Pandemic

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Abstract: Input–output analysis finds widespread application in estimating the shock effects on production networks within both local and global economies. We are developing a new technology for intersectoral analysis that takes into account the substitution of production factors within a complex supply network triggered by external or internal shocks. This technology is based on the explicit solution of a pair of convex programming problems: the resource allocation problem under the assumption of Constant Elasticity of Substitution (CES) technologies and the special dual Young problem. Solving these problems, we can ascertain the equilibrium inputs and price indexes of goods within the production network. In this paper, we apply this technology to analyze the economy of Kazakhstan in the context of the COVID-19 pandemic. Our calculations provide us with the means to discuss the macroeconomic responses of the multi-sectoral production network in Kazakhstan to both external and internal shocks stemming from the pandemic.

Keywords: input–output analysis; CES production function; resource allocation problem; Young transform; competitive equilibrium; pandemic; economic shocks

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1. Introduction

The global COVID-19 pandemic has brought about a period of economic instability and disrupted supply chains worldwide. The adverse impacts on local economies during the 2020–2021 pandemic were the results of disruptions in international economic interactions, along with internal factors related to reduced production in certain sectors due to a higher incidence of illness among the population and quarantine measures. The effectiveness of government measures to support the economy in such a situation depends not only on the characteristics of the region's economy but also on the ability to adapt macroeconomic forecasts in a changing environment. Government decisions aimed at ensuring economic stability under condition of global and local shocks should be grounded in a comprehensive analysis of the intricate modern production network. Intelligent decision support information technologies, rooted in mathematical models capable of describing the evolution of inter-industry flows under changing conditions and the macroeconomic effects stemming from the implementation of diverse economic policies, serve as suitable tools for conducting such analyses.

In this paper, we continue our research [1,2], which is focused on developing a new modification of the input–output balance model that takes into account key features of modern production networks evolution. Our technology has the potential to serve as the foundation for decision support systems in local economies facing shock conditions. The input–output analysis, initiated in the mid-20th century with pioneering works by W. Leontief [3,4], is a well-established method for scrutinizing inter-industry flows. A vast body of literature is dedicated to the theory and applications of input–output analysis [5–8]. Many of these applications pertain to the examination of the economic repercussions of the COVID-19 pandemic, as seen in [9,10]. However, one of the primary criticisms leveled against Leontief's input–output models pertains to the assumption of fixed coefficients of production [11]. This arises due to the diversity of goods and services in modern production networks, which allows for the substitution of production factors [12]. Traditional approaches often describe factor substitution based on econometric dependencies, which complicates economic interpretation (e.g., as seen in [13]).

The evolution of competitive equilibrium in supply networks of different topologies under random shocks with factor substitution is explored in [14–17]. However, explicit calculations are primarily conducted for the special case of Constant Elasticity of Substitution (CES) functions—Cobb–Douglas technologies. Additionally, the framework under consideration does not enable the calculation of changes in input price indexes within the production network resulting from shock impacts.

The novelty of our approach lies in the new mathematical methods that allow us to formalize input substitution and price evolution in production networks [1]. We evaluate equilibrium inter-industry flows and input price indexes within the production network as the solution to a pair of mutually dual convex programming problems: the problem of optimal resource allocation with neoclassical production functions and the dual problem of a special type that we refer to as the Young dual problem. In the case of technologies with constant elasticity of substitution, the model offers an explicit solution and can be identified based on input–output tables officially published by statistical agencies in most countries. The one-time identification of the model enables the swift construction and adjustment of forecasts for input–output tables and major macroeconomic indicators of the economy, accounting for the substitution of intermediate and primary inputs in production processes under various scenario conditions. Scenario conditions are determined by altering the final consumption of products of sectors and prices for primary inputs (such as labor and imported intermediates). Our experience in applying this technology to inter-industry analysis is detailed in [2]. In [2], built upon the developed framework, we constructed and identified a highly aggregated input–output model of Kazakhstan's economy with constant elasticity of substitution technologies. The model identifies six major production complexes of Kazakhstan's economy, grouped based on their involvement in export–import operations and their reaction to the shock of the COVID-19 pandemic. Through model calculations, we compared the economic consequences of external and internal shocks stemming from the COVID-19 pandemic.

Addressing the challenge of analyzing and forecasting inter-industry connections during shock conditions within a large-dimensional production network is of evident interest. For instance, Kazakhstan statistics service publishes input–output tables spanning 68 pure industries. The direct evaluation of inter-industry flows using the constructed model proves to be challenging due to the high dimensionality of the resulting system of nonlinear equations. Additionally, the results of model identification based on historical data are distorted due to significant fluctuations in industry indicators that make a relatively minor contribution to aggregate gross domestic product (GDP) indicators.

In this paper, we use the previous results [2] and introduce a technology for economic shocks analyzing based on input–output balances with input substitution. This technology allows for the calculation of scenario forecasts for detailed input–output tables, facilitating the analysis of economic consequences resulting from various shocks, while considering the features of the local production network. It can be effectively employed in decision support systems for major economic decisions.

The paper is organized as follows. Section 2 presents the foundational mathematical framework and is based on results from [1,2]. Assuming CES technologies, we provide a methodology for model identification based on statistics from input–output tables (IO tables). Section 3 delves into the solution of the identification problem of the CES IO framework based on officially published national input–output statistics data. Section 4 outlines the applicable model-based technology and algorithms for the identification and evaluation of aggregated IO tables. It also introduces the model-based technology for projecting detailed input–output tables under various scenarios. The technology encompasses a staged disaggregation method of the constructed input–output balance, reliant on solving the problem of optimal resource allocation and the Young dual problem for calculating equilibrium price indexes for each production complex. Appendix B provides a description of the corresponding CES IO framework software library. In Section 5, we apply the developed technology for evaluating IO tables to construct a detailed input–output balance for Kazakhstan's economy during the 2020–2021 COVID-19 pandemic period. Through scenario calculations, we analyze the economic ramifications of the pandemic for major industries in Kazakhstan's economy in 2020 and 2021. Section 6 offers a concise summary of the paper's key findings.

2. Baseline Input–Output Framework with CES Technologies

We consider an open production network with m pure industries with the following properties:

- Industry i produces a unique product i ($i = 1, \dots, m$);
- Outputs of products $1, \dots, m$ are distributed to final consumption $X^0 = \left(X_1^0, \dots, X_m^0\right) \geq 0$ and to intermediate inputs of industries $X^j = \left(X_1^j, \dots, X_m^j\right)$, $j = 1, \dots, m$;
- Besides intermediates, each industry j consumes n types of primary inputs $l^j = \left(l_1^j, \dots, l_n^j\right) \geq 0$ for its production process which are not produced by the network;
- Primary inputs are limited, that is:

$$\sum_{j=1}^{m} l^j \leq l,$$

where $l = (l_1, \dots, l_n) \geq 0$;
- The output $F_j\left(X^j, l^j\right)$ of industry j and final consumption utility $F_0\left(X^0\right)$ are given by CES production functions:

$$F_j\left(X^j, l^j\right) = \left(\sum_{i=1}^{m} \left(\frac{X_i^j}{w_i^j}\right)^{-\rho_j} + \sum_{k=1}^{n} \left(\frac{l_k^j}{w_{m+k}^j}\right)^{-\rho_j}\right)^{-\frac{1}{\rho_j}} \quad j = 1, \dots, m, \tag{1}$$

$$F_0\left(X^0\right) = \left(\sum_{i=1}^{m} \left(\frac{X_i^0}{w_i^0}\right)^{-\rho_0}\right)^{-\frac{1}{\rho_0}}, \tag{2}$$

where $\rho_j \in (-1, 0) \cup (0, +\infty)$, $w_1^j > 0, \dots, w_{m+n}^j > 0$, $j = 0, \dots, m$. Note that the constant elasticity of substitution equals to $\sigma_j = \frac{1}{1+\rho_j}$, $j = 0, 1, \dots, m$.

We measure the introduced flows in constant prices of some base year. Assume that the primary input constraint $l = (l_1, \dots, l_n) \geq 0$ is given.

2.1. Optimal Resources Allocation Problem

In [1], we investigated the following problem of optimal resources allocation:

$$F_0\left(X^0\right) \to \max \tag{3}$$

$$F_j\left(X^j, l^j\right) \geq \sum_{i=0}^{m} X_j^i, j = 1, \ldots, m \tag{4}$$

$$\sum_{j=1}^{m} l^j \leq l \tag{5}$$

$$X^0 \geq 0, \ \ X^1 \geq 0, \ldots, X^m \geq 0, \ \ l^1 \geq 0, \ldots, l^m \geq 0. \tag{6}$$

Solving the problems (3)–(6), we obtain vectors X^j, X^0, l^j. Note that functions $F_j(X^j, l^j)$, $F_0(X^0)$ are concave, monotonically nondecreasing, continuous, and positively homogeneous of degree one for nonnegative arguments and vanish at the origin.

We assume that the network is productive, i.e., there exist $\hat{X}^1 \geq 0, \ldots, \hat{X}^m \geq 0$, $\hat{l}^1 \geq 0, \ldots, \hat{l}^m \geq 0$ such that:

$$F_j\left(\hat{X}^j, \hat{l}^j\right) > \sum_{i=1}^{m} \hat{X}_j^i, \ j = 1, \ldots, m.$$

Denote the set:

$$\mathrm{A}(l) = \left\{ X^0 = (X_1^0, \ldots, X_m^0) \geq 0 \middle| X_j^0 \leq F_j(X^j, l^j) - \textstyle\sum_{i=1}^{m} X_j^i, \ j = 1, \ldots, m; \right.$$
$$\left. \textstyle\sum_{j=1}^{m} l^j \leq l, \ X^1 \geq 0, \ldots, X^m \geq 0, \ l^1 \geq 0, \ldots, l^m \geq 0 \right\}.$$

Assume that there exists $\hat{l} = (l_1, \ldots, l_n) > 0$ such that the set $\mathrm{A}\left(\hat{l}\right)$ is bounded. Then, the features of functions $F_j(\cdot)$ imply that the set $\mathrm{A}(l)$ is bounded, convex, and closed for any $l = (l_1, \ldots, l_n) > 0$.

It is obvious that network productivity implies the fulfillment of Slater condition for the optimization problem (3)–(6) for the given $l = (l_1, \ldots, l_n) > 0$. Thus, a limited optimal solution of convex optimization problem (3)–(6) exists. Applying the Karush–Kuhn–Tucker theorem, we obtain the following result.

Theorem 1. *([1]). The set of vectors* $\left\{\hat{X}^0, \hat{X}^1, \ldots, \hat{X}^m, \hat{l}^1, \ldots, \hat{l}^m\right\}$*, which satisfy to the constraints (4)–(6), is the solution of the optimization problem (3)–(6) if and only if there exist Lagrange multipliers* $p_0 > 0$*,* $p = (p_1, \ldots, p_m) \geq 0$ *and* $s = (s_1, \ldots, s_n) \geq 0$ *such that:*

$$\left(\hat{X}^j, \hat{l}^j\right) \in Arg\max\{p_j F_j\left(X^j, l^j\right) - pX^j - sl^j \mid X^j \geq 0, l^j \geq 0\}, \ j = 1, \ldots, m \tag{7}$$

$$p_j\left(F_j\left(\hat{X}^j, \hat{l}^j\right) - \hat{X}_j^0 - \sum_{i=1}^{m} \hat{X}_j^i\right) = 0, \ j = 1, \ldots, m \tag{8}$$

$$s_k\left(l_k - \sum_{j=1}^{m} \hat{l}_k^j\right) = 0, \ k = 1, \ldots, n \tag{9}$$

$$\hat{X}^0 \in Arg\max\{p_0 F_0(X^0) - pX^0 \mid X^0 \geq 0\}. \tag{10}$$

We interpret Lagrange multipliers $p = (p_1, \ldots, p_m)$ to constraint (4) as price indexes of products and Lagrange multipliers $s = (s_1, \ldots, s_n)$ to constraint (5) as price indexes on primary inputs.

The Theorem 1 provides that optimal resources allocation corresponds to a market equilibrium. Thus, demand of j-th product equals to its supply with equilibrium price index p_j if $p_j > 0, j = 1, \ldots, m$.

2.2. Young Transform

Let us assign to each production function $F_j(X^j, l^j)$ and utility function $F_0(X^0)$ a function of a special form $q_j(p,s)$ and $q_0(q)$ correspondingly, which we call the Young transform:

$$q_j(p,s) = \inf\left\{ \frac{pX^j + sl^j}{F_j(X^j, l^j)} \middle| X^j \geq 0, l^j \geq 0, F_j\left(X^j, l^j\right) > 0 \right\},\ j = 1, \dots, m \tag{11}$$

$$q_0(q) = \inf\left\{ \frac{qX^0}{F_0(X^0)} \middle| X^0 \geq 0, F_0\left(X^0\right) > 0 \right\} \tag{12}$$

Definitions (11) and (12) imply that the Young transform of CES function is CES function too [1]. More precisely, Young transforms of functions (1) and (2) have the following form:

$$q_j(p,s) = \left(\sum_{i=1}^{m} \left(w_i^j p_i\right)^{\frac{\rho_j}{1+\rho_j}} + \sum_{k=1}^{n} \left(w_{m+k}^j s_k\right)^{\frac{\rho_j}{1+\rho_j}} \right)^{\frac{1+\rho_j}{\rho_j}},\ j = 1, \dots, m. \tag{13}$$

$$q_0(p) = \left(\sum_{i=1}^{m} \left(w_i^0 p_i\right)^{\frac{\rho_0}{1+\rho_0}} \right)^{\frac{1+\rho_0}{\rho_0}}. \tag{14}$$

We interpret the function $q_j(p,s)$ as a cost function of the industry j and the function $q_0(q)$ as a consumer price index.

2.3. Young Dual Problem for Price Indexes

Theorem 2. *([1]) If Lagrange multipliers $\hat{p} = (\hat{p}_1, \dots, \hat{p}_m) \geq 0$, $\hat{s} = (\hat{s}_1, \dots, \hat{s}_n) \geq 0$ to the problem: (3)–(6) satisfy to (7)–(10), then $\hat{p} = (\hat{p}_1, \dots, \hat{p}_m) \geq 0$ is the solution of the following problem*

$$q_0(p) \to \max_p \tag{15}$$

$$q_j(\hat{s}, p) \geq p_j,\ j = 1, \dots, m \tag{16}$$

$$p = (p_1, \dots, p_m) \geq 0. \tag{17}$$

The convex programming problem (15)–(17) we call the Young dual problem to the problem (3)–(6).

Corollary 1. *Let the set $\left\{\hat{X}^0, \hat{X}^1, \dots, \hat{X}^m, \hat{l}^1, \dots, \hat{l}^m\right\}$ be the solution of the optimization problem (3)–(6) and vectors $\hat{p} = (\hat{p}_1, \dots, \hat{p}_m) \geq 0$, $\hat{s} = (\hat{s}_1, \dots, \hat{s}_n) \geq 0$ satisfy to (7)–(10). If for some industry $j \in \{1, \dots, m\}$ the strict inequality (16) holds, i.e., $q_j(\hat{s}, \hat{p}) > \hat{p}_j$, then the industry j is unprofitable and $F_j\left(\hat{X}^j, \hat{l}^j\right) = 0$.*

Corollary 2. *If the set $\left\{\hat{X}^0, \hat{X}^1, \dots, \hat{X}^m, \hat{l}^1, \dots, \hat{l}^m\right\}$ is the solution of the optimization problem (3)–(6) and strict inequality:*

$$F_j\left(\hat{X}^j, \hat{l}^j\right) > 0 \ \ j = 1, \dots, m \tag{18}$$

holds, then for the given vector of primary inputs price indexes $\hat{s} = (\hat{s}_1, \dots, \hat{s}_n) > 0$, the equilibrium price indexes of intermediates $\hat{p} = (\hat{p}_1, \dots, \hat{p}_m) > 0$ are the solution of the following system of equations:

$$q_j(\hat{s}, \hat{p}) = \hat{p}_j,\ \ j = 1, \dots, m, \tag{19}$$

where $q_j(\cdot)$ has the CES form (13).

3. Identification: Evaluation of IO Tables with the Framework

3.1. Initial Data for Identification of the Model

We identify the model on the base of official input–output (IO) table datasets of the economy. These data are published by national statistical agencies, as well as in major international projects (for example, see [18]). According to the available statistics of national account system, we form a set of symmetric IO tables Z of domestic products over a range of years in the following format.

The IO table is denominated at current prices and includes the following three Quadrants: Quadrant I: $Z_i^j \geq 0,\ i, j = 1, \ldots, m$ is the cash flow from industry j to industry i for the intermediates produced by i and consumed by j.
Quadrant II: $Z^0 = \left(Z_1^0, \ldots, Z_m^0\right) \geq 0$—column vector of total final consumption (households, government, export etc.) in the economy.
Quadrant III: $Z_{m+i}^j \geq 0$ is the cash flow from industry j for primary input $i = 1, \ldots, n$. From the IO statistics we form two rows of primaries ($n = 2$): imported intermediate inputs Z_{m+1}^j and gross value added (GVA) Z_{m+2}^j, $j = 1, \ldots, m$, which can be considered as the measure of labor employed by pure industries.

Note that we operate with IO tables only, where the total output of any industry is positive, i.e., $Y_j > 0$ for any $j = 1, \ldots, m$.

Without loss of generality, we present our mathematical results for the general case of any finite number n of primary inputs with cash flows $Z_{m+i}^j \geq 0$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Remark 1. *The symmetry of an IO table means the equality of supply (total output) and demand of every product in an economy. Thus, the symmetry of the IO table Z implies the following balance equality (see Figure 1):*

$$Y_j = \sum_{i=1}^{m+n} Z_i^j = \sum_{i=1}^{m} Z_j^i + Z_j^0 > 0, \quad j = 1, \ldots, m, \tag{20}$$

$$\sum_{j=1}^{m} \sum_{i=1}^{n} Z_{m+i}^j = \sum_{i=1}^{m} Z_i^0. \tag{21}$$

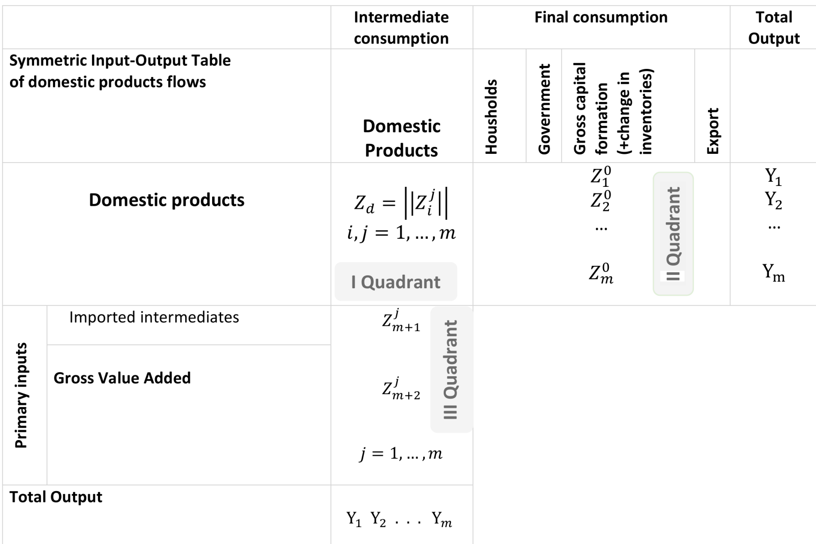

Figure 1. Symmetric input–output table of domestic products flows.

We introduce the following notation:

$$a_{ij} = \frac{Z_i^j}{\sum_{k=1}^{m+n} Z_k^j}, \ \ b_{kj} = \frac{Z_{m+k}^j}{\sum_{k=1}^{m+n} Z_k^j}, \ a_{i0} = \frac{Z_i^0}{\sum_{i=1}^{m} Z_i^0} \ \ i,j = 1,\dots,m, \ k = 1,\dots,n. \tag{22}$$

Obviously, we have:

$$\sum_{i=1}^{m} a_{i0} = 1, \ \sum_{i=1}^{m} a_{ij} + \sum_{k=1}^{n} b_{kj} = 1, \ \sum_{k=1}^{n} b_{kj} > 0, \ i,j = 1,\dots,m, \ k = 1,\dots,n.$$

Remark 2. *Since $\sum_{i=1}^{m} a_{ij} < 1$, $(j = 1,\dots,m)$, then Leontief $(m \times m)$-matrix $A = \|a_{ij}\| \geq 0$ is productive, and hence, the Leontief inverse exists and nonnegative:*

$$(E - A)^{-1} \geq 0, \tag{23}$$

where E is $(m \times m)$-identity matrix.

We denote $B = \left\|b_{kj}\right\| \geq 0$ -$(n \times m)$ matrix.

3.2. Model Evaluation of IO Tables: Identification of CES-Technologies and Utility

Recall that for applications, we operate with a full range of significant sectors of an economy, the output of which is strictly positive. Therefore, the condition (18) holds. Then, the Corollary 2 implies that for the given vector of price indexes $s = (s_1,\dots,s_n) > 0$ of primary inputs for a target year (in relation to the once fixed base year), the corresponding equilibrium price indexes $p = (p_1,\dots,p_m) > 0$ are the solution of the following system of the following algebraic equations:

$$\left(\sum_{i=1}^{m} \left(w_i^j p_i\right)^{\frac{\rho_j}{1+\rho_j}} + \sum_{k=1}^{n} \left(w_{m+k}^j s_k\right)^{\frac{\rho_j}{1+\rho_j}}\right)^{\frac{1+\rho_j}{\rho_j}} = p_j, \ j = 1,\dots,m. \tag{24}$$

Solving the resource allocation problem (3)–(6) coupled with Young dual problem allows us to evaluate equilibrium intersectoral cash flows and price indexes of a considered production network for a target year. Thus, the quadrants I and III of the target IO table Z can be calculated in the case of CES-technologies (1) and CES-utility (2). More precisely, the following theorem holds.

Theorem 3. *Let us fix a base year with IO table $\hat{Z}$ in current prices ($\hat{Z}$ has the form as in Figure 1), which is published by the official national IO statistics. In accordance to Figure 1, we denote the elements of $\hat{Z}$ as $\hat{Z}_i^j$.*

A. *Let us set parameters of CES-technologies (1) and CES-utility (2) as follows:*

$$w_i^j = (a_{ij})^{\frac{1+\rho_j}{\rho_j}}, \quad w_{m+k}^j = \left(b_{kj}\right)^{\frac{1+\rho_j}{\rho_j}}, w_i^0 = (a_{i0})^{\frac{1+\rho_0}{\rho_0}}, \ i,j = 1,\dots,m, \quad k = 1,\dots,n, \tag{25}$$

where a_{ij}, b_{kj}, a_{i0} are evaluated by (22) for the IO table $\hat{Z}$ in the base year. Then, the problem (3)–(6) with primaries restriction:

$$l = (l_1,\dots,l_n), \ \ l_t = \sum_{j=1}^{m} \hat{Z}_{m+t}^j, \ \ t = 1,\dots,n. \tag{26}$$

explains the symmetric IO table $\hat{Z}$ in the base year for any given values of parameters ρ_j, $j = 0, \ldots, m$, i.e., the set of values:

$$\left\{ X_i^0 = \hat{Z}_i^0, X_i^j = \hat{Z}_i^j, l_t^j = \hat{Z}_{m+t}^j, i, j = 1, \ldots, m,\ t = 1, \ldots, n \right\}$$

is the solution of the problem (3)–(6).

B. *Suppose that the following conditions hold.*

a. *The vector of primary inputs price indexes $s = (s_1, \ldots, s_n) > 0$ for the target year in relation to the base year is given.*
b. *The vector of total final consumption $Z^0 = (Z_1^0, \ldots, Z_m^0)^T \geq 0$ for the target year (in current prices of the target year) is given, i.e., we know the Quadrant II (see Figure 1) of the target IO table Z.*

Then, the following statements hold.

1. *The vector of equilibrium price indexes $p = (p_1, \ldots, p_m) > 0$ of intermediate inputs for a target year is a solution of the system (24).*
2. *The elements of the Ist and the IIIrd Quadrants of the target IO table Z (see Figure 1) are evaluated as follows:*

$$\begin{aligned} Z_i^j &= \left(\frac{p_i}{p_j} w_i^j \right)^{\frac{\rho_j}{1+\rho_j}} Y_j = \lambda_{ij} Y_j,\ \ i, j = 1, \ldots, m, \\ Z_{m+k}^j &= \left(\frac{s_k}{p_j} w_{m+k}^j \right)^{\frac{\rho_j}{1+\rho_j}} Y_j,\ \ k = 1, \ldots, n,\ \ j = 1, \ldots, m, \end{aligned} \tag{27}$$

where the vector of Total Output $Y = (Y_1, \ldots, Y_m)^T$ in prices of the target year is evaluated by:

$$Y = (E - \Lambda)^{-1} Z^0, \tag{28}$$

and $m \times m$-matrix $\Lambda = \left\| \lambda_{ij} \right\|$ has elements:

$$\lambda_{ij} = \left(\frac{p_i}{p_j} w_i^j \right)^{\frac{\rho_j}{1+\rho_j}},\ \ i, j = 1, \ldots, m. \tag{29}$$

We give the proof of Theorem 3 in Appendix A.

Corollary 3. *According to the Theorem 3, we identify coefficients w_i^j of CES-technologies (1) by the IO table $\hat{Z}$ of the base year so that (25) holds and fix the vectors of primary inputs price indexes $s = (s_1, \ldots, s_n) > 0$ and total final consumption $Z^0 = (Z_1^0, \ldots, Z_m^0)^T \geq 0$ for a target year. Then:*

- *The system (24) for equilibrium price indexes evaluation takes the form:*

$$\left(\sum_{i=1}^{m} a_i^j (p_i)^{\frac{\rho_j}{1+\rho_j}} + \sum_{k=1}^{n} b_k^j (s_k)^{\frac{\rho_j}{1+\rho_j}} \right)^{\frac{1+\rho_j}{\rho_j}} = p_j,\ j = 1, \ldots, m. \tag{30}$$

- *In the case of $\rho_1 = \rho_2 = \ldots = \rho_m = \rho$, the system (30) has unique solution:*

$$\begin{aligned} &p_j = \pi_j^{\frac{1+\rho}{\rho}},\ \ \pi = (\pi_1, \ldots, \pi_m)^T = \left(E - A^T\right)^{-1} V, \\ &V = (v_1, \ldots, v_m)^T,\ \ v_i = \textstyle\sum_{k=1}^{n} b_k^j s_k^{\frac{\rho}{1+\rho}}, \end{aligned} \tag{31}$$

where $A = \left\| a_i^j \right\|_{i,j=1,\ldots,m}$ is the Leontief matrix.

- *Quadrants I and III for a target IO table Z (see Figure 1) take the form:*

$$Z_i^j = \left(\frac{p_i}{p_j}\right)^{\frac{\rho_j}{1+\rho_j}} a_i^j Y_j = \lambda_{ij} Y_j, \ \ i,j = 1,\ldots,m,$$
$$Z_{m+k}^j = \left(\frac{s_k}{p_j}\right)^{\frac{\rho_j}{1+\rho_j}} b_k^j Y_j, \ \ k = 1,\ldots,n, \ \ j = 1,\ldots,m, \tag{32}$$

where Total Output $Y = (Y_1, \ldots, Y_m)$. *We define from the balance (28) with modified matrix:* Λ

$$Y = (E - \Lambda)^{-1} Z^0, \quad \Lambda = \|\lambda_{ij}\|, \ \lambda_{ij} = a_{ij}\left(\frac{p_i}{p_j}\right)^{\frac{\rho_j}{1+\rho_j}}, \ \ i,j = 1,\ldots,m. \tag{33}$$

3.3. Calibration of the Model: Evaluation of Elasticity of Substitution of Inputs

The Corollary 3 allows us to evaluate the elasticity parameters ρ_j, $j = 1, \ldots, m$ of CES-technologies (1) as a result of model calibration on the base of the known yearly statistics of IO tables Z of an economy for several years. Generally, we can choose various criteria for the evaluation which are based on a comparison of evaluated macroeconomic indicators of industries for to corresponding statistics. A criterion should be chosen depending on the purposes of IO analysis.

In paper [2], we evaluate ρ_j, $j = 1, \ldots, m$ as the solution of the following quadratic optimization problem:

$$\sum_{t=t_1}^{t_2}\Big[(YMod(\rho,t) - YStat_t)^2 + (VMod(\rho,t) - VStat_t)^2 + (IMod(\rho,t) - IStat_t)^2\Big] \to \min_{\rho}, \tag{34}$$

for $\rho = (\rho_1, \ldots, \rho_m) \in (-1,0) \cup (0,+\infty)$. Here, $YStat_t, VStat_t, IStat_t$ are the official statistical Total Output, Gross Value Added (GVA), and Total Imported Intermediates (Import used) of the economy in the year t, and $YMod(\rho,t), VMod(\rho,t), IMod(\rho,t)$ are the corresponding values which we evaluate with the model:

$$YMod(\rho,t) = \sum_{j=1}^{m} Y_j, \ \ VMod(\rho,t) = \sum_{j=1}^{m} Z_j^{i+2}, \ \ IMod(\rho,t) = \sum_{j=1}^{m} Z_j^{i+1}, \tag{35}$$

where Z_i^j, Y_j are scenario model evaluations (32) and (33) for the target year t.

In Section 3, we present the modified algorithm of the full dimension IO table evaluation, which is based on step-by-step disaggregation principles. The algorithm gives a useful tool for IO tables projection including for scenario conditions of shocks.

4. The Method of Full Dimension IO Table Evaluation for IO Analysis of Shocks

The algorithm for IO table evaluation that is presented in this section is relatively easy to implement, even in the case of a high-dimensional production network. It gives a useful tool of IO tables projection calculating in shock scenarios of various nature: pandemics, sanctions, etc. Recall that the model takes into account the substitution of inputs of industries. Moreover, we calculate not only equilibrium inter-industry cash flows, but also equilibrium price indexes of products for a target year within the developed framework. This feature seems to be important in stress scenario conditions.

Note that we need fairly complete database of statistic IO tables of an economy and price indexes of primary inputs of industries for a range of years $t \in [t_1, t_2]$ to identify the model.

Before solving of identification problem of the framework, we identify the input vector of primary price indexes $s = (s_1, s_2)$ for every considered year $t \in [t_1, t_2]$ as the statistics of imported intermediates price index s_1 (for a rough approximation, the exchange rate of the national currency to USD can be considered) and Consumer Price Index s_2. We use the same interpretation of these indexes for scenario evaluations.

Note that statistics IO tables usually include several dozen or even more than a hundred industries. That makes the system (30) for equilibrium prices evaluation with $\rho_j \neq \rho_i$ if $i \neq j$ computationally poorly realizable as well as the optimization problem (34), which transforms to difficult global optimization task in this case.

Our concept of IO analysis with the developed framework is based on the following results.

(A) We identify the developed framework for aggregated production network structure of the economy with several large complexes of sectors. These complexes cover all industries of the considered economy and are formed in connection to the stated purposes of IO analysis. According to (25) we evaluate coefficients of CES-technologies and calculate the elasticity of substitution of inputs ρ_j for each aggregated sector as a solution of optimization problem (34). As a result, we form the framework for IO analysis of shocks in terms of high aggregated production network ($m = N$, where N < 10).

(B) On the base of Corollary 3, we evaluate equilibrium price indexes $P = (P_1, \dots, P_N)$ of sectors as a solution of (30) and the corresponding IO table Z from (32) for given scenario conditions of shocks. Note that these results are significant in their own right, as they have an adequate macroeconomic interpretation and make it possible to analyze the effects of shocks at a highly aggregated level. In addition, the problem of forming scenario conditions for projection year has a more realistic solution in the case of aggregated vector of final consumption. We have positive results of our framework applications for studying shocks in aggregated production networks, which are presented, for example, in [2].

(C) Obviously, the economic shock affects all sectors of the economy. Therefore, we propose a technology for a more detailed IO analysis of shocks on the base of developed framework. Using the results (A) and (B) we form the new structure of the evaluated in (B) IO table Z. We use the staged method of IO table disaggregation. The main idea is to fix the evaluated in (B) price indexes and apply the model to disaggregate each complex by considering the system with more primary inputs for each selected complex: the rest complexes as well as imported intermediates and gross value added. The staged procedure allows us to reconstruct the full-dimension IO table for a projection year with given scenario conditions.

The Algorithm of IO Table Evaluation Using the IO Framework with CES Technologies

In this section, we explain the main steps of the algorithm in terms of the framework. For the specification of corresponding software functions of IO CES Framework Library, see Appendix B.

Step 0: Initialization.

0.1. Prepare input statistical IO tables in the form that we give in Figure 1 for selected range of years $[t_1, t_2]$ (see (34)). In fact, we can exclude some years by solving (34) because of the unstable period, but this is not necessary.

0.2. Select the base year $t_0 \in [t_1, t_2]$.

Step 1: Background for Aggregation.

1.1. Due to the purposes of IO analysis, we configure information for aggregation of full set of industries to several (N, $N < 10$) of the large aggregated sectors of the economy, so that the sector $M \in \{1, \dots, N\}$ includes the set of initial industries $I_M \subset \{1, \dots, m\}$, $I_{M1} \cap I_{M2} = \emptyset$, if $M1 \neq M2$ and $\cup_{M=1}^{N} I_M = \{1, \dots, m\}$.

Result: a map of original industry numbers I_M to aggregated sector numbers $j = 1, \dots, N$.

Step 2: Aggregation.

2.1. Aggregate all initial IO tables of the years from range $[t_1, t_2]$ into IO tables with N industrial complexes and two primary resources: imported intermediates and GVA) (Figure 1).

Step 3: Run model identification of A, B.

3.1. Evaluate Leontief coefficients a_i^j, $i, j = 1, \ldots, N$ and coefficients b_{kj}, $k = 1, 2$, $j = 1, \ldots, N$ by (22) for the aggregated IO table $\hat{Z}$ with complexes $1, \ldots, N$ of the base year t_0.

3.2. Optionally (for Disaggregation Procedure): Evaluate Leontief coefficients a_i^j, $i, j = 1, \ldots, m$ and coefficients b_{kj}, $k = 1, 2$, $j = 1, \ldots, m$ by (22) for the statistical full-size IO table $\hat{Z}$ of the base year t_0.

Step 4: Set scenario input data for identification.

4.1. Form scenario conditions data base by the official national statistics: primary resource price indexes $s = (s_1, s_2)$ and final consumption vector Z^0 (in current prices) for the range of years $[t_1, t_2]$.

Step 5: Identification of ρ_j.

5.1. Set new value of ρ, $\rho = (\rho_1, \ldots, \rho_N) \in (-1, 0) \cup (0, +\infty)$.

5.2. Set scenario primary resource prices $s = (s_1, s_2)$ and final consumption vector Z^0 for the year $t \in [t_1, t_2]$.

5.3. Given ρ_j, $j = 1, \ldots, N$, coefficients a_{ij}, b_{kj} from the step 3.1 and scenario conditions for primary inputs price indexes $s = (s_1, s_2)$ (step 5.2), evaluate equilibrium price indexes $p = (p_1, \ldots, p_N)$ of aggregated production complexes for the year $t \in [t_1, t_2]$ as a solution of the system (30).

5.4. Given ρ_j, $j = 1, \ldots, N$, coefficients a_{ij}, b_{kj} from the step 3.1, final consumption vector Z^0 (step 5.2) and price indexes vector $p = (p_1, \ldots, p_N)$ from the step 5.3, evaluate Quadrants I and III of aggregated IO table Z for the year $t \in [t_1, t_2]$ by (32).

5.5. Given Quadrants I and III of aggregated IO table Z for the year $t \in [t_1, t_2]$ (step 5.4), calculate totals of the whole economy (35): Total Output $YMod(\rho, t)$, Total Gross Value Added (GVA) $VMod(\rho, t)$, Intermediate Inputs of Primary Resources $IMod(\rho, t)$.

5.6. Repeat steps 5.2–5.5 for the range $[t_1, t_2]$ of years.

5.7. Solve the optimization problem (34) by repeating 5.1–5.6 for ρ from allowable range $\rho = (\rho_1, \ldots, \rho_N) \in (-1, 0) \cup (0, +\infty)$ with the enough small step $\Delta\rho$.

Result: the set of values of elasticity of substitution of inputs $\rho = (\rho_1, \ldots, \rho_N)$ for aggregate industrial complexes of the economy.

Step 5 finishes the identification and calibration of the aggregated CES1Model. Given scenario conditions $s = (s_1, s_2)$ and Z^0 for target year t, CES1Model is now ready for evaluations of aggregate N-dimension IO tables and equilibrium price indexes $p = (p_1, \ldots, p_N)$ for target year t.

Step 6: CES1Model Evaluations. Scenario Evaluation of IO table and equilibrium price indexes for aggregated N-dimension network.

6.1. Set target year t.

6.2. Form scenario conditions for the year t: primary resource price indexes $s = (s_1, s_2)$ (in relation to the base year t_0) and final consumption vector Z^0 (in current prices).

6.3. Given ρ_j, $j = 1, \ldots, N$ (result of the step 5), coefficients a_{ij}, b_{kj} (step 3.1), and scenario conditions for primary inputs price indexes $s = (s_1, s_2)$ (step 6.2), evaluate equilibrium price indexes $p = (p_1, \ldots, p_N)$ of aggregated production complexes for the year t as a solution of the system (30).

6.4. Given ρ_j, $j = 1, \ldots, N$ (result of the step 5), coefficients a_{ij}, b_{kj} (step 3.1), final consumption vector Z^0 (step 6.2), and price indexes vector $p = (p_1, \ldots, p_N)$ from step 6.3, evaluate Quadrants I and III of aggregated IO table Z for the year t by (32)

6.5. Given Quadrants I and III of aggregated IO table Z for the year t (step 6.4), we can evaluate main economic indicators of aggregated complexes, such as Total Output, inputs consumption (primaries and intermediate), as well as the totals of the whole economy. Moreover, we can interpret price indexes $p = (p_1, \ldots, p_N)$ (step 6.3) as a measure of inflation level in the economy in scenario conditions of shocks.

Step 7: Optional: disaggregate CES1Model to original IO table size and get CES1ModelO.

7.1. Set the number M of aggregated complex $M \in 1, \ldots, N$ (step 1.1) with the set of industries I_M , $I_m \subset 1, \ldots, m$.

7.2. Given elasticity of substitution ρ_M (step 5) of aggregated complex M, fix the value of elasticity of substitution $\tilde{\rho}_i$ of inputs of any industry from the complex M as:

$$\tilde{\rho}_i = \rho_M, \quad i \in I_M.$$

7.3. Given primary resource price indexes $s = (s_1, s_2)$ (in relation to the base year t_0) (step 6.2) and price indexes vector $p = (p_1, \ldots, p_N)$ of aggregated complexes (step 6.3) for target year t form new vector of price indexes of $(N+1)$ primary inputs:

$$\tilde{s}_M = (\tilde{s}_1, \ldots, \tilde{s}_{N+1}) = (p_1, \ldots, p_{M-1}, p_{M+1}, \ldots, p_N, s_1, s_2)$$

for the network of industries from complex M.

7.4. Given full-size IO table $\hat{Z} = \hat{Z}_i^j$ for m industries of the base year t_0 (steps 0.1, 3.2.), calculate coefficients of (31) to evaluate price indexes vector for industries from M:

- $A_M = \left\| \left(a_i^j \right) \right\|_{i,j \in I_M}$ corresponds to the submatrix for the complex M of the initial mxm Leontief matrix A of base year t_0 (step 3.2).
- $\tilde{b}_k^j = \frac{\sum_{i \in I_k} \hat{Z}_i^j}{\sum_{q=1}^{m+2} Z_q^j}, k = 1, \ldots, N-1, j \in I_M$.
-
$$\tilde{b}_k^j = \begin{cases} \tilde{b}_1^j, \ if \ \ k = N, \ j \in \ I_M \\ \tilde{b}_2^j, \ if \ \ k = N+1, \ j \in I_M \end{cases}$$
  where b_k^j, $k = 1, 2$, $j \in I_M$ are defined from the full size IO table of base year t_0 (step 3.2).

7.5. Evaluate disaggregated equilibrium price indexes $\tilde{p}_j$, $j \in I_M$ for industries of the complex M from (31) with $\rho = \rho_M$, matrix $A = A_M$, $b_k^j = \tilde{b}_k^j$, $k = 1, \ldots, N+1$, $j \in I_M$ defined in step 7.4 and the vector $\tilde{s}_M = (\tilde{s}_1, \ldots, \tilde{s}_{N+1})$ from the step 7.3.

7.6. Repeat steps 7.1–7.5 for every complex M = 1, . . . , N.

7.7. Form the full m-size equilibrium price indexes vector $\tilde{p}_1, \ldots, \tilde{p}_m$ for target year t.

7.8. Given price indexes vector $\tilde{p}_1, \ldots, \tilde{p}_m$ (step 7.7) and scenario full-scale vector of final consumption Z^0 for the year t (step 6.2), calculate Quadrants I and III of target full-scale IO table Z by (32) with $\rho_j = \rho_M$, if $j \in I_M$, $j = 1, \ldots, m$, $M = 1, \ldots, N$ and coefficients $a_{ij}, b_{kj}, i, j = 1, \ldots, m, k = 1, 2$ are evaluated by (22) for full size IO table $\hat{Z}^0$ of base year t_0 (step 3.2).

Result: the CES1ModelO is ready for simulating of the full-dimension IO table Z with initial m industries for target year t with evaluated and fixed price vector $p = (p_1, \ldots, p_m)$ for the year t. CES1ModelO allows us to study shifts of macroeconomic indicators of industries of the economy under conditions of shocks.

5. Applications: IO Analysis of Kazakhstan Economy in Face of COVID-19 Pandemic

In [2], we analyzed the COVID-19 pandemic shock of 2020 in terms of highly aggregated complexes and evaluated responses of Kazakhstan economy to shock of 2020 in terms of aggregated IO framework. In this section, we apply the developed CES model-based disaggregation technology for a more detailed analysis of the impact of COVID-19 pandemic on sectors of Kazakhstan economy in 2020 and 2021. Following the algorithm that we present in Section 4, we use the results of the identification of aggregated model from [2] to solve the disaggregation task. The official IO statistics of Kazakhstan includes 68 pure industries and is published for 2013–2021 as a part of the National Accounts by the Agency for Strategic Planning and Reforms [19]. The aggregated industry complexes in [2] are formed based on the involvement of industries in export–import operations. The reason for such division is, on the one hand, a significant part of the country's budget revenues from the export of raw materials. On the other hand, high-tech (labor-intensive) industries heavily depend on import flows and prices for imported intermediates. When forming industry complexes, we also took into account the impact of COVID-19 shock in 2020 on industry indicators and identified groups of industries with positive (+) and negative (−) pandemic impact. Based on the analysis of Kazakhstan economy in [2], the following industry complexes were identified: Exporting+, Exporting−, Manufacturing, Service+, Service−, Infrastructure. A list of industries included in each complex is provided in Appendix C. The analysis of indicators of identified complexes demonstrated the adequacy of our aggregation principles (see [2]).

In [2], we identified and calibrated aggregated 6-dimension CES1Model on the base of Kazakhstan IO statistics 2013–2019, having 2013 as base year. In Figure 2, we present parameters ρ_j and corresponding elasticities of substitution of inputs for each of six industrial complexes, which are evaluated by the model CES1Model in [2]. Recall that the substitution elasticity of the complex j equals to $\sigma_j = \frac{1}{1+\rho_j}$ (see (1)).

	Manufacturing	Exporting+	Exporting-	Infrastructure	Service+	Service-
Optimal ρ_j	-0.4	0.2	0.3	0.85	0.9	-0.35
Elasticity	1.7	0.8	0.8	0.5	0.5	1.5

Figure 2. Elasticities of substitution of inputs for aggregated complexes of the Kazakhstan production network.

Applying the CES-model-based algorithm with disaggregation step 7 (Section 4), we identify the model CES1Model0 for the full-dimension IO table scenario evaluation of the Kazakhstan production network. For identification problem solution and evaluations (using models CES1Model and CES1Model0), we set the values of primary resources price indexes of 2020, 2021 from statistics [20,21] as $s_{2020} = (s_1, s_2) = (1.07, 1.08)$ and $s_{2021} = (s_1, s_2) = (1.16, 1.11)$, where s_1 is the consumer price index and s_2 is the KZT/USD currency exchange rate index for equilibrium prices.

Using the identified model CES1Model0, we evaluate the responses of Kazakhstan industries to the COVID-19 pandemic shock. The COVID-19 pandemic, first of all, caused a shock to export–import flows and to the final consumption of households and government consumption (social programs, medicine, etc.). We carried out scenario calculations of changes in inter-industry flows in the case of variations in the components of the final consumption vector, which correspond to different components of the final-use shock caused by the pandemic. Scenario calculations using the model make it possible to assess risks for sectors of the economy from different shock components.

By analogue with [2] in this paper, we study the impact of two types of shock of COVID-19: the external shock of export flows due to the reduction in global business activity and change in export structure and the internal shock caused by the shift of domestic final consumption.

We consider two scenarios for 2020, so-called "External shock" and "Internal shock". Shock impact values we take from IO statistics (Quadrant II of the IO tables 2020, 2021).

Export in 2020 (in prices of 2019) compared to 2019 was significantly different for the aggregated complexes. In Table 1, we present the aggregated statistics that we use for the External shock scenario.

Table 1. Export change—2020 vs. 2019.

Sector	Export Change in 2020	Coefficient of 2020 Shock Compensation
Manufacturing	+11%	0.89
Exporting+	+6%	0.94
Exporting−	−30%	1.30
Infrastructure	−18%	1.18
Service+	+27%	0.73
Service−	+7%	0.93

For scenario calculations, we represent the vector of final consumption Z^0 in 2020 and 2021 as the sum of component vectors of it (see Figure 1). Then, we restore export or domestic part of final consumption to the level of 2019 in scenarios for 2020 and 2021 to compensate the external or internal part of shock.

To compensate for the external shock in this scenario, we multiply the export part of final consumption 2020 for every sector on the corresponding coefficient (the last column of Table 1). Note that we remain the part of final domestic consumption at the level of 2020 in this scenario. Note that in 2021, export volumes recovered, so there was no external shock in 2021.

The internal shock includes changes in household consumption and an increase in government spending 2020 and 2021. Households and government consumption of drugs and education services increased, while demand for traveling, food services, and some other services and goods decreased. Domestic final consumption volumes in 2020 and 2021 (in prices of 2019) compared to 2019 are shown in Table 2.

Table 2. Domestic consumption change in 2020 and 2021 vs. 2019.

Sector	Consumption Change in 2020	Coefficient of 2020 Shock Compensation	Consumption Change in 2021	Coefficient of 2021 Shock Compensation
Manufacturing	+11%	0.89	+4%	0.96
Exporting+	+13%	0.87	+0.18%	0.82
Exporting−	−1%	1.01	−40%	0.60
Infrastructure	−26%	1.26	−30%	0.70
Service+	+21%	0.79	+38%	0.62
Service−	−9%	1.09	+4%	0.96

In Figures 3–5, we show the model-based scenario evaluation of shifts of Total Output values of several large sectors of each aggregated complex of Kazakhstan as a result of isolated shock. Sectors in each complex are sorted by their output share. Note that in Figures 3–5, a prefix before the name of a sector denotes the corresponding aggregated complex.

As a result, the "Statistics" columns in Figures 3–5 include both types of shocks, while "Compensated … shock" columns contain the pandemic shock without the corresponding part that is evaluated by the model.

We can see that the external shock 2020 scenario leads to significant fall of Total Output of the following sectors: Crude oil, Ferrous metals, Petroleum products and Wholesale trade dependent on them, Land transport, and Warehousing. There is a negative impact on scientific research and development, Electricity, and Telecommunications (Figure 3).

The internal shock made similar evaluated effect in both 2020 and 2021 (Figures 4 and 5). These changes of domestic consumption obviously have had the most positive significant impact (>25%) on education, medicine, and public administration. Crops and animal production, food products, and motor vehicle manufacturing grew, unlike Telecommunications, Real estate activities, Land transport, Warehousing, and Retail trade.

We see that branches of the economy can be divided into three groups: affected by external shock, internal shock, or both. Land transport, Warehousing, and Telecommunications suffer from both shocks.

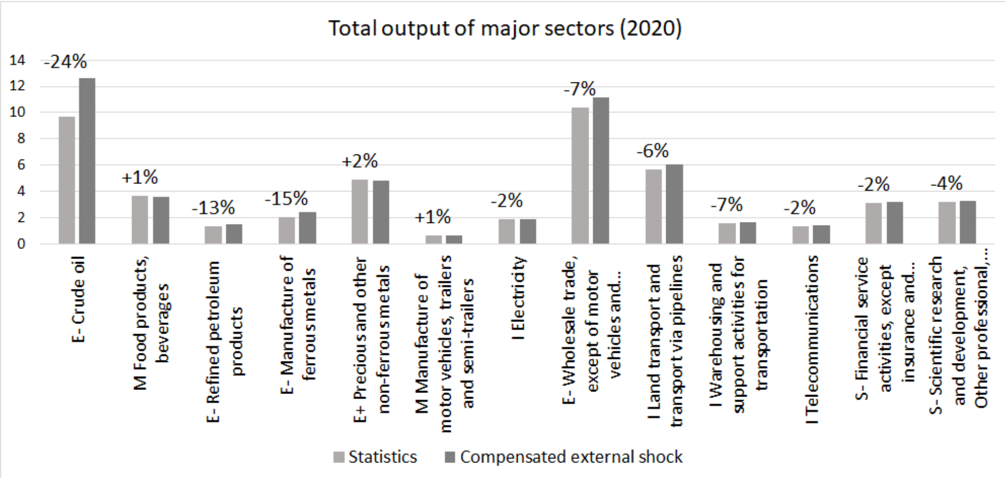

Figure 3. External shock impact on sectors of the Kazakhstan economy, 2020.

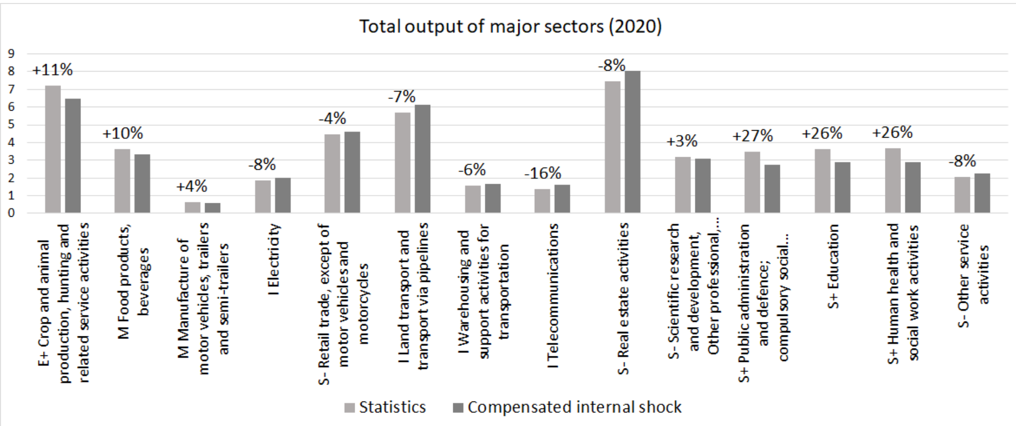

Figure 4. Internal shock impact on sectors of the Kazakhstan economy, 2020.

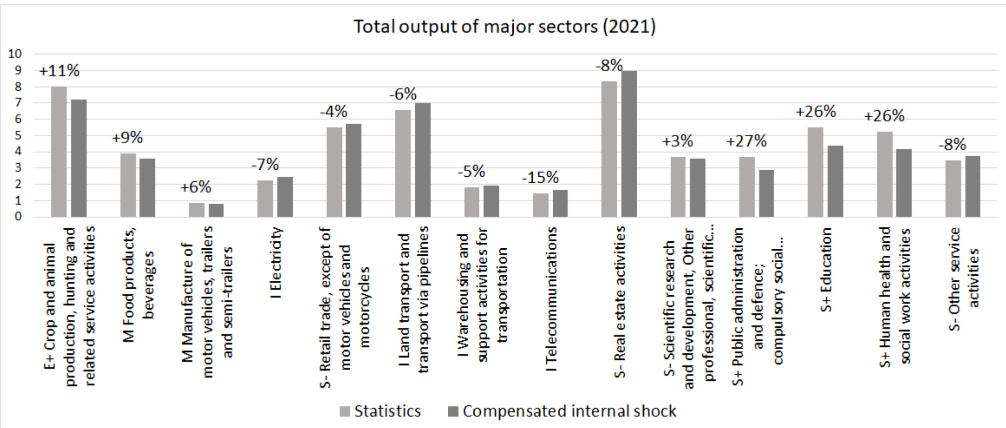

Figure 5. Internal shock impact on sectors of the Kazakhstan economy, 2021.

6. Conclusions

Overall, the results of the analysis of COVID-19 pandemic shock for disaggregated production network of Kazakhstan in 2020 and 2021 confirm the results of [2] for aggregated production complexes. External shocks pose more risks to the sectors of the economy than internal shocks. This result is supported by the absence of reinforced negative dynamics in the industry indicator evaluations for 2021, when the shock in external trade was overcome. The moderate level of decline in some sectors of the economy under internal COVID-19 shock in 2020 and 2021 indicates the effectiveness of state economic decisions made during the pandemic. The modeling results of the full-dimension production network of Kazakhstan under the COVID-19 shock with substitution of inputs allowed to identify groups of top industries that respond differently to internal and external shocks. The IO analysis of Kazakhstan's economy during the pandemic confirms the economic risks associated with the high share of raw material exports.

In this paper, we present a production network optimization framework with CES technologies, which explicitly includes influence on inter-industry links of factors substitution and restrictions on primary inputs. The framework generalizes the classical Leontief scheme and is based on solution of a resource allocation problem and Young dual problem for price indexes. On the base of the framework studying we provide new useful technology of large dimension or highly aggregated IO table projection under conditions of shocks (pandemic, sanctions, etc.). Given the scenario of primary inputs prices and final consumption of products, the technology allows to evaluate equilibrium cash flows and product price indexes in the network, taking into account substitution of inputs. Thus, we obtain a new tool for scenario forecasting of national IO tables.

It should be noted that the algorithm application based on CES model is limited by parameters identification opportunity. Several year IO tables with similar set of products should be available for elasticity of substitution identification. Based on our experience, identification using statistics from 3 to 5 years leads to an acceptable level of accuracy in calculations for the medium-term period using our model. Previous calculations shows that the accuracy of our model with substitution of inputs is noticeably better than using Leontief technologies that do not allow for substitution. Yet, note that identification periods longer than 8–10 years may distort elasticity parameters due to significant changes in production technologies.

The evaluations of the Kazakhstan production network response to COVID-19 pandemic shock presented in Section 5 confirm that the proposed technology of IO analysis of a full-dimension production network of an economy can be used in decision-making support systems under conditions of shocks.

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Abbreviations

The following abbreviations are used in this manuscript:

IO	Input–output
CES	Constant elasticity of substitution

Appendix A. Proof of the Theorem 3

A. We proved statement (1) before the Theorem. Let us prove statement (2). Note that the Total Output Y_j for a target year is evaluated as follows:

$$Y_j = p_j F_j\left(X^j, l^j\right).$$

The CES-form of technologies (1) obviously implies that for any $i, j = 1, \ldots, m, k = 1, \ldots, n$:

$$\begin{aligned} \frac{\partial F_j(X^j,l^j)}{\partial X_i^j} &= \left(\frac{X_i^j}{w_i^j}\right)^{-(1+\rho_j)} \frac{1}{w_i^j}\left(F_j(X^j,l^j)\right)^{1+\rho_j} \\ \frac{\partial F_j(X^j,l^j)}{\partial l_k^j} &= \left(\frac{l_k^j}{w_{m+i}^j}\right)^{-(1+\rho_j)} \frac{1}{w_{m+i}^j}\left(F_j(X^j,l^j)\right)^{1+\rho_j} \end{aligned} \tag{A1}$$

Theorem 1 implies that the equilibrium inter-industry cash flows give maximum industries profits (7) using the equilibrium price indexes $p > 0,\ s > 0$. Then, from (7), we obviously have:

$$\begin{aligned} p_j\frac{\partial F_j(X^j,l^j)}{\partial X_i^j} &= p_i,\ \ i = 1, \ldots, m,\ \ j = 1, \ldots, m, \\ p_j\frac{\partial F_j(X^j,l^j)}{\partial l_i^j} &= s_i,\ \ i = 1, \ldots, n,\ \ j = 1, \ldots, m, \end{aligned} \tag{A2}$$

From (A1) and (A2), we immediately obtain (27):

$$\begin{aligned} Z_{ij} = p_i X_i^j &= \left(w_i^j \frac{p_i}{p_j}\right)^{\frac{\rho_j}{1+\rho_j}} p_j F_j(X^j, l^j) = \lambda_{ij} Y_j,\ i, j = 1, \ldots, m, \\ Z_{m+k}^j = s_k l_k^j &= \left(\frac{s_k}{p_j} w_{m+k}^j\right)^{\frac{\rho_j}{1+\rho_j}} Y_j,\ \ k = 1, \ldots, n,\ \ j = 1, \ldots, m, \end{aligned} \tag{A3}$$

Note that (9) implies the balance of primary resources in the target year:x

$$s_k l_k - \sum_{j=1}^{m} s_k l_k^j = 0,\ k = 1, \ldots, n,$$

from which the primary resources constraint $l = (l_1, \ldots, l_n)$ for a target year can be evaluated.

From the balance (8) with $p_j > 0$, we obtain the equality:

$$p_i F_i\left(X^i, l^i\right) = \sum_{j=1}^{m} p_i X_i^j + p_i X_i^0 = 0,\ i = 1, \ldots, m$$

which immediately implies (28). Finally, the inclusion (9) holds for $p_0 = q_0(p)$, where $q_0(p)$ is the Young transform of $F_0(X^0)$ (for details see [1]), that is:

$$q_0(q) = inf\left\{\frac{qX^0}{F_0(X^0)}\middle| X^0 \geq 0, F_0\left(X^0\right) > 0\right\}$$

This fact concludes the proof of part A. Note that in the base year the all price indexes are equal to one, i.e., $p_1 = \ldots = p_m = s_1 = \ldots = s_n = 1$. Therefore, part B obviously follows from part A. The proof is concluded.

Appendix B. Software CES Framework Library

Appendix B.1. C++ Main Functions Specifications

```
// Read statistical IOT data from file and scale it
int fillAIn(Matrix &m, int mRows, int mCols, FILE *fin, int iRowPass, int ColPass, char Delimeter, double ValueMultiplier = 1.0);
```

// Aggregate IOT into fewer branches
int Aggregate(Matrix &mIn, Matrix &mOut, const CAggInfo &AggInfo);

// Calculate a(i,j) from IOT
int fillA(Matrix &mIn, const CIOTInfo& Info, Matrix &mA);

// Calculate b(i,j) from IOT
int fillB(Matrix &mIn, const CIOTInfo& Info, Matrix &mB);

// Calculate elasticities of substitution through years (nYears)
int FindRo1(Matrix** AIn, int nYears, CIOTInfo &AInInfo, CCES1Model &CES1Model, int, int adjustindex = 0);

// Make sample (roCES) for elastities of substitution search procedure
bool MakeRo1(double* roCES, int len, int ri, int base, double* ro1jMin, double* ro1jMax, int mask = 0xFF);

// Calculate price indexes (prout) on primary resource price indexes (curInput.s) based on CES1ModelOut parameters
int CalcPriceCES1(CCES1Model& CES1ModelOut, CIOTinput& curInput, Matrix& proOut, int index = 0, double plam = 1.0);

// Calculate Ist and IIIrd quadrant on IInd (Z0) based on CES Model matrices LA, LB
int MakeForecastCES(Matrix& Z0, Matrix& LA, Matrix& LB, Matrix& Y, Matrix& XI, Matrix& XIII);

// Calculate price index vector (pout), Ist (CIOT.X1) and IIIrd (CIOT.X3) quadrant on IInd quadrant (iotInput.Z0) and primary resource price indexes (iotInput.s) in fixed year (year) based on CES1Model parameters
int tryCES1Model(int year, CCES1Model& CES1Model, CIOTinput& iotInput, CIOT& out, Matrix &pout, struct tryParams* params = 0, int nparams = 0);

// Calculate Ist (out.X1) and IIIrd (out.X3) quadrant on IInd quadrant (iotInput.Z0) and primary resource price indexes (iotInput.s) and complexes' goods price index vector (prices) based on CES1Model parameters
int tryCES1ModelPrice(CCES1ModelO& CES1Model, CIOTinput& iotInput, Matrix& prices, CIOT& out);

// Calculate Ist (out.X1) and IIIrd (out.X3) quadrant on IInd quadrant (iotInput.Z0) and primary resource prices (iotInput.s) based on CES1Model parameters
int CalcIOT(CCES1Model& CES1Model, CIOTinput& iotInput, Matrix& p, CIOT& iotout, int index = 0, double plam = 1.0);
int CalcIOTO(CCES1ModelO& CES1Model, CIOTinput& iotInput, Matrix& p, CIOT& iotout);

// Price calculation on primary factors price indexes (s) for CES1 production function with the same elasticity value (ro) for any sector based on a_ij (A) and b_ij (B) coefficients
Matrix& priceCES(Matrix& A, Matrix& B, Matrix& s, double ro);

// Fill price index vector (prOut) with price indexes of the branches that are aggregated in fixed (ng) complex with elasticity (rong)
int FillPricesOfOneComplex(Matrix& Ain0, const CAggInfo& agg, int ng, double rong , CIOTInfo &AIn0Info , Matrix& s, Matrix& pr, Matrix &prOut);

Appendix B.2. Matrix Calculation Class with Base Operators: +,−, and Some Other Useful Functions: Matrix Inverting, Transposing, etc.*

```
Matrix(int t0 = 0); // constructor
Matrix(int m0, int n0, int t0 = 0); // constructor with size
Matrix(); // destructor
void SetSize(int m0, int n0);
double * operator[](int i);
Matrix & operator= (Matrix &y);
Matrix & operator+(Matrix &y);
Matrix & operator−(Matrix &y);
Matrix & operator−();
Matrix & operator*(Matrix &y);
Matrix & operator*(double y);
Matrix & operator/(double y);
Matrix & operator/(int y);
Matrix & operator/(Matrix &y); // multiply on inverted y
Matrix& operator%(Matrix &y); // X % A = X / (E−A) = X * E + A + A*A + A*A*A ...
Matrix& operator!(); // Transpose
Matrix& operator||(Matrix& y); // Elementwise maximum
Matrix& operator^(Matrix& y); // Like *, maximum instead of sum
operator double();
void SetE();
void SetValue(double v);
void SetValues(double *v);
void SetCol(Matrix &Col, int nCol); // nCol from 0 to Cols-1
void SetDiag(Matrix &Col, bool bInvert = false);
void NormByColSum();
void NormByDiagElem(double scale);
double trace();
double norm();
Matrix& RelativeDeviation(Matrix&);
double SumAbs();
Matrix & Set01ByTresh(double tr);
```

Appendix B.3. CES Framework Library Simulation Guide

1. Prepare input text files with statistical IOT.
2. Configure information for aggregation: a map of original branch numbers to aggregated sectors numbers: (ComplexNumber[BranchNumber] in CAggInfo).
3. Run reading data (fillAIn).
4. Aggregate IOT into fewer number of industrial complexes (Aggregate).
5. Run model identification (fillA, fillB, FindRo1) and get CES1Model.
6. Optional: disaggregate CES1Model to original IOT size (FillPricesOfOneComplex, fillA, fillB) and obtain CES1ModelO.
7. Set scenario input data: primary resource price indexes s and final consumption vector Z0.
8. Run simulation functions (tryCES1Model, CalcPriceCES1 + tryCES1ModelPrice).
9. Output the calculated values: I and III quadrants, price index vector.
10. Calculate and output derivative values: total output, primary resources consumption (Matrix Class for matrix arithmetic).

Appendix C. Aggregation Map of Kazakhstan Industries

Export— (14 branches): Coal and ignite, Crude oil, Manufacture of chemicals and chemical products , Manufacture of computer, electronic and optical products, Manufacture of electrical equipment, Manufacture of ferrous metals, Manufacture of paper and paper products, Manufacture of textiles, wearing apparel and leather products, Manufacture of wood and of products of wood and cork, except furniture; manufacture of articles of straw and plaiting materials, Mining and quarrying, Natural gas, Refined petroleum products , Wholesale trade, except of motor vehicles and motorcycles, Pipes, profiles

Export+ (8 branches): Crop and animal production, hunting and related service activities, Ferrous metals, Forestry and logging, leather products, Manufacture of machinery and equipment n.e.c., Nonferrous metals, Precious and other nonferrous metals, Tobacco products

Infrasructure (5 branches): Air transport, Electricity, Land transport and transport via pipelines, Telecommunications, Warehousing and support activities for transportation

Manufacturing (14 branches): Fishing and aquaculture, Food products, beverages, Manufacture of basic pharmaceutical products and pharmaceutical preparations, Manufacture of coke, Manufacture of fabricated metal products, except machinery and equipment, Manufacture of furniture, Manufacture of furniture; other manufacturing, Manufacture of motor vehicles, trailers and semi-trailers, Manufacture of other metallic products, Manufacture of other nonmetallic mineral products, Manufacture of other transport equipment, Manufacture of rubber and plastic products, Repair and installation of machinery and equipment, Wearing apparel

Service— (18 branches): Accommodation and food service activities, Activities auxiliary to financial services and insurance activities, Activities of households as employers; undifferentiated goods- and services-producing activities of households for own use, Administrative and support service activities, Casting services, Construction, Financial service activities, except insurance and pension funding, Food products and beverages supply services, Gas distribution services, Mining services, Other service activities, Printing and reproduction of recorded media, Real estate activities, Retail trade, except of motor vehicles and motorcycles, Scientific research and development, Other professional, scientific and technical activities;, Steam and air conditioning supply, Water collection, treatment and supply, Wholesale and retail trade and repair of motor vehicles and motorcycles

Service+ (9 branches): Education, Entertainments, Human health and social work activities, Information services, Insurance, reinsurance and pension funding, except compulsory social security, Legal and accounting activities; activities of head offices; management consultancy activities, Postal and courier activities, Public administration and defence; compulsory social security, Water transport.

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