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Stochastic Differential Game of Sustainable Allocation Strategy for Idle Emergency Supplies in Post-Disaster Management

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Abstract: This study aims to explore allocation strategies for idle emergency supplies in a "demander–platform–supplier" supply chain system along with government regulation during the post-disaster recovery period. Allocation of emergency supplies is a complex task that encompasses resource allocation before and after disasters. It is essential to reduce losses in disaster-stricken areas and support development during post-disaster recovery. However, there is often an excessive supply of emergency materials and a mismatch between supply and demand sides in downstream supply chains, which may lead to severe waste and difficulties in recovering surplus materials. This paper takes idle emergency resource sharing level and corporate social responsibility goodwill as endogenous variables. The allocation approaches are dynamically evaluated by incorporating random elements that influence the endogenous variables. Three stochastic differential games are introduced to examine the interactions between the players. The centralized decision-making satisfies the consistency of overall and individual rationalities at any time in the emergency material allocation process, promoting the optimal sharing levels of emergency materials and overall profits. The decentralized decision-making with cost-sharing contracts achieves local optima and increases the dual marginal effect of the emergency industry chain. This paper incorporates the sharing economy into emergency management, showing how technology-driven sharing platforms can optimize resource utilization. The results suggest introducing cost-sharing contracts between demanders and suppliers can enhance collaboration and effort, leading to better resource allocation and increased efficiency. It contributes to sustainability by promoting efficient resource utilization through idle emergency resource sharing. By optimizing allocation strategies and enhancing corporate social responsibility, the study fosters the long-term viability and resilience of the supply chain system in post-disaster management.

Keywords: stochastic differential game; random disturbance; supply chain; disaster response

Citation: Li, L.; Wu, J.; Zhu, M.; Wang, M.; Li, Y. Stochastic Differential Game of Sustainable Allocation Strategy for Idle Emergency Supplies in Post-Disaster Management. *Sustainability* **2024**, *16*, 10003. https://doi.org/10.3390/su162210003

Received: 24 October 2024
Revised: 9 November 2024
Accepted: 14 November 2024
Published: 16 November 2024

1. Introduction

Natural disasters, driven by environmental issues and climate change, pose significant threats to human life and economic progress due to their frequency, complexity, and unpredictability [1]. Statistics show that extreme weather events like hurricanes, hail, and floods account for 83% of natural disasters, causing over 410,000 deaths and affecting 1.7 billion people globally since the 20th century. In total, 92% of the fatalities occurred in low- and middle-income countries, particularly in the Asia Pacific and Africa. For instance, the 2004 Indian Ocean tsunami, triggered by a massive earthquake, devastated Indonesia and surrounding regions, killing hundreds of thousands. Similarly, the 2008 Wenchuan Earthquake in China caused widespread destruction and loss of life. This highlights the critical need for effective disaster reduction and emergency management by governments [2].

After a disaster, emergency departments at all levels work and coordinate with each other. As the situation improves, the focus should gradually shift from providing disaster relief supplies to the recovery and management of idle emergency resources. Due to the unpredictability of emergency events and the high fluctuations in demand for supplies during the early stages of a natural disaster, emergency reserves often struggle to meet peak demand. Additionally, there is a risk of supply and demand mismatches in the downstream emergency supply chain. Shortages of urgently needed items and overstock of non-emergency supplies take up valuable storage space and add management burdens. Perishable items with short shelf lives, like medications and food, may expire due to overstocking or improper use, leading to significant waste. Additionally, non-essential supplies can occupy storage space and hinder the availability of critical materials. At the same time, specialized emergency equipment, such as rescue boats and high-altitude rescue gear, often remains idle during non-emergency periods, resulting in low resource utilization [3]. These issues highlight the need for improved demand forecasting and dynamic allocation mechanisms within the emergency supply chain to enhance supply–demand matching, minimize resource waste, and avoid delays in emergency response efforts.

The sharing economy is an economic model based on sharing, collaboration, and renting. It aims to connect people who own resources with those who need them through technology such as the Internet and mobile apps [4]. The sharing economy offers a promising approach to addressing the challenges of supply–demand mismatches in post-disaster emergency supply management. By leveraging shared resources and collaborative networks, emergency supply chains can be more flexible and responsive, allowing for quicker and more efficient distribution of critical materials to affected areas. This model encourages collaboration between idle resource suppliers, a resource-sharing platform, and idle resource demanders, using technology to enhance coordination and optimize resource allocation. In the aftermath of disasters, such an approach can significantly reduce operational costs, minimize waste, and ensure that essential supplies reach those in need promptly, thereby enhancing the overall effectiveness of disaster response and recovery efforts.

Inspired by the sharing economy and previous research, this study explores allocation strategies for idle emergency supplies in a "demander–platform–supplier" supply chain under government regulation. Using three stochastic differential game models, it integrates resource-sharing levels with corporate social responsibility goodwill. Our main aim is to identify critical factors and determine optimal strategies that influence the sustainable development of the supply chain.

The dynamic allocation problem can be solved using control theory, which was proposed in the book "Game-Theoretical Control Problems" [5]. This book builds on the pioneering research of Krasovskii and Subbotin, who made significant contributions to differential games and control theory. Their work, along with foundational studies by Isaacs, Pontryagin, and other scholars from the Russian/Soviet schools, shaped modern approaches to optimal control in uncertain environments. The book also integrates key results from Nash's non-cooperative game theory, offering a synthesis of dynamic control principles and strategic decision-making. It provides formal definitions, equilibrium conditions, and optimal strategies, accompanied by discussions of computational methods. This comprehensive framework serves as a resource for mathematicians, engineers, economists, and others applying control and game-theoretical approaches in practical settings.

The following work is arranged as follows: Section 2 offers a review of the relevant literature. Section 3 addresses the problem and provides some useful preliminaries. Section 4 explores the cooperative game model, where players collaborate to achieve a mutually beneficial allocation of resources. Section 5 analyzes the Nash equilibrium model, focusing on how decentralized decision-making among demanders, suppliers, and platforms impacts resource sharing. Section 6 examines the Stackelberg game model, in which a leader, such as the government, makes decisions that influence the actions of other players. Section 7 provides a comparative analysis and offers numerical simulations. Section 8 provides several practical implications. Section 9 contains a summary.

2. Review of the Relevant Research

This section reviews the body of work in related fields in three aspects. The review encompasses three major areas: resource sharing in supply chains, differential game models, and allocation of emergency supplies.

2.1. Resource Sharing in Supply Chains

The research on supply chain resource sharing mainly focuses on the sharing of logistics resources, manufacturing resources, warehousing resources, and others. Lozano, using cooperative game theory, studied the problem of cost allocation when different companies consolidate transportation demands, i.e., when they share transportation resources [6]. Based on the diversity of shared resources in the Internet of Things (IoT) environment, Zhao developed a model of two manufacturers' bidirectional transfer of manufacturing resources considering random demand and aiming to maximize corporate profits and derived the optimal strategy for resource transfer [7]. Zhao examined the impact of whether companies participating in resource sharing within the supply chain use RFID technology to achieve comprehensive real-time sharing of demand information on the manufacturers' capacity-sharing strategies [8]. Qi constructed an evolutionary game model for shared manufacturing resources considering the behavior of two manufacturers and found the equilibrium point in the game population under uniform and non-uniform mixing conditions [9]. He built different value calculation models by analyzing the demand for shared warehousing under different warehousing methods, proving that warehousing sharing helps reduce warehousing management expenses and lower logistics costs [10]. The preceding studies generally treat the sharing platform and the resource demand or supply side as a whole, only studying a two-level supply chain without directly involving a separate sharing platform, which weakens the matching costs generated by the platform in resource collection and allocation. Pan used differential games to study the problem of manufacturing capacity sharing in a two-level supply chain composed of a single resource supplier and a single cloud platform [11]. Pan considered the sharing platform and resource-demanding enterprises together, making unified decisions, which simplified the problem model to some extent and led to some important conclusions. However, in real production and life, the sharing platform plays an important role in resource sharing, and its decisions are independent of resource suppliers and resource-demanding enterprises. Each participant must bear the corresponding matching costs, so considering the optimal allocation decisions of resource suppliers, resource demanders, and the sharing platform separately has practical significance. Mahtab constructed a multi-objective robust-stochastic humanitarian logistics model that incorporates vehicle flow and multi-period planning within a robust-stochastic optimization framework [12]. Moosavi presented a comprehensive set of bibliometric, network, and thematic analyses, offering managerial insights to promote resilient and sustainable supply chains during pandemics [13].

2.2. Differential Game Models

Game theory is a mathematical framework that analyzes dynamic interactions and decision-making processes between multiple agents or players, each with objectives, constraints, and control strategies. It finds applications in a wide range of dynamic decision-making problems, such as resource allocation [14], carbon emission [15,16], advertising cooperation [17], product recycling [18], and water pollution policy coordination [19,20]. The application of differential games in emergency management has gradually grown in recent years. Examples include the cooperative transportation of disaster relief supplies [21], the management synergy of regional coal mine emergencies [22], the post-disaster donation of businesses [23], and the optimization of false information classification systems [24]. Specifically, Chen built a system of Chinese relief reserves and used the bargaining model to analyze such a system [25]. It found that the emergency supplies reserve strategy in the form of ex-ante government and enterprise co-reserve is superior to the strategy in the form of ex-post emergency procurement and emergency production with fund limitation and high social costs. Zhang investigated an emergency supplies joint reserve mode and explored the specific conditions and influencing factors of realizing government–enterprise cooperation by establishing a tripartite evolutionary game model of the government, enterprise, and society [26]. As regards the transport and allocation of emergency supplies, Qiu proved that the intense supervision of higher-level administration of emergency (HAE) has a critical impact on the realization of cross-regional coordinated dispatch of emergency supplies. Additionally, the financial rewards and punishments imposed by HAE on other entities can accelerate or delay the achievement of the equilibrium strategy [27]. Moreover, Li considered a three-party evolutionary game model involving the government, potential catastrophe insurance participants, and insurance companies [28]. Du explored the government's mobilization strategy with the help of government-owned and grassroots non-profit organizations [29].

2.3. Allocation of Emergency Supplies

Regarding the research on the allocation of emergency supplies before disasters occur, Dodo established a linear programming model targeting earthquake mitigation activities to solve the problem of disaster prevention funding allocation before earthquakes [30]. Generally, after large-scale emergencies, issues such as severe damage to infrastructure, temporary transportation interruptions, and insufficient emergency transport capacity arise, increasing the difficulty of post-disaster rescue efforts [31]. Zhang developed a two-stage stochastic programming model for emergency resource allocation to address the uncertainty of demand after disasters, using compensation variables to establish a linear relationship with uncertain demand variables, making the results more realistic [32]. Zhao analyzed traffic congestion caused by highway emergencies and proposed an emergency resource scheduling model for highways, considering the delay caused by using emergency lanes [33]. Li studied the issue of securing repair resources for power distribution networks after earthquakes and established an optimization model for post-earthquake recovery considering geographical features to help restore the power distribution network quickly [34]. In terms of building a collaborative response mechanism for emergencies, Liu analyzed the government's reputation mechanism in addressing group incidents related to environmental pollution and explained the diffusion effect of PX incidents that erupted across the country under local government stability maintenance strategies [35]. PX events refer to paraxylene (PX)-related environmental pollution incidents, where local communities protest the construction or expansion of PX chemical plants due to concerns over health and environmental safety. Whether before or after disasters, these studies on emergency resource allocation mostly aim to solve issues such as emergency resource location path, optimal scheduling of resources, and collaborative response mechanisms to meet the demand for emergency resources as much as possible. However, such allocation models do not fully align with actual emergency rescue situations. The main reason is that during post-disaster emergency supply allocation, there exists an asymmetry in supply and demand information, and post-disaster allocation rarely considers recyclable emergency supplies, often leading to inefficient and costly initial allocations. In practice, Guo proposed corresponding recycling and emergency supply strategies to address the high expiry loss and stockout loss of perishable emergency supplies. Post-disaster recyclable or surplus emergency supplies still hold a certain value [36]. They can be shared with other enterprises in urgent need. Therefore, the allocation decision of recyclable emergency supplies after disasters presents a significant challenge and opportunity.

On the practical level of emergency supply sharing, Qiu analyzed the structure and elements of existing metadata standards in the emergency field concerning information sharing and interoperability and proposed a general metadata standard and extension mechanism for the emergency field [37]. Kapucu believes that the use of information technology can impact the sharing of information and resources, enhancing the collaboration and effectiveness of communities in responding to disasters. Furthermore, some scholars have integrated information sharing with cloud platforms [38]. Kochan found that cloud-based information sharing could improve the responsiveness of the medical supply chain [39]. Due to the loss of information after disasters leading to failed emergency decision-making and adjustment difficulties, Xue designed a post-disaster emergency supply allocation method based on a real-time information sharing platform, significantly improving the feasibility and timeliness of supply distribution and transportation decisions [40]. It can be observed that little work has been accomplished regarding overall emergency supply sharing allocation, especially in platform practice, because the emergency industry is unique. The sharing and co-construction of emergency supplies involve different departmental interests, making actual implementation challenging. The research preceding provides a solid foundation for studying emergency supply allocation and the shared economy. However, in practical production and operations, the cross-network externalities of the platform are often overlooked. This includes the mutual influence of decision-making by both supply and demand sides, the long-term dynamics and complexity of enterprise operations, and the fact that the cooperative relationships among participants can often better stimulate consumption demand. A common research method for coordinating the supply chain and achieving efficient resource allocation is the introduction of cost-sharing mechanisms.

The preceding research provides a solid foundation for studying emergency material allocation. There are still some questions that need to be sorted out. First, whether before or after a disaster, most of the research on emergency resource allocation aims to address issues such as the location-routing of emergency resources, the optimal quantity of resources to be dispatched, and coordinated response mechanisms to maximize the satisfaction of emergency resource needs. However, such allocation models do not fully align with the actual conditions of emergency rescue operations. The main shortage is the information asymmetry between supply and demand in the post-disaster emergency material allocation process. Such allocation strategies often lead to inefficiencies and high costs. In reality, post-disaster recyclable or surplus emergency materials still hold certain utility. Therefore, the allocation decision of post-disaster recyclable emergency materials presents both a significant challenge and an opportunity. Second, in the actual production and operation process, they overlook the cross-network externalities, such as the mutual influence of supply–demand decisions and the long-term dynamics and complexity of enterprise operations. It can be observed that existing research on cost-sharing mainly focuses on incentive strategies for supply chain members, with few studies applying cost-sharing mechanisms to resource sharing research [41].

To deal with the information asymmetry and recycle surplus emergency materials, this research introduces a resource sharing platform to achieve the allocation of idle emergency resources during the recovery phase. As for the cross-network externalities, we introduce cost-sharing mechanisms to coordinate the supply chain and achieve efficient resource allocation. More importantly, the cooperation among participants often stimulates consumer demand more effectively. This study aims to establish stochastic models to explore allocation strategies for idle emergency supplies in post-disaster management that incorporate corporate social responsibility goodwill and idle emergency resource sharing level. Our main aim is to identify critical factors and determine optimal strategies that influence the sustainable development of the supply chain.

Based on the previously discussed works, the following research gaps can be found:

- The sharing economy offers a promising approach to addressing the challenges of supply–demand mismatches in post-disaster emergency supply management. At the same time, such a concept is rarely considered in related research. By introducing a resource sharing platform, this research studies allocation strategies for idle emergency supplies in the post-disaster recovery period [42,43].
- The previous works focus on emergency management from the perspectives of supervision, cost, demand, response time, etc., but overlook the impact of uncertain factors. Players' optimal strategies and decision-making processes can be greatly impacted by random factors, which accords with the uncertainty of emergency events [44,45].

This research employs stochastic differential game theory to model the optimization of emergency material allocation decisions. It examines the long-term effects of emergency material sharing and matching strategies [46,47].

- Existing research on cost-sharing mainly focuses on incentive strategies for supply chain members [41], with few research works applying cost-sharing mechanisms in the study of resource sharing. This research incorporates a cost-sharing mechanism to improve the efficiency of emergency material allocation, identifies the prerequisites for its existence, and explores its impact on improving the efficiency of supply–demand allocation in downstream emergency supply chain systems [48,49].

3. Problem Description and Model Assumptions

Section 3 explains the stochastic differential game models employed in this study. Section 3.1 describes the studied problem and the mechanism between each player. Section 3.2 provides several model assumptions. Section 3.3 offers some useful preliminaries.

3.1. Problem Description

The ability to provide emergency supplies is one of the fundamental guarantees for emergency management. Due to the unpredictability of emergencies and the large fluctuations in demand during the early stages of a natural disaster, emergency supply reserves often fall short of peak demand. Additionally, there is a risk of supply–demand mismatches in the downstream emergency supply chain. To better address the above problems, this research introduces random factors and stochastic differential equations (SDEs) to explore allocation strategies for emergency supplies from a dynamic perspective:

- The SDE incorporates random factors, such as unexpected surges in demand or delays in supply, which are common in post-disaster scenarios. For example, sudden needs for specific resources may arise due to evolving conditions, while supplies may degrade or become misallocated.
- The equation accounts for the unpredictable nature of emergencies, where resource availability and demand are not constant but vary based on factors like the level of damage, logistics challenges, or weather conditions.
- The model uses an SDE to allow for continuous adjustments in resource-sharing strategies. As more information becomes available or conditions change (e.g., new damage reports or the arrival of supplies), stakeholders can update their decisions in real time to optimize resource allocation.

This research considers a three-tier supply chain system consisting of a resource sharing platform (denoted as P), an emergency supply supplier (denoted as S), and an emergency supply demander (denoted as D) along with government (denoted as A), see Figure 1. The supplier and the demander are manufacturing enterprises of the same type that produce homogeneous emergency products. They use this platform to achieve supply and demand allocation of idle emergency resources. There also exists a government that regulates the platform. The key to achieving emergency resource sharing is how the supplier and demander form resource sharing and matching decisions and choose appropriate cooperation modes. In this process, shared resources include both the transmission of intangible information and the exchange of tangible resources. The supplier sends the intangible information to the platform first, and the platform searches and matches it based on the user's need and dynamically provides the information to the demander. Then, the demander purchases the corresponding resources from the supplier.

This research uses the "matching effort" $D(t), P(t), S(t)$ to represent the degree of effort that each party paid to achieve supply–demand matching in the supply chain and the "regulation effort" $A(t)$ to represent the degree of effort that the government paid to regulate the platform, which are positive decision variables. The matching efforts include purchasing intelligent perception devices and information uploading and maintenance for the supply side; production planning, multi-party coordination, and information collection and uploading for the demand side; data collection, searching and matching information, maintenance and operation for the platform. The regulation efforts include protecting data privacy and security, maintaining fair competition, and protecting consumer rights. In addition, the demander pays a commission fee c to the platform and a resource usage fee ω to the supplier per unit of the resources. The platform bears a portion of the demander's matching costs to facilitate the achievement of transactions. The government also provides subsidies to the platform to keep an active market. The subsidies from the platform and the government are denoted as ξ_d and ξ_p, respectively.

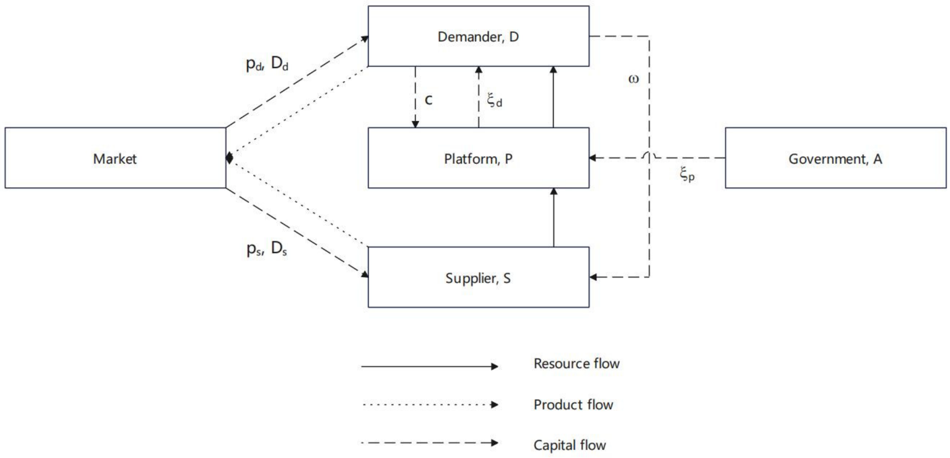


Figure 1. The network flow between different supply chain partnerships.

First of all, we list all of the variables and parameters that are used in the following Tables 1 and 2.

Table 1. List of variables.

Variable	Description
$D(t)$	The matching effort of the demander
$P(t)$	The matching effort of the platform
$S(t)$	The matching effort of the supplier
$A(t)$	The regulation effort of the government
$G(t)$	The CSR goodwill
$\chi(t)$	The idle emergency resource sharing level

Table 2. List of parameters.

Parameter	Description
μ	The natural decay rate of emergency supplies
σ	The natural decay rate of CSR goodwill
ρ	The diffusion coefficient in (1)
δ	The diffusion coefficient in (2)
r	The discount rate
λ_1	The marginal impact coefficient of D in (1)
λ_2	The marginal impact coefficient of P in (1)
λ_3	The marginal impact coefficient of S in (1)
λ_4	The marginal impact coefficient of A in (1)
γ_1	The marginal impact coefficient of D in (2)
γ_2	The marginal impact coefficient of P in (2)
γ_3	The marginal impact coefficient of S in (2)
ϵ	The marginal impact coefficient of χ in (2)
C_D	The cost of the demander
C_P	The cost of the platform
C_S	The cost of the supplier
C_A	The cost of the government

Table 2. *Cont.*

Parameter	Description
u_d	The cost coefficient of the demander
u_p	The cost coefficient of the platform
u_s	The cost coefficient of the supplier
u_g	The cost coefficient of the government
D_s	The demand function of the supplier
D_d	The demand function of the demander
a	The potential size of the market
p_s	The marginal revenue of the supplier
p_d	The marginal revenue of the demander
θ	The substitute coefficient
η_{s1}	The marginal impact coefficient of external overflow effect of χ in (3)
η_{s2}	The marginal impact coefficient of external overflow effect of χ in (4)
η_{d1}	The marginal impact coefficient of external overflow effect of G in (3)
η_{d2}	The marginal impact coefficient of external overflow effect of G in (4)
c	The commission fee from the demander to the platform
ω	The resource usage fee from the demander to the supplier
ξ_d	The subsidy from the platform to the demander
ξ_p	The subsidy from the government to the platform
π_d	The marginal profit coefficients of the supplier
π_p	The marginal profit coefficients of the platform
π_s	The marginal profit coefficients of the demander
π_g	The marginal profit coefficients of the government

3.2. Model Assumption

(1) Players are assumed to act rationally, meaning they aim to maximize their benefits with full knowledge of the game [50].

(2) The amount of emergency supplies is dynamic progress that can be improved by the matching and regulation efforts of the players. According to the goodwill model [51–53], we assume the idle emergency resource sharing level(χ) satisfies

$$d\chi(t) = (\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi)dt + \rho\sqrt{\chi(t)}dz(t), \chi(0) = \chi_0 > 0, \quad (1)$$

where $D(t), P(t), S(t)$ represent the matching efforts that each party paid to achieve supply–demand matching in the supply chain and $A(t)$ represents the regulation effort that the government paid to regulate the platform. $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the marginal impact coefficients of the efforts. μ denotes the natural decay rate, reflecting the timeliness of shared emergency supplies. $dz(t)$ is the Brownian motion. ρ represents the diffusion coefficient, quantifying the intensity of random disturbances or uncertainties in the system.

(3) The term "CSR goodwill" indicates the social trust and positive reputation that a business cultivates as a result of its Corporate Social Responsibility activities. According to [54], CSR goodwill is a dynamic progress and is determined by diverse components, such as the matching efforts of the demander, platform, supplier, and random factors. Additionally, it is also affected by the idle emergency resource sharing level. Therefore, we assume the CSR goodwill (G) satisfies

$$dG(t) = (\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G)dt + \delta\sqrt{G(t)}dz(t), \chi(0) = \chi_0 > 0, G(0) = G_0 > 0, \quad (2)$$

where $D(t), P(t), S(t)$ represent the matching efforts that each party paid to achieve supply–demand matching in the supply chain and $A(t)$ represents the regulation effort that the government paid to regulate the platform. $\gamma_1, \gamma_2, \gamma_3$ are the marginal impact coefficients of the efforts. σ denotes the natural decay rate, $\mathrm{d}z(t)$ is the Brownian motion. δ is the diffusion coefficient. ϵ denotes the marginal impact coefficient of the idle emergency resource sharing level. To be specific, when companies share idle emergency resources to support communities during a crisis, they are perceived as responsible and community-oriented. Such actions boost the public's view of the company's commitment to social responsibility, thereby increasing CSR goodwill. χ captures this "spillover effect" of resource sharing on CSR goodwill, indicating that active resource-sharing behavior positively impacts the company's reputation and trustworthiness.

(4) The costs of regulation and matching efforts of participants are positively related to their efforts and follow a convex function [55,56]. Thus, the costs of the efforts are

$$C_D = \frac{1}{2}u_d D^2, \quad C_P = \frac{1}{2}u_p P^2, \quad C_S = \frac{1}{2}u_s S^2, \quad C_A = \frac{1}{2}u_g A^2,$$

where u_d, u_p, u_s, u_g denote the cost coefficients.

(5) Resource sharing and corporate social responsibility can generate a positive external overflow of the provider and demander in the supply chain system, increasing market demand [57]. Considering the existence of competitive relationships among similar enterprises [58], the demand functions for the supplier and the demander in a competitive market can be written as

$$D_s = a - p_s + \theta(p_d - p_s) + \eta_{s1}\chi + \eta_{s2}G, \tag{3}$$

$$D_d = a - p_d + \theta(p_s - p_d) + \eta_{d1}\chi + \eta_{d2}G \tag{4}$$

where a is the market's potential size, p_s, p_d are the supplier's and the demander's marginal revenues. θ is the substitute coefficient. The larger the θ, the greater the impact of price on the demand. η_{s1}, η_{d1} are the marginal impact coefficients of external overflow effects of the idle emergency resource sharing level. When the amount of shared resources increases, the supplier and the demander can meet more needs, such that the market trusts them more. Hence, the market demand is proportional to the idle emergency resource sharing level. The larger η_{s1}, η_{d1} are, the greater the impact of the idle emergency resource sharing level on demand. η_{s2}, η_{d2} represent the marginal impact coefficients of external overflow effects of the CSR goodwill that similarly affect the market demand.

Hence, the objective functions are

$$\begin{cases} J_s = \int_0^{\infty} e^{-rt}\left\{p_s D_s + \omega\chi + \pi_s G - \dfrac{u_s S^2}{2}\right\}\mathrm{d}t \\ s.t. \begin{cases} \mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z_1(t), & \chi(0) = \chi_0 > 0, \\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z_2(t), & G(0) = G_0 > 0; \end{cases} \end{cases}$$

The objective J_s represents the present value of the expected benefits for the supplier s from their efforts in contributing resources to the system, taking into account CSR-related goodwill and resource-sharing. e^{-rt} represents the discounting over time, where r is the discount rate, reflecting that future benefits or costs are valued less than present ones. $p_s D_s$ captures the direct benefit that the supplier s gains from their demand-driven efforts D_s. The coefficient p_s represents the price or benefit per unit of demand effort. $\omega\chi$ reflects the positive impact of idle emergency resource sharing on CSR goodwill. χ represents the level of shared resources, and ω is a coefficient showing how much resource sharing contributes to goodwill. $\pi_s G$ This term captures the impact of CSR goodwill G on the supplier's benefits, with π_s being the sensitivity of the supplier's payoff to goodwill levels.

$-\frac{u_s S^2}{2}$ is a cost term, representing the effort cost associated with S, the supplier's level of resource sharing. The parameter u_s indicates the intensity of this cost.

$$\begin{cases} J_d = \int_0^\infty e^{-rt}\left\{p_d D_d + \pi_d G - (\omega + c)\chi - \frac{u_d D^2}{2}\right\}\mathrm{d}t \\ s.t. \begin{cases} \mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z_1(t), & \chi(0) = \chi_0 > 0, \\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z_2(t), & G(0) = G_0 > 0; \end{cases} \end{cases}$$

The objective J_d represents the present value of expected benefits for the demander, which could be an organization or community benefiting from resource sharing. This objective includes gains from CSR goodwill and resource availability. e^{-rt} discounts future benefits, acknowledging that immediate benefits are valued more highly than future ones. $p_d\chi$ captures the benefit derived from the level of resource sharing χ for the demander. Here, p_d represents the importance or utility that the demander places on each unit of shared resources. $\pi_d G$ reflects the positive impact of CSR goodwill G on the demander's benefits, where π_d is a coefficient indicating how much the demander's utility is influenced by the company's CSR reputation. $-\frac{u_d D^2}{2}$ represents the cost of the demander's own effort D to access or engage in resource-sharing activities. The parameter u_d indicates the intensity of this cost, which grows with the square of the effort, meaning that higher effort levels come with higher incremental costs.

$$\begin{cases} J_p = \int_0^\infty e^{-rt}\left\{c\chi + \pi_p G - \frac{u_p P^2}{2}\right\}\mathrm{d}t \\ s.t. \begin{cases} \mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z_1(t), & \chi(0) = \chi_0 > 0, \\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z_2(t), & G(0) = G_0 > 0; \end{cases} \end{cases}$$

The objective J_p represents the present value of expected benefits for the platform. As an intermediary facilitating resource sharing, the platform benefits from both CSR goodwill and the efficiency of resource distribution. e^{-rt} is the discount factor, which assingns greater weight to current benefits and costs relative to future ones. $p_p\chi$ denotes the benefit to the platform from the level of resource sharing χ. The coefficient p_p indicates the platform's valuation of shared resources and their distribution impact. $p_p G$ captures the benefit to the platform from CSR goodwill G. A higher goodwill level improves the platform's reputation or appeal, with π_p representing how strongly the platform's utility is tied to CSR perception. $-\frac{u_p P^2}{2}$ reflects the cost associated with the platform's effort P to facilitate resource sharing. The cost increases quadratically with P, meaning that increased effort in managing resources or coordinating efforts becomes increasingly costly for the platform.

$$\begin{cases} J_g = \int_0^\infty e^{-rt}\left\{\pi_g\chi - \frac{u_g A^2}{2}\right\}\mathrm{d}t \\ s.t. \begin{cases} \mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z_1(t), & \chi(0) = \chi_0 > 0, \\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z_2(t), & G(0) = G_0 > 0; \end{cases} \end{cases}$$

The objective J_g reflects the present value of expected benefits for the government from encouraging resource-sharing levels and CSR activities, factoring in the influence on public welfare and CSR goodwill. e^{-rt} discounts future benefits and costs. $\pi_g\chi$ represents the government's benefit from resource-sharing activities χ. The coefficient π_g reflects the government's valuation of public welfare improvements that result from resource sharing. $-\frac{u_g A^2}{2}$ represents the cost of the government's actions in encouraging resource-sharing initiatives. Here, A denotes the government's intervention effort level and u_g represents the cost intensity of this intervention.

To solve the Hamilton–Jacobi–Bellman (HJB) equation, the general approach involves using dynamic programming [5]. First, the problem is formulated as a continuous-time optimal control problem, where the HJB equation represents the necessary condition for optimality. Based on the current state and control variables, the solution process begins by expressing the value function—the maximum utility or payoff a decision-maker can achieve. The next step involves solving the partial differential equation (PDE), where the value function satisfies the HJB equation. Analytical solutions can sometimes be obtained for simple cases. Still, numerical methods like finite difference schemes, value iteration, or policy iteration are often used to approximate the solution. Once the value function is found, the optimal control policy is derived by maximizing the Hamiltonian with respect to the control variables, ensuring that the policy satisfies the optimality condition.

3.3. Preliminary

In this section, this research offers some useful preliminaries. First, we recall the Kolmogorov's extension theory to define Brownian motion. Then, we introduce the Itô process and the Itô formula, which are utilized in the following deductions.

Lemma 1 (Kolmogorov's extension theory [59])**.** *For all $t_1, \ldots, t_k \in T, k \in \mathbf{N}$, let $v_{t_1,\ldots,t_k}$ be probability measures on $\mathbf{R}^{nk}$ subject to*

$$v_{t_{\sigma(1)},\ldots,t_{\sigma(k)}}(F_1 \times \cdots \times F_k) = v_{t_1,\ldots,t_k}(F_{\sigma^{-1}(1)} \times \cdots \times F_{\sigma^{-1}(k)}) \tag{5}$$

for all permutations σ on $1, 2, \ldots, k$ and

$$v_{t_1,\ldots,t_k}(F_1 \times \cdots \times F_k) = v_{t_1,\ldots,t_k,t_{k+1},\ldots,t_{k+m}}(F_1 \times \cdots \times F_k \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n) \tag{6}$$

for all $m \in \mathbf{N}$, where the set on the right hand side has a total of $k+m$ factors. Therefore, there exists a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a stochastic process X_t on Ω, $X_t : \Omega \to \mathbf{R}^n$ subject to

$$v_{t_1,\ldots,t_k}(F_1 \times \cdots \times F_k) = P[X_{t_1} \in F_1, \cdots, X_{t_k} \in F_k]$$

for all $t_i \in \mathbf{T}, k \in \mathbf{N}$ and all Borel sets F_i.

Fix $x \in \mathbf{R}^n$ and define

$$p(t,x,y) = (2\pi t)^{-n/2} \exp\left(-\frac{|x-y|^2}{2t}\right) \text{ for } y \in \mathbf{R}^n, t > 0.$$

If $0 \le t_1 \le t_2 \le \cdots \le t_k$, define measure $v_{t_1,\ldots,t_k}$ on $\mathbf{R}^{nk}$ by

$$\begin{aligned} &v_{t_1,\ldots,t_k}(F_1 \times \cdots \times F_k) \\ &= \int_{F_1 \times \cdots \times F_k} p(t_1,x,x_1)p(t_2-t_1,x_1,x_2)\cdots p(t_k-t_{k-1},x_{k-1},x_k)d_{x_1}\cdots d_{x_k}, \end{aligned} \tag{7}$$

where notation $dy = dy_1 \cdots dy_k$ is used for Lebesgue measure and the convention $p(0,x,y)dy = \delta_x(y)$, the unit point mass at x. The above definition can be extended to all finite sequences of t_i's in terms of (5). Since $\int_{\mathbf{R}^n} p(t,x,y)dy = 1$ for all $t \ge 0$, (6) holds. Then, by Kolmogorov's extension theorem, there exists a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a stochastic process $B_{t t \ge 0}$ on Ω such that the finite-dimensional distributions of B_t are given by (7), i.e.,

$$\begin{aligned} &P^x(B_{t_1} \in F_1, \cdots, B_{t_k} \in F_k) \\ &= \int_{F_1 \times \cdots \times F_k} p(t_1,x,x_1)p(t_2-t_1,x_1,x_2)\cdots p(t_k-t_{k-1},x_{k-1},x_k)d_{x_1}\cdots d_{x_k}. \end{aligned}$$

Definition 1 (Brownian motion)**.** *Such a process is a version of Brownian motion starting at x.*

Definition 2 (One-dimensional Itô process)**.** *Let B_t be 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$. A 1-dimensional Itô process or stochastic integral is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathcal{P})$ of the form*

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s, \tag{8}$$

where $v \in \mathcal{W}_{\mathcal{H}}$, such that

$$P\left[\int_0^t v(s,\omega)^2 ds < \infty \text{ for all } t > 0\right] = 1$$

and u is $\mathcal{H}_t$-adapted satisfying

$$P\left[\int_0^t |u(s,\omega)| ds < \infty \text{ for all } t > 0\right] = 1.$$

Remark 1. *If X_t is an Itô process of (8), it can be written as*

$$dX_t = udt + vdB_t.$$

Lemma 2 (One-dimensional Itô formula [59])**.** *Let X_t be an Itô process given by*

$$dX_t = udt + vdB_t.$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbf{R})$($g$ is twice continuously differentiable on $[0,\infty) \times \mathbf{R}$). Then

$$Y_t = g(t, X_t)$$

is also an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2.$$

4. Centralized Decision-Making

In the context of collaborative decision-making, the enterprises are aware of the impact of sharing surplus resources on the profits. The government realizes the importance of maintaining a high emergency resource sharing level. Under this sense, the demander, the platform, the supplier, and the government form a community of interest, actively engaging in collaborative cooperation. Both parties jointly decide their efforts to maximize the benefits of the system. The such model is referred to as "c" for convenience. Consequently, the optimal control equation has the following form:

$$\begin{aligned}&\max_{D,P,S,A} J^c(D,P,S,A) \\ &= \int_0^\infty e^{-rt}\left\{p_d D_d + p_s D_s + (\pi_d + \pi_p + \pi_s)G + \pi_g \chi - \frac{u_d D^2 + u_p P^2 + u_s S^2 + u_g A^2}{2}\right\}\mathrm{d}t\end{aligned}$$

subject to

$$\begin{aligned}&\mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z(t), \quad \chi(0) = \chi_0 > 0, \\ &\mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z(t), \quad G(0) = G_0 > 0,\end{aligned} \tag{9}$$

where ρ, δ are the diffusion coefficients. $\rho\sqrt{\chi}\mathrm{d}z(t)$ dictates the amplitude of random fluctuations driven by Brownian motion $\mathrm{d}z(t)$, a common model used to describe random processes. The larger the ρ, the greater the impact of randomness on the system, meaning the volatility of the idle emergency resource sharing level, $\chi(t)$, increases. This can model the effect of unpredictable events, such as natural disasters or policy shifts, on resource allocation. Similarly, a higher value of δ means that process $G(t)$ experiences greater volatility due to random disturbance, while a lower value of δ implies less sensitivity to random disturbances.

The above objective function contain four parts. The first part is the revenue of selling emergency products without the help of the resource sharing platform along with the revenue comes from the external overflow effects from the emergency resource sharing and corporate social responsibility. The second part is the profit of obtaining CSR goodwill for the supplier, the platform, the demander, and the profit of keeping an active emergency resource sharing market for the government. The third part is the cost of matching and regulation efforts.

Theorem 3. *(1) The optimal efforts are*

$$\begin{aligned}
D^c &= \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+\pi_p+\pi_s+p_d\eta_{d2}+p_s\eta_{s2})+\lambda_1(r+\sigma)(\pi_g+p_d\eta_{d1}+p_s\eta_{s1})}{u_d(r+\sigma)(r+\mu)}, \\
P^c &= \frac{(\gamma_2(r+\mu)+\epsilon\lambda_2)(\pi_d+\pi_p+\pi_s+p_d\eta_{d2}+p_s\eta_{s2})+\lambda_2(r+\sigma)(\pi_g+p_d\eta_{d1}+p_s\eta_{s1})}{u_p(r+\sigma)(r+\mu)}, \\
S^c &= \frac{(\gamma_3(r+\mu)+\epsilon\lambda_3)(\pi_d+\pi_p+\pi_s+p_d\eta_{d2}+p_s\eta_{s2})+\lambda_3(r+\sigma)(\pi_g+p_d\eta_{d1}+p_s\eta_{s1})}{u_s(r+\sigma)(r+\mu)}, \\
A^c &= \frac{\epsilon\lambda_4(\pi_d+\pi_p+\pi_s+p_d\eta_{d2}+p_s\eta_{s2})+\lambda_4(r+\sigma)(\pi_g+p_d\eta_{d1}+p_s\eta_{s1})}{u_g(r+\sigma)(r+\mu)}.
\end{aligned} \tag{10}$$

(2) The optimal performance function of the system is

$$\begin{aligned}
V^c &= \frac{\pi_d+\pi_p+\pi_s+p_d\eta_{d2}+p_s\eta_{s2}}{r+\sigma}G \\
&+\frac{\epsilon(\pi_d+\pi_p+\pi_s+p_d\eta_{d2}+p_s\eta_{s2})+(r+\sigma)(\pi_g+p_d\eta_{d1}+p_s\eta_{s1})}{r+\sigma}\chi+pol_c,
\end{aligned}$$

where pol_c is a polynomial. (See Appendix A).

Proof. Denoting V^c as the value function of the system, one has $V^c = \max_{D,P,S,A} J^c$. According to the optimal control theory [60], it should observe the HJB equation

$$\begin{aligned}
rV^c(G,\chi) &= \max_{D,P,S,A}\Big\{p_dD_d+p_sD_s+(\pi_d+\pi_p+\pi_s)G+\pi_g\chi-\frac{u_dD^2+u_pP^2+u_sS^2+u_gA^2}{2} \\
&+(\lambda_1D+\lambda_2P+\lambda_3S+\lambda_4A-\mu\chi)\frac{\partial V^c(G,\chi)}{\partial\chi}+(\gamma_1D+\gamma_2P+\gamma_3S+\epsilon\chi-\sigma G)\frac{\partial V^c(G,\chi)}{\partial G} \\
&+\frac{\rho^2}{2}\frac{\partial^2V^c(G,\chi)}{\partial\chi^2}+\frac{\delta^2}{2}\frac{\partial^2V^c(G,\chi)}{\partial G^2}+\rho\delta\sqrt{G}\sqrt{\chi}\frac{\partial^2V^c(G,\chi)}{\partial G\partial\chi}\Big\}.
\end{aligned} \tag{11}$$

Taking the first-order derivatives of the right-hand side of (11) with respect to D, P, S, A, and setting them to zero, one has

$$
\begin{aligned}
D^c &= \frac{1}{u_d}\Big[\lambda_1 \frac{\partial V^c(G,\chi)}{\partial \chi} + \gamma_1 \frac{\partial V^c(G,\chi)}{\partial G}\Big], \\
P^c &= \frac{1}{u_p}\Big[\lambda_2 \frac{\partial V^c(G,\chi)}{\partial \chi} + \gamma_2 \frac{\partial V^c(G,\chi)}{\partial G}\Big], \\
S^c &= \frac{1}{u_s}\Big[\lambda_3 \frac{\partial V^c(G,\chi)}{\partial \chi} + \gamma_3 \frac{\partial V^c(G,\chi)}{\partial G}\Big], \\
A^c &= \frac{\lambda_4}{u_g}\frac{\partial V^c(G,\chi)}{\partial \chi}.
\end{aligned}
\tag{12}
$$

Assuming V^c has the form

$$V^c(G,\chi) = a_{c1}G + a_{c2}\chi + a_{c3}, \tag{13}$$

where a_{c1}, a_{c2}, a_{c3} are constants, substituting (12) and (13) into (11), we obtain

$$
\begin{aligned}
r(a_{c1}G + a_{c2}\chi + a_{c3}) \quad &= ap_d - p_d^2 + ap_s - p_s^2 + p_s\theta(p_d - p_s) + p_d\theta(p_s - p_d) \\
&+(\pi_d + \pi_p + \pi_s + p_d\eta_{d2} + p_s\eta_{s2} - a_1\sigma)G + (\pi_g + p_d\eta_{d1} + p_s\eta_{s1} + a_1\epsilon - a_2\mu)\chi \\
&+a_1(\gamma_1 D + \gamma_2 P + \gamma_3 S) + a_2(\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A) \\
&-\frac{u_d D^2 + u_p P^2 + u_s S^2 + u_g A^2}{2}.
\end{aligned}
$$

Comparing the coefficients on the left and right side of the above equation for G, χ and the constant term, we obtain

$$
\begin{aligned}
a_{c1} &= \frac{\pi_d + \pi_p + \pi_s + \pi_d\eta_{d2} + \pi_s\eta_{s2}}{r+\sigma}, \\
a_{c2} &= \frac{\lambda_3}{(r+\sigma)(r+\mu)}\Big[\epsilon(\pi_d + \pi_p + \pi_s + p_d\eta_{d2} + p_s\eta_{s2}) + (r+\sigma)(\pi_g + p_d\eta_{d1} + p_s\eta_{s1})\Big], \\
a_{c3} &= pol_c.
\end{aligned}
$$

□

On the one hand, the optimal matching efforts increase with the marginal impact coefficients of the CSR goodwill, which suggests as the size of emergency resource sharing market increases, each player is more willing to participate in resource sharing activities. The optimal regulation effort increases not only with the marginal impact coefficients of the CSR goodwill but also with the marginal impact coefficient of the idle emergency resource sharing level. It indicates that the government is willing to put more energy into regulating the market if the resource sharing market increases and enterprises fulfill their corporate social responsibilities. Furthermore, the optimal efforts amplify with their marginal impact coefficients, which means the efforts made by each party in the process of emergency supplies allocation increase with the units of the efforts. Moreover, the optimal efforts demonstrate the same tendency to the overflow effect coefficients $\eta_{s1}, \eta_{d1}, \eta_{s2}, \eta_{d2}$. That is, as the amount of shared resources and the level of CSR goodwill increase, the players win more trust from the public and receive greater demand from the market. Therefore, they are more willing to engage in resource sharing activities.

On the other hand, the optimal efforts D^c, P^c, S^c, A^c decrease with σ. This indicates that elements such as public awareness and policy change not only adversely affect CSR goodwill but also diminish the readiness of suppliers, platforms, demanders, and the government to engage in resource-sharing activities. Similarly, these efforts show a negative relation with μ. Thus, the timeliness of emergency supplies inhibits the effectiveness of resource sharing. In addition, the optimal matching efforts D^c, P^c, S^c decrease with the matching cost coefficients u_d, u_p, u_s, and the optimal regulation effort A^c decreases with the regulation cost coefficients u_g, which means reducing the matching and regulation cost improves their enthusiasm to participate in resource sharing activities. Lastly, the optimal performance value of the entire system positively relates to the idle emergency resource sharing level and the CSR goodwill.

For the sensitivity analysis of the key parameters, the readers can refer to Table 3.

Table 3. Sensitivity analysis of key parameters.

	p_s	p_d	η_{s1}	η_{s2}	η_{d1}	η_{d2}	ω	c	μ	σ	u_d	u_p	u_s	u_g
D^c	↗	↗	↗	↗	↗	↗	—	—	↘	↘	↘	—	—	—
P^c	↗	↗	↗	↗	↗	↗	—	—	↘	↘	—	↘	—	—
S^c	↗	↗	↗	↗	↗	↗	—	—	↘	↘	—	—	↘	—
A^c	↗	↗	↗	↗	↗	↗	—	—	↘	↘	—	—	—	↘
$\lim_{t\to\infty} \mathrm{E}[\chi^c(t)]$	↗	↗	↗	↗	↗	↗	—	—	↘	↘	↘	↘	↘	↘
$\lim_{t\to\infty} \mathrm{E}[G^c(t)]$	↗	↗	↗	↗	↗	↗	—	—	↘	↘	↘	↘	↘	↘

↗ represents positive influence, ↘ represents negative influence, — represents no influence

Proposition 1. *(1) The expectation and variance of the emergency resource sharing level are*

$$E[\chi^c(t)] = e^{-\mu t}\chi_0 - \frac{\Delta_1^c}{\mu}e^{-\mu t} + \frac{\Delta_1^c}{\mu},$$

$$Var[\chi^c(t)] = \frac{\rho^2}{\mu}\chi_0\left(e^{-\mu t} - e^{-2\mu t}\right) + \frac{\Delta_1^c\rho^2}{2\mu^2}\left(1 - 2e^{-\mu t} + e^{-2\mu t}\right),$$

$$\lim_{t\to\infty} E[\chi^c(t)] = \frac{\Delta_1^c}{\mu}, \quad \lim_{t\to\infty} Var[\chi^c(t)] = \frac{\Delta_1^c\rho^2}{2\mu^2},$$

where $\Delta_1^c = \lambda_1 D^c + \lambda_2 P^c + \lambda_3 S^c + \lambda_4 A^c$.

(2) The expectation and variance of the CSR goodwill are

$$E[G^c(t)] = e^{-\sigma t}(G_0 + \chi_0) + \frac{\Delta_2^c + \epsilon\chi^c}{\sigma}(1 - e^{-\sigma t}),$$

$$Var[G^c(t)] = \frac{\delta^2}{\sigma}(G_0 + \chi_0)\left(e^{-\sigma t} - e^{-2\sigma t}\right) + \frac{(\Delta_2^c + \epsilon\chi^c)\delta^2}{2\sigma^2}\left(1 - 2e^{-\sigma t} + e^{-2\sigma t}\right),$$

$$\lim_{t\to\infty} E[G^c(t)] = \frac{\mu\Delta_2^c + \epsilon\Delta_1^c}{\mu\sigma}, \quad \lim_{t\to\infty} Var[G^c(t)] = \frac{2\mu^2\delta^2\Delta_2^c + \epsilon\rho^2\delta^2\Delta_1^c}{4\mu^2\sigma^2},$$

where $\Delta_2^c = \gamma_1 D^c + \gamma_2 P^c + \gamma_3 S^c$.

Proof. Substituting (10) into (9), the optimal emergency resource sharing level $\chi^c(t)$ has the form

$$\begin{cases} \mathrm{d}\chi^c(t) = (\Delta_1^c - \mu\chi(t))\mathrm{d}t + \rho\sqrt{\chi(t)}\mathrm{d}z_1(t), \\ \chi^c(0) = \chi_0 > 0, \\ \Delta_1^c = \lambda_1 D^c + \lambda_2 P^c + \lambda_3 S^c + \lambda_4 A^c. \end{cases}$$

Assuming $f(t,x) = e^{\mu t}x$, in terms of Lemma 2, $f(t,\chi^c(t))$ turns to

$$\begin{aligned} \mathrm{d}f(t,\chi^c(t)) &= f_t(t,\chi^c(t)) + f_x(t,\chi^c(t))\mathrm{d}\chi^c(t) + \frac{f_{xx}\chi^c(t)}{2}(\mathrm{d}\chi^c(t))^2 \\ &= \mu e^{\mu t}\chi^c(t)\mathrm{d}t + e^{\mu t}\Big((-\mu\chi^c(t)+\Delta_1^c)\mathrm{d}t + \rho\sqrt{\chi^c(t)}\mathrm{d}z_1(t)\Big) \\ &= \Delta_1^c e^{\mu t}\mathrm{d}t + \rho e^{\mu t}\sqrt{\chi^c(t)}\mathrm{d}z_1(t). \end{aligned} \tag{14}$$

Integrating (14), one has

$$\begin{aligned} e^{\mu t}\chi^c(t) &= \chi^c(0) + \int_0^t \Delta_1^c e^{\mu u}\mathrm{d}u + \int_0^t \rho e^{\mu u}\sqrt{\chi^c(u)}dz_1(u) \\ &= \chi_0 + \frac{\Delta_1^c}{\mu}(e^{\mu u}-1) + \rho\int_0^t e^{\mu u}\sqrt{\chi^c(u)}dz_1(u). \end{aligned} \tag{15}$$

Taking expectations on (15), one obtains

$$\mathrm{E}[\chi^c(t)] = e^{-\mu t}\chi_0 - \frac{\Delta_1^c}{\mu}e^{-\mu t} + \frac{\Delta_1^c}{\mu}. \tag{16}$$

For $\mu > 0$, one has

$$\lim_{t\to\infty}\mathrm{E}[\chi^c(t)] = \frac{\Delta_1^c}{\mu}.$$

Assuming $Y(t) = e^{\mu t}\chi^c(t)$. Based on (14), one has

$$\mathrm{d}Y(t) = \Delta_1^c e^{\mu t}\mathrm{d}t + \rho e^{\mu t}\sqrt{\chi^c(t)}\mathrm{d}\omega(t) = \Delta_1^c e^{\mu t}\mathrm{d}t + \rho e^{\frac{\mu t}{2}}\sqrt{Y(t)}\mathrm{d}z_1(t).$$

According to Lemma 2, it has

$$\mathrm{d}Y^2(t) = 2Y(t)\mathrm{d}Y(t) + \mathrm{d}Y(t)\mathrm{d}Y(t) = 2\Delta_1^c e^{\mu t}Y(t)\mathrm{d}t + 2\rho e^{\frac{\mu t}{2}}Y^{\frac{3}{2}}(t)\mathrm{d}\omega(t) + \rho^2 e^{\mu t}Y(t)\mathrm{d}t. \tag{17}$$

Integrating (17), one obtains

$$Y^2(t) = \chi_0^2 + (2\Delta_1^c + \rho^2)\int_0^t e^{\mu u}Y(u)\mathrm{d}u + 2\rho\int_0^t e^{\frac{\mu u}{2}}Y^{\frac{3}{2}}(u)\mathrm{d}z_1(u). \tag{18}$$

Taking mathematical expectations on (15) and (18), it becomes

$$\mathrm{E}[Y(t)] = \chi_0 + \frac{\Delta_1^c}{\mu}(e^{\mu t-1})$$

and

$$\mathrm{E}[Y^2(t)] = \chi_0^2 + \frac{2\Delta_1^c+\rho^2}{\mu}\left(\chi_0 - \frac{\Delta_1^c}{\mu}\right)(e^{\mu t}-1) + \frac{\tau^c(2\Delta_1^c+\rho^2)}{2\mu^2}\left(e^{2\mu t}-1\right).$$

Since $Y(t) = e^{\mu t}\chi^c(t)$, one has

$$\mathrm{E}[(\chi^c)^2(t)] = e^{-2\mu t}\chi_0^2 + \frac{\tau^c(2\Delta_1^c+\rho^2)}{2\mu^2}\left(1-e^{-2\mu t}\right) + \frac{2\Delta_1^c+\rho^2}{\mu}\left(\chi_0 - \frac{\Delta_1^c}{\mu}\right)\left(e^{-\mu t}-e^{-2\mu t}\right). \tag{19}$$

Thus, in terms of (16) and (19), one has

$$\mathrm{Var}[\chi^c(t)] = \frac{\rho^2}{\mu}\chi_0\left(e^{-\mu t}-e^{-2\mu t}\right) + \frac{\Delta_1^c\rho^2}{2\mu^2}\left(1-2e^{-\mu t}+e^{-2\mu t}\right)$$

and

$$\lim_{t\to\infty} \mathrm{Var}[\chi^c(t)] = \frac{\Delta_1^c \rho^2}{2\mu^2}.$$

□

Proposition 1 indicates that the expected idle emergency resource sharing level is more substantial the more inputs it receives from the players. Both the limits of expectation and the variance decrease with the natural decay rate, and the limit of variance escalates with random disturbances. Compared with the idle emergency resource sharing level, the expectation and the variance of the CSR goodwill not only interrelate with Δ_2^c but also bear on Δ_1^c. The corresponding limitations become greater as the efforts from Δ_1^c and Δ_2^c increase, but decrease with the nature decay rates μ, σ.

Remark 2. *The expectation of the idle emergency resource sharing level has the property*

$$\frac{\partial E[\chi^c(t)]}{\partial t} > 0 \quad \textit{for} \quad \mu < \frac{\Delta_1^c}{\chi_0},$$
$$\frac{\partial E[\chi^c(t)]}{\partial t} < 0 \quad \textit{for} \quad \mu > \frac{\Delta_1^c}{\chi_0}.$$

5. Decentralized Decision-Making Without Cost-Sharing Contract

In the context of decentralized decision-making, the demander, platform, supplier, and government operate individually. Both parties seek to optimize their benefits and determine their efforts independently. Furthermore, there are no subsidies from either the government or supplier. Such model is referred to as "n" for convenience. Consequently, the optimal control equations have the following forms:

$$\begin{cases} \max\limits_{D} J_d^n(D) = \int_0^\infty e^{-rt}\left\{p_d D_d + \pi_d G - (\omega + c)\chi - \dfrac{u_d D^2}{2}\right\}\mathrm{d}t \\ s.t. \begin{cases} \mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z(t), & \chi(0) = \chi_0 > 0, \\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z(t), & G(0) = G_0 > 0; \end{cases} \end{cases}$$

$$\begin{cases} \max\limits_{P} J_p^n(P) = \int_0^\infty e^{-rt}\left\{c\chi + \pi_p G - \dfrac{u_p P^2}{2}\right\}\mathrm{d}t \\ s.t. \begin{cases} \mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z(t), & \chi(0) = \chi_0 > 0, \\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z(t), & G(0) = G_0 > 0; \end{cases} \end{cases}$$

$$\begin{cases} \max\limits_{S} J_s^n(S) = \int_0^\infty e^{-rt}\left\{p_s D_s + \omega\chi + \pi_s G - \dfrac{u_s S^2}{2}\right\}\mathrm{d}t \\ s.t. \begin{cases} \mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z(t), & \chi(0) = \chi_0 > 0, \\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z(t), & G(0) = G_0 > 0; \end{cases} \end{cases}$$

$$\begin{cases} \max\limits_{A} J_g^n(A) = \int_0^\infty e^{-rt}\left\{\pi_g\chi - \dfrac{u_g A^2}{2}\right\}\mathrm{d}t \\ s.t. \begin{cases} \mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z(t), & \chi(0) = \chi_0 > 0, \\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z(t), & G(0) = G_0 > 0, \end{cases} \end{cases}$$

where ρ, δ are the diffusion coefficients. $\rho\sqrt{\chi}\mathrm{d}z(t)$ dictates the amplitude of random fluctuations driven by Brownian motion $\mathrm{d}z(t)$, a common model used to describe random processes. The larger the ρ, the greater the impact of randomness on the system, meaning the volatility of the idle emergency resource sharing level, $\chi(t)$, increases. This can model the effect of unpredictable events, such as natural disasters or policy shifts, on resource allocation. Similarly, a higher value of δ means that the process $G(t)$ experiences greater volatility due to random disturbance, while a lower value of δ implies less sensitivity to random disturbances.

Theorem 4. *(1) The optimal efforts are*

$$D^n = \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+p_d\eta_{d2})+\lambda_1(r+\sigma)(p_d\eta_{d1}-\omega-c)}{u_d(r+\mu)(r+\sigma)},$$

$$P^n = \frac{\pi_p(\gamma_2(r+\mu)+\epsilon\lambda_2)+c\lambda_2(r+\sigma)}{u_p(r+\mu)(r+\sigma)},$$

$$S^n = \frac{\gamma_3(\pi_s+p_s\eta_{s2})(r+\mu)+\lambda_3(\epsilon(\pi_s+p_s\eta_{s2})+(p_s\eta_{s1}+\omega)(r+\sigma))}{u_s(r+\sigma)(r+\mu)},$$

$$A^n = \frac{\pi_g\lambda_4}{u_g(r+\mu)}.$$

(2) The optimal performance function of the demander is

$$V_d^n = \frac{\pi_d+p_d\eta_{d2}}{r+\sigma}G + \frac{\epsilon\pi_d+p_d\eta_{d1}(r+\sigma)+\epsilon p_d\eta_{d2}-(c+\omega)(r+\sigma)}{(r+\mu)(r+\sigma)}\chi + pol_{nd},$$

The optimal performance function of the platform is

$$V_p^n = \frac{\pi_p}{r+\sigma}G + \frac{\epsilon\pi_p+c(r+\sigma)}{(r+\mu)(r+\sigma)}\chi + pol_{np},$$

The optimal performance function of the supplier is

$$V_s^n = \frac{\pi_s+p_s\eta_{s2}}{r+\sigma}G + \frac{\epsilon\pi_s+p_s(\eta_{s1}(r+\sigma)+\epsilon\eta_{s2})+\omega(r+\sigma)}{(r+\mu)(r+\sigma)}\chi + pol_{ns},$$

The optimal performance function of the government is

$$V_g^n = \frac{\pi_g}{r+\mu}\chi + pol_{ng},$$

where $pol_{nd}, pol_{np}, pol_{ns}, pol_{ng}$ are polynomials.

Proof. Setting the value functions of the demander, platform, supplier, government as $V_d^n, V_p^n, V_s^n, V_g^n$, one has $V_d^n = \max_D J_d^n(D), V_p^n = \max_P J_p^n(P), V_s^n = \max_S J_s^n(S)$ and $V_g^n = \max_A J_g^n(A)$. According to the optimal control theory [60], V_d^n should observe the HJB equation,

$$\begin{aligned} rV_d^n(G,\chi) &= \max_D \Big\{ p_dD_d + \pi_dG - (\omega+c)\chi - \frac{u_dD^2}{2} + \rho\delta\sqrt{G}\sqrt{\chi}\frac{\partial^2 V_d^n(G,\chi)}{\partial G\partial\chi} \\ &+(\lambda_1D+\lambda_2P+\lambda_3S+\lambda_4A-\mu\chi)\frac{\partial V_d^n(G,\chi)}{\partial\chi} + \frac{\rho^2}{2}\chi\frac{\partial^2 V_d^n(G,\chi)}{\partial\chi^2} \\ &+(\gamma_1D+\gamma_2P+\gamma_3S+\epsilon\chi-\sigma G)\frac{\partial V_d^n(G,\chi)}{\partial G} + \frac{\delta^2}{2}G\frac{\partial^2 V_d^n(G,\chi)}{\partial G^2}\Big\}. \end{aligned} \tag{20}$$

Taking the first-order derivatives of the right-hand side of (20) with respect to D and setting it to zero, one has

$$D^n = \frac{1}{u_d}\Big[\lambda_1 \frac{\partial V_d^n(G,\chi)}{\partial \chi} + \gamma_1 \frac{\partial V_d^n(G,\chi)}{\partial G}\Big]. \tag{21}$$

Under the same manners, V_p^n, V_s^n and V_g^n satisfy the following equation:

$$\begin{aligned} rV_p^n(G,\chi) &= \max_P \Big\{ c\chi + \pi_p G - \frac{u_p P^2}{2} + \rho\delta\sqrt{G}\sqrt{\chi}\frac{\partial^2 V_p^n(G,\chi)}{\partial G \partial \chi} \\ &+ (\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi)\frac{\partial V_p^n(G,\chi)}{\partial \chi} + \frac{\rho^2}{2}\chi\frac{\partial^2 V_p^n(G,\chi)}{\partial \chi^2} \\ &+ (\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G)\frac{\partial V_p^n(G,\chi)}{\partial G} + \frac{\delta^2}{2}G\frac{\partial^2 V_p^n(G,\chi)}{\partial G^2} \Big\}, \end{aligned} \tag{22}$$

$$\begin{aligned} rV_s^n(G,\chi) &= \max_S \Big\{ p_s D_s + \omega\chi + \pi_s G - \frac{u_s S^2}{2} + \rho\delta\sqrt{G}\sqrt{\chi}\frac{\partial^2 V_s^n(G,\chi)}{\partial G \partial \chi} \\ &+ (\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi)\frac{\partial V_s^n(G,\chi)}{\partial \chi} + \frac{\rho^2}{2}\chi\frac{\partial^2 V_s^n(G,\chi)}{\partial \chi^2} \\ &+ (\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G)\frac{\partial V_s^n(G,\chi)}{\partial G} + \frac{\delta^2}{2}G\frac{\partial^2 V_s^n(G,\chi)}{\partial G^2} \Big\}, \end{aligned} \tag{23}$$

$$\begin{aligned} rV_g^n(G,\chi) &= \max_A \Big\{ \pi_g\chi - \frac{u_g A^2}{2} + \rho\delta\sqrt{G}\sqrt{\chi}\frac{\partial^2 V_g^n(G,\chi)}{\partial G \partial \chi} \\ &+ (\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi)\frac{\partial V_g^n(G,\chi)}{\partial \chi} + \frac{\rho^2}{2}\chi\frac{\partial^2 V_g^n(G,\chi)}{\partial \chi^2} \\ &+ (\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G)\frac{\partial V_g^n(G,\chi)}{\partial G} + \frac{\delta^2}{2}G\frac{\partial^2 V_g^n(G,\chi)}{\partial G^2} \Big\}, \end{aligned} \tag{24}$$

and P, S, A have the following forms:

$$\begin{aligned} P^n &= \frac{1}{u_p}\Big[\lambda_2 \frac{\partial V_p^n(G,\chi)}{\partial \chi} + \gamma_2 \frac{\partial V_p^n(G,\chi)}{\partial G}\Big], \\ S^n &= \frac{1}{u_s}\Big[\lambda_3 \frac{\partial V_s^n(G,\chi)}{\partial \chi} + \gamma_3 \frac{\partial V_s^n(G,\chi)}{\partial G}\Big], \\ A^n &= \frac{\lambda_4}{u_a}\frac{\partial V_g^n(G,\chi)}{\partial \chi}. \end{aligned} \tag{25}$$

Assume that $V_d^n, V_p^n, V_s^n, V_g^n$ observe the following ansatz,

$$\begin{aligned} V_d^n(G,\chi) &= a_{n1}G + a_{n2}\chi + a_{n3}, \\ V_p^n(G,\chi) &= b_{n1}G + b_{n2}\chi + b_{n3}, \\ V_s^n(G,\chi) &= c_{n1}G + c_{n2}\chi + c_{n3}, \\ V_g^n(G,\chi) &= d_{n1}G + d_{n2}\chi + d_{n3}, \end{aligned} \tag{26}$$

where $a_{n1} \ldots a_{n3}, b_{n1}...b_{n3}, c_{n1}...c_{n3}, d_{n1}...d_{n3}$ are constants. Substiting into (20), (22)–(24) and zeroing all the coefficients of $G^m\chi^n$, one has

$$a_{n1}=\frac{\pi_d+p_d\eta_{d2}}{r+\sigma},\quad a_{n2}=\frac{\epsilon\pi_d+p_d\eta_{d1}(r+\sigma)+\epsilon p_d\eta_{d2}-(c+\omega)(r+\sigma)}{(r+\mu)(r+\sigma)},\quad a_{n3}=pol_{nd},$$
$$b_{n1}=\frac{\pi_p}{r+\sigma},\quad b_{n2}=\frac{\epsilon\pi_p+c(r+\sigma)}{(r+\mu)(r+\sigma)},\quad b_{n3}=pol_{np},$$
$$c_{n1}=\frac{\pi_s+p_s\eta_{s2}}{r+\sigma},\quad c_{n2}=\frac{\epsilon\pi_s+p_s(\eta_{s1}(r+\sigma)+\epsilon\eta_{s2})+\omega(r+\sigma)}{(r+\mu)(r+\sigma)},\quad c_{n3}=pol_{ns},$$
$$d_{n1}=0,\quad d_{n2}=\frac{\pi_g}{r+\mu},\quad d_{n3}=pol_{ng}.$$

□

In contrast to the centralized decision-making, the optimal efforts are only specified by their own marginal profits. For example, the optimal matching effort of the demander contains λ_1, γ_1, u_d, π_d, D_d, ω, c. However, its optimal effort also concludes π_p, π_s, π_g, D_s, λ_3 in the centralized decision-making. Therefore, the optimal effort of the demander only increases with its own overflow effect, and the marginal impact of the CSR goodwill as well as the overflow effect of the idle emergency resource sharing level, decreases with the commission fee c, the resource usage fee ω and the matching cost u_d. Although the supplier's optimal matching effort has similar dynamics, it increases with the commission fee c and the resource usage fee ω charged from the demander.

For the sensitivity analysis of the key parameters, the readers can refer to Table 4.

Table 4. Sensitivity analysis of key parameters.

	p_s	p_d	η_{s1}	η_{s2}	η_{d1}	η_{d2}	ω	c	μ	σ	u_d	u_p	u_s	u_g
D^n	—	↗	—	—	↗	↗	↘	↘	↘	↘	↘	—	—	—
P^n	—	—	—	—	—	—	—	—	↘	↘	—	↘	—	—
S^n	↗	—	↗	↗	—	—	↗	—	↘	↘	—	—	↘	—
A^n	—	—	—	—	—	—	—	—	↘	—	—	—	—	↘
$\lim_{t\to\infty} \mathrm{E}[\chi^n(t)]$	↗	↗	↗	↗	↗	↗	∼	↘	↘	↘	↘	↘	↘	↘
$\lim_{t\to\infty} \mathrm{E}[G^n(t)]$	↗	↗	↗	↗	↗	↗	∼	↘	↘	↘	↘	↘	↘	↘

↗ represents positive influence, ↘ represents negative influence, — represents no influence, ∼ represents not clear.

The value functions of the demander, the platform, and the supplier are positively related to the idle emergency resource sharing level and the CSR goodwill, while the government's value function only connects with the idle emergency resource sharing level.

Proposition 2. *(1) The expectation and variance of the emergency resource sharing level are*

$$E[\chi^n(t)]=e^{-\mu t}\chi_0-\frac{\Delta_1^n}{\mu}e^{-\mu t}+\frac{\Delta_1^n}{\mu},$$
$$Var[\chi^n(t)]=\frac{\rho^2}{\mu}\chi_0\left(e^{-\mu t}-e^{-2\mu t}\right)+\frac{\Delta_1^n\rho^2}{2\mu^2}\left(1-2e^{-\mu t}+e^{-2\mu t}\right),$$
$$\lim_{t\to\infty}E[\chi^n(t)]=\frac{\Delta_1^n}{\mu},\quad \lim_{t\to\infty}Var[\chi^n(t)]=\frac{\Delta_1^n\rho^2}{2\mu^2},$$

where $\Delta_1^n=\lambda_1D^n+\lambda_2P^n+\lambda_3S^n+\lambda_4A^n$.

(2) The expectation and variance of the CSR goodwill are

$$
\begin{aligned}
&E[G^n(t)] = e^{-\sigma t}(G_0+\chi_0) + \frac{\Delta_2^n + \epsilon\chi^n}{\sigma}(1-e^{-\sigma t}),\\
&Var[G^n(t)] = \frac{\delta^2}{\sigma}(G_0+\chi_0)\left(e^{-\sigma t} - e^{-2\sigma t}\right) + \frac{(\Delta_2^n + \epsilon\chi^n)\delta^2}{2\sigma^2}\left(1 - 2e^{-\sigma t} + e^{-2\sigma t}\right),\\
&\underset{t\to\infty}{lim} E[G^n(t)] = \frac{\mu\Delta_2^n + \epsilon\Delta_1^n}{\mu\sigma}, \quad \underset{t\to\infty}{lim} Var[G^n(t)] = \frac{2\mu^2\delta^2\Delta_2^n + \epsilon\rho^2\delta^2\Delta_1^n}{4\mu^2\sigma^2},
\end{aligned}
$$

where $\Delta_2^n = \gamma_1 D^n + \gamma_2 P^n + \gamma_3 S^n$.

6. Decentralized Decision-Making with Cost-Sharing Contract

This section examines a Stackelberg primary–secondary game governed by the government and the platform. Initially, the government determines the level of its effort and the degree of subsidies ($0 < \xi_p < 1$) to allocate to the platform. Then, once the government determines its subsidies, the platform allocates the subsidy ($0 < \xi_d < 1$) to the demander. Similar to the previous model, the demander, the platform, the supplier, and the government strive to optimize their benefits and make decisions regarding their efforts independently. This model is referred to as "s" for convenience. Consequently, the optimal control equations have the following forms:

$$
\begin{cases}
\max\limits_{D} J_d^s(D) = \int_0^\infty e^{-rt}\left\{p_d D_d + \pi_d G - (\omega + c)\chi - \frac{(1-\xi_d)u_d D^2}{2}\right\}\mathrm{d}t\\
s.t.\begin{cases}\mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z(t), & \chi(0) = \chi_0 > 0,\\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z(t), & G(0) = G_0 > 0;\end{cases}
\end{cases}
$$

$$
\begin{cases}
\max\limits_{P} J_p^s(P) = \int_0^\infty e^{-rt}\left\{c\chi + \pi_p G - \frac{(1-\xi_p)u_p P^2}{2} - \frac{\xi_d u_d D^2}{2}\right\}\mathrm{d}t\\
s.t.\begin{cases}\mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z(t), & \chi(0) = \chi_0 > 0,\\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z(t), & G(0) = G_0 > 0;\end{cases}
\end{cases}
$$

$$
\begin{cases}
\max\limits_{S} J_s^s(S) = \int_0^\infty e^{-rt}\left\{p_s D_s + \omega\chi + \pi_s G - \frac{u_s S^2}{2}\right\}\mathrm{d}t\\
s.t.\begin{cases}\mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z(t), & \chi(0) = \chi_0 > 0,\\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z(t), & G(0) = G_0 > 0;\end{cases}
\end{cases}
$$

$$
\begin{cases}
\max\limits_{A} J_g^s(A) = \int_0^\infty e^{-rt}\left\{\pi_g\chi - \frac{u_g A^2}{2} - \frac{\xi_p u_p P^2}{2}\right\}\mathrm{d}t\\
s.t.\begin{cases}\mathrm{d}\chi(t) = \left[\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi\right]\mathrm{d}t + \rho\sqrt{\chi}\mathrm{d}z(t), & \chi(0) = \chi_0 > 0,\\ \mathrm{d}G(t) = \left[\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G\right]\mathrm{d}t + \delta\sqrt{G}\mathrm{d}z(t), & G(0) = G_0 > 0,\end{cases}
\end{cases}
$$

where ρ, δ are the diffusion coefficients. $\rho\sqrt{\chi}\mathrm{d}z(t)$ dictates the amplitude of random fluctuations driven by Brownian motion $\mathrm{d}z(t)$, a common model used to describe random processes. The larger the ρ, the greater the impact of randomness on the system, meaning the volatility of the idle emergency resource sharing level, $\chi(t)$, increases. This can model the effect of unpredictable events, such as natural disasters or policy shifts, on resource allocation. Similarly, a higher value of δ means that the process $G(t)$ experiences greater volatility due to random disturbance, while a lower value of δ implies less sensitivity to random disturbances.

Theorem 5. *(1) The optimal efforts are*

$$D^s = \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+2\pi_p+p_d\eta_{d2})+\lambda_1(r+\sigma)(p_d\eta_{d1}-\omega+c)}{2u_d(r+\mu)(r+\sigma)},$$

$$P^s = \frac{\pi_p(\gamma_2(r+\mu)+\epsilon\lambda_2)+\lambda_2(c+2\pi_g)(r+\sigma)}{2u_p(r+\mu)(r+\sigma)},$$

$$S^s = \frac{\gamma_3(\pi_s+p_s\eta_{s2})(r+\mu)+\lambda_3(\epsilon(\pi_s+p_s\eta_{s2})+(p_s\eta_{s1}+\omega)(r+\sigma))}{u_s(r+\sigma)(r+\mu)},$$

$$A^s = \frac{\pi_g\lambda_4}{u_g(r+\mu)}.$$

(2) The optimal cost sharing rates are

$$\xi_d = -\frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(p_d\eta_{d2}+\pi_d+2\pi_p)+\lambda_1(r+\sigma)(p_d\eta_{d1}+c-\omega)}{(\gamma_1(r+\mu)+\epsilon\lambda_1)(p_d\eta_{d2}+\pi_d+2\pi_p)+\lambda_1(r+\sigma)(p_d\eta_{d1}-3c-\omega)},$$

$$\xi_p = -\frac{\pi_p(\gamma_2(r+\mu)+\epsilon\lambda_2)+\lambda_2(c-2\pi_g)(r+\sigma)}{\pi_p(\gamma_2(r+\mu)+\epsilon\lambda_2)+\lambda_2(c+2\pi_g)(r+\sigma)}.$$

(3) The optimal performance function of the demander is

$$V_d^s = \frac{\pi_d+p_d\eta_{d2}}{r+\sigma}G + \frac{\epsilon\pi_d+p_d\eta_{d1}(r+\sigma)+\epsilon p_d\eta_{d2}-(c+\omega)(r+\sigma)}{(r+\mu)(r+\sigma)}\chi + pol_{sd},$$

The optimal performance function of the platform is

$$V_p^s = \frac{\pi_p}{r+\sigma}G + \frac{\epsilon\pi_p+c(r+\sigma)}{(r+\mu)(r+\sigma)}\chi + pol_{sp},$$

The optimal performance function of the supplier is

$$V_s^s = \frac{\pi_s+p_s\eta_{s2}}{r+\sigma}G + \frac{\epsilon\pi_s+p_s(\eta_{s1}(r+\sigma)+\epsilon\eta_{s2})+\omega(r+\sigma)}{(r+\mu)(r+\sigma)}\chi + pol_{ss},$$

The optimal performance function of the government is

$$V_a^s = \frac{\pi_g}{r+\mu}\chi + pol_{sa},$$

where $pol_{sd}, pol_{sp}, pol_{ss}, pol_{sa}$ are polynomials.

Proof. Setting the value function of the demander, platform, supplier and government as $V_d^s, V_p^s, V_s^s, V_g^s$, one has $V_d^s = \max_D J_d^s(D), V_p^s = \max_P J_p^s(P), V_s^s = \max_S J_s^s(S)$ and $V_g^s = \max_A J_g^n(A)$. According to the optimal control theory [60], V_d^s should observe the HJB equation

$$\begin{aligned} rV_d^s(G,\chi) &= \max_D \Big\{ p_d D_d + \pi_d G - (\omega+c)\chi - \frac{(1-\xi_d)u_d D^2}{2} + \rho\delta\sqrt{G}\sqrt{\chi}\frac{\partial^2 V_d^s(G,\chi)}{\partial G\partial\chi} \\ &+(\lambda_1 D+\lambda_2 P+\lambda_3 S+\lambda_4 A-\mu\chi)\frac{\partial V_d^s(G,\chi)}{\partial\chi} + \frac{\rho^2}{2}\chi\frac{\partial^2 V_d^s(G,\chi)}{\partial\chi^2} \\ &+(\gamma_1 D+\gamma_2 P+\gamma_3 S+\epsilon\chi-\sigma G)\frac{\partial V_d^s(G,\chi)}{\partial G} + \frac{\delta^2}{2}G\frac{\partial^2 V_d^s(G,\chi)}{\partial G^2}\Big\}. \end{aligned} \tag{27}$$

Taking the first-order derivatives of the right-hand side of (27) with respsect to D and setting it to zero, one has

$$D^s = \frac{1}{u_d(1-\xi_d)}\Big[\lambda_1 \frac{\partial V_d^s(G,\chi)}{\partial \chi} + \gamma_1 \frac{\partial V_d^s(G,\chi)}{\partial G}\Big]. \tag{28}$$

Similarly, V_s^s has the form

$$\begin{aligned} rV_s^s(G,\chi) &= \max_S \Big\{ p_s D_s + \omega\chi + \pi_s G - \frac{u_s S^2}{2} + \rho\delta\sqrt{G}\sqrt{\chi}\frac{\partial^2 V_s^s(G,\chi)}{\partial G \partial \chi} \\ &+(\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi)\frac{\partial V_s^s(G,\chi)}{\partial \chi} + \frac{\rho^2}{2}\chi\frac{\partial^2 V_s^s(G,\chi)}{\partial \chi^2} \\ &+(\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G)\frac{\partial V_s^s(G,\chi)}{\partial G} + \frac{\delta^2}{2}G\frac{\partial^2 V_s^s(G,\chi)}{\partial G^2} \Big\}, \end{aligned} \tag{29}$$

and S has the form

$$S^s = \frac{1}{u_s}\Big[\lambda_3 \frac{\partial V_s^s(G,\chi)}{\partial \chi} + \gamma_3 \frac{\partial V_s^s(G,\chi)}{\partial G}\Big]. \tag{30}$$

Moreover, the platform's and government's value functions read

$$\begin{aligned} rV_p^s(G,\chi) &= \max_P \Big\{ c\chi + \pi_p G - \frac{(1-\xi_p)u_p P^2}{2} + \rho\delta\sqrt{G}\sqrt{\chi}\frac{\partial^2 V_p^s(G,\chi)}{\partial G \partial \chi} \\ &+(\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi)\frac{\partial V_p^s(G,\chi)}{\partial \chi} + \frac{\rho^2}{2}\chi\frac{\partial^2 V_p^s(G,\chi)}{\partial \chi^2} \\ &+(\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G)\frac{\partial V_p^s(G,\chi)}{\partial G} + \frac{\delta^2}{2}G\frac{\partial^2 V_p^s(G,\chi)}{\partial G^2} \Big\}, \end{aligned} \tag{31}$$

and

$$\begin{aligned} rV_g^n(G,\chi) &= \max_A \Big\{ \pi_g\chi - \frac{u_g A^2}{2} - \frac{\xi_p u_p P^2}{2} + \rho\delta\sqrt{G}\sqrt{\chi}\frac{\partial^2 V_g^s(G,\chi)}{\partial G \partial \chi} \\ &+(\lambda_1 D + \lambda_2 P + \lambda_3 S + \lambda_4 A - \mu\chi)\frac{\partial V_g^s(G,\chi)}{\partial \chi} + \frac{\rho^2}{2}\chi\frac{\partial^2 V_g^s(G,\chi)}{\partial \chi^2} \\ &+(\gamma_1 D + \gamma_2 P + \gamma_3 S + \epsilon\chi - \sigma G)\frac{\partial V_g^s(G,\chi)}{\partial G} + \frac{\delta^2}{2}G\frac{\partial^2 V_g^s(G,\chi)}{\partial G^2} \Big\}. \end{aligned} \tag{32}$$

Substituting (28) into (31) and taking the first-order derivatives of the right-hand side of (31) with respect to P, ξ_d, a straightforward computation indicates

$$P^s = \frac{1}{u_p(1-\xi_p)}\Big[\lambda_2 \frac{\partial V_p^s(G,\chi)}{\partial \chi} + \gamma_2 \frac{\partial V_p^s(G,\chi)}{\partial G}\Big] \tag{33}$$

and

$$\xi_d = \frac{-\lambda_1 \frac{\partial V_d^s(G,\chi)}{\partial \chi} + 2\lambda_1 \frac{\partial V_p^s(G,\chi)}{\partial \chi} - \gamma_1 \frac{\partial V_d^s(G,\chi)}{\partial G} + 2\gamma_1 \frac{\partial V_p^s(G,\chi)}{\partial G}}{\lambda_1 \frac{\partial V_d^s(G,\chi)}{\partial \chi} + 2\lambda_1 \frac{\partial V_p^s(G,\chi)}{\partial \chi} + \gamma_1 \frac{\partial V_d^s(G,\chi)}{\partial G} + 2\gamma_1 \frac{\partial V_p^s(G,\chi)}{\partial G}}. \tag{34}$$

Substituting (28), (30), (33) and (34) into (32) and taking the first-order derivatives of the right-hand side of (32) with respect to A, ξ_p, one has

$$
A^s = \frac{\lambda_4}{u_a}\frac{\partial V_g^s(G,\chi)}{\partial \chi},
$$
$$
\xi_p = \frac{2\lambda_2 \frac{\partial V_g^s(G,\chi)}{\partial \chi} - \lambda_2 \frac{\partial V_p^s(G,\chi)}{\partial \chi} + 2\gamma_2 \frac{\partial V_g^s(G,\chi)}{\partial G} - \gamma_2 \frac{\partial V_g^s(G,\chi)}{\partial G}}{2\lambda_2 \frac{\partial V_g^s(G,\chi)}{\partial \chi} + \lambda_2 \frac{\partial V_p^s(G,\chi)}{\partial \chi} + 2\gamma_2 \frac{\partial V_g^s(G,\chi)}{\partial G} + \gamma_2 \frac{\partial V_g^s(G,\chi)}{\partial G}}. \tag{35}
$$

Assuming $V_s^s, V_p^s, V_s^s, V_g^s$ observe the following ansatz:

$$
\begin{aligned}
V_d^s(G,\chi) &= a_{s1}G + a_{s2}\chi + a_{s3},\\
V_p^s(G,\chi) &= b_{s1}G + b_{s2}\chi + b_{s3},\\
V_s^s(G,\chi) &= c_{s1}G + c_{s2}\chi + c_{s3},\\
V_g^s(G,\chi) &= d_{s1}G + d_{s2}\chi + d_{s3},
\end{aligned} \tag{36}
$$

where $a_{s1} \ldots a_{s3}, b_{s1} \ldots b_{s3}, c_{s1} \ldots c_{s3}, d_{s1} \ldots d_{s3}$ are constants. Substituting them into (27), (29), (31), (32) and zeroing all the coefficients of $G^m\chi^n$, one has

$$
\begin{aligned}
a_{s1} &= \frac{\pi_d + p_d\eta_{d2}}{r+\sigma}, & a_{s2} &= \frac{\epsilon\pi_d + p_d\eta_{d1}(r+\sigma) + \epsilon p_d\eta_{d2} - (c+\omega)(r+\sigma)}{(r+\mu)(r+\sigma)}, & a_{s3} &= pol_{sd},\\
b_{s1} &= \frac{\pi_p}{r+\sigma}, & b_{s2} &= \frac{\epsilon\pi_p + c(r+\sigma)}{(r+\mu)(r+\sigma)}, & b_{s3} &= pol_{sp},\\
c_{s1} &= \frac{\pi_s + p_s\eta_{s2}}{r+\sigma}, & c_{s2} &= \frac{\epsilon\pi_s + p_s(\eta_{s1}(r+\sigma) + \epsilon\eta_{s2}) + \omega(r+\sigma)}{(r+\mu)(r+\sigma)}, & c_{s3} &= pol_{ss},\\
d_{s1} &= 0, & d_{s2} &= \frac{\pi_g}{r+\mu}, & d_{s3} &= pol_{sg}.
\end{aligned}
$$

□

The cost-sharing contract between the platform and the government allows the platform's optimal matching effort to admit the government's marginal profits. It is obvious that the platform's optimal matching effort is greater than that of the decentralized decision-making without cost-sharing contract. This is due to the fact that as the emergency resource sharing market expands, the government gains more profits from it and put more subsidies to the platform. Similarly, the cost-sharing contract between the platform and the demander allows the demander's optimal matching effort to admit the platform's marginal profits. However, we cannot figure out whether the demander's optimal effort increases or not from the result directly, as the demander has to pay the commission fee to the platform whether there is contract or not.

In this sense, the cost-sharing agreements encourage the demander and the platform to implement resource sharing activities of emergency supplies, thereby fulfilling their corporate social responsibilities. It is worth noting that, compared to the case without cost-sharing contract, the government's and the supplier's optimal efforts remain the same. Different from the dynamic of the government, the supplier's optimal effort does not change since it does not provide or receive subsidy from or to the other parties. In spite of that, the government-led cost-sharing contract does not undermine its own profit.

For the sensitivity analysis of the key parameters, the readers can refer to Table 5.

Table 5. Sensitivity analysis of key parameters.

	p_s	p_d	η_{s1}	η_{s2}	η_{d1}	η_{d2}	ω	c	μ	σ	u_d	u_p	u_s	u_g
D^s	—	$\nearrow$	—	—	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$	—	—	—
P^s	—	—	—	—	—	—	—	$\nearrow$	$\searrow$	$\searrow$	—	$\searrow$	—	—
S^s	$\nearrow$	—	$\nearrow$	$\nearrow$	—	—	$\nearrow$	—	$\searrow$	$\searrow$	—	—	$\searrow$	—
A^s	—	—	—	—	—	—	—	—	—	—	—	—	—	$\searrow$
$\lim_{t\to\infty} \mathrm{E}[\chi^s(t)]$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\sim$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$
$\lim_{t\to\infty} \mathrm{E}[G^s(t)]$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\sim$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$

$\nearrow$ represents positive influence, $\searrow$ represents negative influence, — represents no influence, $\sim$ represents not clear.

Proposition 3. *(1) The expectation and variance of the emergency resource sharing level are*

$$E[\chi^s(t)] = e^{-\mu t}\chi_0 - \frac{\Delta_1^s}{\mu}e^{-\mu t} + \frac{\Delta_1^s}{\mu},$$

$$Var[\chi^s(t)] = \frac{\rho^2}{\mu}\chi_0\left(e^{-\mu t} - e^{-2\mu t}\right) + \frac{\Delta_1^s\rho^2}{2\mu^2}\left(1 - 2e^{-\mu t} + e^{-2\mu t}\right),$$

$$\lim_{t\to\infty} E[\chi^s(t)] = \frac{\Delta_1^s}{\mu}, \quad \lim_{t\to\infty} Var[\chi^s(t)] = \frac{\Delta_1^s\rho^2}{2\mu^2},$$

where $\Delta_1^s = \lambda_1 D^s + \lambda_2 P^s + \lambda_3 S^s + \lambda_4 A^s$.

(2) The expectation and variance of the CSR goodwill are

$$E[G^s(t)] = e^{-\sigma t}(G_0 + \chi_0) + \frac{\Delta_2^s + \epsilon\chi^s}{\sigma}(1 - e^{-\sigma t}),$$

$$Var[G^s(t)] = \frac{\delta^2}{\sigma}(G_0 + \chi_0)\left(e^{-\sigma t} - e^{-2\sigma t}\right) + \frac{(\Delta_2^s + \epsilon\chi^s)\delta^2}{2\sigma^2}\left(1 - 2e^{-\sigma t} + e^{-2\sigma t}\right),$$

$$\lim_{t\to\infty} E[G^s(t)] = \frac{\mu\Delta_2^s + \epsilon\Delta_1^s}{\mu\sigma}, \quad \lim_{t\to\infty} Var[G^s(t)] = \frac{2\mu^2\delta^2\Delta_2^s + \epsilon\rho^2\delta^2\Delta_1^s}{4\mu^2\sigma^2},$$

where $\Delta_2^s = \gamma_1 D^s + \gamma_2 P^s + \gamma_3 S^s$.

The cooperative attitudes primarily rely on the government's and the platform's willingness. Meanwhile, the supplier is unlikely to reject the Stackelberg game, assuming they possess the rationality to acknowledge its advantages. Considering the optimal matching efforts D^n, D^s and the subsidy ξ_d, they should observe

$$D^n = \frac{(\gamma_1(r+\mu)\epsilon\lambda_1)(\pi_d + p_d\eta_{d2}) + \lambda_1(r+\sigma)(p_d\eta_{d1} - \omega - c)}{u_d(r+\mu)(r+\sigma)} > 0,$$

$$D^s = \frac{(\gamma_1(r+\mu) + \epsilon\lambda_1)(\pi_d + 2\pi_p + p_d\eta_{d2}) + \lambda_1(r+\sigma)(p_d\eta_{d1} - \omega + c)}{2u_d(r+\mu)(r+\sigma)} > 0,$$

$$0 < \xi_d = -\frac{(\gamma_1(r+\mu) + \epsilon\lambda_1)(p_d\eta_{d2} + \pi_d + 2\pi_p) + \lambda_1(r+\sigma)(p_d\eta_{d1} + c - \omega)}{(\gamma_1(r+\mu) + \epsilon\lambda_1)(p_d\eta_{d2} + \pi_d + 2\pi_p) + \lambda_1(r+\sigma)(p_d\eta_{d1} - 3c - \omega)} < 1,$$

$$0 < \xi_p = -\frac{\pi_p(\gamma_2(r+\mu) + \epsilon\lambda_2) + \lambda_2(c - 2\pi_g)(r+\sigma)}{\pi_p(\gamma_2(r+\mu) + \epsilon\lambda_2) + \lambda_2(c + 2\pi_g)(r+\sigma)} < 1,$$

that comes

$$\omega < \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+p_d\eta_{d2})+\lambda_1(r+\sigma)(p_d\eta_{d1}-c)}{\lambda_1(r+\sigma)},$$

$$\omega < \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+p_d\eta_{d2}+2\pi_p)+\lambda_1(r+\sigma)(p_d\eta_{d1}+c)}{\lambda_1(r+\sigma)},$$

$$\omega > \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+p_d\eta_{d2}-2\pi_p)+\lambda_1(r+\sigma)(p_d\eta_{d1}-3c)}{\lambda_1(r+\sigma)}.$$

Setting

$$W_1 = \max\Big\{0, \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+p_d\eta_{d2}-2\pi_p)+\lambda_1(r+\sigma)(p_d\eta_{d1}-3c)}{\lambda_1(r+\sigma)}\Big\},$$

$$W_2 = \max\Big\{\frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+p_d\eta_{d2})+\lambda_1(r+\sigma)(p_d\eta_{d1}-c)}{\lambda_1(r+\sigma)}, \frac{(\gamma_3(r+\mu)+\epsilon\lambda_3)(\pi_d+\pi_p+p_d\eta_{d2})+\lambda_3(r+\sigma)(\pi_g+p_d\eta_{d1})}{\lambda_3(r+\sigma)}\Big\},$$

ω has the following relation:

$$W_1 < \omega < W_2. \tag{37}$$

7. Comparative Analysis and Numerical Simulations

7.1. Comparative Analysis

Above, we discussed three emergency resource sharing strategies. Next, we compare and analyze the obtained results. To save space, we only present the complete proof of the first corollary.

Corollary 1. *The optimal efforts in each models have the following relations:*

$$D^c > D^s > D^n,$$

$$P^c > P^s > P^n,$$

$$S^c > S^s = S^n,$$

$$A^c > A^s = A^n.$$

Proof. The optimal matching efforts for the demander are

$$D^c = \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+\pi_p+\pi_s+p_d\eta_{d2}+p_s\eta_{s2})+\lambda_1(r+\sigma)(\pi_g+p_d\eta_{d1}+p_s\eta_{s1})}{u_d(r+\sigma)(r+\mu)},$$

$$D^n = \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+p_d\eta_{d2})+\lambda_1(r+\sigma)(p_d\eta_{d1}-\omega-c)}{u_d(r+\mu)(r+\sigma)},$$

$$D^s = \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+2\pi_p+p_d\eta_{d2})+\lambda_1(r+\sigma)(p_d\eta_{d1}-\omega+c)}{2u_d(r+\mu)(r+\sigma)}.$$

It is clear that $D^c > D^s$, $D^c > D^n$. Hence, we only have to consider $D^s > D^n$. On the basis of (37), we obtain

$$
\begin{aligned}
D^s - D^n &= \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(-\pi_d+2\pi_p-p_d\eta_{d2})+\lambda_1(r+\sigma)(-p_d\eta_{d1}+3c+\omega)}{2u_d(r+\mu)(r+\sigma)} \\
&= \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(-\pi_d+2\pi_p-p_d\eta_{d2})+\lambda_1(r+\sigma)(-p_d\eta_{d1}+3c)}{2u_d(r+\mu)(r+\sigma)} + \frac{\lambda_1(r+\sigma)\omega}{2u_d(r+\mu)(r+\sigma)} \\
&> \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(-\pi_d+2\pi_p-p_d\eta_{d2})+\lambda_1(r+\sigma)(-p_d\eta_{d1}+3c)}{2u_d(r+\mu)(r+\sigma)} \\
&+ \frac{\lambda_1}{2u_d(r+\mu)} \frac{(\gamma_1(r+\mu)+\epsilon\lambda_1)(\pi_d+p_d\eta_{d2}-2\pi_p)+\lambda_1(r+\sigma)(p_d\eta_{d1}-3c)}{\lambda_1(r+\sigma)} \\
&= 0.
\end{aligned}
$$

The optimal matching efforts for the platform are

$$
\begin{aligned}
P^c &= \frac{(\gamma_2(r+\mu)+\epsilon\lambda_2)(\pi_d+\pi_p+\pi_s+p_d\eta_{d2}+p_s\eta_{s2})+\lambda_2(r+\sigma)(\pi_g+p_d\eta_{d1}+p_s\eta_{s1})}{u_p(r+\sigma)(r+\mu)}, \\
P^s &= \frac{\pi_p(\gamma_2(r+\mu)+\epsilon\lambda_2)+\lambda_2(c+2\pi_g)(r+\sigma)}{2u_p(r+\mu)(r+\sigma)}, \\
P^n &= \frac{\pi_p(\gamma_2(r+\mu)+\epsilon\lambda_2)+c\lambda_2(r+\sigma)}{u_p(r+\mu)(r+\sigma)}.
\end{aligned}
$$

It is obvious that $P^c > P^s > P^n$.

The optimal matching efforts for the supplier are

$$
\begin{aligned}
S^c &= \frac{(\gamma_3(r+\mu)+\epsilon\lambda_3)(\pi_d+\pi_p+\pi_s+p_d\eta_{d2}+p_s\eta_{s2})+\lambda_3(r+\sigma)(\pi_g+p_d\eta_{d1}+p_s\eta_{s1})}{u_s(r+\sigma)(r+\mu)}, \\
S^s &= \frac{(\gamma_3(r+\mu)+\epsilon\lambda_3)(\pi_s+p_s\eta_{s2})+\lambda_3(r+\sigma)(p_s\eta_{s1}+\omega)}{u_s(r+\sigma)(r+\mu)}, \\
S^n &= \frac{(\gamma_3(r+\mu)+\epsilon\lambda_3)(\pi_s+p_s\eta_{s2})+\lambda_3(r+\sigma)(p_s\eta_{s1}+\omega)}{u_s(r+\sigma)(r+\mu)}.
\end{aligned}
$$

It is clear that $S^s = S^n$. Here, we only have to verify $S^c > S^n$. Based on Constraint (37), we have

$$
\begin{aligned}
S^c - S^n &= \frac{(\gamma_3(r+\mu)+\epsilon\lambda_3)(\pi_d+\pi_p+p_d\eta_{d2})+\lambda_3(r+\sigma)(\pi_g+p_d\eta_{d1})}{u_s(r+\sigma)(r+\mu)} - \frac{\lambda_3(r+\sigma)\omega}{u_s(r+\sigma)(r+\mu)} \\
&> \frac{(\gamma_3(r+\mu)+\epsilon\lambda_3)(\pi_d+\pi_p+p_d\eta_{d2})+\lambda_3(r+\sigma)(\pi_g+p_d\eta_{d1})}{u_s(r+\sigma)(r+\mu)} \\
&- \frac{\lambda_3(r+\sigma)}{u_s(r+\sigma)(r+\mu)} \frac{(\gamma_3(r+\mu)+\epsilon\lambda_3)(\pi_d+\pi_p+p_d\eta_{d2})+\lambda_3(r+\sigma)(\pi_g+p_d\eta_{d1})}{\lambda_3(r+\sigma)} = 0.
\end{aligned}
$$

The optimal regulation efforts for the government are

$$A^c = \frac{\epsilon\lambda_4(\pi_d + \pi_p + \pi_s + p_d\eta_{d2} + p_s\eta_{s2}) + \lambda_4(r+\sigma)(\pi_g + p_d\eta_{d1} + p_s\eta_{s1})}{u_g(r+\sigma)(r+\mu)},$$

$$A^s = \frac{\pi_g\lambda_4}{u_g(r+\mu)},$$

$$A^n = \frac{\pi_g\lambda_4}{u_g(r+\mu)}.$$

It is evident that $A^c > A^s = A^n$. □

The cooperative game shows the highest optimal efforts. Comparing the Nash non-cooperative game with the government-platform-led Stackelberg game, although the platform's and government's optimal effects remain the same by introducing a cost-sharing contract, the others tremendously increase.

Corollary 2. *The optimal performance values have the following relations:*

$$V_d^n(G,\chi) < V_d^s(G,\chi),$$
$$V_p^n(G,\chi) < V_p^s(G,\chi),$$
$$V_s^n(G,\chi) < V_s^s(G,\chi),$$
$$V_g^n(G,\chi) < V_g^s(G,\chi).$$

Compared to the Nash non-cooperative game, the government-platform-dominated game maximizes the benefits for each participant. Additionally, this model enables Pareto improvement and is self-executing. Under the government-platform-dominated game model, the overall system value is clearly higher than that of the Nash non-cooperative game, since $V_d^n + V_p^n + V_s^n + V_g^n < V_d^s + V_p^s + V_s^s + V_g^s$. This result indicates a fair cost-sharing contract may serve as an incentive.

Corollary 3. *The expectation and variance of the CSR goodwill and the idle emergency resource sharing level have the following relations*

$$\lim_{t\to\infty} E[G^c(t)] > \lim_{t\to\infty} E[G^s(t)] > \lim_{t\to\infty} E[G^n(t)],$$
$$\lim_{t\to\infty} Var[G^c(t)] > \lim_{t\to\infty} Var[G^s(t)] > \lim_{t\to\infty} Var[G^n(t)];$$

$$\lim_{t\to\infty} E[\chi^c(t)] > \lim_{t\to\infty} E[\chi^s(t)] > \lim_{t\to\infty} E[\chi^n(t)],$$
$$\lim_{t\to\infty} Var[\chi^c(t)] > \lim_{t\to\infty} Var[\chi^s(t)] > \lim_{t\to\infty} Var[\chi^n(t)].$$

The cooperative game exhibits the most significant expected emergency resource sharing level and CSR goodwill among the three models. Nevertheless, it possesses the highest risk level. In contrast to the Nash non-cooperative game, the government-platform-led game derives higher CSR goodwill and emergency resource sharing level. It also results in increased volatility. Consequently, firms must embrace greater risks to sustain elevated profits. Consequently, manufacturing businesses may choose different games based on risk tolerance. A firm inclined towards high risk may choose the cooperative game. Conversely, persons averse to risk may favour the Nash non-cooperative game, while those inclined towards moderate risk may prefer the Stackelberg game.

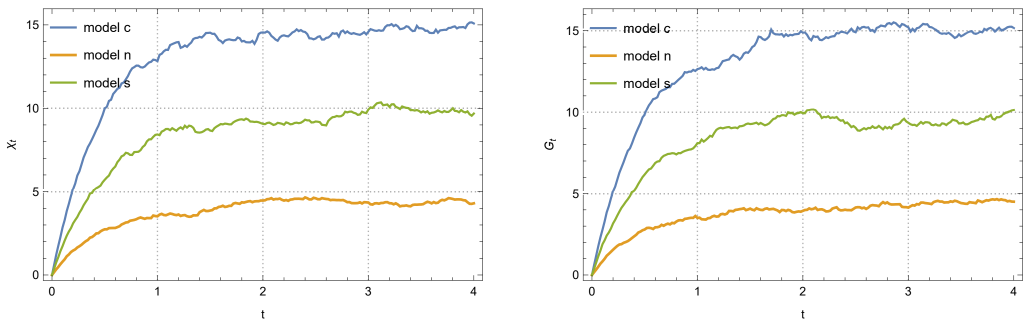
7.2. Numerical Simulations

This part offers numerical simulations for the three models discussed above and presents a straightforward comparison.

Realistic parameter values in China can be drawn from current data across various sectors. Regarding Corporate Social Responsibility (CSR), large manufacturing, energy, and technology enterprises typically contribute significant efforts, with moderate to high CSR impact coefficients ranging from 0.4 to 0.6 [61]. The decay rate σ, which represents the natural decline in CSR goodwill over time, typically ranges between 0.05 and 0.1 for most industries, reflecting a slow natural loss of goodwill without further CSR efforts. The diffusion coefficient δ values of 0.1 to 0.2 could serve as a baseline for industries with stable and consistent CSR practices where goodwill volatility is relatively low. This range suits sectors such as financial services or consumer goods, where CSR initiatives are well-established, and goodwill shifts are moderate and predictable [61]. For emergency supply chains, the natural decay rate of emergency supplies μ varies between 0.1 and 0.3, depending on the perishability of the goods [62], while the diffusion coefficient ρ related to resource sharing can be set between 0.2 and 0.4 to account for moderate volatility [63]. The potential market size a in sectors like logistics and healthcare can range from 10 to 100, reflecting China's large and growing market. At the same time, marginal revenues for suppliers and demanders typically fall between 1 and 10 [64]. Government policies also play a crucial role, with subsidies for platforms and enterprises –ξ_d, ξ_p–ranging from 0.2 to 0.6 to support emergency response and CSR initiatives [65].

Based on the above discussion and the parameter chosen of [45], the parameters are set to $p_s = 4, p_d = 5, u_d = 0.1, u_p = 0.15, u_s = 0.1, u_g = 0.2, a = 10, \eta_{s1} = 0.9, \eta_{s2} = 1.2, \eta_{d1} = 1.7, \eta_{d2} = 1.5, \theta = 0.5, \rho = 0.2, \omega = 5, c = 3, \epsilon = 0.4, \mu = 0.2, r = 0.2, \sigma = 0.1, \delta = 0.1, \lambda_1 = 0.2, \lambda_2 = 0.25, \lambda_3 = 0.3, \lambda_4 = 0.3, \gamma_1 = 0.3, \gamma_2 = 0.4, \gamma_3 = 0.3, \gamma_4 = 0.2, \pi_d = \pi_s = \pi_p = \pi_g = 1$.

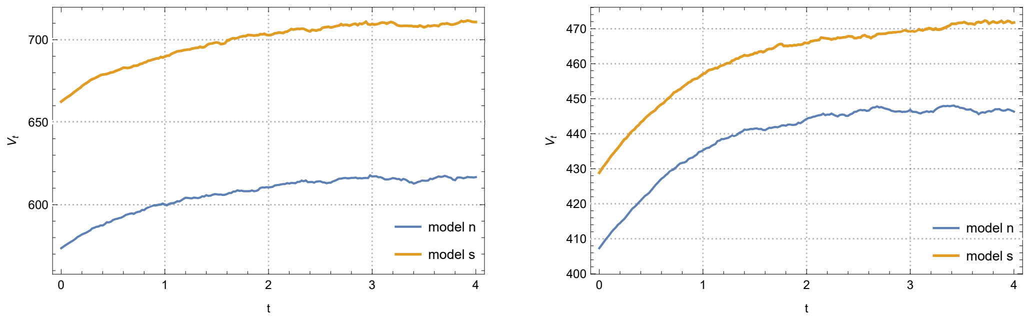
In all three models, the idle emergency resource sharing level and CSR goodwill increase with time t and ultimately attain a stable state, as illustrated in Figure 2. The cooperative model yields the highest level, while the Nash non-cooperative model yields the lowest level. The result of the Stackelberg primary–secondary model is located in the middle.



(**a**) (**b**)

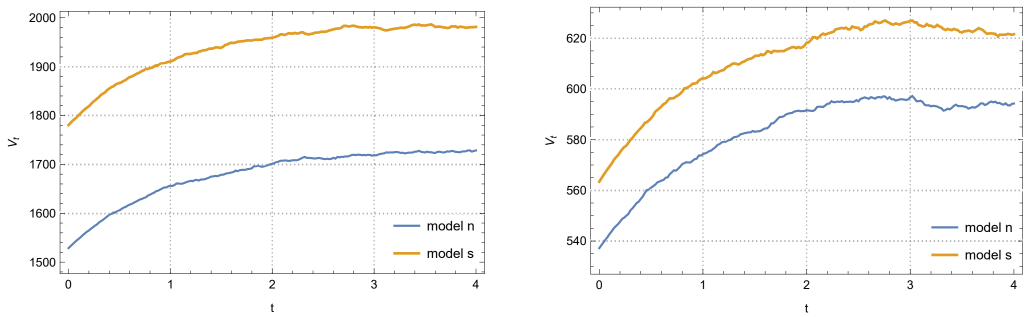
Figure 2. The trajectory of the idle emergency resource sharing level (**a**) and the CSR goodwill (**b**).

Figures 3 and 4 show the demander, platform, supplier, and government profits of the Nash non-cooperative model and Stackelberg primary–secondary model.



(**a**) (**b**)

Figure 3. The demander's value (**a**) and platform's value (**b**) in Model n and Model s.



(**a**) (**b**)

Figure 4. The supplier' value (**a**) and government's value (**b**) in Model n and Model s.

Figure 5 demonstrates the comparison result of the total profit of the system under the three models. In the initial moment, the total profit under the cooperative game falls behind the Stackelberg primary–secondary game, where the participants are not likely to form a community of interest. This shows that free market-oriented decision-making is better than that of the ideal centralized decision-making in the early stage of post-disaster emergency resource sharing progress. As time goes by, the total profit of the cooperative game eventually moves ahead of the others.

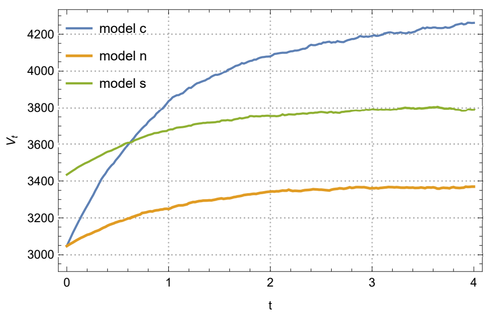


Figure 5. The total value of the supply chain in the three models.

In the cooperative game, the total value exhibits rapid growth at the beginning, while the total values of the other models increase at a slower rate. Regardless, applying cooperative games to real life presents many challenges because of benefit distribution and players' non-rationality. However, the government-platform-led Stackelberg game maximizes the entire system's profit, emergency resource sharing level, and CSR goodwill.

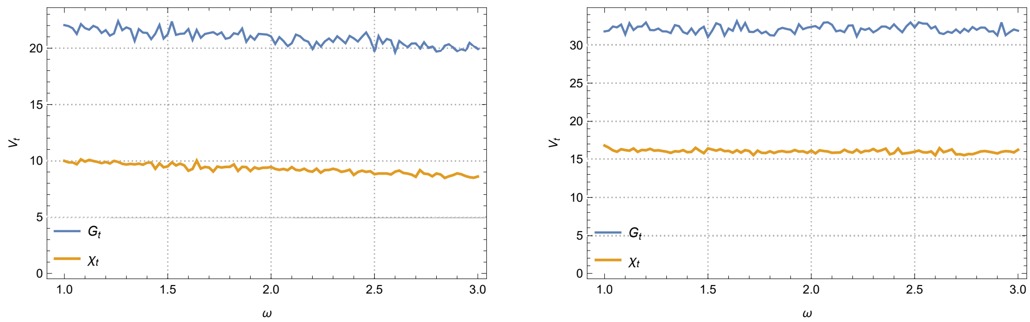
We summarized Figures 2–5 below; the readers can refer to Table 6.

Table 6. Summary of the values.

	V_{total}	V_d	V_p	V_s	V_a	χ_t	G_t
model n		✓	✓	✓	✓		
model s							
model c	✓					✓	✓

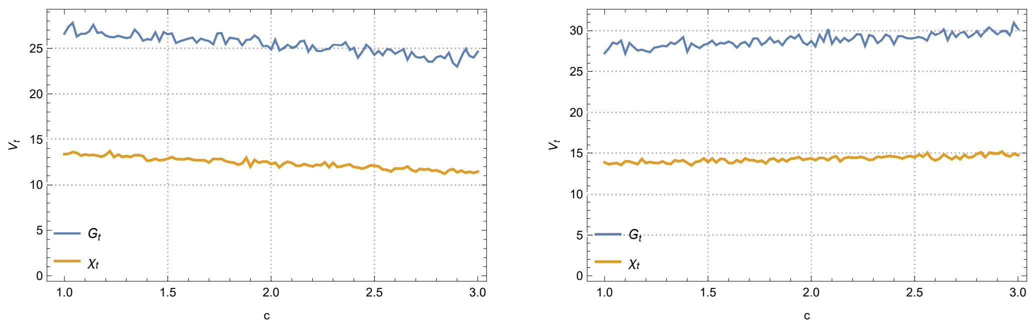
✓ represents the highest value.

Figures 6 and 7 depict the impacts of the resource usage fee ω and commission fee c on the idle emergency resource sharing level and the CSR goodwill in the Stackelberg and Nash non-cooperative models. As we can see, the resource usage fee ω diminishes the idle emergency resource sharing level as well as the CSR goodwill in the Nash non-cooperative game model while they are barely affected in the Stackelberg model. For the commission fee c, it undermines the idle emergency resource sharing level and the CSR goodwill in Nash non-cooperative game model. Oppositely, these two indexes are promoted in the Stackelberg primary–secondary model. Thus, in order to keep an active emergency resource sharing market, the government should encourage the enterprises to cooperate with each other with a fair cost-sharing contract.



(**a**) (**b**)

Figure 6. The impact of ω on the CSR goodwill and the idle emergency resource sharing level (**a**) in Model n, (**b**) in Model s.



(**a**) (**b**)

Figure 7. The impact of c on the CSR goodwill and the idle emergency resource sharing level (**a**) in Model n, (**b**) in Model s.

In spite of the idle emergency resource sharing level and CSR goodwill, it is also necessary to examine the values of the supplier and platform since the demander has to pay commission fee c to the platform and resource usage fee ω to the supplier. According to Figures 8 and 9, the commission fee c and resource usage fee ω decrease the value of the demander and increase the values of the platform and supplier. Compared to the Nash non-cooperative model, the value of each player is higher in the Stackelberg model.

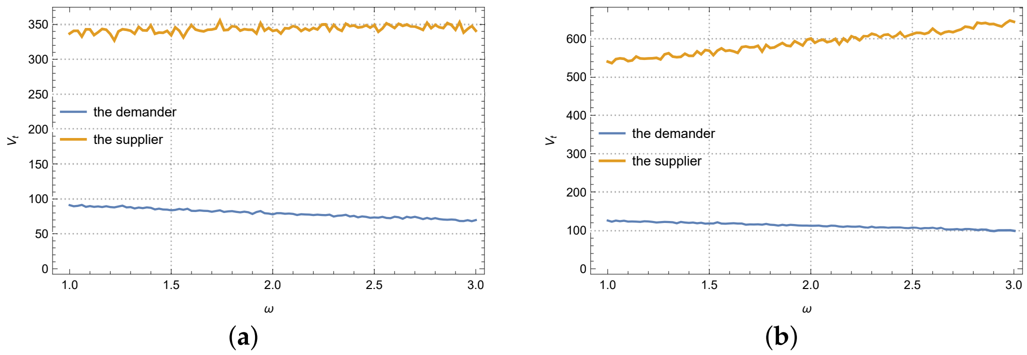


Figure 8. The impact of ω on the demander's and supplier's profits (**a**) in Model n, (**b**) in Model s.

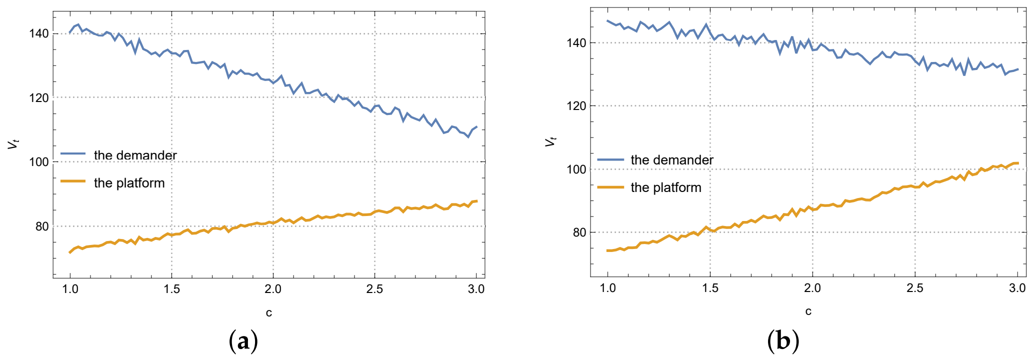


Figure 9. The impact of c on the demander's and platform's profits (**a**) in Model n, (**b**) in Model s.

Although the commission fee c and resource usage fee ω have different impacts on the values of players, the commission fee enhances the total value while the resource usage fee diminishes the total value (see Figure 10). Therefore, to enhance the emergency resource sharing level and keep an active emergency resource sharing market, it is better to distribute more profit to the platform. However, the government should also pay attention to avoiding rent-seeking and the monopoly of it.

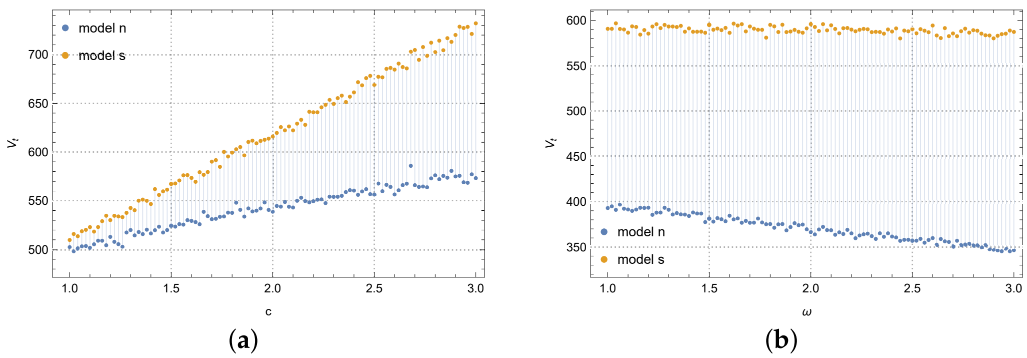


Figure 10. The impacts of c (**a**) and ω (**b**) on the total value in Model n and Model s.

We summarized Figures 6–10 below; the readers can refer to Table 7.

Table 7. The performance comparison between the cooperative and non-cooperative models.

		ω	c
model n	$G(t)$	$\searrow$	$\searrow$
	$\chi(t)$	$\searrow$	$\searrow$
	V_d	$\searrow$	$\searrow$
	V_s	$\nearrow$	
	V_p		$\nearrow$
	V_{total}	$\searrow$	$\nearrow$
model s	$G(t)$	$\sim$	$\nearrow$
	$\chi(t)$	$\sim$	$\nearrow$
	V_d	$\searrow$	$\searrow$
	V_s	$\nearrow$	
	V_p		$\nearrow$
	V_{total}	$\sim$	$\nearrow$

$\nearrow$ represents positive influence, $\searrow$ represents negative influence, $\sim$ represents not clear.

8. Discussion

This paper offers distinct but complementary perspectives on resource allocation and decision-making under uncertainty, enhancing the practical and theoretical understanding of resource utilization.

Practical implications for decision-makers:

(1) The study provides a flexible framework for government agencies and platforms to dynamically adjust resource-sharing strategies in real time, responding to unpredictable shifts in supply and demand during post-disaster recovery.
Implementing real-time adjustments requires accurate, up-to-date data on supply and demand, which might not be readily available in post-disaster environments. Data infrastructure may be damaged, making it difficult for decision-makers to obtain the information needed for dynamic resource sharing. Second, multiple stakeholders, including governments, private companies, and NGOs, need to coordinate their efforts in a fast-paced environment. Achieving seamless collaboration, particularly across different sectors and regions, presents significant logistical challenges.
Therefore, investment in disaster-resilient communication infrastructure and pre-established coordination protocols can help overcome these obstacles, ensuring that accurate information is available and decision-making processes are streamlined.

(2) By modeling centralized and decentralized decision-making, the approaches empower decision-makers to evaluate the benefits of collaboration versus competition, helping them optimize resource allocation based on situational needs.
The decision to implement centralized or decentralized strategies can be complex and depends on the context. For example, centralized models might work better in regions with strong government control, while decentralized models may be more effective in regions with strong private sector involvement. The challenge lies in assessing which model works best under specific disaster scenarios. Governments can create hybrid models that offer flexibility, combining the strengths of both centralized and decentralized approaches.

(3) The game models offer valuable insights into government regulation and intervention strategies, showing how subsidies or leadership roles can effectively coordinate efforts among private sector actors, enhancing overall system resilience.
Government interventions, such as subsidies or regulations, need to strike a balance between incentivizing resource-sharing without creating dependencies or inefficiencies. Excessive regulation could stifle innovation, while too little oversight might lead to free-riding or monopolistic behaviors. On the other hand, offering subsidies to encourage collaboration in resource sharing requires substantial funding, which might not be sustainable in the long run, especially for governments with limited resources.

A phased approach to subsidies, where incentives decrease over time, can encourage self-sustainability. Additionally, governments can foster public–private partnerships to share the financial burden of maintaining robust resource-sharing systems.

Theoretical implications for scholars:

(1) This research contributes to applying stochastic differential games in emergency management, highlighting the role of uncertainty and randomness in decision-making processes.
(2) By integrating three distinct game-theoretic models, the study provides a more comprehensive framework for understanding interactions between various players in a post-disaster context, allowing future researchers to extend these models to other uncertain environments.
(3) The findings encourage further exploration into collaborative decision-making models and how they impact sustainability, resource efficiency, and long-term recovery in uncertain and dynamic environments.

The scalability of the model across different types of disasters and national contexts presents several challenges and considerations. The model, as described in the article, primarily focuses on post-disaster resource allocation and CSR (Corporate Social Responsibility) dynamics. Different types of disasters—such as natural disasters (earthquakes, hurricanes), human-made disasters (industrial accidents, terrorist attacks), or health crises (pandemics)—present varying demands for resource sharing, logistical coordination, and stakeholder engagement.

(1) Varying Resource Needs: Different disasters require vastly different types of resources. For instance, in a pandemic, medical supplies and healthcare resources are critical, while in a natural disaster like a hurricane, food, shelter, and infrastructure repair materials may take precedence. The model may need adjustments to account for the specific resources and supply chain needs of each disaster type.
(2) Response Timelines: Some disasters, like earthquakes, demand immediate and urgent responses, while others, such as pandemics, unfold over a longer period. The model may require modifications to address the varying time scales and urgency associated with different types of crises.
(3) Unpredictability and Scope: Natural disasters like earthquakes or floods tend to be geographically constrained, while pandemics are global in scope. The scalability of the model must account for both localized and wide-scale impacts, ensuring flexibility in resource distribution across multiple regions and time zones.

To address the above challenges, the model can incorporate disaster-specific customization by adjusting resource prioritization and timeline variables based on the unique needs of each crisis. For example, in health emergencies, the model could focus on healthcare resources, while in natural disasters, it could emphasize logistical support and infrastructure repair. Additionally, a flexible classification system for resources can be introduced, allowing decision-makers to adapt quickly to the evolving demands of different disaster scenarios. This flexibility can enable the model to function more effectively, regardless of the type or scale of the disaster.

For the scalability of national contexts, the current work assumes a degree of coordination between government bodies, private entities, and platforms in a disaster management scenario. However, the nature of these relationships and the extent of governmental involvement vary widely across different national contexts.

(1) Differences in Governance: In countries with strong centralized governments, such as China or Saudi Arabia, the model's centralized decision-making processes may be more effective. However, in decentralized democracies like the United States or India, where local governments and private entities play significant roles, the model may need to incorporate more decentralized decision-making mechanisms.
(2) Regulatory Environments: National regulatory frameworks differ, impacting how CSR initiatives, resource sharing, and subsidies are implemented. In some countries, stringent regulations may make it difficult to mobilize private sector resources

quickly, while in others, a lack of regulation might lead to uncoordinated or inefficient resource allocation.

(3) Economic and Social Contexts: The availability of financial resources for subsidies, the level of technology adoption, and the culture of public–private collaboration vary greatly across countries. Developing countries may face greater challenges in implementing technologically driven platforms for resource sharing or in providing subsidies to incentivize CSR-driven resource distribution.

To address the challenges of scaling the model across different national contexts, a hybrid approach can be implemented that accommodates both centralized and decentralized decision-making frameworks. This allows the model to adapt to countries with varying governance structures, ensuring effectiveness in both hierarchical and participatory systems. Additionally, the model can be customized to align with local regulatory frameworks, enabling governments to apply either strict oversight or more flexible self-regulation based on their national context. Promoting international cooperation and technology transfer, particularly for developing countries, can also help enhance the model's effectiveness in resource-limited settings, fostering cross-border resource sharing and capacity building.

The model's assignment of matching efforts (e.g., $D(t), P(t), S(t)$ for each player) is simplified by assuming a homogeneous impact. This assumption may overlook the variability that exogenous factors, such as technology adoption or resource accessibility, could introduce to each player's efforts. In reality, these external factors can significantly influence the effectiveness and allocation of matching efforts across players. For further extension, we could consider introducing variable influence coefficients, i.e.,

$$D(t) = \alpha_D D(t), \quad P(t) = \alpha_P P(t), \quad S(t) = \alpha_S S(t),$$

where $\alpha_D, \alpha_P, \alpha_S$ are coefficients reflecting the impact of technology or resource availability for each player. For example, if a player has high technology adoption or superior resource access, their matching effort would be more effective, and thus a higher coefficient could be applied. These coefficients can be determined based on empirical data or industry benchmarks, with higher values indicating greater efficacy in effort due to better external conditions.

Moreover, in cooperative game strategies, the players' efforts often exhibit interactive or synergistic effects, where one player's effort enhances or supports the efforts of another. To make the model more realistic and applicable, it could be extended with additional terms that capture these interdependencies.

(1) Adding Synergistic Terms: To reflect how one player's effort can positively or negatively influence another's, the model could incorporate synergistic terms that account for the mutual reinforcement or dependency of efforts. For example, let us assume efforts $D(t), P(t), S(t)$ are exerted by different players, including cross-terms

$$\beta_{DP} D(t) P(t), \quad \beta_{PS} P(t) S(t), \quad \beta_{DS} D(t) S(t),$$

where $\beta_{DP}, \beta_{PS}, \beta_{DS}$ are coefficients capturing the synergistic or complementary impact of one player's effort on another. Positive values indicate synergy, while negative values could capture competition or interference.

(2) Introducing a Combined Effort Term for Joint Contributions: A combined term representing a joint contribution of efforts can be introduced to model situations where players pool resources or synchronize strategies. This term could be denoted as $J(t)$, a weighted combination of each player's effort

$$J(t) = \alpha_D D(t) + \alpha_P P(t) + \alpha_S S(t),$$

where $\alpha_D, \alpha_P, \alpha_S$ are weights representing each player's contribution to the joint effort. This combined term $J(t)$ can then be included in the main equations as an additional factor affecting outcomes, reflecting how collective efforts influence results.

The model assumptions outlined in this research provide a structured foundation for analyzing the sustainable allocation of idle emergency resources in post-disaster management. However, extending the discussion to real-world challenges and limitations can highlight some critical considerations that may affect the model's applicability and results.

(1) Rationality of Players: In reality, the assumption of full rationality is often unrealistic. Decision-makers, especially in emergency situations, may be driven by incomplete information, bounded rationality, or emotional responses such as urgency and fear. Stakeholders may lack the time or resources to make fully informed decisions, leading to suboptimal outcomes. This assumption may limit the model's application to real-world situations, where human behavior can deviate from rationality. The model's predictions may be overly optimistic regarding optimal resource allocation and collaboration.
(2) Dynamic Progress of Emergency Supplies: While the assumption of dynamic progress is sound, in reality, supply chains are often constrained by logistical bottlenecks, infrastructure damage, and unpredictable external factors (e.g., weather, transportation breakdowns). Emergency supplies may not always reach their intended targets efficiently, regardless of the matching efforts. The assumption may overestimate the players' ability to influence supply levels. If logistical challenges are not fully accounted for, the model's results might be overly optimistic regarding the impact of matching efforts on resource allocation.
(3) Influence of CSR Goodwill: CSR goodwill, while important, may not always translate directly into economic benefits, particularly in crisis situations where immediate needs often take priority over long-term reputation. Furthermore, CSR initiatives can vary significantly in effectiveness depending on cultural and regional contexts. Some stakeholders may not prioritize CSR to the extent the model assumes. The model assumes that CSR goodwill has a direct and measurable impact on demand and supply chain behavior, but in practice, this effect may be weaker or harder to quantify. This can reduce the model's predictive power in situations where CSR is not a strong influencing factor.
(4) Cost Functions and Effort Levels: In practice, the relationship between effort and cost may not always be convex. For example, some economies of scale could reduce costs with increased effort, while in other situations, diminishing returns may set in more quickly than expected. Moreover, financial constraints in post-disaster scenarios can limit the willingness or ability of players to invest in efforts, no matter the potential return. The model may overestimate the players' ability to scale efforts if financial or logistical constraints are more severe than predicted. The cost–benefit analysis in real-world settings might not align perfectly with the convex cost structure assumed in the model.
(5) Cross-Network Externalities and Cooperation: In practice, the external overflow effects generated by resource sharing and CSR may be more complex than assumed. Not all shared resources directly translate into market demand. For example, some types of shared resources might only create significant external benefits under specific conditions or in certain markets, while in other contexts, the effect could be minimal or nonexistent. If the real-world external overflow effects are not as strong as assumed, the model may overestimate the growth in market demand. This could lead to overly optimistic predictions, especially regarding how quickly resource sharing and CSR can stimulate demand.

9. Conclusions

This paper provides a novel approach to dynamic resource allocation in post-disaster settings by utilizing three stochastic differential game models to account for uncertainty. These three models—Nash equilibrium, cooperative game, and Stackelberg game—offer distinct but complementary perspectives on resource allocation and decision-making under uncertainty. By modeling the interactions between key players—demanders, platforms, suppliers, and the government—each approach enhances decision-making in resource utilization by incorporating randomness and dynamic feedback into the process.

(1) Nash equilibrium ensures that each actor maximizes their benefit, essential for understanding decentralized decision-making in competitive environments.
(2) The cooperative game model fosters collaboration between players, demonstrating how coordinated efforts can lead to optimal resource sharing and increased system-wide benefits.
(3) The Stackelberg game provides insights into hierarchical decision-making, where leaders (such as the government) set strategies that followers (such as suppliers and demanders) react to, allowing for more efficient control and regulation of emergency resources.

This study offers a framework for managing resource sharing in post-disaster environments by integrating these three approaches. It enhances the flexibility and robustness of resource allocation decisions in the face of uncertainty. The models' ability to adapt to changing conditions and incorporate stochastic elements into decision-making is crucial for improving real-time resource management. Decision-makers can refine their post-disaster response strategies by adopting these models to forecast potential challenges in real-time resource allocation, helping to balance efficiency with fairness during emergency management.

Future research could explore how bounded rationality and behavioral factors alter decision-making processes and resource outcomes in real-world crises. Expanding the framework to account for heterogeneity and adaptive behaviors can provide a more comprehensive understanding of how decisions are made under uncertainty, further enhancing the practical applications of this model. On the other hand, this framework can be used to study other disaster types or dynamic systems, applying the principles of stochastic differential games to complex supply chains or sustainability challenges.

Author Contributions: Methodology, Y.L.; Software, J.W.; Investigation, M.Z.; Writing—original draft, L.L.; Supervision, M.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

$$pol_c = \frac{1}{2r(r+\mu)(r+\sigma)}(-2p_s^2r^2 - (A^c)^2r^2u_g - (D^c)^2r^2u_d - (P^c)^2r^2u_p - r^2(S^c)^2u_s + 2D^c\pi_d r\gamma_1 +$$
$$2D^c\pi_p r\gamma_1 + 2D^c\pi_s r\gamma_1 + 2P^c\pi_d r\gamma_2 + 2P^c\pi_p r\gamma_2 + 2P^c\pi_s r\gamma_2 + 2\pi_d rS^c\gamma_3 + 2\pi_p rS^c\gamma_3 + 2\pi_s rS^c\gamma_3 +$$
$$2D^c p_s r\gamma_1\eta_{s2} + 2P^c p_s r\gamma_2\eta_{s2} + 2p_s rS^c\gamma_3\eta_{s2} - 2p_s^2r^2\theta + 2D^c\pi_g r\lambda_1 + 2D^c\pi_d\epsilon\lambda_1 + 2D^c\pi_p\epsilon\lambda_1 + 2D^c\pi_s\epsilon\lambda_1 +$$
$$2D^c p_s r\eta_{s1}\lambda_1 + 2D^c p_s\epsilon\eta_{s2}\lambda_1 + 2P^c\pi_g r\lambda_2 + 2P^c\pi_d\epsilon\lambda_2 + 2P^c\pi_p\epsilon\lambda_2 + 2P^c\pi_s\epsilon\lambda_2 + 2P^c p_s r\eta_{s1}\lambda_2 +$$
$$2P^c p_s\epsilon\eta_{s2}\lambda_2 + 2\pi_g rS^c\lambda_3 + 2\pi_d S^c\epsilon\lambda_3 + 2\pi_p S^c\epsilon\lambda_3 + 2\pi_s S^c\epsilon\lambda_3 + 2p_s rS^c\eta_{s1}\lambda_3 + 2p_s S^c\epsilon\eta_{s2}\lambda_3 + 2A^c\pi_g r\lambda_4 +$$
$$2A^c\pi_d\epsilon\lambda_4 + 2A^c\pi_p\epsilon\lambda_4 + 2A^c\pi_s\epsilon\lambda_4 + 2A^c p_s r\eta_{s1}\lambda_4 + 2A^c p_s\epsilon\eta_{s2}\lambda_4 - 2p_s^2r\mu - (A^c)^2ru_a\mu - (D^c)^2ru_d\mu -$$
$$(P^c)^2ru_p\mu - r(S^c)^2u_s\mu + 2D^c\pi_d\gamma_1\mu + 2D^c\pi_p\gamma_1\mu + 2D^c\pi_s\gamma_1\mu + 2P^c\pi_d\gamma_2\mu + 2P^c\pi_p\gamma_2\mu + 2P^c\pi_s\gamma_2\mu +$$
$$2\pi_d S^c\gamma_3\mu + 2\pi_p S^c\gamma_3\mu + 2\pi_s S^c\gamma_3\mu + 2D^c p_s\gamma_1\eta_{s2}\mu + 2P^c p_s\gamma_2\eta_{s2}\mu + 2p_s S^c\gamma_3\eta_{s2}\mu - 2p_s^2r\theta\mu - 2p_s^2r\sigma -$$
$$(A^c)^2ru_a\sigma - (D^c)^2ru_d\sigma - (P^c)^2ru_p\sigma - r(S^c)^2u_s\sigma - 2p_s^2r\theta\sigma + 2D^c\pi_g\lambda_1\sigma + 2D^c p_s\eta_{s1}\lambda_1\sigma + 2P^c\pi_g\lambda_2\sigma +$$
$$2P^c p_s\eta_{s1}\lambda_2\sigma + 2\pi_g S^c\lambda_3\sigma + 2p_s S^c\eta_{s1}\lambda_3\sigma + 2A^c\pi_g\lambda_4\sigma + 2A^c p_s\eta_{s1}\lambda_4\sigma - 2p_s^2\mu\sigma - (A^c)^2u_a\mu\sigma - (D^c)^2u_d\mu\sigma -$$
$$(S^c)^2u_s\mu\sigma - 2p_s^2\theta\mu\sigma + 2a(p_d+p_s)(r+\mu)(r+\sigma) - 2p_d^2(1+\theta)(r+\mu)(r+\sigma) - (P^c)^2u_p\mu\sigma +$$
$$2p_d(rS^c\gamma_3\eta_{d2} + 2p_s r^2\theta + rS^c\eta_{d1}\lambda_3 + S^c\epsilon\eta_{d2}\lambda_3 + A^c r\eta_{d1}\lambda_4 + A^c\epsilon\eta_{d2}\lambda_4 + S^c\gamma_3\eta_{d2}\mu + 2p_s r\theta\mu + 2p_s r\theta\sigma +$$
$$S^c\eta_{d1}\lambda_3\sigma + A^c\eta_{d1}\lambda_4\sigma + 2p_s\theta\mu\sigma + D^c(r\gamma_1\eta_{d2} + r\eta_{d1}\lambda_1 + \epsilon\eta_{d2}\lambda_1 + \gamma_1\eta_{d2}\mu + \eta_{d1}\lambda_1\sigma) +$$
$$P^c(r\gamma_2\eta_{d2} + r\eta_{d1}\lambda_2 + \epsilon\eta_{d2}\lambda_2 + \gamma_2\eta_{d2}\mu + \eta_{d1}\lambda_2\sigma))).$$

$$pol_{nd} = \frac{1}{2ru_g u_d u_p u_s(r+\mu)^2(r+\sigma)^2}(\pi_d^2r^2u_g u_p u_s\gamma_1^2 + 2\pi_d\pi_p r^2u_g u_d u_s\gamma_2^2 + 2\pi_d\pi_s r^2u_g u_d u_p\gamma_3^2 -$$
$$2c\pi_d r^2u_g u_p u_s\gamma_1\lambda_1 + 2\pi_d^2ru_g u_p u_s\gamma_1\epsilon\lambda_1 + c^2r^2u_g u_p u_s\lambda_1^2 - 2c\pi_d ru_g u_p u_s\epsilon\lambda_1^2 + \pi_d p_s ru_g u_d u_p\gamma_3\eta_{d2}\eta_{s2}\lambda_3\sigma$$
$$\pi_d^2u_g u_p u_s\epsilon^2\lambda_1^2 + 2c\pi_d r^2u_g u_d u_s\gamma_2\lambda_2 - 2c\pi_p r^2u_g u_d u_s\gamma_2\lambda_2 + 4\pi_d\pi_p ru_g u_d u_s\gamma_2\epsilon\lambda_2 + 2\pi_d p_s r^2u_g u_d u_p\gamma_3^2\eta_{s2} -$$
$$2c^2r^2u_g u_d u_s\lambda_2^2 + 2c\pi_d ru_g u_d u_s\epsilon\lambda_2^2 - 2c\pi_p ru_g u_d u_s\epsilon\lambda_2^2 + 2\pi_d\pi_p u_g u_d u_s\epsilon^2\lambda_2^2 - 2c\pi_s r^2u_g u_d u_p\gamma_3\lambda_3 +$$
$$4\pi_d\pi_s ru_g u_d u_p\gamma_3\epsilon\lambda_3 + 2\pi_d p_s r^2u_g u_d u_p\gamma_3\eta_{s1}\lambda_3 - 2cp_s r^2u_g u_d u_p\gamma_3\eta_{s2}\lambda_3 + 4\pi_d p_s ru_g u_d u_p\gamma_3\epsilon\eta_{s2}\lambda_3 -$$
$$2c\pi_s ru_g u_d u_p\epsilon\lambda_3^2 + 2\pi_d p_s u_g u_d u_p\epsilon\eta_{s1}\lambda_3^2 - 2cp_s ru_g u_d u_p\epsilon\eta_{s2}\lambda_3^2 + 2\pi_d p_s u_g u_d u_p\epsilon^2\eta_{s2}\lambda_3^2 - 2c\pi_g r^2u_d u_p u_s\lambda_4^2 +$$
$$2\pi_d\pi_g ru_d u_p u_s\epsilon\lambda_4^2 + 2\pi_d^2ru_g u_p u_s\gamma_1^2\mu + 4\pi_d\pi_p ru_g u_d u_s\gamma_2^2\mu + 4\pi_d\pi_s ru_g u_d u_p\gamma_3^2\mu + 4\pi_d p_s ru_g u_d u_p\gamma_3^2\eta_{s2}\mu -$$
$$2c\pi_d ru_g u_p u_s\gamma_1\lambda_1\mu + 2\pi_d^2u_g u_p u_s\gamma_1\epsilon\lambda_1\mu + 2c\pi_d ru_g u_d u_s\gamma_2\lambda_2\mu - 2c\pi_p ru_g u_d u_s\gamma_2\lambda_2\mu + 4\pi_d\pi_p u_g u_d u_s\gamma_2\epsilon\lambda_2\mu -$$
$$2c\pi_s ru_g u_d u_p\gamma_3\lambda_3\mu + 4\pi_d\pi_s u_g u_d u_p\gamma_3\epsilon\lambda_3\mu + 2\pi_d p_s ru_g u_d u_p\gamma_3\eta_{s1}\lambda_3\mu - 2cp_s ru_g u_d u_p\gamma_3\eta_{s2}\lambda_3\mu +$$
$$4\pi_d p_s u_g u_d u_p\gamma_3\epsilon\eta_{s2}\lambda_3\mu + \pi_d^2u_g u_p u_s\gamma_1^2\mu^2 + 2\pi_d\pi_p u_g u_d u_s\gamma_2^2\mu^2 + 2\pi_d\pi_s u_g u_d u_p\gamma_3^2\mu^2 + 2\pi_d p_s u_g u_d u_p\gamma_3^2\eta_{s2}\mu^2 -$$
$$2c\pi_d ru_g u_p u_s\gamma_1\lambda_1\sigma + 2c^2ru_g u_p u_s\lambda_1^2\sigma - 2c\pi_d u_g u_p u_s\epsilon\lambda_1^2\sigma + 2c\pi_d ru_g u_d u_s\gamma_2\lambda_2\sigma - 2c\pi_p ru_g u_d u_s\gamma_2\lambda_2\sigma -$$
$$4c^2ru_g u_d u_s\lambda_2^2\sigma + 2c\pi_d u_g u_d u_s\epsilon\lambda_2^2\sigma - 2c\pi_p u_g u_d u_s\epsilon\lambda_2^2\sigma - 2c\pi_s ru_g u_d u_p\gamma_3\lambda_3\sigma + 2\pi_d p_s ru_g u_d u_p\gamma_3\eta_{s1}\lambda_3\sigma -$$
$$2cp_s ru_g u_d u_p\gamma_3\eta_{s2}\lambda_3\sigma - 2c\pi_s u_g u_d u_p\epsilon\lambda_3^2\sigma + 4cp_s ru_g u_d u_p\eta_{s1}\lambda_3^2\sigma + 2\pi_d p_s u_g u_d u_p\epsilon\eta_{s1}\lambda_3^2\sigma - 2c\pi_s u_g u_d u_p\epsilon\eta_{s2}\lambda_3^2\sigma -$$
$$4c\pi_g ru_d u_p u_s\lambda_4^2\sigma + 2\pi_d\pi_g u_d u_p u_s\epsilon\lambda_4^2\sigma + 4\pi_d\pi_s ru_g u_d u_p\theta\mu - c\pi_d u_g u_p u_s\gamma_1\eta_{d2}\lambda_1\mu\sigma + c\pi_d u_g u_d u_s\gamma_2\eta_{d2}\lambda_2\mu\sigma -$$
$$2c\pi_p u_g u_d u_s\gamma_2\eta_{d2}\lambda_2\mu\sigma - 2c\pi_s u_g u_d u_p\gamma_3\eta_{d2}\lambda_3\mu\sigma + \pi_d p_s u_g u_d u_p\gamma_3\eta_{d2}\eta_{s1}\lambda_3\sigma + 2\pi_d p_s u_g u_d u_p\gamma_3\eta_{d2}\eta_{s2}\lambda_3\sigma +$$
$$2\pi_d p_s r^3u_g u_d u_p\theta\mu - cru_g u_p u_s\gamma_1\eta_{d2}\lambda_1\mu\sigma + cru_g u_d u_s\gamma_2\eta_{d2}\lambda_2\mu\sigma + \pi_d u_g u_d u_p\gamma_3\eta_{d1}\lambda_3\mu\sigma + 2\pi_d u_g u_d u_p\gamma_3\epsilon\eta_{d2}\lambda_3\mu\sigma).$$

$$pol_{np} = \frac{1}{2ru_g u_d u_p u_s (r+\mu)^2 (r+\sigma)^2}(\pi_p^2 u_g u_d u_s (\epsilon\lambda_2 + \gamma_2(r+\mu))^2 +$$
$$2u_g u_p u_s \gamma_1 (\pi_d + p_d \eta_{d2})(r+\mu)(\pi_p \epsilon\lambda_1 + \pi_p \gamma_1 (r+\mu) + c\lambda_1 (r+\sigma)) +$$
$$c(r+\sigma)(cu_g u_d u_s \lambda_2^2 (r+\sigma) + 2u_g u_p u_s \lambda_1^2 (\pi_d \epsilon + p_d \epsilon \eta_{d2} - c(r+\sigma) + p_d \eta_{d1}(r+\sigma) - (r+\sigma)\omega) +$$
$$2u_d u_p \big(u_g \gamma_3 (\pi_s + p_s \eta_{s2})\lambda_3 (r+\mu) + \pi_g u_s \lambda_4^2 (r+\sigma) + u_g \lambda_3^2 (\pi_s \epsilon + p_s \epsilon \eta_{s2} + p_s \eta_{s1}(r+\sigma) + (r+\sigma)\omega))) +$$
$$2\pi_p (u_g u_d u_p \gamma_3 (\pi_s + p_s \eta_{s2})(r+\mu)(\epsilon\lambda_3 + \gamma_3 (r+\mu)) + cu_g u_d u_s \lambda_2 (\epsilon\lambda_2 + \gamma_2 (r+\mu))(r+\sigma) +$$
$$u_g u_p u_s \lambda_1 (\epsilon\lambda_1 + \gamma_1 (r+\mu))(\pi_d \epsilon + p_d \epsilon \eta_{d2} - c(r+\sigma) + p_d \eta_{d1}(r+\sigma) - (r+\sigma)\omega) +$$
$$u_d u_p (\pi_g u_s \epsilon \lambda_4^2 (r+\sigma) + u_g \lambda_3 (\epsilon\lambda_3 + \gamma_3 (r+\mu))(\pi_s \epsilon + p_s \epsilon \eta_{s2} + p_s \eta_{s1}(r+\sigma) + (r+\sigma)\omega)))).$$

$$pol_{ns} = \frac{1}{2ru_g u_d u_p u_s (r+\mu)^2 (r+\sigma)^2} 2\pi_d \pi_s r^2 u_g u_p u_s \gamma_1^2 + 2\pi_p \pi_s r^2 u_g u_d u_s \gamma_2^2 + \pi_s^2 r^2 u_g u_d u_p \gamma_3^2 +$$
$$4\pi_d \pi_s r u_g u_p u_s \gamma_1 \epsilon \lambda_1 + 2p_d \pi_s r^2 u_g u_p u_s \gamma_1 \eta_{d1} \lambda_1 + 4p_d \pi_s r u_g u_p u_s \gamma_1 \epsilon \eta_{d2} \lambda_1 - 2c\pi_s r u_g u_p u_s \epsilon \lambda_1^2 + 2p_d \pi_s r^2 u_g u_p u_s \gamma_1^2 \eta_{d2} +$$
$$2\pi_d \pi_s u_g u_p u_s \epsilon^2 \lambda_1^2 + 2p_d \pi_s r u_g u_p u_s \epsilon \eta_{d1} \lambda_1^2 + 2p_d \pi_s u_g u_p u_s \epsilon^2 \eta_{d2} \lambda_1^2 + 2c\pi_s r^2 u_g u_d u_s \gamma_2 \lambda_2 +$$
$$4\pi_p \pi_s r u_g u_d u_s \gamma_2 \epsilon \lambda_2 + 2c\pi_s r u_g u_d u_s \epsilon \lambda_2^2 + 2\pi_p \pi_s u_g u_d u_s \epsilon^2 \lambda_2^2 + 2\pi_s^2 r u_g u_d u_p \gamma_3 \epsilon \lambda_3 + \pi_s^2 u_g u_d u_p \epsilon^2 \lambda_3^2 +$$
$$2\pi_g \pi_s r u_d u_p u_s \epsilon \lambda_4^2 + 4\pi_d \pi_s r u_g u_p u_s \gamma_1^2 \mu + 4\pi_p \pi_s r u_g u_d u_s \gamma_2^2 \mu + 2\pi_s^2 r u_g u_d u_p \gamma_3^2 \mu + 4p_d \pi_s r u_g u_p u_s \gamma_1^2 \eta_{d2} \mu -$$
$$2c\pi_s r u_g u_p u_s \gamma_1 \lambda_1 \mu + 4\pi_d \pi_s u_g u_p u_s \gamma_1 \epsilon \lambda_1 \mu + 2p_d \pi_s r u_g u_p u_s \gamma_1 \eta_{d1} \lambda_1 \mu + 4p_d \pi_s u_g u_p u_s \gamma_1 \epsilon \eta_{d2} \lambda_1 \mu +$$
$$2c\pi_s r u_g u_d u_s \gamma_2 \lambda_2 \mu + 4\pi_p \pi_s u_g u_d u_s \gamma_2 \epsilon \lambda_2 \mu + 2\pi_s^2 u_g u_d u_p \gamma_3 \epsilon \lambda_3 \mu + 2\pi_d \pi_s u_g u_p u_s \gamma_1^2 \mu^2 + 2\pi_p \pi_s u_g u_d u_s \gamma_2^2 \mu^2 +$$
$$\pi_s^2 u_g u_d u_p \gamma_3^2 \mu^2 + 2p_d \pi_s u_g u_p u_s \gamma_1^2 \eta_{d2} \mu^2 - 2c\pi_s r u_g u_p u_s \gamma_1 \lambda_1 \sigma + 2p_d \pi_s r u_g u_p u_s \gamma_1 \eta_{d1} \lambda_1 \sigma - 2c\pi_s u_g u_p u_s \epsilon \lambda_1^2 \sigma +$$
$$2p_d \pi_s u_g u_p u_s \epsilon \eta_{d1} \lambda_1^2 \sigma + 2c\pi_s r u_g u_d u_s \gamma_2 \lambda_2 \sigma + 2c\pi_s u_g u_d u_s \epsilon \lambda_2^2 \sigma + 2\pi_g \pi_s u_d u_p u_s \epsilon \lambda_4^2 \sigma - 2c\pi_s u_g u_p u_s \gamma_1 \lambda_1 \mu \sigma +$$
$$2p_d \pi_s u_g u_p u_s \gamma_1 \eta_{d1} \lambda_1 \mu \sigma + 2c\pi_s u_g u_d u_s \gamma_2 \lambda_2 \mu \sigma + 2ap_s u_g u_d u_p u_s (r+\mu)^2 (r+\sigma)^2 + p_s^2 u_g u_d u_p - 2r^4 u_s (1+\theta) +$$
$$\eta_{s2}^2 (\epsilon\lambda_3 + \gamma_3 \mu)^2 + 2\eta_{s1} \eta_{s2} \lambda_3 (\epsilon\lambda_3 + \gamma_3 \mu)\sigma + (\eta_{s1}^2 \lambda_3^2 - 2u_s (1+\theta)\mu^2)\sigma^2 - 4r^3 u_s (1+\theta)(\mu+\sigma) +$$
$$r^2 (\gamma_3 \eta_{s2} + \eta_{s1} \lambda_3)^2 - 2u_s (1+\theta)(\mu^2 + 4\mu\sigma + \sigma^2) + 2r(\epsilon \eta_{s1} \eta_{s2} \lambda_3^2 + \gamma_3^2 \eta_{s2}^2 \mu + \eta_{s1}^2 \lambda_3^2 \sigma - 2u_s (1+\theta)\mu\sigma(\mu+\sigma) +$$
$$\gamma_3 \eta_{s2} \lambda_3 (\epsilon \eta_{s2} + \eta_{s1} (\mu+\sigma))) + 2(r+\sigma)(\pi_d u_g u_p u_s \lambda_1 (\epsilon\lambda_1 + \gamma_1 (r+\mu)) + \pi_s u_g u_p (r(-u_s \gamma_1 \lambda_1 + u_d \gamma_3 \lambda_3) -$$
$$u_s \lambda_1 (\epsilon\lambda_1 + \gamma_1 \mu) + u_d \lambda_3 (\epsilon\lambda_3 + \gamma_3 \mu)) + u_s (\pi_p u_g u_d \lambda_2 (\epsilon\lambda_2 + \gamma_2 (r+\mu)) - cu_g (u_p \lambda_1^2 - u_d \lambda_2^2)(r+\sigma) +$$
$$\pi_g u_d u_p \lambda_4^2 (r+\sigma) + p_d u_g u_p \lambda_1 (r\gamma_1 \eta_{d2} + r\eta_{d1} \lambda_1 + \epsilon \eta_{d2} \lambda_1 + \gamma_1 \eta_{d2} \mu + \eta_{d1} \lambda_1 \sigma))\omega - u_g u_p (2u_s \lambda_1^2 - u_d \lambda_3^2)(r+\sigma)^2 \omega^2 +$$
$$2p_s (\pi_s r^2 u_g u_d u_p \gamma_3^2 \eta_{s2} + p_d r^2 u_g u_p u_s \gamma_1^2 \eta_{d2} \eta_{s2} + p_d r^4 u_g u_d u_p u_s \theta + p_d r^2 u_g u_p u_s \gamma_1 \eta_{d1} \lambda_1 \eta_{s2} +$$
$$r^2 u_g u_d u_p u_s \theta \eta_{s2} (\eta_{s2}^2 - 2u_s (1+\theta)(\mu+\sigma)) + \eta_{s2} (p_d u_g u_p u_s \gamma_1 \eta_{d2} \lambda_1 \eta_{s2} +$$
$$2r^2 u_g u_d u_p \gamma_3 \lambda_3 \eta_{s2} + 2r u_g u_d u_p u_s \theta (\mu+\sigma)) - 2c\pi_s r^2 u_g u_p u_s \gamma_1 \lambda_1.$$

$$pol_{ng} = \frac{1}{2ru_g u_d u_p u_s (r+\mu)^2 (r+\sigma)^2} \pi_g (2u_g u_p u_s \gamma_1 (\pi_d + p_d \eta_{d2}) \lambda_1 (r+\mu) + 2u_g u_p u_s \lambda_1^2 (\pi_d \epsilon +$$
$$p_d \epsilon \eta_{d2} - c(r+\sigma) + p_d \eta_{d1} (r+\sigma) - (r+\sigma)\omega)) + u_d (2\pi_p u_g u_s \gamma_2 \lambda_2 (r+\mu) +$$
$$2u_g u_s \lambda_2^2 (\pi_p \epsilon + c(r+\sigma)) + u_p (2u_g \gamma_3 (\pi_s + p_s \eta_{s2}) \lambda_3 (r+\mu) + \pi_g u_s \lambda_4^2 (r+\sigma) +$$
$$2u_g \lambda_3^2 (\pi_s \epsilon + p_s \epsilon \eta_{s2} + p_s \eta_{s1} (r+\sigma) + (r+\sigma)\omega))).$$

$$pol_{sd} = -\frac{1}{4ru_g u_d u_p u_s (r+\mu)^2 (r+\sigma)^2} (-\pi_d^2 r^2 u_g u_p u_s \gamma_1^2 - 2\pi_d \pi_p r^2 u_g u_p u_s \gamma_1^2 - 2\pi_d \pi_p r^2 u_g u_d u_s \gamma_2^2 -$$
$$4\pi_d \pi_s r^2 u_g u_d u_p \gamma_3^2 - 4\pi_d p_s r^2 u_g u_d u_p \gamma_3^2 \eta_{s2} + 2c\pi_p r^2 u_g u_p u_s \gamma_1 \lambda_1 - 2\pi_d^2 r u_g u_p u_s \gamma_1 \epsilon \lambda_1 - 4\pi_d \pi_p r u_g u_p u_s \gamma_1 \epsilon \lambda_1 +$$
$$c^2 r^2 u_g u_p u_s \lambda_1^2 + 2c\pi_p r u_g u_p u_s \epsilon \lambda_1^2 - \pi_d^2 u_g u_p u_s \epsilon^2 \lambda_1^2 - 2\pi_d \pi_p u_g u_p u_s \epsilon^2 \lambda_1^2 - 2c\pi_d r^2 u_g u_d u_s \gamma_2 \lambda_2 -$$
$$4\pi_d \pi_g r^2 u_g u_d u_s \gamma_2 \lambda_2 + 2c\pi_p r^2 u_g u_d u_s \gamma_2 \lambda_2 - 4\pi_d \pi_p r u_g u_d u_s \gamma_2 \epsilon \lambda_2 + 2c^2 r^2 u_g u_d u_s \lambda_2^2 + 4c\pi_g r^2 u_g u_d u_s \lambda_2^2 -$$
$$2c\pi_d r u_g u_d u_s \epsilon \lambda_2^2 - 4\pi_d \pi_g r u_g u_d u_s \epsilon \lambda_2^2 + 2c\pi_p r u_g u_d u_s \epsilon \lambda_2^2 - 2\pi_d \pi_p u_g u_d u_s \epsilon^2 \lambda_2^2 + 4c\pi_s r^2 u_g u_d u_p \gamma_3 \lambda_3 -$$
$$8\pi_d \pi_s r u_g u_d u_p \gamma_3 \epsilon \lambda_3 - 4\pi_d p_s r^2 u_g u_d u_p \gamma_3 \eta_{s1} \lambda_3 + 4c p_s r^2 u_g u_d u_p \gamma_3 \eta_{s2} \lambda_3 - 8\pi_d p_s r u_g u_d u_p \gamma_3 \epsilon \eta_{s2} \lambda_3 +$$
$$4c\pi_s r u_g u_d u_p \epsilon \lambda_3^2 - 4\pi_d \pi_s u_g u_d u_p \epsilon^2 \lambda_3^2 + 4c p_s r^2 u_g u_d u_p \eta_{s1} \lambda_3^2 - 4\pi_d p_s r u_g u_d u_p \epsilon \eta_{s1} \lambda_3^2 -$$
$$4\pi_d \pi_g r u_g u_d u_s \gamma_2 \lambda_2 \mu + 2c\pi_p r u_g u_d u_s \gamma_2 \lambda_2 \mu + 4c p_s r u_g u_d u_p \epsilon \eta_{s2} \lambda_3^2 - 4\pi_d p_s u_g u_d u_p \epsilon^2 \eta_{s2} \lambda_3^2 +$$
$$4c\pi_g r^2 u_d u_p u_s \lambda_4^2 - 4\pi_d \pi_g r u_d u_p u_s \epsilon \lambda_4^2 - 2\pi_d^2 r u_g u_p u_s \gamma_1^2 \mu - 4\pi_d \pi_p r u_g u_p u_s \gamma_1^2 \mu - 4\pi_d \pi_p r u_g u_d u_s \gamma_2^2 \mu -$$
$$8\pi_d \pi_s r u_g u_d u_p \gamma_3^2 \mu - 8\pi_d p_s r u_g u_d u_p \gamma_3^2 \eta_{s2} \mu + 2c\pi_p r u_g u_p u_s \gamma_1 \lambda_1 \mu - 2\pi_d^2 u_g u_p u_s \gamma_1 \epsilon \lambda_1 \mu -$$
$$4\pi_d \pi_p u_g u_p u_s \gamma_1 \epsilon \lambda_1 \mu - 2c\pi_d r u_g u_d u_s \gamma_2 \lambda_2 \mu - 4\pi_d \pi_p u_g u_d u_s \gamma_2 \epsilon \lambda_2 \mu + 4c\pi_s r u_g u_d u_p \gamma_3 \lambda_3 \mu - 8\pi_d \pi_s u_g u_d u_p \gamma_3 \epsilon \lambda_3 \mu -$$
$$4\pi_d p_s r u_g u_d u_p \gamma_3 \eta_{s1} \lambda_3 \mu - 2\pi_d \pi_p u_g u_d u_s \gamma_2^2 \mu^2 \sigma^2 - 4\pi_d \pi_s u_g u_d u_p \gamma_3^2 \mu^2 \sigma^2 - 4\pi_d p_s u_g u_d u_p \gamma_3^2 \eta_{s2} \mu^2 \sigma^2 +$$
$$4c p_s r u_g u_d u_p \gamma_3 \eta_{s2} \lambda_3 \mu - 8\pi_d p_s u_g u_d u_p \gamma_3 \epsilon \eta_{s2} \lambda_3 \mu - \pi_d^2 u_g u_p u_s \gamma_1^2 \mu^2 - 2\pi_d \pi_p u_g u_p u_s \gamma_1^2 \mu^2 - 2\pi_d \pi_p u_g u_d u_s \gamma_2^2 \mu^2 -$$
$$4\pi_d \pi_s u_g u_d u_p \gamma_3^2 \mu^2 - 4\pi_d p_s u_g u_d u_p \gamma_3^2 \eta_{s2} \mu^2 + 2c\pi_p r u_g u_p u_s \gamma_1 \lambda_1 \sigma + 2c^2 r u_g u_p u_s \lambda_1^2 \sigma + 2c\pi_p u_g u_p u_s \epsilon \lambda_1^2 \sigma -$$
$$2c\pi_d r u_g u_d u_s \gamma_2 \lambda_2 \sigma - 4\pi_d \pi_g r u_g u_d u_s \gamma_2 \lambda_2 \sigma + 2c\pi_p r u_g u_d u_s \gamma_2 \lambda_2 \sigma + 4c^2 r u_g u_d u_s \lambda_2^2 \sigma + 8c\pi_g r u_g u_d u_s \lambda_2^2 \sigma -$$
$$2c\pi_d u_g u_d u_s \epsilon \lambda_2^2 \sigma - 4\pi_d \pi_g u_g u_d u_s \epsilon \lambda_2^2 \sigma + 2c\pi_p u_g u_d u_s \epsilon \lambda_2^2 \sigma + 4c\pi_s r u_g u_d u_p \gamma_3 \lambda_3 \sigma + 4c p_s r u_g u_d u_p \gamma_3 \eta_{s2} \lambda_3 \sigma -$$
$$8\pi_d \pi_s u_g u_d u_p \gamma_3 \epsilon \lambda_3 \sigma - 8\pi_d p_s u_g u_d u_p \gamma_3 \epsilon \eta_{s2} \lambda_3 \sigma + 4c\pi_s u_g u_d u_p \epsilon \lambda_3^2 \sigma + 4c p_s r u_g u_d u_p \eta_{s1} \lambda_3^2 \sigma -$$
$$4\pi_d p_s u_g u_d u_p \epsilon \eta_{s1} \lambda_3^2 \sigma + 4c p_s u_g u_d u_p \epsilon \eta_{s2} \lambda_3^2 \sigma - 4\pi_d p_s u_g u_d u_p \epsilon^2 \eta_{s2} \lambda_3^2 \sigma - 2\pi_d^2 r u_g u_p u_s \gamma_1^2 \sigma^2 -$$
$$4\pi_d \pi_p r u_g u_p u_s \gamma_1^2 \sigma^2 - 4\pi_d \pi_p r u_g u_d u_s \gamma_2^2 \sigma^2 - 8\pi_d \pi_s r u_g u_d u_p \gamma_3^2 \sigma^2 - 8\pi_d p_s r u_g u_d u_p \gamma_3^2 \eta_{s2} \sigma^2 -$$
$$2\pi_d^2 u_g u_p u_s \gamma_1^2 \mu \sigma - 4\pi_d \pi_p u_g u_p u_s \gamma_1^2 \mu \sigma - 4\pi_d \pi_p u_g u_d u_s \gamma_2^2 \mu \sigma - 8\pi_d \pi_s u_g u_d u_p \gamma_3^2 \mu \sigma -$$
$$8\pi_d p_s u_g u_d u_p \gamma_3^2 \eta_{s2} \mu \sigma - 2\pi_d^2 u_g u_p u_s \gamma_1^2 \mu \sigma^2 - 4\pi_d \pi_p u_g u_p u_s \gamma_1^2 \mu \sigma^2 - 4\pi_d \pi_p u_g u_d u_s \gamma_2^2 \mu \sigma^2 -$$
$$8\pi_d \pi_s u_g u_d u_p \gamma_3^2 \mu \sigma^2 - 8\pi_d p_s u_g u_d u_p \gamma_3^2 \eta_{s2} \mu \sigma^2 - \pi_d^2 u_g u_p u_s \gamma_1^2 \mu^2 \sigma^2 - 2\pi_d \pi_p u_g u_p u_s \gamma_1^2 \mu^2 \sigma^2).$$

$$pol_{sp} = \frac{1}{8ru_gu_du_pu_s(r+\mu)^2(r+\sigma)^2}(u_gu_pu_s\gamma_1^2(\pi_d+p_d\eta_{d2})^2(r+\mu)^2+2\pi_p^2u_gu_s(r^2(2u_p\gamma_1^2+u_d\gamma_2^2)+$$
$$4ru_p\gamma_1(\epsilon\lambda_1+\gamma_1\mu)+2u_p(\epsilon\lambda_1+\gamma_1\mu)^2+2ru_d\gamma_2(\epsilon\lambda_2+\gamma_2\mu)+u_d(\epsilon\lambda_2+\gamma_2\mu)^2)+$$
$$2c^2u_gu_s(2u_p\lambda_1^2+u_d\lambda_2^2)(r+\sigma)^2+u_gu_pu_s\lambda_1^2(\pi_d\epsilon+p_dr\eta_{d1}+p_d\epsilon\eta_{d2}+p_d\eta_{d1}\sigma-c(r+\sigma)-r\omega-\sigma\omega)^2+$$
$$2u_gu_pu_s\gamma_1(\pi_d+p_d\eta_{d2})\lambda_1(r+\mu)(\pi_d\epsilon+p_dr\eta_{d1}+p_d\epsilon\eta_{d2}+p_d\eta_{d1}\sigma+c(r+\sigma)-r\omega-\sigma\omega)+$$
$$\frac{1}{r+\mu}(4\pi_p(2r^2u_gu_du_p\gamma_3^2(\pi_s+p_s\eta_{s2})(r+\mu)+2ru_gu_du_p\gamma_3\epsilon(\pi_s+p_s\eta_{s2})\lambda_3(r+\mu)+$$
$$4ru_gu_du_p\gamma_3^2(\pi_s+p_s\eta_{s2})\mu(r+\mu)+2u_gu_du_p\gamma_3\epsilon(\pi_s+p_s\eta_{s2})\lambda_3\mu(r+\mu)+$$
$$2u_gu_du_p\gamma_3^2(\pi_s+p_s\eta_{s2})\mu^2(r+\mu)+u_gu_pu_s\gamma_1(\pi_d+p_d\eta_{d2})(r+\mu)^2(r\gamma_1+\epsilon\lambda_1+\gamma_1\mu)+$$
$$\pi_gr^2u_gu_du_s\gamma_2\lambda_2(r+\sigma)+\pi_gru_gu_du_s\epsilon\lambda_2^2(r+\sigma)+2\pi_gru_du_pu_s\epsilon\lambda_4^2(r+\sigma)+2\pi_gru_gu_du_s\gamma_2\lambda_2\mu(r+\sigma)+$$
$$\pi_gu_gu_du_s\epsilon\lambda_2^2\mu(r+\sigma)+2\pi_gu_du_pu_s\epsilon\lambda_4^2\mu(r+\sigma)+\pi_gu_gu_du_s\gamma_2\lambda_2\mu^2(r+\sigma)+$$
$$2cru_gu_pu_s\gamma_1\lambda_1(r+\mu)(r+\sigma)+2cu_gu_pu_s\epsilon\lambda_1^2(r+\mu)(r+\sigma)+cru_gu_du_s\gamma_2\lambda_2(r+\mu)(r+\sigma)+$$
$$cu_gu_du_s\epsilon\lambda_2^2(r+\mu)(r+\sigma)+2cu_gu_pu_s\gamma_1\lambda_1\mu(r+\mu)(r+\sigma)+cu_gu_du_s\gamma_2\lambda_2\mu(r+\mu)(r+\sigma)+$$
$$r^2u_gu_pu_s\gamma_1\lambda_1(\pi_d\epsilon+p_dr\eta_{d1}+p_d\epsilon\eta_{d2}+p_d\eta_{d1}\sigma-c(r+\sigma)-r\omega-\sigma\omega)+$$
$$ru_gu_pu_s\epsilon\lambda_1^2(\pi_d\epsilon+p_dr\eta_{d1}+p_d\epsilon\eta_{d2}+p_d\eta_{d1}\sigma-c(r+\sigma)-r\omega-\sigma\omega)+$$
$$2ru_gu_pu_s\gamma_1\lambda_1\mu(\pi_d\epsilon+p_dr\eta_{d1}+p_d\epsilon\eta_{d2}+p_d\eta_{d1}\sigma-c(r+\sigma)-r\omega-\sigma\omega)+$$
$$u_gu_pu_s\epsilon\lambda_1^2\mu(\pi_d\epsilon+p_dr\eta_{d1}+p_d\epsilon\eta_{d2}+p_d\eta_{d1}\sigma-c(r+\sigma)-r\omega-\sigma\omega)+$$
$$u_gu_pu_s\gamma_1\lambda_1\mu^2(\pi_d\epsilon+p_dr\eta_{d1}+p_d\epsilon\eta_{d2}+p_d\eta_{d1}\sigma-c(r+\sigma)-r\omega-\sigma\omega)+$$
$$2r^2u_gu_du_p\gamma_3\lambda_3(\pi_s\epsilon+p_s(r\eta_{s1}+\epsilon\eta_{s2}+\eta_{s1}\sigma)+(r+\sigma)\omega)+$$
$$2ru_gu_du_p\epsilon\lambda_3^2(\pi_s\epsilon+p_s(r\eta_{s1}+\epsilon\eta_{s2}+\eta_{s1}\sigma)+(r+\sigma)\omega)+$$
$$4ru_gu_du_p\gamma_3\lambda_3\mu(\pi_s\epsilon+p_s(r\eta_{s1}+\epsilon\eta_{s2}+\eta_{s1}\sigma)+(r+\sigma)\omega)+$$
$$2u_gu_du_p\epsilon\lambda_3^2\mu(\pi_s\epsilon+p_s(r\eta_{s1}+\epsilon\eta_{s2}+\eta_{s1}\sigma)+(r+\sigma)\omega)+$$
$$2u_gu_du_p\gamma_3\lambda_3\mu^2(\pi_s\epsilon+p_s(r\eta_{s1}+\epsilon\eta_{s2}+\eta_{s1}\sigma)+(r+\sigma)\omega))))+$$
$$4c(r+\sigma)(u_gu_pu_s\lambda_1^2(\pi_d\epsilon+p_dr\eta_{d1}+p_d\epsilon\eta_{d2}+p_d\eta_{d1}\sigma-c(r+\sigma)-r\omega-\sigma\omega)+$$
$$u_d(\pi_gu_s(u_g\lambda_2^2+2u_p\lambda_4^2)(r+\sigma)+2u_gu_p\lambda_3(\gamma_3(\pi_s+p_s\eta_{s2})(r+\mu)+\lambda_3(\pi_s\epsilon+p_s(r\eta_{s1}+\epsilon\eta_{s2}+\eta_{s1}\sigma)+(r+\sigma)\omega))))).$$

$$pol_{ss} = \frac{1}{2ru_g u_d u_p u_s (r+\mu)^2 (r+\sigma)^2}((\pi_d \pi_s r^2 u_g u_p u_s \gamma_1^2 + 2\pi_p \pi_s r^2 u_g u_p u_s \gamma_1^2 + \pi_p \pi_s r^2 u_g u_d u_s \gamma_2^2$$
$$+\pi_s^2 r^2 u_g u_d u_p \gamma_3^2 + p_d \pi_s r^2 u_g u_p u_s \gamma_1^2 \eta_{d2} + c\pi_s r^2 u_g u_p u_s \gamma_1 \lambda_1 + 2\pi_d \pi_s r u_g u_p u_s \gamma_1 \epsilon \lambda_1 + 4\pi_p \pi_s r u_g u_p u_s \gamma_1 \epsilon \lambda_1$$
$$+p_d \pi_s r^2 u_g u_p u_s \gamma_1 \eta_{d1} \lambda_1 + 2p_d \pi_s r u_g u_p u_s \gamma_1 \epsilon \eta_{d2} \lambda_1 + c\pi_s r u_g u_p u_s \epsilon \lambda_1^2 + \pi_d \pi_s u_g u_p u_s \epsilon^2 \lambda_1^2$$
$$+2\pi_p \pi_s u_g u_p u_s \epsilon^2 \lambda_1^2 + p_d \pi_s r u_g u_p u_s \epsilon \eta_{d1} \lambda_1^2 + p_d \pi_s u_g u_p u_s \epsilon^2 \eta_{d2} \lambda_1^2 + c\pi_s r^2 u_g u_d u_s \gamma_2 \lambda_2 + 2\pi_g \pi_s r^2 u_g u_d u_s \gamma_2 \lambda_2$$
$$+2\pi_p \pi_s r u_g u_d u_s \gamma_2 \epsilon \lambda_2 + c\pi_s r u_g u_d u_s \epsilon \lambda_2^2 + 2\pi_g \pi_s r u_g u_d u_s \epsilon \lambda_2^2 + \pi_p \pi_s u_g u_d u_s \epsilon^2 \lambda_2^2 + 2\pi_s^2 r u_g u_d u_p \gamma_3 \epsilon \lambda_3$$
$$+\pi_s^2 u_g u_d u_p \epsilon^2 \lambda_3^2 + 2\pi_g \pi_s r u_d u_p u_s \epsilon \lambda_4^2 + 2\pi_d \pi_s r u_g u_p u_s \gamma_1^2 \mu + 4\pi_p \pi_s r u_g u_p u_s \gamma_1^2 \mu + 2\pi_p \pi_s r u_g u_d u_s \gamma_2^2 \mu$$
$$+2\pi_s^2 r u_g u_d u_p \gamma_3^2 \mu + 2p_d \pi_s r u_g u_p u_s \gamma_1^2 \eta_{d2} \mu + c\pi_s r u_g u_p u_s \gamma_1 \lambda_1 \mu + 2\pi_d \pi_s u_g u_p u_s \gamma_1 \epsilon \lambda_1 \mu + 4\pi_p \pi_s u_g u_p u_s \gamma_1 \epsilon \lambda_1 \mu$$
$$+p_d \pi_s r u_g u_p u_s \gamma_1 \eta_{d1} \lambda_1 \mu + 2p_d \pi_s u_g u_p u_s \gamma_1 \epsilon \eta_{d2} \lambda_1 \mu + c\pi_s r u_g u_d u_s \gamma_2 \lambda_2 \mu + 2\pi_g \pi_s r u_g u_d u_s \gamma_2 \lambda_2 \mu$$
$$+2\pi_p \pi_s u_g u_d u_s \gamma_2 \epsilon \lambda_2 \mu + 2\pi_s^2 u_g u_d u_p \gamma_3 \epsilon \lambda_3 \mu + \pi_d \pi_s u_g u_p u_s \gamma_1^2 \mu^2 + 2\pi_p \pi_s u_g u_p u_s \gamma_1^2 \mu^2 + \pi_p \pi_s u_g u_d u_s \gamma_2^2 \mu^2$$
$$+\pi_s^2 u_g u_d u_p \gamma_3^2 \mu^2 + p_d \pi_s u_g u_p u_s \gamma_1^2 \eta_{d2} \mu^2 + c\pi_s r u_g u_p u_s \gamma_1 \lambda_1 \sigma + p_d \pi_s r u_g u_p u_s \gamma_1 \eta_{d1} \lambda_1 \sigma + c\pi_s u_g u_p u_s \epsilon \lambda_1^2 \sigma$$
$$+p_d \pi_s u_g u_p u_s \epsilon \eta_{d1} \lambda_1^2 \sigma + c\pi_s r u_g u_d u_s \gamma_2 \lambda_2 \sigma + 2\pi_g \pi_s r u_g u_d u_s \gamma_2 \lambda_2 \sigma + c\pi_s u_g u_d u_s \epsilon \lambda_2^2 \sigma + 2\pi_g \pi_s u_g u_d u_s \epsilon \lambda_2^2 \sigma$$
$$+2\pi_g \pi_s u_d u_p u_s \epsilon \lambda_4^2 \sigma + c\pi_s u_g u_p u_s \gamma_1 \lambda_1 \mu \sigma + p_d \pi_s u_g u_p u_s \gamma_1 \eta_{d1} \lambda_1 \mu \sigma + c\pi_s u_g u_d u_s \gamma_2 \lambda_2 \mu \sigma + 2\pi_g \pi_s u_g u_d u_s \gamma_2 \lambda_2 \mu \sigma$$
$$+2ap_s u_g u_d u_p u_s (r+\mu)^2 (r+\sigma)^2 + p_s^2 u_g u_d u_p (-2r^4 u_s (1+\theta) + \epsilon^2 \eta_{s2}^2 \lambda_3^2 + 2\gamma_3 \epsilon \eta_{s2}^2 \lambda_3 \mu + \gamma_3^2 \eta_{s2}^2 \mu^2$$
$$+2\epsilon \eta_{s1} \eta_{s2} \lambda_3^2 \sigma + 2\gamma_3 \eta_{s1} \eta_{s2} \lambda_3 \mu \sigma + \eta_{s1}^2 \lambda_3^2 \sigma^2 - 2u_s \mu^2 \sigma^2 - 2u_s \theta \mu^2 \sigma^2 - 4r^3 u_s (1+\theta)(\mu+\sigma)$$
$$+r^2 (\gamma_3^2 \eta_{s2}^2 + 2\gamma_3 \eta_{s1} \eta_{s2} \lambda_3 + \eta_{s1}^2 \lambda_3^2 - 2u_s (1+\theta)(\mu^2 + 4\mu\sigma + \sigma^2))$$
$$+2r(\epsilon \eta_{s1} \eta_{s2} \lambda_3^2 + \gamma_3^2 \eta_{s2}^2 \mu + 2\gamma_3 \eta_{s1} \eta_{s2} \lambda_3 \mu + \eta_{s1}^2 \lambda_3^2 \mu \sigma - 2u_s (1+\theta)(\mu^3 + \sigma(\mu^2 + \mu\sigma)))$$
$$+\epsilon \eta_{s1}^2 \lambda_3^2 \sigma + 2\epsilon \eta_{s2} \gamma_3 \eta_{s2} \lambda_3 (\mu+\sigma) + 2\epsilon \eta_{s1} \eta_{s2} \lambda_3 (\mu^2 + \sigma(\mu+\sigma))$$
$$+2\gamma_3 \eta_{s1} \eta_{s2} \lambda_3 (\mu^2 + \sigma(\mu+\sigma)) + \eta_{s1}^2 \lambda_3^2 (\mu^2 + \sigma^2 + 2\mu\sigma)))).$$

$$pol_{sg} = \frac{1}{8ru_g u_d u_p u_s (r+\mu)^2 (r+\sigma)^2}(\pi_p^2 u_g u_d u_s (r\gamma_2 + \epsilon\lambda_2 + \gamma_2\mu)^2 + 2\pi_p u_g u_s (cu_d \lambda_2 (r\gamma_2 + \epsilon\lambda_2 + \gamma_2\mu) +$$
$$2\pi_g (r(2u_p \gamma_1 \lambda_1 + u_d \gamma_2 \lambda_2) + 2u_p \lambda_1 (\epsilon\lambda_1 + \gamma_1 \mu) + u_d \lambda_2 (\epsilon\lambda_2 + \gamma_2 \mu)))(r+\sigma) + (r+\sigma)(4c\pi_g r u_g u_p u_s \lambda_1^2 +$$
$$c^2 r u_g u_d u_s \lambda_2^2 + 4c\pi_g r u_g u_d u_s \lambda_2^2 + 4\pi_g^2 r u_g u_d u_s \lambda_2^2 + 8\pi_g \pi_s r u_g u_d u_p \gamma_3 \lambda_3 + 8\pi_g p_s r u_g u_d u_p \gamma_3 \eta_{s2} \lambda_3 +$$
$$8\pi_g \pi_s u_g u_d u_p \epsilon \lambda_3^2 + 8\pi_g p_s r u_g u_d u_p \eta_{s1} \lambda_3^2 + 8\pi_g p_s u_g u_d u_p \epsilon \eta_{s2} \lambda_3^2 + 4\pi_g^2 r u_d u_p u_s \lambda_4^2 + 8\pi_g \pi_s u_g u_d u_p \gamma_3 \lambda_3 \mu +$$
$$8\pi_g p_s u_g u_d u_p \gamma_3 \eta_{s2} \lambda_3 \mu + 4\pi_d \pi_g u_g u_p u_s \lambda_1 (r\gamma_1 + \epsilon\lambda_1 + \gamma_1 \mu) + 4c\pi_g u_g u_p u_s \lambda_1^2 \sigma + c^2 u_g u_d u_s \lambda_2^2 \sigma +$$
$$4c\pi_g u_g u_d u_s \lambda_2^2 \sigma + 4\pi_g^2 u_g u_d u_s \lambda_2^2 \sigma + 8\pi_g p_s u_g u_d u_p \eta_{s1} \lambda_3^2 \sigma + 4\pi_g^2 u_d u_p u_s \lambda_4^2 \sigma +$$
$$4p_d \pi_g u_g u_p u_s \lambda_1 (r\gamma_1 \eta_{d2} + r\eta_{d1} \lambda_1 + \epsilon \eta_{d2} \lambda_1 + \gamma_1 \eta_{d2} \mu + \eta_{d1} \lambda_1 \sigma) -$$
$$4\pi_g r u_g u_p u_s \lambda_1^2 \omega + 8\pi_g r u_g u_d u_p \lambda_3^2 \omega - 4\pi_g u_g u_p u_s \lambda_1^2 \sigma \omega + 8\pi_g u_g u_d u_p \lambda_3^2 \sigma \omega)).$$

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