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A Turnpike Property of Trajectories of Dynamical Systems with a Lyapunov Function

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Received: 29 September 2020; Accepted: 7 December 2020; Published: 14 December 2020

Abstract: In this paper, we study the structure of trajectories of discrete disperse dynamical systems with a Lyapunov function which are generated by set-valued mappings. We establish a weak version of the turnpike property which holds for all trajectories of such dynamical systems which are of a sufficient length. This result is usually true for models of economic growth which are prototypes of our dynamical systems.

Keywords: compact metric space; global attractor; lyapunov function; set-valued mapping; turnpike

1. Introduction

In [1,2] A. M. Rubinov introduced a discrete disperse dynamical system determined by a set-valued mapping acting on a compact metric space, which was studied in [1–7]. This disperse dynamical system has prototype in the mathematical economics [1,8,9]. In particular, it is an abstract extension of the classical von Neumann–Gale model [1,8,9]. Our dynamical system is determined by a compact metric space of states and a transition operator. In [1–7] and in the present paper, this transition operator is set-valued. Such dynamical systems correspond to certain models of economic dynamics [1,8,9].

Assume that (X, ρ) is a compact metric space and that $a : X \to 2^X \setminus \{\emptyset\}$ is a set-valued mapping whose graph

$$\text{graph}(a) = \{(x, y) \in X \times X : \ y \in a(x)\}$$

is a closed set in $X \times X$. For every nonempty set $E \subset X$ define

$$a(E) = \cup\{a(x) : \ x \in E\} \text{ and } a^0(E) = E.$$

By induction we define $a^n(E)$ for every integer $n \geq 1$ and every nonempty subset $E \subset X$ as follows:

$$a^n(E) = a(a^{n-1}(E)).$$

In the present paper, we analyze the structure of trajectories of the dynamical system determined by a which is called a discrete dispersive dynamical system [1,2].

We say that a sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is a trajectory of a (or just a trajectory if a is understood) if

$$x_{t+1} \in a(x_t), \ t = 0, 1, \ldots .$$

Let $T_2 > T_1$ be integers. We say that $\{x_t\}_{t=T_1}^{T_2} \subset X$ is a trajectory of a (or just a trajectory if a is understood) if

$$x_{t+1} \in a(x_t), \ t = T_1, \ldots, T_2 - 1.$$

Define

$$\Omega(a) = \{\xi \in X : \text{ for every positive number } \epsilon \text{ there exists a trajectory}$$

$$\{y_t\}_{t=0}^{\infty} \text{ for whicht } \liminf_{t\to\infty} \rho(\xi, y_t) \le \epsilon\}. \tag{1}$$

Evidently, $\Omega(a)$ is a nonempty closed set in the metric space (X, ρ). In the literature, the set $\Omega(a)$ is called a global attractor of a. Note that in [1,2] $\Omega(a)$ is called a turnpike set of a. This terminology is motivated by mathematical economics [1,8,9].

For every point $x \in X$ and every nonempty closed set $E \subset X$ define

$$\rho(x, E) = \inf\{\rho(x, y) : y \in E\}.$$

Let $\phi : X \to R^1$ be a continuous function satisfying

$$\phi(z) \ge 0 \text{ for every } z \in X, \tag{2}$$

$$\phi(y) \le \phi(x) \text{ for every } x \in X \text{ and every } y \in a(x). \tag{3}$$

It is clear that ϕ is a Lyapunov function for the dynamical system determined by the map a. It should be mentioned that in mathematical economics usually X is a subset of the finite-dimensional Euclidean space and ϕ is a linear functional on this space [1,8,9]. Our goal in [7] was to study approximate solutions of the problem

$$\phi(x_T) \to \max,$$

$$\{x_t\}_{t=0}^{T} \text{ is a program satisfying } x_0 = x,$$

where $x \in X$ and $T \in \{1, 2, \dots\}$ are given.

The following result was obtained in [7].

Theorem 1. *The following properties are equivalent:*

(1) If a sequence $\{x_t\}_{t=-\infty}^{\infty} \subset X$, $x_{t+1} \in a(x_t)$ and $\phi(x_{t+1}) = \phi(x_t)$ for every integer t, then

$$\{x_t\}_{t=-\infty}^{\infty} \subset \Omega(a).$$

(2) For every positive number ϵ there exists an integer $T(\epsilon) \ge 1$ such that for every trajectory $\{x_t\}_{t=0}^{\infty} \subset X$ which satisfies $\phi(x_t) = \phi(x_{t+1})$ for every nonnegative integer t the relation $\rho(x_t, \Omega(a)) \le \epsilon$ is valid for every integer $t \ge T(\epsilon)$.

Put

$$\|\phi\| = \sup\{|\phi(z)| : z \in X\}.$$

We denote by $\mathrm{Card}(A)$ the cardinality of a set A and suppose that the sum over the empty set is zero.

In this paper, we establish a weak version of the turnpike property which hold for all trajectories of our dynamical system which are of a sufficient length and which are not necessarily approximate solutions of the problem above. This result as well as the turnpike results of [7] is usually true for models of economic growth which are prototypes of our dynamical system [1,8,9].

Namely, we prove the following result.

Theorem 2. *Let property (1) of Theorem 1 hold and let ϵ be a positive number. Then there exists an integer $L \ge 1$ such that for every natural number $T > L$ and every trajectory $\{x_t\}_{t=0}^{T}$ the inequality*

$$Card(\{t \in \{0, \dots, T\} : \rho(x_t, \Omega(a)) > \epsilon\}) \le L$$

is valid.

This result is proved in Section 3. Its proof is based on an auxiliary result which is proved in Section 2.

Assume that $\{x_t\}_{t=0}^{\infty}$ is a trajectory. By (3), there exists

$$c = \lim_{t\to\infty} \phi(x_t).$$

Evidently, the sequence $\{x_t\}_{t=0}^{\infty}$ converges to the set $\Omega \cap \phi^{-1}(c)$. This fact is well-know in the dynamical systems theory as LaSalle's invariance principle [10–13]. In the present paper, we are interested in the structure of trajectories on finite intervals of a sufficiently large length and their turnpike property established in Theorem 1.2, which was not considered in [10–13].

It should be mentioned that turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [14]), where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path and a turnpike). This property was further investigated for optimal trajectories of models of economic dynamics. See, for example, [2,8,9] and the references mentioned there. Recently it was shown that the turnpike phenomenon holds for many important classes of problems arising in various areas of research [15–23]. For related infinite horizon problems see [9,24–31].

2. An Auxiliary Result

Lemma 1. *Let property (1) of Theorem 1 hold and ϵ be a positive number. Then there exist a positive number δ and an integer $L \geq 1$ such that for every natural number $T > 2L$ and every trajectory $\{x_t\}_{t=0}^{T}$ satisfying*

$$\phi(x_0) \leq \phi(x_T) + \delta$$

the inequality

$$\rho(x_t, \Omega(a)) \leq \epsilon, \; t = L, \dots, T - L$$

is valid.

Proof. Assume the contrary. Then for every integer $n \geq 1$ there are a natural number $T_n > 2n$ and a trajectory $\{x_t^{(n)}\}_{t=0}^{T_n}$ which satisfy

$$\phi(x_0^{(n)}) \leq \phi(x_{T_n}^{(n)}) + 1/n, \tag{4}$$

$$\max\{\rho(x_t^{(n)}, \Omega(a)) : \; t = n, \dots, T_n - n\} > \epsilon. \tag{5}$$

By of (5), for every $n \in \{1, 2, \dots\}$ there is

$$S_n \in \{n, \dots, T_n - n\} \tag{6}$$

for which

$$\rho(x_{S_n}^{(n)}, \Omega(a)) > \epsilon. \tag{7}$$

Assume that $n \in \{1, 2, \dots\}$. Set

$$y_t^{(n)} = x_{t+S_n}^{(n)}, \; t = -S_n, \dots, T_n - S_n. \tag{8}$$

In view of (8), $\{y_t^{(n)}\}_{t=-S_n}^{T_n - S_n}$ is a trajectory. By (4) and (8),

$$\phi(y_{T_n - S_n}^{(n)}) - \phi(y_{-S_n}^{(n)}) = \phi(x_{T_n}^{(n)}) - \phi(x_0^{(n)}) \geq -1/n. \tag{9}$$

Equations (3) and (9) imply that for every integer $t \in \{-S_n, \dots, T_n - S_n - 1\}$, we have

$$\phi(y_{t+1}^{(n)}) - \phi(y_t^{(n)}) \geq \phi(y_{T_n - S_n}^{(n)}) - \phi(y_{-S_n}^{(n)}) \geq -1/n. \tag{10}$$

Equations (7) and (8) imply that

$$\rho(y_0^{(n)}, \Omega(a)) = \rho(x_{S_n}^{(n)}, \Omega(a)) > \epsilon. \tag{11}$$

Clearly, there is a strictly increasing sequence of positive integers $\{n_j\}_{j=1}^{\infty}$ such that for every integer t there exists

$$y_t = \lim_{j \to \infty} y_t^{(n_j)}. \tag{12}$$

By Equations (11) and (12),

$$\rho(y_0, \Omega(a)) \geq \epsilon. \tag{13}$$

By (12) and the closedness of the graph of a, we have

$$y_{t+1} \in a(y_t) \text{ for all integers } t. \tag{14}$$

By (10) and (12), for all integers t,

$$\phi(y_{t+1}) - \phi(y_t) = \lim_{j \to \infty} \phi(y_{t+1}^{(n_j)}) - \lim_{j \to \infty} \phi(y_t^{(n_j)}) \geq \lim_{j \to \infty} (-n_j^{-1}) = 0.$$

Combining with (3) this implies that

$$\phi(y_{t+1}) = \phi(y_t) \text{ for all integers } t. \tag{15}$$

Property (1) of Theorem 1, (14), (15) imply the inclusion

$$y_t \in \Omega(a)$$

for every integer t. This inclusion contradicts Equation (13). The contradiction we have reached completes the proof of Lemma 1. □

3. Proof of Theorem 2

Lemma 1 implies that there are a positive number $\delta < \epsilon$ and $L_0 \in \{1, 2, \dots\}$ for which the following property holds:

(a) for every integer $T > 2L_0$ and every trajectory $\{x_t\}_{t=0}^{T}$ satisfying

$$\phi(x_0) \leq \phi(x_T) + \delta$$

we have

$$\rho(x_t, \Omega(a)) \leq \epsilon, \ t = L_0, \dots, T - L_0.$$

Choose an integer

$$L > 2L_0 + 2 + (4L_0 + 7)(1 + 2\delta^{-1} \|\phi\|). \tag{16}$$

Suppose that $T > L$ is a natural number and that a sequence $\{x_t\}_{t=0}^{T}$ is a trajectory. By induction we define a strictly increasing finite sequence $t_i \in \{0, \dots, T\}$, $i = 0, \dots, q$. Set

$$t_0 = 0. \tag{17}$$

If
$$\phi(x_T) \geq \phi(x_0) - \delta,$$
then set
$$t_1 = T$$
and complete to construct the sequence.

Assume that
$$\phi(x_T) < \phi(x_0) - \delta.$$

Evidently, there is an integer $t_1 \in (t_0, T]$ satisfying
$$\phi(x_{t_1}) < \phi(x_0) - \delta \tag{18}$$
and that if an integer S satisfies
$$t_0 < S < t_1,$$
then
$$\phi(x_S) \geq \phi(x_0) - \delta. \tag{19}$$

If $t_1 = T$, then we complete to construct the sequence.

Assume that $k \in \{1, 2, \dots\}$ and that we defined a strictly increasing sequence $t_0, \dots, t_k \in \{0, \dots\}$ such that
$$t_0 = 0, \ t_k \leq T$$
and that for each $i \in \{0, \dots, k-1\}$,
$$\phi(x_{t_{i+1}}) < \phi(x_{t_i}) - \delta$$
and if an integer S satisfies $t_i < S < t_{i+1}$, then
$$\phi(x_S) \geq \phi(x_{t_i}) - \delta.$$

(In view of (18) and (19), the assumption is true with $k = 1$).

If $t_k = T$, then we complete to construct the sequence. Assume that $t_k < T$. If
$$\phi(x_T) \geq \phi(x_{t_k}) - \delta,$$
then we set $t_{k+1} = T$ and complete to construct the sequenced.

Assume that
$$\phi(x_T) < \phi(x_{t_k}) - \delta. \tag{20}$$

Evidently, there is a natural number
$$t_{k+1} \in (t_k, T]$$
for which
$$\phi(x_{t_{k+1}}) < \phi(x_{t_k}) - \delta$$
and that if an integer S satisfies
$$t_k < S < t_{k+1},$$
then
$$\phi(x_S) \geq \phi(x_{t_k}) - \delta.$$

Evidently, the assumption made for k is true for $k + 1$ too. Therefore by induction, we constructed the strictly increasing finite sequence of integers $t_i \in [0, T]$, $i = 0, \dots, q$ such that

$$t_0 = 0,\ t_q = T$$

and that for every i satisfying $0 \le i < q - 1$,

$$\phi(x_{t_{i+1}}) < \phi(x_{t_i}) - \delta \tag{21}$$

and for each $i \in \{0, \ldots, q-1\}$ and each integer S satisfies $t_i < S < t_{i+1}$, we have

$$\phi(x_S) \ge \phi(x_{t_i}) - \delta. \tag{22}$$

By (21),

$$2\|\phi\| \ge \phi(x_{t_0}) - \phi(x_{t_{q-1}})$$
$$\sum\{\phi(x_{t_i}) - \phi(x_{t_{i+1}}) :\ i \text{ is an integer},\ 0 \le i \le q-2\} \ge \delta(q-1)$$

and

$$q \le 1 + 2\delta^{-1}\|\phi\|. \tag{23}$$

Set

$$E = \{i \in \{0, \ldots, q-1\} :\ t_{i+1} - t_i \ge 2L_0 + 4\}. \tag{24}$$

Let

$$i \in E. \tag{25}$$

By (24) and (25),

$$t_{i+1} - 1 - t_i \ge 2L_0 + 3. \tag{26}$$

Equations (22) and (26) imply that

$$\phi(x_{t_{i+1}-1}) \ge \phi(x_{t_i}) - \delta. \tag{27}$$

Equations (26), (27) and property (a) applied to the program $\{x_t\}_{t=t_i}^{t_{i+1}-1}$ imply that

$$\rho(x_t, \Omega(a)) \le \epsilon,\ t = t_i + L_0, \ldots, t_{i+1} - 1 - L_0. \tag{28}$$

Equation (28) implies that

$$\{t \in \{0, \ldots, T\} :\ \rho(x_t, \Omega(a)) > \epsilon\}$$
$$\subset \cup\{\{t_i, \ldots, t_{i+1}\} :\ i \in \{0, \ldots, q-1\} \setminus E\}$$
$$\cup\{\{t_i, \ldots, t_i + L_0 - 1\} \cup \{t_{i+1} - L_0, \ldots, t_{i+1}\} :\ i \in E\}. \tag{29}$$

By (23), (24) and (29),

$$\mathrm{Card}(\{t \in \{0, \ldots, T\} :\ \rho(x_t, \Omega(a)) > \epsilon\})$$
$$\le q(2L_0 + 5) + (2L_0 + 2)q = q(4L_0 + 7)$$
$$(4L_0 + 7)(1 + 2\delta^{-1}\|\phi\|) \le L.$$

Theorem 2 is proved.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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