games

MDPI

Article

Egalitarian Allocations and Convexity

Irinel C. Dragan

Department of Mathematics, University of Texas, Arlington, TX 76019-0408, USA; dragan@uta.edu

Abstract: In the Inverse Set, relative to the Shapley Value of a non-convex cooperative game, we derive a procedure to find out a convex game in which the Egalitarian Allocation is a coalitional rational value. The procedure depends on the relationship between two parameters called the Convexity Threshold and the Coalitional Rationality Threshold. Some examples follow and illustrate the procedure. We discussed a similar problem for other efficient values, the Shapley Value and the Egalitarian Nonseparable Contribution, in earlier work.

Keywords: inverse set; convex game; convexity threshold; coalitional rationality threshold

check for updates

Citation: Dragan, I.C. Egalitarian Allocations and Convexity. *Games* **2022**, *13*, 27. https://doi.org/10.3390/g13020027

Academic Editors: Maria Gallego and Ulrich Berger

Received: 2 December 2021
Accepted: 22 March 2022
Published: 31 March 2022

1. Threshold of Convexity

Let (N, v) be a cooperative transferable utilities game, with a Shapley Value L. The game is a convex game if we have

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \tag{1}$$

for all pairs of coalitions S and T. It is well known that if the game is convex, then the Shapley Value is coalitional rational, that is belongs to the Core of the game. This last fairness property was a good reason for considering in earlier work, the problem: for the given game (N, v), which is not a convex game, find out in the Inverse Set, relative to the Shapley Value, a new game (N, w), that is convex. This game has been found in the so-called Almost Null Family of the Inverse Set, depending on one parameter a, (see [1]). If the parameter is taken in the interval given by the number

$$\kappa = \frac{1}{2(n-1)} \sum_{k \in N} L_k + \frac{1}{2} MIN_{ij}(L_i + L_j), \tag{2}$$

that is $a \in [0, \kappa]$, and the expression of the games in the Almost Null Family from the Inverse Set, given by formula

$$w_a(N - \{j\}) = (n-1)(a - L_j), \forall j \in N, w_a(N) = \sum_{j \in N} L_j, w_a(S) = 0, \forall S : |S| \leq n-2, \tag{3}$$

is used, then we get a convex game. Moreover, in this new game, the Shapley Value is coalitional rational. The number Equation (2) is called the threshold of convexity.

Example 1. *Consider the game*

$$v(1) = v(2) = v(3) = 0, v(1,2) = v(1,3) = 6, v(2,3) = 14, v(1,2,3) = 17. \tag{4}$$

It is easy to see that this is not a convex game, because for some pairs of coalitions we have

$$v(1,2) + v(2,3) = v(1,3) + v(2,3) = 20 \geq v(1,2,3) = 17, \tag{5}$$

that is the inequalities Equation (1) do not hold. The Shapley Value is computed by means of the Shapley formula: $SH(N, v) = (3, 7, 7)$. It is easy to see that the Shapley Value is coalitional rational, as all the inequalities expressing the appurtenance to the Core, that is the following

$$L_1 + L_2 = L_1 + L_3 = 10 \geq v(1,2) = v(1,3) = 6, L_2 + L_3 = 14 \geq v(2,3) = 14, \quad (6)$$

hold. Now, compute the convexity threshold $\kappa = \frac{37}{4}$, from Formula (2), and take the parameter at the maximal value in the interval $[0, \kappa]$, in the Formulas (3), to get the new game

$$w(1) = w(2) = w(3) = 0, w(1,2) = w(1,3) = \frac{9}{2}, w(2,3) = \frac{25}{2}, w(1,2,3) = 17. \quad (7)$$

Now, we may check that Equation (1) holds, so that this game is convex, then we can compute the Shapley Value to see that the value is unchanged, and check the coalitional rationality in this game:

$$L_1 + L_2 = 10 \geq w(1,2) = \frac{9}{2}, L_1 + L_3 = 10 \geq w(1,3) = \frac{9}{2}, L_2 + L_3 = 14 \geq w(2,3) = \frac{25}{2}, \quad (8)$$

and the efficiency. The property of the Shapley Value has been verified. In fact, it was no need to check the coalitional rationality, because we introduced a coalitional rationality threshold, namely

$$\alpha = \frac{1}{n-1} \sum_{i \in N} L_i + \frac{n-2}{n-1} MIN_i L_i, \quad (9)$$

and proved that $\alpha = 10 \geq \kappa = \frac{37}{4}$, so that the convexity implies the coalitional rationality for the new game, (see [1]).

In other earlier works (see [2,3]), we discussed similar problems for two other efficient values: the Egalitarian Allocation and the Egalitarian Nonseparable Contribution. However, in each of the two cases, we were looking for the new game in the same Inverse Set, relative to the Shapley Value, namely in the Almost Null Family of the Inverse Set. Of course, we had to prove that these two values are unchanged for the games in the Inverse Set, relative to the Shapley Value. Then, we had to introduce the coalitional rationality thresholds, for the two new values, given, respectively, by

$$\beta = \frac{1}{n} \sum_{i \in N} L_i + MIN_i L_i = \frac{26}{3}, \gamma = \frac{2}{n} \sum_{i \in N} L_i = \frac{34}{3}, \quad (10)$$

and to see the relationship with the threshold α. We proved the inequalities $\gamma \geq \alpha \geq \beta$, which hold; it follows that for the Egalitarian Nonseparable Contribution, the convexity implies the coalitional rationality, like for the Shapley Value. An open question has been left for further research: is it the same thing true for the Egalitarian Allocation? In other words, we were supposed to find some inequality connecting β and κ. This is the purpose of the present work, in order to close the topic of convexity thresholds. After this introduction, the second section gives the main result, and computational discussions will be shown in the last section.

2. Egalitarian Allocations and Coalitional Rationality

As explained above, for a cooperative TU game (N, v), which is not a convex game, to find a new convex game (N, w), such that $SH(N, w) = SH(N, v) = L \in R^n_+$, and having also the property that the Shapley Value is coalitional rational, we should choose in Equation (3) the parameter $a* \in [0, \kappa]$, where κ is the convexity threshold. From $[0.\kappa] \subseteq [0, \alpha]$, we conclude that the Shapley Value and the Egalitarian Nonseparable Contribution, will be coalitional rational. Now, we can anticipate that the same fact is not true for the Egalitarian Allocations. To show how this can be carried out, we shall prove the following result:

Theorem 1. *In a cooperative TU game which is not convex and has the Shapley Value $L \in R^n_+$, denote the smallest component of the value by L^* and the second smallest component by L^{**}. Now, if we have*

$$L^{**} \leq L^* + \frac{n-2}{n(n-1)} w(N), \tag{11}$$

then, we get $\kappa \leq \beta$, so that the Egalitarian Allocation is also coalitional rational. In case that the opposite inequality holds, the Egalitarian Allocation is coalitional rational only if we take the parameter of the Almost Null Family of games $a^* \in [0, \kappa]$, such that we have $a^* \in [0, \beta]$, and $a^* \notin (\beta, \kappa]$.

Proof. As the new game is taken from the Inverse Set, relative to the Shapley Value, where all three coalitional rationality thresholds for the three values considered above are the same as for the initial game, from Equations (2) and (10), we get:

$$\beta - \kappa = \frac{1}{2}[\frac{n-2}{n(n-1)} w(N) + L^* - L^{**}]. \tag{12}$$

so that if Equation (11) holds, then we have $\beta \geq \kappa$, hence the Egalitarian Allocation will be coalitional rational in the new game. If Equation (11) does not hold, then the Egalitarian Allocation is either coalitional rational, when we take $a^* \in (0, \beta] \subset (0, \kappa]$, in Equation (3), or it is not coalitional rational, when we take $a* \in (0, \kappa]$, but $a^* \notin (0, \beta]$. The theorem is proved, and it will be illustrated by two examples in which will appear the two cases discussed above. □

Example 2. *Return to Example 1, shown in Equation (4), and compute the right hand side in Equation (11), as $L^* = 3$ and $L^{**} = 7$. We have $L^* + \frac{n-2}{n(n-1)} w(N) = \frac{35}{6}$, and this is smaller than $L^{**} = 7$. It follows that we got the coalitional rationality threshold $\beta = \frac{26}{3}$ smaller than the convexity threshold $\kappa = \frac{37}{4}$, so that the Egalitarian Allocation will not be coalitional rational in the case that $a^* \in (\beta, \kappa]$. Indeed, take for the parameter $a^* \in (\beta, \kappa]$, for example the maximal value. Now, in Equation (3) by taking $a^* = \frac{37}{4}$, we obtain a convex game, namely the game Equation (7), for which we may check the coalitional rationality of the Egalitarian Allocation. Among the inequalities required for the value for the appurtenance to the core, the one for the coalition {2,3}, does not hold:*

$$EA_2(N, w) + EA_3(N, w) = \frac{34}{3} \geq w(2, 3) = \frac{25}{2}, \tag{13}$$

while the other two are holding. Hence, the Egalitarian Allocation is not coalitional rational in the new convex game.

Example 3. *Consider a second game:*

$$v(1) = v(2) = v(3) = 0, v(1, 2) = 22, v(1.3) = v(2.3) = 18, v(1.2, 3) = 25. \tag{14}$$

From definition Equation (1), it follows that the game is not convex, as this inequality does not hold for any pair of coalitions of size two. The Shapley Value is $SH(N, v) = (9, 9, 7)$, so that the inequalities required for the coalitional rationality do not hold either, for all coalitions of size two. Compute the right hand side in Equation (11), where $L^* = 7$ and $L^{**} = 9$. Now, the inequality of the theorem

$$L^* + \frac{n-2}{n(n-1)} v(N) = \frac{67}{6} \geq L^{**} = 9, \tag{15}$$

holds, so that in the game obtained from Equation (3) for $a^* = \kappa = \frac{46}{3}$, as

$$w(1) = w(2) = w(3) = 0, w(1, 2) = \frac{29}{2}, w(1, 3) = w(2, 3) = \frac{21}{2}, w(1, 2, 3) = 25, \tag{16}$$

the Egalitarian Allocation $EA(N,w) = (\frac{25}{3}, \frac{25}{3}, \frac{25}{3})$, *will be coalitional rational. It is easy to check that this is true, and this is the result of the fact that the thresholds satisfy the inequality* $\beta \geq \kappa$. *These two examples show the two possible cases, for the three values discussed in this paper. Some computational facts will be discussed in the last section.*

3. Discussion on Computational Issues

Let us summarize the procedure for solving the convexity problem stated in the first section. The steps are the followings:

(1) Check the convexity of the given game, by using Equation (1). If the given game is convex, then allocate the win of the grand coalition by using the Shapley Value, as this value is known to be coalitional rational (see [4]). It is possible to use also the Egalitarian Nonseparable Contribution value, which is also a coalitional rational value (see [2]).

(2) If the game is not convex, compute the Shapley Value, to find out the components L^*, the smallest component, and L^{**}, the second smallest component. Further, check the inequality Equation (11) of the above Theorem. If the inequality holds, this means that $\beta \geq \kappa$, so that it is enough to use Equation (3) for building a new game, which will be convex and can be used for all three values, as it is coalitional rational. Hence, take the parameter $a^* \in [0, \kappa]$, and make the computation. Otherwise, if the inequality Equation (11) does not hold, we should go on:

(3) If we take in Equation (3) $a^* \in (\beta, \kappa]$, then, we get a convex game in which the Egalitarian Allocation is not coalitional rational, as the parameter is greater than the coalitional rational threshold. Hence, we take $a^* \in [0, \beta]$. We conclude that the Egalitarian Allocation has properties different of those met at the Shapley Value, in what concerns the connection the convexity and the coalitional rationality.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The above result is solving the research problem mentioned in [3], so that it is a good idea to start by reading that paper first.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Dragan, I. The Inverse Problem relative to the Shapley Value and the Convexity. *J. Math. Stat. Sci.* **2021**, *7*, 18–24.
2. Dragan, I. On the Coalitional Rationality and the Egalitarian Nonseparable Contribution. *Appl. Math.* **2019**, *10*, 363–370. [CrossRef]
3. Dragan, I. Egalitarian Allocation and the Inverse Problem for the Shapley Value. *Am. J. Oper. Res.* **2018**, *8*, 448–456. [CrossRef]
4. Owen, G. *Game Theory*; Academic Press: Boston, MA, USA, 1995.