Online Appendix to "Stationary Bayesian Markov Equilibria in Bayesian Stochastic Games with Periodic Revelation"

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A Application: A Duopoly in the Pharmaceutical Industry

The patent/innovation races have been studied extensively. A usual assumption of the literature is that once discovery or invention is made, then the market is monopolized, and the race terminates. However, in real life, the other firms do not give up differentiation and innovation even if there is a patent-awarded leader. In addition, consumers' tastes may vary over time, and firms keep improving their products and innovating to defend their customers. Recall that when big screen cell phones such as Galaxy note 2 were launched, they were popular but could not monopolize the smartphone market. Apple developed iPhone 6s to keep their old customers from leaving for "a cell phone with a larger screen". So I use the Bayesian stochastic game with periodic revelation to generalize the structure of an innovation race using the notion of 'loyal consumer'. The pharmaceutical industry exemplifies this model, but it can be applied to any industry where firms are dependent on its loyal consumer base.

The notable literature that used a discrete time stochastic game is Judd, Schmedders, and Yeltekin (2012). They showed an innovation race using a discounted stochastic game in complete information. Heterogeneous firms compete for a patent while they know their competitors' current states. They assumed that the one which obtains the patent monopolizes the market, and the race ends. They investigated optimal patent policy balancing overinvestment and quicker innovation by a discounted stochastic game with finite states. In contrast, this paper deals with a continuum of states. Also, it is assumed that the type of competitor is not observed during the innovation process, and there is no winner who monopolizes the market after the race. Instead, every period, both firms stochastically draw non-empty new consumer bases depending on the results of innovation. Moreover, I focus on the stochastic evolution of market share, while Judd, Schmedders, and Yeltekin (2012) explored the dynamic nature of innovation process.

Suppose there are two pharmaceutical companies 1 and 2 that produce a certain category of medication for a particular use. For example, firm 1 is Bayer, producing Aspirin, and firm 2 is Johnson & Johnson McNeil, producing Tylenol. Hereafter, $i \in \{1,2\}$. Each firm uses special ingredients for its own medication, for example, aspirin and acetaminophen. Each medication is contraindicated in a certain set of patients because of potential side effects. I assume that a potential patient-consumer is represented by a vector of positive real numbers according to their biological characteristics and denote the set of patients for whom the medication i is non-allergic and efficacious as $E_i \subset \mathfrak{R}^n$. Technically, I assume that E_i belongs to σ-algebra of m-measurable set in $\mathfrak{R}^n$ which is the space of patient-consumers (characteristics). I implicitly construct a measure m on the set of patient-consumers and obtain the type space S_i of firm i. Among E_i, there are some patients who cannot use the other medication because of the ingredients. That is, $E_i \backslash E_{-i} \in \mathfrak{R}^n$ is nonempty. I define the measure of this group of patients $s_i = m(E_i \backslash E_{-i})$ as the type of the firm i. For simplicity, assume that $s_i \in [0,1]$. These patients are perfectly inelastic to the price of the their medication such that $\frac{\partial \log s_i}{\partial \log p_i} = 0$. In contrast, the rest of patients in E_i can use either medication such that $E_i \cap E_{-i}$ is nonempty and those patients' price elasticity is nonzero. For simplicity, I normalize the measure of this group of patients as $m(E_1 \cap E_2) = 1$ for every period. This normalization works as an adjustment for market growth rate. And by doing so, I restrict the measure of $E_i \backslash E_{-i}$ not to exceed the measure of $E_1 \cap E_2$.

At each period, firms launch a new product line of their medication. As ingredients of each medication change, I assume that E_1, E_2 both stochastically evolve according to these new medications. As a firm launches its new product, the firm learns that for how many patients, i.e., $m(E_i \backslash E_{-i})$, its medication will exclusively work for. But they do not know what kind of chemical compound the competitor has developed. So, each firm knows its own type s_i but does not know the type of competitor. However, they observe the previous market performances of their own product and that of competitors, so each firm has beliefs about the competitor's type. Given their realized type, firms choose their prices for the their currently available products at $p_1, p_2 \in [c, \overline{P}]$, respectively, where $0 < c < \overline{P} < \infty$. Firms also decide how many researchers they hire for the new development for next period (R & D investment).

I normalize the absolute size of investment such that $h_1, h_2 \in [0,1]$ Here, the action space and the admissible action correspondence for each type are equivalent. Depending on the size of the investment and the previous type, the current type is realized stochastically. The law of motion for the next period type of firm i is given by

$$s_i^+ = (1-\rho_i)\cdot s_i + \rho_i \cdot h_i. \tag{A.1}$$

That is, the current type s_i is depreciated by ρ_i, but it is compensated proportionally to the investment size. The parameter for next period ρ_i will be drawn at the beginning of the next period. It is expected to be $\mathbb{E}\rho_i = \frac{1}{2}$. Therefore $\frac{\partial \mathbb{E}\rho_i}{\partial h_i} = 0$. Then $\frac{\partial \mathbb{E} s_i^+}{\partial h_i} = 1 - \mathbb{E}\rho_i = \frac{1}{2} > 0$. Thus assuming that h_{-i} is given, as the size of the current investment h_i increases, the next period type s_i^+ is more likely to be larger ('monotone likelihood transition'). This means that the larger the investment size h_i, the higher the chances that "today's underdog to be tomorrow's champion (leapfrog)." However, I assume that the depreciation rate ρ_i is a random variable, which follows a uniform distribution $\rho_2^- \sim \mathfrak{u}[0,1]$, and it is unobservable by the competitor. The effectiveness of investment is also random, and for simplicity, it is set equal to the depreciation rate. This random effectiveness rate of investment is assumed reflecting the stochastic process of innovation in real life. In the model, an increase in h_i has a random effect on s_i^+. Even if there were a lot of investments for innovation, the result might be meager so that the consumer base does not increase. In contrast, it can be the case that the result becomes a huge success despite little investment, and the firm takes a big market share at once.

The law of motion is common knowledge, so the belief of firm 1 about the current type of firm 2, $\eta_1(\cdot \mid s^-, (p^-, h^-), s_1)$, is as follows:

$$s_2 = (1-\rho_2^-)\cdot s_2^- + \rho_2^- \cdot h_2^-, \tag{A.2}$$

where $\rho_2^- \sim \mathfrak{u}[0,1]$. The stochastic parameter ρ_2^- is realized at the beginning of the current period, but it is unknown to firm 1 until periodic revelation at the end of current period. The belief of firm 1 about the current type of firm 2 $\eta_1(\cdot \mid s^-, (p^-, h^-), s_1)$ is given by

$$s_2 \sim \mathfrak{u}\left[\min\{s_2^-, h_2^-\}, \quad \max\{s_2^-, h_2^-\}\right]. \tag{A.3}$$

Notice that in any case, if $s_2^- = h_2^-$, beliefs degenerate to $s_2 = s_2^-$. Since I want to illustrate a Bayesian stochastic game, I will focus on non-degenerate cases. The belief of firm 2 about the current type of firm 1 $\eta_2(\cdot \mid s^-, (p^-, h^-), s_2)$ is defined symmetrically.

The period profit function of firm 1 is given by

$$\pi_1(E_1, E_2, p, h) \tag{A.4}$$
$$= [m(E_1 \backslash E_2) + m(E_1 \cap E_2) L_1(p_1(s_1), p_2(s_2))] \cdot (p_1(s_1) - c) - f(h_1(s_1), h_2(s_2)) \tag{A.5}$$

Replacing $m(E_1 \backslash E_2)$ with s_1 and $m(E_1 \cap E_2) = 1$, I have

$$\pi_1(s, p, h) = [s_1 + L_1(p_1(s_1), p_2(s_2))] \cdot (p_1(s_1) - c) - f(h_1(s_1), h_2(s_2)) \tag{A.6}$$

I assume that period payoff function $\pi(\cdot)$ is differentiable in p_1, p_2, h_1, h_2. As I mentioned before, the exclusive users of medication 1, elements of $E_1 \backslash E_2$, are perfectly price-inelastic. I assume that firm 1 knows only the size of $E_1 \backslash E_2$. The firm cannot implement price discrimination against individual consumers. Recall that $m(E_1 \cap E_2)$ is normalized to 1. $L_1(p_1, p_2)$ is the demand function for the product of firm 1 among the patients who can use both medications, and it reflects the price elasticity of these patients. I assume that for patients who can choose from both medications, the demand of a medication is more elastic with respect to its price conditional that its price is higher than the other medication: $\frac{\partial \log L_1(p_1, p_2)}{\partial \log p_1} < -1$ when $p_1 > p_2$, and if $p_1 \leq p_2$, $-1 < \frac{\partial \log L_1(p_1, p_2)}{\partial \log p_1} < 0$. The production cost c is assumed as a constant. $f_1(h_1, h_2)$ is the cost of R & D investment for firm 1. Since R & D investment involves hiring researchers and buying equipment, given the investment of firm 1 h_1, as the competitor invests more, the factor market prices will rise. This pecuniary externality causes the costs of investment $f_1(h_1, h_2)$ for firm 1 to depend on not only the investment level of firm 1 h_1 but also that of firm 2 h_2. Moreover, it is a convex function: $\frac{\partial^2 f_1}{\partial h_1^2} > 0$ and $\frac{\partial^2 f_1}{\partial h_1 \partial h_2} > 0$. All assumptions are symmetrically applied to firm 2.

I refer to this model as an incomplete information version of an innovation race with periodic revelation in the pharmaceutical duopoly. Even though I assume the pharmaceutical industry, this particular model can be applied to any industry involved in continuous innovation that is based on a group of loyal customers. In the innovation race with periodic revelation, there exists a stationary Bayesian-Markov equilibrium.

Proposition 1. *In an incomplete information version of the innovation race with periodic revelation in the pharmaceutical duopoly, there exists a stationary Bayesian-Markov equilibrium.*

Proof. The type space for each player $[0,1]$ is clearly complete separable metric space. Equipped with the uniform probability measure, denoted by ϕ_i, it is a complete measure space. The action space for price $[c, \overline{P}]$ and the action space for investment $[0,1]$ are compact metric space. The action correspondence is equivalent to the action space for each realized type: Nonempty, compact and lower measurable. The period profit function is continuous in price and investment. It is measurable in type. The transition probability is given by the uniform distribution generated from the law of motion $s_i^+ = (1-\rho_i)s_i + \rho_i h_i$ where $\rho_i \sim \mathfrak{u}[0,1]$ for each $i \in \{1,2\}$. Then it is clearly absolutely continuous with respect to ϕ_i and norm-continuous in (p,h). Since the law of motion is common knowledge, the beliefs are given by the corresponding law of motion for the competitor and the information from periodic revelation. Notice that for each $i \in \{1,2\}$, s_{-i} is independent from the realization of s_i. These then satisfy the conditions for the existence of stationary Bayesian-Markov equilibria. □

To be more concrete, I can construct a symmetric equilibrium. Let $i \in \{1,2\}$. Assume that $L_i(p_i, p_{-i})$ satisfies that [1]

$$1 + L_i(c,c)\ (1 + \frac{\partial \log L_i(p_i,c)}{\partial \log p_i})\Big|_{p_i=c} - c \cdot \frac{\partial L_i(p_i,c)}{\partial p_i}\Big|_{p_i=c} \leq 0, \tag{A.7}$$

and that $\frac{\partial f_1(h_1,h_2)}{\partial h_1} = \frac{\partial f_2(h_2,h_1)}{\partial h_2}$. In an incomplete version of an innovation race with periodic revelation in pharmaceutical duopoly, there is a symmetric stationary Bayesian-Markov equilibrium such that $p_i(s^-, h^-, s_i) = \overline{P}$ for $s_i \geq \theta_i(s^-, h^-)$ and $p_i(s^-, h^-, s_i) = c$ for $s_i < \theta_i(s^-, h^-)$. The change in cutoff θ_i has perfect positive correlation with the previous state s_i^- and the previous investment decision h_i^-. Meanwhile, in this equilibrium, each firm increases their investment $h_i(s^-, h^-, s_i)$ as its type s_i rises if their type is less than or equal to their own threshold of investment $s_i \leq \xi_i(s^-, h^-)$. Firm i decreases the investment as its type s_i rises if the type of firm i is greater than the threshold, $s_i > \xi_i(s^-, h^-)$. The threshold $\xi_i(s^-, h^-)$ shows perfect positive correlation with the previous type s_i^- and the previous investment level h_i^-.

[1] If $\left| \frac{\partial \log L_i(p_i,c)}{\partial \log p_i}\Big|_{p_i=c} \right|$ is sufficiently large, this assumption is easily satisfied. For example, the demand function of typical Bertrand price competition model gives us $\left| \frac{\partial \log L_i(p_i,c)}{\partial \log p_i}\Big|_{p_i=c} \right| = \infty$.

B K-period Lagged Revelation

The basic model assume that player i's type remains as private information for one period. As an extension, I investigate the case where the information about player i's types and actions remains private for k periods ($k \geq 2$, hereafter, "blackout period"). Instead, each player receives their realized payoff as a real number at the end of period. Players can form their beliefs by the history of their payoffs and their own types during the blackout period.

This extension of k-period lagged revelation shares the rest of the economic environment with the basic model. As each player's type evolves stochastically according to a first-order Markov process, if the previous type and action profiles were revealed, the common prior for the current type profile would be given by the information from the previous period $(t-1)$. However, the information from the period $(t-1)$ is not accessible until the blackout period passes. So players' beliefs are formed by the information available which includes k-period lagged information and the history of player i's own types and payoffs during the blackout period. Notice that the past actions are not monitored. So, players use information from the history of player i's periodic payoffs, where each periodic payoff is a function of the then-current type profile and the then-current action profile.

The difficulties of the extension stem from the aspects that player i's beliefs about the other players' types are built on the player i's information set of the current period, and that each player can make their own inferences the likely histories of type profiles and action profiles from the history of payoffs. It contrasts to the case of periodic revelation where the only source of differences in beliefs is differences in each player's current type. Given the possibility of multiple stationary Bayesian-Markov equilibria under periodic revelation (the basic model), in the most general case, players will have inconsistent beliefs. This is because players will consider all the possible histories of equilibria that could have been realized during the blackout period to form their beliefs about the other players' current types.

To maintain tractability of the framework, I assume that players' beliefs are consistent in the sense that the only source of differences in beliefs is the difference in each player's history of types during the blackout period. Nature plays a role in keeping the track of the history of type profiles and action profiles $\left\{(s_\tau, a_\tau)\right\}_{\tau=t-k}^{t-1}$ during the blackout period. Nature draws the type profile for the current period t using the information of period $(t-1)$ and also informs a transition function that enables player i to form consistent beliefs conditional on the revealed information $((s_{t-k}, a_{t-k}))$ of period $(t-k)$ and the history of one's own types and received payoffs during the blackout period $\{u(s_\tau, a_\tau),\ s_{i,\tau}\}_{\tau=t-k}^{t-1}$.

B.1 The Primitives

I use subscript $-k$ to denote the relevant timing of the symbol (k is a natural number). Subscript $-k$ indicates that the relevant information is produced k periods prior to the current period. I also use subscript i for an individual player. Hence, $s_{i,-1}$ indicates player i's type in the previous period. As usual, I use superscript for the product, that is, $S^k = \prod_{z=1}^k S$. A discounted Bayesian stochastic game with k-period lagged revelation is a tuple,

$$\Big(\mathcal{I}, \Big(\ (S_i, \mathfrak{S}_i),\ (H_i, \mathcal{H}_i),\ X_i, A_i, u_i, \delta_i, \mu, \eta\ \Big)_{i\in\mathcal{I}}, \xi\Big). \tag{B.8}$$

The notations are the same as in Sections 2 and 2 except for $(H_i, \mathcal{H}_i)$ and ξ:

- For each player $i \in \mathcal{I}$, $(H_i, \mathcal{H}_i)$ is a measurable space of player i's exclusive information except for one's type, i.e., $H_i = \mathfrak{R}^{k-1} \times S_i^{k-1}$ regardless of the calendar time,
- $\xi : S \times X \times \mathfrak{S} \to [0, 1]$ is a transition function.

The roles of $A_i,\ \mu,\ \eta$ are similar as in Sections 2 and 3, but the correspondence and functions include additional arguments related to $h_i \in H_i$:

for each player $i \in \mathcal{I}$,

- $A_i : H_i \times S_i \rightrightarrows X_i$ is the feasible action correspondence,
- $\mu : S \times X \times H_i \times \mathcal{H}_i \times \mathfrak{S}_i \to [0, 1]$ is a transition function for player i's next period type[2],
- $\eta : S \times X \times H_i \times S_i \times \times \mathcal{H}_{-i} \times \mathfrak{S}_{-i} \to [0, 1]$ is a transition function which is used to form player i's beliefs on the other players' history of types and realized payoffs during the blackout period.

On top of the assumptions that are made in Section 2, I add the following assumptions regarding $(H_i, \mathcal{H}_i)$ and $\xi(\cdot)$. Also, I replace assumptions for $\eta(\cdot)$ and $\mu(\cdot)$ as follows.

[2]For the formal definition of a transition function, see Stokey and Lucas with Prescott (1989, p.212)

($\mathcal{B}1$) For each $i \in \mathcal{I}$, H_i is a Borel subset of a complete separable metric space, and $\mathcal{H}_i$ is its Borel σ-algebra. Endowed with the product topology, the Cartesian product H is a Borel subset of a complete separable metric space. A product of σ-algebras $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ is its Borel σ-algebra.

($\mathcal{B}2$) For each $i \in \mathcal{I}$, there is an atomless probability measure ζ_i such that $(H_i, \mathcal{H}_i, \zeta_i)$ is a complete measure space of player i's types. ζ is a product probability measure such that $\zeta = \zeta_1 \times \cdots \times \zeta_n$.

($\mathcal{B}3$) For each i, define $T_i \equiv S \times X \times H_i \times S_i$. A typical element is denoted by $(s_{-k}, a_{-k}, h_i, s_i)$. Notice that T_i is a complete separable metric space. Let $\mathfrak{T}_i$ be its Borel-σ algebra. There is an atomless probability measure λ_i such that $(T_i, \mathfrak{T}_i, \lambda_i)$ is complete measure space. Endowed with the product topology, the Cartesian product T is also a complete separable metric space and a product of σ-algebras $\mathfrak{T}$ is its Borel-σ algebra. λ is a product probability measure such that $\lambda = \lambda_1 \times \cdots \lambda_n$.

($\mathcal{B}4$) For each $i \in \mathcal{I}$, A_i is nonempty, compact valued, and lower measurable.

($\mathcal{B}5$) For each (s_{-1}, a_{-1}), there is common prior $\xi(\cdot \mid s_{-1}, a_{-1}) \in \Delta(S)$ about the current type. For each $Z \in \mathfrak{S}$, $\xi(Z \mid \cdot, \cdot)$ is jointly measurable. $\xi(\cdot \mid s_{-1}, a_{-1})$ is absolutely continuous with respect to the atomless measure ϕ.

($\mathcal{B}6$) For each $(s_{-k}, a_{-k}, h_i, s_i)$ tuple, there are beliefs about of player i's on the other players' current private information, $\eta\left(\cdot \mid s_{-k}, a_{-k},\ h_i, s_i\right) \in \Delta\left(H_{-i} \times S_{-i}\right)$. [3]

[3] First of all, only the realized value of $u_i(s, a)$ is revealed to player i. That is, the realized type profile and action profile (s, a) are not revealed until k periods pass from then. Player i can conceive beliefs using $\eta(\cdot)$ based on player i's own information. Second, since the tuple of (h_i, s_i) remains private whereas the lagged information (s_{-k}, a_{-k}) is known to all the players, the private tuple can be redefined as a new "type" , i.e., $\widehat{s}_i \overset{def}{=} (h_i, s_i)$, in the sense that it gives player i a specific belief over the current type profile. This idea relies on the universality of a belief space (Mertens and Zamir, **?**; Zamir, 2009, p.433). Then, $\eta\left(\cdot \mid s_{-k}, a_{-k}, \widehat{s}_i\right) \in \Delta\left(\widehat{S}_{-i}\right)$ is equivalent to $\eta\left(\cdot \mid s_{-k}, a_{-k}, h_i, s_i\right) \in \Delta\left(H_{-i} \times S_{-i}\right)$. Third, observe that by contrast to the case of one-period lagged revelation in Sections 2 and 3, the beliefs are allowed to be different from the s_i-section of $\xi(\cdot \mid s_{-1}, a_{-1})$. Finally, as the information revelation proceeds, in the next period, each player will be informed about (s_{-k+1}, a_{-k+1}). Beliefs remain stationary, because $\eta(\cdot)$ is established to reflect the stationarity of equilibrium strategy and the ergodicity of the evolution of types.

($\mathcal{B}7$) Given (s_{-k}, a_{-k}) and for each $\widehat{s}_i \stackrel{def}{=} (h_i, s_i)$, the mapping $(s_{-k}, a_{-k}, \widehat{s}_i) \mapsto \eta(\cdot \mid s_{-k}, a_{-k}, \widehat{s}_i)$ is a regular conditional probability on $\widehat{S}_{-i} = H_{-i} \times S_{-i}$. For each subset $Z_{-i} \subset \widehat{S}_{-i}$, $\eta(Z_{-i} \mid \cdot)$ is jointly measurable in $(s_{-k}, a_{-k}, \widehat{s}_i)$. $\eta(\cdot \mid s_{-k}, a_{-k}, \widehat{s}_i)$ is absolutely continuous with respect to the atomless product measure $\prod_{j\in\mathcal{I}\setminus i} \phi_j \times \prod_{j\in\mathcal{I}\setminus i} \zeta_j$.

($\mathcal{B}8$) For each (s_{-k+1}, a_{-k+1}), for each player i, there is another belief $\widehat{\mu}(\cdot \mid s_{-k+1}, a_{-k+1}, h_{i,+1}) \in \Delta(\widehat{S}_i)$ about the evolution of private information of player i for time $(t+1)$. For each $G_i \in \mathcal{H}_i$ and $Z_i \in \mathfrak{S}_i$, $\widehat{\mu}(G_i \times Z_i \cdot \mid s_{-k+1}, a_{-k+1}, h_{i,+1})$ is jointly measurable in $(s_{-k+1}, a_{-k+1}, h_{i,+1})$. $\widehat{\mu}(\cdot \mid s_{-k+1}, a_{-k+1}, h_{i,+1})$ is absolutely continuous with respect to the complete, atomless measure $\phi_i \times \zeta_i$. For ϕ -almost all s, the mapping $a_{-k+1} \mapsto \mu(\cdot \mid s_{-k+1}, a_{-k+1}, h_{i,+1})$ is norm-continuous.[4]

($\mathcal{B}9$) For each player i, I assume that the following probability is well defined: for any $(s_{-k+1}, a_{-k+1}) \in S \times X$, and any $\widehat{Z} \in \widehat{\mathfrak{S}}$,

$$\begin{aligned} & Prob\left(\widehat{Z} \mid s_{-k+1}, a_{-k+1}\right) \\ &= \int_{\widehat{s}_{i,+1}} \int_{\widehat{s}_{-i,+1}} \mathbb{1}_{[(\widehat{s}_{i,+1}, \widehat{s}_{-i,+1}) \in \widehat{Z}]}\, \eta(\, d\widehat{s}_{-i,+1} \mid s_{-k+1}, a_{-k+1}, \widehat{s}_{i,+1}) \\ &\qquad \times \widehat{\mu}\left(d\widehat{s}_{i,+1} \mid s_{-k+1}, a_{-k+1}, h_{i,+1}\right). \end{aligned} \tag{B.9}$$

The above probability measure can be regarded as *common prior conditional on k-periods lagged information*, in the sense that the probability measure induces consistent beliefs.

[4]Observe that $\widehat{s}_{i,+1} = (h_{i,+1}, s_{i,+1})$, hence I can replace $\widehat{s}_{i,+1}$ with $s_{i,+1}$ if $h_{i,+1}$ is conditioned. That is, $\widehat{\mu}(\widehat{s}_{i,+1} \mid s_{-k+1}, a_{-k+1}, h_{i,+1}) = \widehat{\mu}(s_{i,+1} \mid s_{-k+1}, a_{-k+1}, h_{i,+1})$.

B.2 Timing

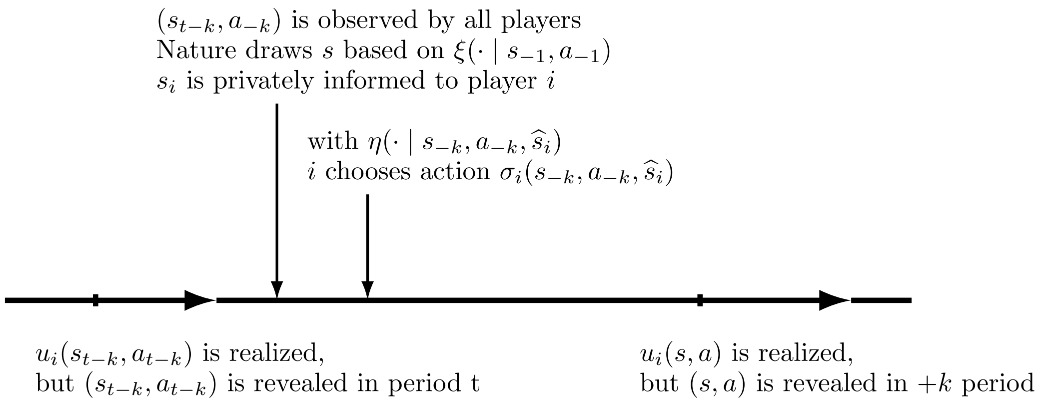

Figure 1: Timeline

1. At the end of the previous period, the realized payoffs $u_i(s_{-1}, a_{-1})$ of the stage game are allocated to player i for all $i \in \mathcal{I}$. However, the realized type profile and action profile (s_{-1}, a_{-1}) remain concealed to players until $(k-1)$ period. Only Nature observes the realized type and action profiles.

2. Before the current period stage game begins, its k-period prior type profile and action profile (s_{-k}, a_{-k}) are revealed.

3. Nature moves to draw each player's type based on the Markov process $\xi(\cdot \mid s_{-1}, a_{-1})$ for all i.

4. For each i, player i whose type is $\widehat{s}_i$ chooses actions based on their beliefs $\eta(\cdot \mid s_{-k}, a_{-k}, \widehat{s}_i)$, with maximizing their discounted sum of expected payoffs.

5. At the end of the current period, the realized payoffs $u_i(s, a)$ of the stage game are allocated to player i for all i. However, the realized type profile and action profile (s, a) remain concealed to players until (k) period. Only Nature observes the realized type and action profiles.

6. Before the next period stage game begins, its k-period prior type profile and action profile (s_{-k+1}, a_{-k+1}) are revealed.

B.3 Stationary Bayesian Markov Equilibrium

A stationary Bayesian-Markov strategy for player i in a k-periodic revelation game is a measurable mapping $\sigma_i : S \times X \times \widehat{S}_i \to \Delta(X_i)$. For each $(s_{-k}, a_{-k}, \widehat{s}_i)$, a probability measure $\sigma_i(s_{-k}, a_{-k}, \widehat{s}_i)$ assigns probability one to $A_i(h_i, s_i) \equiv A_i(\ \widehat{s}_i\ )$. Let Σ_i denote the set of stationary Bayesian-Markov strategies:

$$\Sigma_i = \{\sigma_i \mid \ \sigma_i \in \mathcal{M}(S \times X \times \widehat{S}_i, \Delta(X_i)),\ \sigma_i(s_{-k}, a_{-k}, \widehat{s}_i)(A_i(\ \widehat{s}_i\ )) = 1\} \tag{B.10}$$

For each $s \in S$, let $\sigma(s_{-k}, a_{-k}, \widehat{s})$ denote the product probability measure $\sigma_1(s_{-k}, a_{-k}, \widehat{s}_1) \times \cdots \times \sigma_n(s_{-k}, a_{-k}, \widehat{s}_n)$. I let σ also denote a profile of mappings $(\sigma_1, \cdots, \sigma_n)$ and Σ denote the set of stationary Bayesian-Markov strategy profiles σ.

For each σ, player i's interim expected continuation value function $v_i(\cdot \mid \sigma) : S \times X \times \widehat{S}_i \to \mathfrak{R}$ is a measurable function defined as follows:

for each $(s_{\tau-k}, a_{\tau-k}, \widehat{s}_{i,\tau})$,

$$\begin{aligned}
& v_i(s_{\tau-k}, a_{\tau-k}, \widehat{s}_{i,\tau} \mid \sigma) \qquad \text{(B.11)}\\
&= (1-\delta_i) \sum_{t=\tau}^{\infty} \delta_i^{t-\tau} \int_{\widehat{s}_{-i,t}} \int_{a_t} u_i(s_t, a_t)\ \sigma(s_{t-k}, a_{t-k}, \widehat{s}_{i,t})(da)\ \eta(d\widehat{s}_{-i,t} \mid s_{t-k}, a_{t-k}, \widehat{s}_{i,t})\\
&= (1-\delta_i) \sum_{t=\tau}^{\infty} \delta_i^{t-\tau} \int_{s_{-i,t}} \int_{a_t} u_i(s_t, a_t)\ \sigma(s_{t-k}, a_{t-k}, \widehat{s}_{i,t})(da)\\
&\qquad \times \int_{h_{-i,t}} \eta\left(d\left(s_{-i,t}, h_{-i,t}\right) \mid s_{t-k}, a_{t-k}, \widehat{s}_{i,t}\right)
\end{aligned}$$

The second line is to show that $\eta(\cdot)$ gives more detailed information than it is required to compute the interim expected present discounted payoffs. By recursion, the above payoffs can be exhibited, for each $(s_{\tau-k}, a_{\tau-k}, \widehat{s}_{i,\tau})$,

$$\begin{aligned}
& v_i(s_{\tau-k}, a_{\tau-k}, \widehat{s}_{i,\tau} \mid \sigma) \qquad \text{(B.12)}\\
&= \int_{\widehat{s}_{-i}} \int_a \left[\begin{array}{l} (1-\delta_i) u_i(s,a) \\ +\delta_i \displaystyle\int_{s_{i,\tau+1}} \int_{h_{i,\tau+1}} \left[\begin{array}{l} v_i(s_{\tau-k+1}, a_{\tau-k+1}, \widehat{s}_{i,\tau+1} \mid \sigma) \\ \quad \times\ \widehat{\mu}\ (d\widehat{s}_{i,\tau+1} \mid s_{\tau-k+1}, a_{\tau-k+1}, h_{i,\tau+1}) \end{array} \right] \end{array} \right]\\
&\qquad \times \sigma(s_{\tau-k}, a_{\tau-k}, \widehat{s}_\tau)(da)\ \eta(d\widehat{s}_{-i} \mid s_{\tau-k}, a_{\tau-k}, \widehat{s}_{i,\tau}).
\end{aligned}$$

A profile of stationary Bayesian-Markov strategies in k-periodic revelation σ is a stationary Bayesian-Markov equilibrium in k-periodic revelation if for each $(s_{-k}, a_{-k}, \widehat{s}_i)$, each player i's strategy σ_i maximizes i's interim expected continuation values. That is, given (s_{-k}, a_{-k}), for each $\widehat{s}_i$, $\sigma(s_{-k}, a_{-k}, \widehat{s}_i)$ puts probability one on the set of solution to

$$\max_{a_i \in A_i(\widehat{s}_i)} \int_{s_{-i}} \int_{h_{-i}} \int_a \left[\begin{array}{l} (1-\delta_i) u_i(s,a) \\ +\delta_i \displaystyle\int_{\widehat{s}_{i,\tau+1}} \left[\begin{array}{l} v_i(s_{\tau-k+1}, a_{\tau-k+1}, \widehat{s}_{i,\tau+1} \mid \sigma) \\ \quad \times \widehat{\mu}\ (d\widehat{s}_{i,\tau+1} \mid s_{\tau-k+1}, a_{\tau-k+1}, h_{i,\tau+1}) \end{array} \right] \end{array} \right]$$
$$\times \quad \sigma(s_{\tau-k}, a_{\tau-k}, \widehat{s}_\tau)(da)\ \eta(d(s_{-i}, h_{-i}) \mid s_{\tau-k}, a_{\tau-k}, \widehat{s}_{i,\tau}). \qquad \text{(B.13)}$$

By the one-shot deviation principle, every stationary Bayesian-Markov equilibrium in k-periodic revelation is subgame perfect.

B.4 Existence Theorem in K-Periodic Revelation

Observe that players share common prior and consistent beliefs conditional on (s_{-k}, a_{-k}). I can reformulate the player i's current type by $\widehat{s}_i = (h_i, s_i)$ and index with (s_{-k}, a_{-k}) the set of equilibria, the set of *ex post* payoffs, and the set of interim expected payoffs. Then the same logic of the proof in Section 3 establishes existence of stationary Bayesian-Markov equilibrium in k-periodic revelation.

To proceed, it is required to confirm that $\widehat{S}_i = H_i \times S_i$ is a complete separable metric space. Any n-dimensional Euclidean space is a complete separable metric space. As $H_i \subset \mathfrak{R}^{k-1}$, and S_i is a Borel subset of a complete separable metric space, the product $H_i \times S_i \equiv \widehat{S}_i$ is also a Borel subset of a complete separable metric space.

Corollary (Existence in K-Periodic Revelation)**.** *For every Bayesian stochastic game with k-periodic revelation, there exists a stationary Bayesian-Markov equilibrium.*

Proof of Corollary. Hereafter, I use (h_i, s_i) explicitly, instead of $\widehat{s}_i$ for clearer exposition. For each fixed (s_{-k}, a_{-k}), redefine $\widehat{U}_i^{(s_{-k}, a_{-k})}(\cdot \mid v) : H_i \times S_i \times \Sigma \to \mathfrak{R}$ for each interim expected continuation value function profile v as follows:

$$\widehat{U}_i^{(s_{-k}, a_{-k})}(h_i, s_i, \sigma \mid v) \qquad \text{(B.14)}$$
$$= \int_{h_{-i}} \int_{s_{-i}} \int_a \left[\begin{array}{l} (1-\delta_i) \cdot u_i(s,a) \\ +\delta_i \displaystyle\int_{s_{i,+1}} \int_{h_{i,+1}} \left[\begin{array}{l} v_i(s_{-k+1}, a_{-k+1}, h_{i,+1}, s_{i,+1} \mid \sigma) \\ \quad \times \widehat{\mu}\ (ds_{i,+1} \mid s_{-k+1}, a_{-k+1}, h_{i,+1}) \end{array} \right] \end{array} \right]$$
$$\times \quad \sigma(s_{-k}, a_{-k}, h, s)(da)\ \eta\,(d(h_{-i}, s_{-i}) \mid s_{-k}, a_{-k}, h_i, s_i)$$

Since $\widehat{s}_{i,+1} = (h_{i,+1}, s_{i,+1})$, it holds that $\widehat{\mu}\ (ds_{i,+1} \mid s_{-k+1}, a_{-k+1}, h_{i,+1}) = \widehat{\mu}\ (d\widehat{s}_{i,+1} \mid s_{-k+1}, a_{-k+1}, h_{i,+1})$. Therefore, $\widehat{U}_i^{(s_{-k}, a_{-k})}(h_i, s_i, \sigma \mid v)$ is player i's interim expected continuation value given an interim expected continuation value function profile v for the next period fixed. Then it can be viewed as an interim stage of a static Bayesian game indexed by (s_{-k}, a_{-k}), where player i's type is realized as (h_i, s_i) and behavioral strategy profile of σ is played. As in Section 3, let $\Gamma_v^{(s_{-k},a_{-k})}$ denote the induced static Bayesian game where the type profile is realized as (h, s):

$$\begin{aligned} &\Gamma_v^{(s_{-k},a_{-k})}(h,s) \\ &= \Big(\mathcal{I}, \Big((S_i,\mathfrak{S}_i),(H_i,\mathcal{H}_i),\ X_i, A_i, \widehat{U}_i^{(s_{-k},a_{-k})}(h_i,s_i,\cdot \mid v), \delta_i, \mu, \eta(\cdot \mid s_{-k},a_{-k},h_i,s_i)\Big)_{i\in\mathcal{I}}, \xi\Big). \end{aligned} \tag{B.15}$$

Observe that a behavioral strategy profile σ is unknown in the above game $\Gamma_v^{(s_{-k},a_{-k})}(h,s)$. Now let $B_v^{(s_{-k},a_{-k})}(h,s)$ denote the set of mixed action profiles induced by Bayesian Nash equilibria of $\Gamma_v^{(s_{-k},a_{-k})}$. That is,

$$\begin{aligned} &B_v^{(s_{-k},a_{-k})}(h,s) \\ &= \left\{ \begin{array}{l|l} & \sigma\in\Sigma;\ \text{for each } j, \\ & \text{and for each } i\in\mathcal{I}\backslash\{j\}, \\ \sigma(s_{-k},a_{-k},h,s) & \ \ \sigma_j \text{ satisfies} \\ \in \Delta\Big(A_1(h_1,s_1)\Big)\times\cdots\times\Delta\Big(A_n(h_n,s_n)\Big) & \ \ \widehat{U}_i^{(s_{-k},a_{-k})}(h_i,s_i,\sigma \mid v) \\ & = \max\limits_{\forall a_i\in A_i(s_i)} \widehat{U}_i^{(s_{-k},a_{-k})}(h_i,s_i,a_i,\sigma_{-i} \mid v) \end{array} \right\}. \end{aligned} \tag{B.16}$$

Then, similarly to Lemma 7 in Section 3, for each v, $(s_{-k}, a_{-k}, h, s) \mapsto B_v^{(s_{-k},a_{-k})}(h,s)$ is nonempty, compact valued, and lower measurable. Define the set of *ex post* payoffs for player i from $B_v^{(s_{-k},a_{-k})}(h,s)$ as $P_{v,i}^{(s_{-k},a_{-k})}(h,s)$. Then, $(s_{-k}, a_{-k}, h, s) \mapsto P_{v,i}^{(s_{-k},a_{-k})}(h,s)$ is nonempty, compact valued and lower measurable. The set of interim expected payoffs for player i is denoted by $E_{v,i}^{(s_{-k},a_{-k})}(h_i,s_i)$, where

$$E_{v,i}^{(s_{-k},a_{-k})}(h_i,s_i) \equiv \int_{s_{-i}}\int_{h_{-i}} P_{v,i}^{(s_{-k},a_{-k})}(h,s)\ \eta\Big(\ d(h_{-i},s_{-i}) \mid s_{-k},a_{-k},h_i,s_i\Big).$$

Let $E_v^{(s_{-k},a_{-k})}(h,s)$ denote the Cartesian product $E_{v,1}^{(s_{-k},a_{-k})}(h_1,s_1)\times\cdots\times E_{v,n}^{(s_{-k},a_{-k})}(h_n,s_n)$. Then similarly to Lemma 8, for each v and each (s_{-k},a_{-k},h,s), $E_v^{(s_{-k},a_{-k})}(h,s)$ is convex, i.e., $E_v^{(s_{-k},a_{-k})}(h,s) = coE_v^{(s_{-k},a_{-k})}(h,s)$. Similarly to Lemma 9, for each v, $(s_{-k},a_{-k},h,s) \mapsto E_v^{(s_{-k},a_{-k})}(h,s)$ is lower measurable, nonempty, compact and convex valued. Fix v and let M_v denote the set of all $\lambda-$equivalence classes of measurable selectors of $(s_{-k},a_{-k},h,s) \mapsto$

$E_v^{(s_{-k},a_{-k})}(h,s)$. Similarly to Lemma 10, the mapping $v \mapsto M_v$ is nonempty, closed-graph and convex-valued.

Let V be constructed similarly to Section 3. That is, V_i is set of interim expected continuation value functions v_i where $\mid v_i(s_{-k}, a_{-k}, h_i, s_i) \mid \leq C_i$ for the real number $C_i \in \Re$ that is defined in ($\mathcal{A}6$). The Cartesian product of $\prod_{i \in \mathcal{I}} V_i$ is denoted by V. Obviously $M_v \subset V$. By Kakutani-Fan-Glicksberg theorem (see Theorem 17.55, AB, p.583), I have a fixed point of $v \mapsto M_v$. From this step, I repeat the same process to extract a measurable mapping $f(\cdot)$ such that given (s_{-k}, a_{-k}), for each (h, s), $\quad f(s_{-k}, a_{-k}, h, s) \in B_v^{(s_{-k},a_{-k})}(h,s)$ and for all i, almost all s_i, the following holds:

$$
\begin{aligned}
& w_i(s_{-k}, a_{-k}, h_i, s_i) \\
= & \int_{h_{-i}} \int_{s_{-i}} \left[\begin{array}{l} (1-\delta_i) \cdot u_i(s, f(s_{-k}, a_{-k}, h, s)) \\ +\delta_i \int_{s_{i,+1}} w_i\big(s_{-k+1}, a_{-k+1}, h_{i,+1}, s_{i,+1}\big) \ \mu\Big(ds_{i,+1} \mid s_{-k+1}, f(s_{-k}, a_{-k}, h, s), h_{i,+1} \Big) \end{array} \right] \\
& \quad \times \quad \eta\left(d(h_{-i}, s_{-i}) \mid s_{-k}, a_{-k}, h_i, s_i\right).
\end{aligned} \tag{B.17}
$$

Such $f(\cdot)$ can be obtained using Filippov's implicit function theorem. Then I have a stationary Bayesian-Markov equilibrium strategy profile $f(\cdot)$ in a Bayesian Stochastic game with $k-$periodic revelation. □

References

Judd, Kenneth L, Karl Schmedders and Sevin Yeltekin. 2012. "Optimal Rules for Patent Races." *International Economic Review* 53(1):23–52.

Stokey, Nancy L. and Robert E. Lucas Jr with Edward C. Prescott. 1989. *Recursive Methods in Economic Dynamics*. Harvard University Press.

Zamir, Shmuel. 2009. Bayesian Games: Games with Incomplete Information. In *Encyclopedia of Complexity and Systems Science*. Springer pp. 426–441.